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Co-tabulations, Bicolimits and Van-Kampen Squares in Collagories

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Abstract: We previously defined collagories essentially as "distributive allegories without zero morphisms". Collagories are sufficient for accommodating the relation-algebraic approach to graph transformation, and closely correspond to the adhesive categories important for the categorical DPO approach to graph transformation.
Heindel and Sobociński have recently characterised the Van-Kampen colimits used in adhesive categories as bicolimits in span categories.
In this paper, we study both bicolimits and lax colimits in collagories. We show that the relation-algebraic co-tabulation concept is equivalent to lax colimits of difunctional morphisms and to bipushouts, but much more concise and accessible. From this, we also obtain an interesting characterisation of Van-Kampen squares in collagories.
Keywords: Relation-algebraic graph transformation, Collagories, Allegories, Pushout, Adhesive categories

1 Introduction

One of the hallmarks of the relation-algebraic approach to graph transformation [Kaw90, Kah01, Kah04] is that it allows an abstract characterisation of the gluing condition for the double pushout approach. Nevertheless, the categorical approach to graph transformation has continued to use the node-and-edge-based formulation of the gluing condition even in the handbook chapter [CMR+97]. Recently, the literature of the categorical approach, starting essentially with [EPPH06] has adopted the "adhesive categories" of Lack and Sobociński [LS04], where however the details of the gluing condition are completely sidestepped.

In [Kah09a], we introduced *collagories* essentially as "distributive allegories without zero morphisms". We redeveloped in collagories the fundamentals of the relation-algebraic approach to graph transformation, and showed that adhesive categories arise, and also that bitabular collagories share the most important construction principles, such as slice and co-slice category constructions, with adhesive categories.

Inspired by Heindel and Sobociński's characterisation of van Kampen squares as bicolimits in the bicategory of spans [HS09], we establish in this paper (Sect. 6) the connections between our co-tabulations and bicolimits in collagories, succeding to show that the co-tabulation characterisation of pushouts, which essentially goes back to Kawahara [Kaw90], has a *precise* categorical counterpart in bipushouts, and, even more closely, in lax colimits of difunctional morphisms in a collagory context.

We also succeed in providing, in Sect. 7, an original collagory-theoretic characterisation of van Kampen squares, significantly advancing over the results of [Kah09a, Kah09b].

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2 Categories, Allegories

This section only serves to fix notation and terminology for standard concepts, see [FS90, SS93, Kah04]. Like Freyd and Scedrov and a slowly increasing number of categorists, we denote composition in "diagram order" not only in relation-algebraic contexts, where this is customary, but also in the context of categories. We will always use the infix operator ";" to make composition explicit: $R;S = \mathscr{A} \xrightarrow{R} \mathscr{B} \xrightarrow{S} \mathscr{C}$.

Definition 2.1. A *category* $\mathbf{C}$ is a tuple $(\mathsf{Obj}_\mathbf{C}, \mathsf{Mor}_\mathbf{C}, \mathsf{src}, \mathsf{trg}, \mathbb{I}, ;)$ where

- $\mathsf{Obj}_\mathbf{C}$ is a collection of *objects*.
- $\mathsf{Mor}_\mathbf{C}$ is a collection of *arrows* or *morphisms*.
- src (resp. trg) maps each morphism to its source (resp. target) object.
  Instead of $\mathsf{src}(f) = \mathscr{A} \wedge \mathsf{trg}(f) = \mathscr{B}$ we write $f : \mathscr{A} \to \mathscr{B}$.
  The collection of all morphisms f with $f : \mathscr{A} \to \mathscr{B}$ is denoted as $\mathsf{Mor}_\mathbf{C}[\mathscr{A}, \mathscr{B}]$ and also called a *homset*.
- ";" is the binary *composition* operator, and composition of two morphisms $f : \mathscr{A} \to \mathscr{B}$ and $g : \mathscr{B}' \to \mathscr{C}$ is defined iff $\mathscr{B} = \mathscr{B}'$, and then $(f;g) : \mathscr{A} \to \mathscr{C}$; composition is associative.
- $\mathbb{I}$ associates with every object $\mathscr{A}$ a morphism $\mathbb{I}_\mathscr{A}$ which is both a right and left unit for composition. □

Definition 2.2. An *ordered category* is a category $\mathbf{C}$ such that

- for each two objects $\mathscr{A}$ and $\mathscr{B}$, the relation $\sqsubseteq_{\mathscr{A},\mathscr{B}}$ is a partial order on $\mathsf{Mor}_\mathbf{C}[\mathscr{A}, \mathscr{B}]$ (the indices will usually be omitted), and
- composition is monotonic with respect to $\sqsubseteq$ in both arguments. □

For homsets that have least or greatest elements, we introduce corresponding notation:

Definition 2.3. In an ordered category, for each two objects $\mathscr{A}$ and $\mathscr{B}$ we introduce the following notions:

- If the homset $\mathsf{Mor}_\mathbf{C}[\mathscr{A}, \mathscr{B}]$ contains a greatest element, this is denoted $\mathbb{T}_{\mathscr{A},\mathscr{B}}$.
- If the homset $\mathsf{Mor}_\mathbf{C}[\mathscr{A}, \mathscr{B}]$ contains a least element, this is denoted $\mathbb{L}_{\mathscr{A},\mathscr{B}}$. □

For these extremal morphisms and for identities we frequently omit indices where these can be induced from the context.

Definition 2.4. An *ordered category with converse*, or *OCC*, is an ordered category such that

- each morphism $R : \mathscr{A} \to \mathscr{B}$ has a *converse* $R^\smile : \mathscr{B} \to \mathscr{A}$,
- the *involution equations* hold for all $R : \mathscr{A} \to \mathscr{B}$ and $S : \mathscr{B} \to \mathscr{C}$:
$$(R^\smile)^\smile = R \qquad \mathbb{I}_\mathscr{A}^\smile = \mathbb{I}_\mathscr{A} \qquad (R;S)^\smile = S^\smile;R^\smile$$
- conversion is monotonic with respect to $\sqsubseteq$. □

Many standard properties of relations can be characterised in the context of OCCs [Kah04]:

Definition 2.5. A morphism $R : \mathscr{A} \to \mathscr{B}$ in an OCC is called:

- *univalent* iff $R^\smile;R \sqsubseteq \mathbb{I}_\mathscr{B}$,
- *total* iff $\mathbb{I}_\mathscr{A} \sqsubseteq R;R^\smile$,
- *injective* iff $R;R^\smile \sqsubseteq \mathbb{I}_\mathscr{A}$,
- *surjective* iff $\mathbb{I}_\mathscr{B} \sqsubseteq R^\smile;R$,
- a *mapping* iff it is univalent and total,

- *bijective* iff it is injective and surjective,
- *difunctional* iff $R ; R^{\smile} ; R \sqsubseteq R$. $\square$

For an OCC $\mathbf{C}$, we write $\mathsf{Map}\,\mathbf{C}$ for the sub-category of $\mathbf{C}$ that contains only the mappings as arrows.

