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NONSMOOTH AND NONCONVEX MODELS OF THE BUSINESS PROCESS

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Abstract: The linear model can in some cases be used in the optimization of the business process. In this article an example is employed to show that, using the liner mixed integer model, the business process can be presented more authentically. Nonconvex polyhedrons and some classes of nonsmooth functions can be expressed using zero-one variables. Using these expressions some nonsmooth programming problems can be transformed into liner mixed integer programming problems and solved by available computer programs.

1. INTRODUCTION

The linear programming problem can be used as an approximation of the model for the business process [4]. Because of the development of computational techniques for liner mixed integer programming, the liner mixed integer model can be used. What needs time is the analysis of the business process and transmission and control of data and not the solving of mixed integer linear programming problems [7], [8], [9]. A Petri-net can be advantageous for the analysis of the business process [6]. The construction of such a net also takes time, but it can be useful for the technological analysis of the production process and for other purposes.

Some applicable nonlinear optimization models can be expressed in the form of liner mixed integer programming problems. Using these expressions we are not in need of computer programs for solving all original problems even if suitable algorithms exist for these problems. This result can be useful since many nonlinear functions important for practice, can be approximated by liner mixed integer functions [5]. An example is the linear fractional function [3] which can be expressed in a separable form subject to additional separable constraints and then approximated by liner mixed integer functions.

The product uf (x), where $f: R^n \to R$ is a bounded function and u is a zero-one variable, can be expressed in the form

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$$uf(x) = y \tag{1.1}$$

subject to

$$f(x) = y + z \tag{1.2}$$

$$-du \leq y \leq du \tag{1.3}$$

$$d(u - 1) \leq z \leq d(1 - u). \tag{1.4}$$

Here d is a positive large enough constant. If f (x) is linear then the term uf (x) can be linearized.

The function f [L (x)], where f: R→R and L: R