Difunctionality will play an important rôle in this paper; a concrete relation, understood as a Boolean matrix, is difunctional iff it can be rearranged into "loose block-diagonal form", with full rectangular blocks such that there is no overlap between different blocks in either direction. (See [SS93, 4.4] for more about difunctionality).

For endomorphisms, there are a few additional properties of interest:

Definition 2.6. A morphism $R : \mathscr{A} \to \mathscr{A}$ in an OCC is called:

- *reflexive* iff $\mathbb{I} \sqsubseteq R$,
- *transitive* iff $R ; R \sqsubseteq R$, and *idempotent* iff $R ; R = R$,
- *co-reflexive* or a *sub-identity* iff $R \sqsubseteq \mathbb{I}_{\mathscr{A}}$,
- *symmetric* iff $R^{\smile} \sqsubseteq R$,
- an *equivalence* iff it is symmetric, reflexive and transitive. $\square$

Lemma 2.7. If $\mathscr{B} \xleftarrow{P} \mathscr{A} \xrightarrow{Q} \mathscr{C}$ is a span and $P^{\smile} ; Q$ is difunctional, then $P ; P^{\smile} ; Q ; Q^{\smile}$ is idempotent. If P and Q are moreover total, then $P ; P^{\smile} ; Q ; Q^{\smile}$ is an equivalence.

PROOF: The first claim is immediate: $P ; P^{\smile} ; Q ; Q^{\smile} ; P ; P^{\smile} ; Q ; Q^{\smile} = P ; P^{\smile} ; Q ; Q^{\smile}$.

For the second claim, reflexivity is obvious from totality, and the first claim implies transitivity, and, together with totality, also symmetry:

$$Q ; Q^{\smile} ; P ; P^{\smile} = \mathbb{I}_{\mathscr{A}} ; Q ; Q^{\smile} ; P ; P^{\smile} ; \mathbb{I}_{\mathscr{A}} \sqsubseteq P ; P^{\smile} ; Q ; Q^{\smile} ; P ; P^{\smile} ; Q ; Q^{\smile} = P ; P^{\smile} ; Q ; Q^{\smile} \qquad \square$$

While Freyd and Scedrov [FS90] derive the homset ordering in their allegories from the meet operation, we define allegories on top of ordered categories — the composition operator has higher precedence than all other binary operators.

Definition 2.8. An *allegory* is an OCC such that

- each homset is a lower semilattice with binary meet $\sqcap$.
- for all $Q : \mathscr{A} \to \mathscr{B}$, $R : \mathscr{B} \to \mathscr{C}$, and $S : \mathscr{A} \to \mathscr{C}$, the *modal rule* holds:

$$Q ; R \sqcap S \sqsubseteq (Q \sqcap S ; R^{\smile}) ; R \ . \qquad \square$$

The most well-known allegory is the category *Rel* of sets with relations and standard relational operations. Logical theories give rise to allegories of *derived predicates* [FS90, App. B]. A simpler case of that are the allegories arising from Σ-algebras (over some signature Σ) as objects, and with "relational Σ-homomorphisms", i.e. bisimulations in the sense of [Kah04], as morphisms.

In allegories, one can define domain and range operators:

Definition 2.9. For every morphism $R : \mathscr{A} \leftrightarrow \mathscr{B}$ in an allegory, we define $\mathsf{dom}\,R : \mathscr{A} \leftrightarrow \mathscr{A}$ and $\mathsf{ran}\,R : \mathscr{B} \leftrightarrow \mathscr{B}$ as:

$$\mathsf{dom}\,R := \mathbb{I}_{\mathscr{A}} \sqcap R ; R^{\smile} \qquad\qquad \mathsf{ran}\,R := \mathbb{I}_{\mathscr{B}} \sqcap R^{\smile} ; R \qquad \square$$

3 Collagories

$\kappa\acute{o}\lambda\lambda\alpha$: glue

In Freyd and Scedrov's treatment, although allegories are not required to have zero-ary meets, distributive allegories are required to have zero-ary joins (least elements) together with distributivity of composition over them, that is, the zero law $\perp\!\!\!\perp ; R = \perp\!\!\!\perp$. In [Kah09a], we introduced an intermediate concept that does not assume anything about zero-ary joins:

Definition 3.1. A *collagory* is an allegory where each homset is a distributive lattice with binary join $\sqcup$, and composition distributes over binary joins from both sides. □

We directly axiomatise difunctional closure, without introducing Kleene star:

Definition 3.2. A *difunctionally closed collagory* is a collagory where, there is an additional unary operation $\_^{\boxast}$ which satisfies the following axioms for all $R : \mathscr{A} \to \mathscr{B}$, $Q : \mathscr{C} \to \mathscr{A}$, and $S : \mathscr{B} \to \mathscr{C}$: $Q' : \mathscr{C} \to \mathscr{B}$, and $S' : \mathscr{A} \to \mathscr{C}$:

$$\begin{array}{ccccccccccl} & & & R^{\boxast} & = & R \sqcup R^{\boxast} ; (R^{\boxast})^{\smile} ; R^{\boxast} & & & & & \text{recursive definition} \\ Q ; R & \sqsubseteq & Q' & \wedge & Q' ; R^{\smile} ; R & \sqsubseteq & Q' & \Rightarrow & Q ; R^{\boxast} & \sqsubseteq \; Q' & \text{right induction} \\ R ; S & \sqsubseteq & S' & \wedge & R ; R^{\smile} ; S' & \sqsubseteq & S' & \Rightarrow & R^{\boxast} ; S & \sqsubseteq \; S' & \text{left induction} \end{array}$$

We further define $R^{\rhd} : \mathscr{A} \to \mathscr{A}$ and $R^{\lhd} : \mathscr{B} \to \mathscr{B}$ as:

$$R^{\rhd} := \mathbb{I} \sqcup R^{\boxast} ; (R^{\boxast})^{\smile} \qquad \text{and} \qquad R^{\lhd} := \mathbb{I} \sqcup (R^{\boxast})^{\smile} ; R^{\boxast} \ .$$ □

In a difunctionally closed collagory, the operation $\_^{\boxast}$ produces difunctional closures [Kah09a].

Requiring least morphisms satisfying zero laws turns collagories into distributive allegories, which still heave a much weaker theory than relations in a topos, so graph structures (unary algebras) with relational graph homomorphism in particular also form collagories.

In [Kah09a], we showed that the absence of the zero laws enables the presence of constant symbols (allowing for example pointed sets), and also that restrictions to sub-collagories in signature reducts (for example fixing label sets) and nested algebra constructions (interpreting signatures in the mapping categories of arbitrary collagories instead of just in Map *Rel*) both construct new collagories. These constructions are directly useful for concrete modelling tasks, and for implementation of the resulting models as data structures; they also subsume the construction methods presented by Lack and Sobociński [LS04] for adhesive categories, in particular comprising clice and co-slice category construction.

4 Tabulations and Co-tabulations

Central to the connection between pullbacks and pushouts in categories of mappings on the one hand and constructions in relational theories on the other hand is the fact that a square of mappings commutes iff the "relation" induced by the source span is contained in that induced by the target co-span. The proof of this does not need the modal rule.

$$\begin{tikzcd} & \mathscr{A} \arrow[dl, "P"'] \arrow[dr, "Q"] & \\ \mathscr{B} \arrow[dr, "R"'] & & \mathscr{C} \arrow[dl, "S"] \\ & \mathscr{D} & \end{tikzcd}$$

Lemma 4.1. [FS90, 2.146] Given a square of mappings in an allegory as drawn above, we have $P;R = Q;S$ iff $P^{\smile};Q \sqsubseteq R;S^{\smile}$. □

This provides a first hint that in the relational setting, the identity of the two mappings P and Q does not matter when looking for a pushout of the span $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ — we only need to consider the diagonal $P^{\smile};Q$. Dually, when looking for a pullback of the co-span $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$, only $R;S^{\smile}$ needs to be considered. The gap between the two ways of calculating the horizontal diagonal can be significant since $R;S^{\smile}$ is always difunctional. In fact, Lemma 4.1 can be strenghtened:

Lemma 4.2. Given a square of mappings in an allegory as drawn above, and existence of the difunctional closure of $P^{\smile};Q$, we have $P;R = Q;S$ iff $(P^{\smile};Q)^{\boxast} \sqsubseteq R;S^{\smile}$.

PROOF: The “if” direction follows immediately from $P^{\smile};Q \sqsubseteq (P^{\smile};Q)^{\boxast}$ and the “if” direction of Lemma 4.1.

For “only if”, assume $P;R = Q;S$. Then $P^{\smile};Q \sqsubseteq R;S^{\smile}$ by Lemma 4.1, and

$$\begin{array}{rcll} R;S^{\smile};Q^{\smile};P;P^{\smile};Q & = & R;R^{\smile};P^{\smile};P;P^{\smile};Q & \text{commutativity} \\ & = & R;R^{\smile};P^{\smile};Q & P \text{ unival.} \\ & = & R;S^{\smile};Q^{\smile};Q & \text{commutativity} \\ & \sqsubseteq & R;S^{\smile} & Q \text{ unival.} \end{array}$$

By left-induction for difunctional closure we therefore have $(P^{\smile};Q)^{\boxast} \sqsubseteq R;S^{\smile}$. □

Producing the result span of a pullback (respectively the result co-span of a pushout) from the horizontal diagonal alone is, in some sense, a generalisation of Freyd and Scedrov’s splitting of idempotents; [Kah04] contains more discussion of this aspect.

Definition 4.3. [FS90, 2.14] In an allegory, let a morphism $V : \mathcal{B} \to \mathcal{C}$ be given. The span $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ of *mappings* P and Q is called a *tabulation of* V iff the following equations hold:

$$P^{\smile};Q = V \qquad P;P^{\smile} \sqcap Q;Q^{\smile} = \mathbb{I}_{\mathcal{A}} \ . \qquad \square$$

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "P"'] \arrow[dr, "Q"] & \\ \mathcal{B} \arrow[rr, "V"] & & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B} \arrow[rr, "W"] \arrow[dr, "R"'] & & \mathcal{C} \arrow[dl, "S"] \\ & \mathcal{D} & \end{tikzcd}$$

Definition 4.4. [Kah04] In a collagory, let a morphism $W : \mathcal{B} \to \mathcal{C}$ be given. The co-span $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ of *mappings* R and S is called a *co-tabulation of* W iff the following equations hold:

$$R;S^{\smile} = W \qquad R^{\smile};R \sqcup S^{\smile};S = \mathbb{I}_{\mathcal{D}} \ . \qquad \square$$

The first equation implies $W;W^{\smile};W = R;S^{\smile};S;R^{\smile};R;S^{\smile} \sqsubseteq R;S^{\smile} = W$ (using univalence of R and S), so if W has a co-tabulation, it has to be difunctional.

Furthermore, from univalence of R and S we also obtain the lax cocone conditions $R^{\smile};W = R^{\smile};R;S^{\smile} \sqsubseteq S^{\smile}$ and $W;S = R;S^{\smile};S \sqsubseteq R$.

The following equivalent characterisations provided by [Kah04] have the advantage that they are fully equational, without the implicit inclusions in the mapping conditions. This frequently facilitates calculations. Note that $\mathbb{I} \sqcap V ; V^{\smile} = \mathsf{dom}\, V$; we use the expanded form to emphasise the duality.

Proposition 4.5. In an allegory, the span $\mathscr{B} \xleftarrow{P} \mathscr{A} \xrightarrow{Q} \mathscr{C}$ is a tabulation of $V : \mathscr{B} \to \mathscr{C}$ if and only if the following equations hold:

$$P^{\smile} ; Q = V \qquad \begin{array}{rcl} P^{\smile} ; P & = & \mathbb{I} \sqcap V ; V^{\smile} \\ Q^{\smile} ; Q & = & \mathbb{I} \sqcap V^{\smile} ; V \end{array} \qquad P ; P^{\smile} \sqcap Q ; Q^{\smile} = \mathbb{I}_{\mathscr{A}} \ . \qquad \square$$

Proposition 4.6. In a collagory, the co-span $\mathscr{B} \xrightarrow{R} \mathscr{D} \xleftarrow{S} \mathscr{C}$ is a co-tabulation of $W : \mathscr{B} \to \mathscr{C}$ iff the following equations hold:

$$R ; S^{\smile} = W \qquad \begin{array}{rcl} R ; R^{\smile} & = & \mathbb{I} \sqcup W ; W^{\smile} \\ S ; S^{\smile} & = & \mathbb{I} \sqcup W^{\smile} ; W \end{array} \qquad R^{\smile} ; R \sqcup S^{\smile} ; S = \mathbb{I}_{\mathscr{D}} \ . \qquad \square$$

Definition 4.7. If an allegory has a tabulation for each morphism, we call it *tabular*.

If a collagory has a co-tabulation for each morphism, we call it *co-tabular*, and if it is furthermore tabular, we call it *bi-tabular*. □

Tabulations in an allegory are unique up to isomorphism (this uses the modal rule), and include the following special cases:

- In a tabulation of a sub-identity, both tabulation morphisms are the induced *sub-object* injection [FS90, 2.145].
- We can define a *direct product* of $\mathscr{A}$ and $\mathscr{B}$ to be a tabulation of a $\mathbb{T}_{\mathscr{A},\mathscr{B}}$, provided that greatest morphism exists. The resulting direct product definition differs from that of [SS93] in extending naturally to "empty" objects (e.g., empty sets) by not demanding surjectivity of the projections, but only $\pi^{\smile} ; \pi = \mathsf{dom}\, \mathbb{T}_{\mathscr{A},\mathscr{B}}$ and $\rho^{\smile} ; \rho = \mathsf{ran}\, \mathbb{T}_{\mathscr{A},\mathscr{B}}$.
- If a co-span $\mathscr{B} \xrightarrow{R} \mathscr{D} \xleftarrow{S} \mathscr{C}$ of mappings is given, then each tabulation of $R ; S^{\smile}$ (there might be none) is a *pullback* in $\mathsf{Map}\, \mathbf{A}$ [FS90, 2.147].
  For a tabular allegory $\mathbf{A}$, this implies that each pullback in $\mathsf{Map}\, \mathbf{A}$ is isomorphic to a tabulation, and therefore is itself a tabulation. However, if $\mathbf{A}$ is not tabular, then a co-span $\mathscr{B} \xrightarrow{R} \mathscr{D} \xleftarrow{S} \mathscr{C}$ of mappings for which no tabulation of $R ; S^{\smile}$ exists may still have a pullback in $\mathsf{Map}\, \mathbf{A}$, which then cannot be a tabulation.

If an allegory is known to have all direct products and subobjects, then these can be used to construct a tabulation for each morphism.

In a collagory, we have the following special cases of co-tabulations, dual to the special tabulations above:

- In a co-tabulation of an equivalence relation, both R and S are the induced *quotient* projections.
- We can define a *direct sum* of $\mathscr{A}$ and $\mathscr{B}$ to be a co-tabulation of $\perp_{\mathscr{A},\mathscr{B}}$, if that least morphism exists.
- If a span $\mathscr{B} \xleftarrow{P} \mathscr{A} \xrightarrow{Q} \mathscr{C}$ of mappings is given, and the difunctional closure $W := (P^{\smile} ; Q)^{\boxtimes}$ exists then each co-tabulation of W (there might be none) is a *pushout* in $\mathsf{Map}\, \mathbf{A}$ [Kah09a].
  The situation is, except for the addition of the difunctional closure, perfectly dual to the situation for pullbacks described above: For a co-tabular collagory $\mathbf{C}$, each pushout in $\mathsf{Map}\, \mathbf{C}$ is isomorphic to a co-tabulation, and therefore is itself a co-tabulation. However, if $\mathbf{C}$ is not

co-tabular, then a span $\mathscr{B} \xleftarrow{P} \mathscr{A} \xrightarrow{Q} \mathscr{C}$ of mappings for which no co-tabulation of $(P^{\smile};Q)^{\boxast}$ exists may still have a pushout in $\mathsf{Map}\,\mathbf{C}$, which then cannot be a co-tabulation.

If direct sums and quotients are available, then a co-tabulation can be constructed for each difunctional morphism.

A co-tabulation for a difunctional closure $Z^{\boxast}$ satisfies the following equations:

$$R;S^{\smile} = Z^{\boxast} \qquad R;R^{\smile} = Z^{\rhd} \qquad S;S^{\smile} = Z^{\lhd} \qquad R^{\smile};R \sqcup S^{\smile};S = \mathbb{I}_{\mathscr{D}} \ .$$

This was introduced as a *gluing for* the morphism Z in [Kah01]. Kawahara is the first to have characterised pushouts relation-algebraically in essentially this way [Kaw90]; he used relation-algebraic operations on relations arising in toposes.

Convention 4.8. For a square of morphisms as drawn at the beginning of this section, we say that

- it *is a tabulation* iff $\mathscr{B} \xleftarrow{P} \mathscr{A} \xrightarrow{Q} \mathscr{C}$ is a tabulation for $R;S^{\smile}$,
- it *is a (direct) co-tabulation* iff $\mathscr{B} \xrightarrow{R} \mathscr{D} \xleftarrow{S} \mathscr{C}$ is a co-tabulation for $P^{\smile};Q$,
- it *is a gluing* iff $\mathscr{B} \xrightarrow{R} \mathscr{D} \xleftarrow{S} \mathscr{C}$ is a gluing for $P^{\smile};Q$, that is, if it is a co-tabulation for $(P^{\smile};Q)^{\boxast}$. □

5 The Gluing Condition in Collagories

We can now state a relational variant of the gluing condition, first introduced by Kawahara [Kaw90]:

Definition 5.1. Let two morphisms[1] $\Phi : \mathscr{G} \to \mathscr{L}$ and $X : \mathscr{L} \to \mathscr{A}$ in a collagory with pseudo-complements on subidentities be given.[2]

- We say that the *identification condition* holds iff $X;X^{\smile} \sqsubseteq \mathbb{I} \sqcup (\mathsf{ran}\,\Phi);X;X^{\smile};\mathsf{ran}\,\Phi$.
- We say that the *dangling condition* holds iff $\mathsf{ran}\,X \sqcup (\mathsf{ran}\,X \to \mathsf{ran}\,(\Phi;X)) = \mathbb{I}$. □

$$\begin{tikzcd} \mathscr{L} \arrow[d, "X"'] & \mathscr{G} \arrow[l, "\Phi"'] \arrow[d, dotted, "\Xi"] \\ \mathscr{A} & \mathscr{H} \arrow[l, dotted, "\Psi"] \end{tikzcd}$$

The proofs that the gluing condition is sufficient for the existence of a pushout complement [Kaw90], and that injectivity of Φ is sufficient for unambiguity of the pushout complement [Kah01] carry over to the collagory setting, but are outside the scope of this paper.

Another related condition is important in the context of the single-pushout approach [Löw90, LE91]:

Definition 5.2. In an allegory, we call X *conflict-free for* Φ iff $\mathsf{ran}\,(\Phi;X;X^{\smile}) \sqsubseteq \mathsf{ran}\,\Phi$. □

[1] Note that "X" is a capital "χ".

[2] *Pseudo-complements* are residuation of meet in lower semilattice categories; where pseudo-complements exist, we denote the pseudo-complement or R with respect to S as $R \to S$, and we have:

$$X \sqcap R \sqsubseteq S \qquad \Leftrightarrow \qquad X \sqsubseteq (R \to S)$$

For example, the pseudo-complement of a subgraph R of a graph G with respect to another subgraph S consists of all nodes of G that are in S or not in R, and all edges in S or not in R that are also nor incident with nodes in R. Intuitively, $R \to S$ therefore is G with the parts of R outside S removed, and then also all dangling edges removed.

For a node-and-edges-level formulation of conflict-freeness it is well-known that the induced single-pushout squares have a total embedding of the right-hand side into the application graph [Löw90, Cor. 3.18.5]. The component-free formulation above was first given in [Kah01], where it is also shown (Thm. 5.4.11) that a restricting derivation step for a conflict-free redex produces a pushout of partial functions.

6 Co-tabulations as Bicolimits and Lax Colimits

Ordered categories are a simple example of 2-categories and bicategories: For two morphisms $R, S : \mathscr{A} \to \mathscr{B}$ of an ordered category, there is at most one two-cell from R to S, and there is a two-cell from R to S iff $R \sqsubseteq S$. Therefore, there is an invertible two-cell between R and S if and only if $R = S$.

6.1 OC-Colimits: Bicolimits in Ordered Categories

The general notion of bicolimits takes as its point of departure a *diagram* defined via a functor from a category. We introduce a specialised variant of the definition used in [HS09] by restricting our attention to ordered categories.

Definition 6.1. Given a category $\mathbf{C}$, an (index) category $\mathbf{J}$, a functor $\mathbf{D} : \mathbf{J} \to \mathbf{C}$ defining a diagram, and an object $\mathscr{D}$, a *cocone* η from $\mathbf{D}$ to $\mathscr{D}$ consists of a morphism $\eta_{\mathscr{A}} : \mathbf{D}\mathscr{A} \to \mathscr{D}$ in $\mathbf{C}$ for each object $\mathscr{A}$ of $\mathbf{J}$, satisfying the following *cocone commutativity* condition:

$$\mathbf{D}F \mathbin{;} \eta_{\mathscr{B}} = \eta_{\mathscr{A}} \qquad \text{for each morphism } F : \mathscr{A} \to \mathscr{B} \text{ in } \mathbf{J}. \qquad \square$$

Definition 6.2. Given an ordered category $\mathbf{C}$, an (index) category $\mathbf{J}$, and a functor $\mathbf{D} : \mathbf{J} \to \mathbf{C}$, an *OC-colimit of* $\mathbf{D}$ is given by an object $\mathscr{D}$ of $\mathbf{C}$, and a cocone η from $\mathbf{D}$ to $\mathscr{D}$, satisfying the following conditions:

1. *factorisation:* for any other object $\mathscr{D}'$ of $\mathbf{C}$ with cocone κ from $\mathbf{D}$ to $\mathscr{D}'$, there is a morphism $h : \mathscr{D} \to \mathscr{D}'$ in $\mathbf{C}$ with

$$\eta_{\mathscr{A}} \mathbin{;} h = \kappa_{\mathscr{A}} \qquad \text{for each object } \mathscr{A} \text{ in } \mathbf{J}.$$

2. *isotony:* for any other object $\mathscr{D}'$ of $\mathbf{C}$ and any two morphisms $h, h' : \mathscr{D} \to \mathscr{D}'$, if $\eta_{\mathscr{A}} \mathbin{;} h \sqsubseteq \eta_{\mathscr{A}} \mathbin{;} h'$ for all objects $\mathscr{A}$ in $\mathbf{J}$, then $h \sqsubseteq h'$. $\square$

OC-colimits are unique up to isomorphism.

6.2 Lax Colimits in OCCs

For lax cocones, we only need the concept of lax functor, which differs from the functor concept in that a *lax functor* $\mathbf{D}$ only needs to satisfy $\mathbb{I}_{\mathbf{D}\mathscr{A}} \sqsubseteq \mathbf{D}\mathbb{I}_{\mathscr{A}}$ and $(\mathbf{D}f) \mathbin{;} (\mathbf{D}g) \sqsubseteq \mathbf{D}(f \mathbin{;} g)$, see, e.g., [Stu05, Sect. 8, p. 37ff]. Again, we provide specialised definition of lax cocones and lax colimits for the ordered category case:

Definition 6.3. Given an *ordered* category $\mathbf{C}$, an (index) category $\mathbf{J}$, a lax functor $\mathbf{D} : \mathbf{J} \to \mathbf{C}$ defining a diagram, and an object $\mathscr{D}$, a *lax cocone* η from $\mathbf{D}$ to $\mathscr{D}$ consists of a morphism

$\eta_{\mathscr{A}} : \mathbf{D}\mathscr{A} \to \mathscr{D}$ in $\mathbf{C}$ for each object $\mathscr{A}$ of $\mathbf{J}$, satisfying the following *cocone subcommutativity* condition:

$$\mathbf{D}F ; \eta_{\mathscr{B}} \sqsubseteq \eta_{\mathscr{A}} \qquad \text{for each morphism } F : \mathscr{A} \to \mathscr{B} \text{ in } \mathbf{J}.$$ □

Definition 6.4. Given an ordered category $\mathbf{C}$, an (index) category $\mathbf{J}$, and a lax functor $\mathbf{D} : \mathbf{J} \to \mathbf{C}$, a *lax colimit of* $\mathbf{D}$ is given by an object $\mathscr{D}$ of $\mathbf{C}$, and a lax cocone η from $\mathbf{D}$ to $\mathscr{D}$ satisfying the following conditions

1. *factorisation:* for any object $\mathscr{D}'$ of $\mathbf{C}$ with lax cocone κ from $\mathbf{D}$ to $\mathscr{D}'$, there is a morphism $U : \mathscr{D} \to \mathscr{D}'$ in $\mathbf{C}$ with

$$\eta_{\mathscr{A}} ; U = \kappa_{\mathscr{A}} \qquad \text{for each object } \mathscr{A} \text{ in } \mathbf{J},$$

2. *isotony:* for any object $\mathscr{D}'$ of $\mathbf{C}$ and any two morphisms $U, U' : \mathscr{D} \to \mathscr{D}'$, if $\eta_{\mathscr{A}} ; U \sqsubseteq \eta_{\mathscr{A}} ; U'$ for each object $\mathscr{A}$ in $\mathbf{J}$, then $U \sqsubseteq U'$. □

Lax colimits are unique up to isomorphism, too.

We now add the converse operator to our consideration of lax colimits, and when we use "$\bullet \to \bullet$" to denote an OCC, that OCC has the homset from the first object $\mathscr{A}$ to the second, different object $\mathscr{B}$ contain exactly one morphism, say F, from $\mathscr{A}$ to $\mathscr{B}$. As an OCC, it needs to also have $F^{\smile}$, which will be the only morphism from $\mathscr{B}$ to $\mathscr{A}$. Since in this OCC, also $F ; F^{\smile} ; F$ needs to exist as a morphism from $\mathscr{A}$ to $\mathscr{B}$, it has to be equal to F, which therefore is difunctional.

If a lax functor $\mathbf{D}$ maps $F : \mathscr{A} \to \mathscr{B}$ to $W : \mathscr{A}' \to \mathscr{B}'$, then

$$W ; W^{\smile} ; W = \mathbf{D}F ; (\mathbf{D}F)^{\smile} ; \mathbf{D}F \sqsubseteq \mathbf{D}(F ; F^{\smile} ; F) = \mathbf{D}F = W \ ,$$

so it can map $\mathscr{F}$ only to difunctional morphisms.

Furthermore, if, for a lax cocone, its source $\mathbf{J}$ is considered as an OCC, this implies that for each morphism $F : \mathscr{A} \to \mathscr{B}$ in $\mathbf{J}$, also the converse morphism $F^{\smile} : \mathscr{B} \to \mathscr{A}$ needs to be considered. Such a lax cocone therefore automatically has to satisfy both the following conditions:

$$\left.\begin{array}{rcl} \mathbf{D}F ; \eta_{\mathscr{B}} & \sqsubseteq & \eta_{\mathscr{A}} \\ (\mathbf{D}F)^{\smile} ; \eta_{\mathscr{A}} & \sqsubseteq & \eta_{\mathscr{B}} \end{array}\right\} \qquad \text{for each morphism } F : \mathscr{A} \to \mathscr{B} \text{ in } \mathbf{J}.$$

Convention 6.5. Given a morphism $W : \mathscr{B} \to \mathscr{C}$ in the OCC $\mathbf{C}$, we will frequently identify W with the functor $\mathbf{D}$ mapping the single morphism explicitly mentioned in the OCC $\bullet \to \bullet$ to W.

(Since we are dealing with an OCC, that morphism also has a converse, which then must be mapped to $W^{\smile}$.) □

A lax cocone from W to $\mathscr{D}$ therefore is a cospan $\mathscr{B} \xrightarrow{R} \mathscr{D} \xleftarrow{S} \mathscr{C}$ satisfying $W ; S \sqsubseteq R$ and $W^{\smile} ; R \sqsubseteq S$.

$$\begin{tikzcd} \mathscr{C} \arrow[dr, "S"] & \\ & \mathscr{D} \\ \mathscr{B} \arrow[uu, "W"] \arrow[ur, "R"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathscr{C} \arrow[dr, "S"] \arrow[drr, "S'"] & & \\ & \mathscr{D} \arrow[r, dotted, "U"] & \mathscr{D}' \\ \mathscr{B} \arrow[uu, "W"] \arrow[ur, "R"'] \arrow[urr, "R'"'] & & \end{tikzcd}$$

We explicitly state the definition of resulting special case of lax colimits:

Definition 6.6. An *OCC-colimit* of $W : \mathscr{B} \to \mathscr{C}$ in the OCC $\mathbf{C}$ is a lax cocone $\mathscr{B} \xrightarrow{R} \mathscr{D} \xleftarrow{S} \mathscr{C}$ from W to $\mathscr{D}$ (with $W;S \sqsubseteq R$ and $W^{\smile};R \sqsubseteq S$) satisfying the following conditions:

1. *factorisation:* for any object $\mathscr{D}'$ of $\mathbf{C}$ with lax cocone $\mathscr{B} \xrightarrow{R'} \mathscr{D}' \xleftarrow{S'} \mathscr{C}$ from W to $\mathscr{D}'$, there is a morphism $U : \mathscr{D} \to \mathscr{D}'$ in $\mathbf{C}$ with $R;U = R'$ and $S;U = S'$;
2. *isotony:* for any object $\mathscr{D}'$ of $\mathbf{C}$ and any two morphisms $U, U' : \mathscr{D} \to \mathscr{D}'$, if $R;U \sqsubseteq R;U'$ and $S;U \sqsubseteq S;U'$, then $U \sqsubseteq U'$. $\square$

The crucial aspect of the following theorem (proof in [Kah10]) is that it connects the respecive O*-limits for spans $\mathscr{B} \xleftarrow{P} \mathscr{A} \xrightarrow{Q} \mathscr{C}$ of mappings with those for the single difunctional morphisms $(P^{\smile};Q)^{\circledast}$ (which do not need to be mappings).

Theorem 6.7. If a span $\mathscr{B} \xleftarrow{P} \mathscr{A} \xrightarrow{Q} \mathscr{C}$ of mappings in a collagory is given, then a cospan $\mathscr{B} \xrightarrow{R} \mathscr{D} \xleftarrow{S} \mathscr{C}$ is an OCC-colimit for $(P^{\smile};Q)^{\circledast}$ iff it is an OC-pushout (i.e., OC-colimit for a span) for $\mathscr{B} \xleftarrow{P} \mathscr{A} \xrightarrow{Q} \mathscr{C}$. $\square$

6.3 OCC-Colimits are Co-tabulations

In a collagory $\mathbf{C}$ that is not co-tabular, categorical pushouts in $\mathsf{Map}\,\mathbf{C}$ are not necessarily gluings — the pushout conditions establish no connection between mappings and other morphisms, and pathological cases cannot be excluded.

However, OC-colimits and OCC-colimits do establish the necessary connections; one direction is easy to see (details in [Kah10]):

Theorem 6.8. If a cospan $\mathscr{B} \xrightarrow{R} \mathscr{D} \xleftarrow{S} \mathscr{C}$ in a collagory is a co-tabulation of $W : \mathscr{B} \to \mathscr{C}$, then it is also an OCC-colimit for W. $\square$

We now show that all OCC-colimits (of necessarily difunctional morphisms) are in fact co-tabulations. The proof needs to rely on the lax colimit properties, and therefore needs to use appropriate lax cocones constructed from the morphisms known to exist for a given OCC-colimit. The following lemma already follows this pattern:

Lemma 6.9. If, in an allegory, $\mathscr{B} \xrightarrow{R} \mathscr{D} \xleftarrow{S} \mathscr{C}$ is an OCC-colimit for W, then

$$\begin{aligned} W^{\smile};R &= S;\mathsf{ran}\,R & \qquad W^{\smile};R;R^{\smile} &= S;R^{\smile} \\ W;S &= R;\mathsf{ran}\,S & W;S;S^{\smile} &= R;S^{\smile} \end{aligned}$$

PROOF: Let $R_0 = W;S$ and $S_0 = S$. This defines a lax cocone $\mathscr{B} \xrightarrow{R_0} \mathscr{D} \xleftarrow{S_0} \mathscr{C}$ from W to $\mathscr{D}$, since:

$$\begin{aligned} W^{\smile};R_0 &= W^{\smile};W;S \sqsubseteq W^{\smile};R \sqsubseteq S = S_0 \ ; \\ W;S_0 &= W;S = R_0 \ . \end{aligned}$$

Then factorisation gives us a $U_0 : \mathscr{D} \to \mathscr{D}$ such that $R_0 = W;S = R;U_0$ and $S_0 = S = S;U_0$. Since $R;U_0 = W;S \sqsubseteq R = R;\mathbb{I}_{\mathscr{D}}$ and $S;U_0 = S \sqsubseteq S;\mathbb{I}_{\mathscr{D}}$, isotony gives us $U_0 \sqsubseteq \mathbb{I}_{\mathscr{D}}$.

So U_0 is a sub-identity, and $S = S;U_0$ implies $\mathsf{ran}\,S \sqsubseteq U_0$. Since composition of sub-identities is meet, we obtain the following (which implies $U_0 = \mathsf{ran}\,S$):

$$W;S = W;S;\mathsf{ran}\,S = R;U_0;\mathsf{ran}\,S = R;\mathsf{ran}\,S$$

Analogously, $W^{\smile} ; R = S ; \mathsf{ran}\, R$ also holds, and these further imply

$$W^{\smile} ; R ; R^{\smile} = S ; R^{\smile} \qquad \text{and} \qquad W ; S ; S^{\smile} = R ; S^{\smile} \ . \qquad \square$$

Lemma 6.9 does not use difunctionality of W, and implies:

$$\begin{aligned} W^{\smile} ; W ; W^{\smile} ; R &= W^{\smile} ; W ; S ; \mathsf{ran}\, R = W^{\smile} ; R ; \mathsf{ran}\, S ; \mathsf{ran}\, R \\ &= W^{\smile} ; R ; \mathsf{ran}\, S = S ; \mathsf{ran}\, R ; \mathsf{ran}\, S = S ; \mathsf{ran}\, R = W^{\smile} ; R \end{aligned}$$

and, analogously, $W ; W^{\smile} ; W ; S = W ; S$. Therefore, even with a weaker concept of OCC-colimit, we would still have, in some sense, “almost-difunctionality” of W.

Lemma 6.9 did use allegory properties (for sub-identities); to show the opposite inclusion to Theorem 6.8 we need full collagories (detailed proof in [Kah10]):

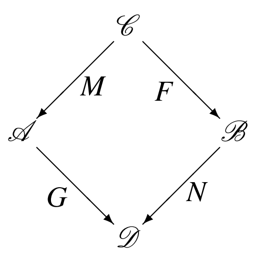
Theorem 6.10. If, in a collagory, $W : \mathscr{B} \to \mathscr{C}$ is a difunctional morphism and $\mathscr{B} \xrightarrow{R} \mathscr{D} \xleftarrow{S} \mathscr{C}$ is an OCC-colimit for W, then it is also a co-tabulation for W. $\square$

In summary, we have shown in Theorem 6.7 that OC-pushouts (i.e., OC-colimits) of a span are the same as OCC-colimits of the difunctional closure of the composition across that span. Furthermore, OCC-colimits of difunctional morphisms are the same as co-tabulations, as shown in Theorems 6.8 and 6.10.

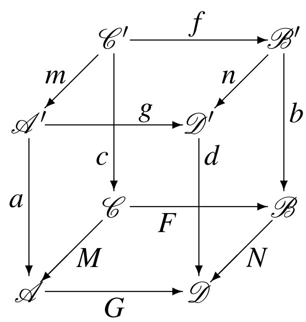
7 Van Kampen Squares in Collagories

Adhesive categories as a more specific setting for double-pushout graph rewriting have been introduced by Lack and Sobociński [LS04, LS05]; the following two definitions are taken from there:

Definition 7.1. A *van Kampen square* (i) is a pushout which satisfies the following condition: given a commutative cube (ii) of which (i) forms the bottom face and the back faces are pullbacks (where $\mathscr{C}$ is considered to be in the back), the front faces are pullbacks if and only if the top face is a pushout.



(i)



(ii) $\square$

Definition 7.2. A category **C** is said to be *adhesive* if

1. **C** has pushouts along monomorphisms;
2. **C** has pullbacks;
3. pushouts along monomorphisms are van Kampen squares. $\square$

For more concise formulations, we define:

Definition 7.3. A *van Kampen setup* in a collagory $\mathbf{C}$ for a square as in Def. 7.1(i) is a commuting cube in $\mathsf{Map}\,\mathbf{C}$ as in Def. 7.1(ii) where the bottom square is a gluing and the two back squares are tabulations. □

In [Kah09b], the following two lemmas were only shown for co-tabulations (i.e., assuming that $M\breve{}\,;F$ is difunctional, and also of $m\breve{}\,;f$ where it is assumed to be a gluing), not for general gluings. In [Kah10], we show the following significantly strengthened versions.

Lemma 7.4. In a collagory, if the front squares of a van Kampen setup are tabulations, then the top square is a gluing. If furthermore $M\breve{}\,;F$ is difunctional, then $m\breve{}\,;f$ is difunctional, too. □

Lemma 7.5. In a van Kampen setup where the top square is a gluing, the front squares are tabulations iff the following holds:

$$m\,;(m\breve{}\,;f)^{\circledast}\,;f\breve{} \sqcap c\,;c\breve{} \sqsubseteq \mathbb{I}_{\mathscr{C}'}$$ □

The condition here is equivalent to the following inclusion in the lattice of equivalences on $\mathscr{C}'$:

$$(m\,;m\breve{} \vee f\,;f\breve{}) \wedge c\,;c\breve{} = \mathbb{I}_{\mathscr{C}'}$$

Since equivalence lattices are not necessarily distributive, we cannot derive this from the tabulation equations $m\,;m\breve{} \wedge c\,;c\breve{} = \mathbb{I}_{\mathscr{C}'}$ and $f\,;f\breve{} \wedge c\,;c\breve{} = \mathbb{I}_{\mathscr{C}'}$.

From Lemmas 7.4 and 7.5, we also directly obtain a characterisation of van Kampen squares in bitabular collagories:

Theorem 7.6. A gluing square (as in Def. 7.1(i)) in a bitabular collagory is van Kampen iff all its van Kampen setups (as in Def. 7.3) where the top square is a gluing satisfy the following:

$$m\,;(m\breve{}\,;f)^{\circledast}\,;f\breve{} \sqcap c\,;c\breve{} \sqsubseteq \mathbb{I}_{\mathscr{C}'}$$ □

The bitabularity condition could be weakened, but even then, this characterisation theorem is still very different from the appropriate diagram instance of Heindel and Sobociński's characterisation theorem [HS09, Theorem 22], due to the fact that, by assuming a gluing, we already restricted ourselves to "well-behaved" pushouts.

Our theorem also stays more in the typical relation-algebraic spirit: instead of Heindel and Sobociński's condition "a colimit exists", we have a local inclusion to check. The universal quantification this is embedded in is essentially the same as in [HS09, Theorem 22].

An interesting question is whether there is a useful characterisation that employs a local condition only on the candidate square, beyond injectivity of one M and F, as used in the definition of adhesive categories.

First we observe (proof in [Kah10]):

Lemma 7.7. In a van Kampen setup where $M\,;M\breve{} \sqcap F\,;F\breve{} \sqsubseteq \mathbb{I}_{\mathscr{C}}$, the following hold:

1. $f\,;f\breve{} \sqcap m\,;m\breve{}\,;c\,;c\breve{} \sqsubseteq \mathbb{I}_{\mathscr{C}'}$
2. $c\,;c\breve{} \sqcap m\,;m\breve{}\,;f\,;f\breve{} \sqsubseteq \mathbb{I}_{\mathscr{C}'}$ □

Injectivity of M makes $M\breve{}\,;F$ difunctional and also enforces injectivity of m and therewith difunctionality of $m\breve{}\,;f$.

In the general case, however, we have seen above that difunctionality of $m\breve{}\,;f$ requires not only difunctionality of $M\breve{}\,;F$, but also the front tabulation conditions.

This failure of difunctionality propagation can be understood as coming from the fact that in the difunctionality inclusion $M^{\smile} ; F ; F^{\smile} ; M ; M^{\smile} ; F \sqsubseteq M^{\smile} ; F$, the right-hand side passes through a "$\mathscr{C}$ element" that may be distinct from the three "$\mathscr{C}$ elements" of the left-hand side.

This distinct "$\mathscr{C}$ element" gives rise to a "$\mathscr{C}'$ element" that is, in the absence of the front tabulation conditions, determined only up to $c ; c^{\smile}$.

One way to avoid this unwanted factor is to specify that in any chain diagram documenting $M ; M^{\smile} ; F ; F^{\smile} ; M ; M^{\smile}$, the fourth (i.e., last) $\mathscr{C}$ element needs to be one of the previous three $\mathscr{C}$ elements. Referring to so many elements simultaneously in a relation-algebraic way requires direct products — we use π and ρ as the projections. The following is one formulation of this condition:

$$M ; M^{\smile} ; (\pi^{\smile} \sqcap F ; F^{\smile} ; M ; M^{\smile} ; \rho^{\smile}) \sqsubseteq M ; M^{\smile} ; (\pi^{\smile} \sqcap (F ; F^{\smile} \sqcup M ; M^{\smile}) ; \rho^{\smile})$$

However, it is not hard to see that this is equivalent to the following, much simpler condition:

$$F ; F^{\smile} ; M ; M^{\smile} \sqsubseteq F ; F^{\smile} \sqcup M ; M^{\smile}$$

This is obviously satisfied if one of M and F is injective. It can also be strengthened to an equality, since M and F are both total. This implies symmetry:

$$F ; F^{\smile} ; M ; M^{\smile} = F ; F^{\smile} \sqcup M ; M^{\smile} = M ; M^{\smile} ; F ; F^{\smile}$$

and, furthermore, difunctionality of $M^{\smile} ; F$:

$$M^{\smile} ; F ; F^{\smile} ; M ; M^{\smile} ; F = M^{\smile} ; M ; M^{\smile} ; F ; F^{\smile} ; F = M^{\smile} ; F \quad .$$

Assuming also $M ; M^{\smile} \sqcap F ; F^{\smile} \sqsubseteq \mathbb{I}_{\mathscr{C}}$, we obtain $f ; f^{\smile} ; m ; m^{\smile} = f ; f^{\smile} \sqcup m ; m^{\smile}$:

$$\begin{array}{lll} & f ; f^{\smile} ; m ; m^{\smile} & \\ = & f ; f^{\smile} ; m ; m^{\smile} \sqcap c ; F ; F^{\smile} ; M ; M^{\smile} ; c^{\smile} & \\ \sqsubseteq & f ; f^{\smile} ; m ; m^{\smile} \sqcap c ; (F ; F^{\smile} \sqcup M ; M^{\smile}) ; c^{\smile} & \text{assumption} \\ = & f ; f^{\smile} ; m ; m^{\smile} \sqcap (c ; c^{\smile} ; f ; f^{\smile} \sqcup c ; c^{\smile} ; m ; m^{\smile}) & \\ = & (f ; f^{\smile} ; m ; m^{\smile} \sqcap c ; c^{\smile} ; f ; f^{\smile}) \sqcup (f ; f^{\smile} ; m ; m^{\smile} \sqcap c ; c^{\smile} ; m ; m^{\smile}) & \\ \sqsubseteq & f ; f^{\smile} \sqcup m ; m^{\smile} & \text{Lemma 7.7} \end{array}$$

Therefore, $m^{\smile} ; f$ is difunctional, too, and together with Lemma 7.7 we obtain

$$m ; (m^{\smile} ; f)^{\circledast} ; f^{\smile} \sqcap c ; c^{\smile} = m ; m^{\smile} ; f ; f^{\smile} \sqcap c ; c^{\smile} \sqsubseteq \mathbb{I}_{\mathscr{C}'} \quad .$$

Altogether we have shown the following:

Theorem 7.8. In the category $\mathsf{Map}\,\mathbf{C}$ of maps over a bi-tabular collagory $\mathbf{C}$, pushouts for spans $\mathscr{A} \xleftarrow{M} \mathscr{C} \xrightarrow{F} \mathscr{B}$ that satisfy also

$$F ; F^{\smile} \sqcap M ; M^{\smile} \sqsubseteq \mathbb{I}_{\mathscr{C}} \qquad \text{and} \qquad F ; F^{\smile} ; M ; M^{\smile} \sqsubseteq F ; F^{\smile} \sqcup M ; M^{\smile}$$

are van Kampen squares. □

Both inclusions can be strengthened to equalities, and since the second condition implies difunctionality, both together imply that such pushouts are also pullbacks.

8 Conclusion

We have shown that, in collagories, lax colimits of single morphisms are the same as co-tabulations, and bicolimits of spans (bipushouts) are the same as gluings. Furthermore, the move from a span $\mathscr{B} \xleftarrow{P} \mathscr{A} \xrightarrow{Q} \mathscr{C}$ to the difunctional closure of $P^{\smallsmile} ; Q$ preserves both kinds of colimits. (The opposite move could be achieved via a tabulation, and may still deserve to be spelt out.)

We also strengthened our previous results about the two implications involved in van Kampen squares from difunctional spans to arbitrary spans, extracted a precise relation-algebraic condition for van Kampen squares in collagories, and gave a new, purely local sufficient condition for van Kampen squares that is more general than the "pushouts along monomorphisms" used in adhesive categories.

These two results together with the fact that the equational characterisation of co-tabulations enables a nice, calculational proof style make a strong case to employ collagories as a convenient basis for theoretical investigations of graph structure transformations. In addition, relation-algebraic formulations and reasoning are accessible to a wide audience due to the fact that in the intuitive special case of *Rel*, they can be understood as Boolean matrix operations.

Future investigations will explore how these new conditions for van Kampen squares can be combined with the different variations of adhesive categories in a collagory setting, including the quasiadhesive categories of [LS05], and their applications.

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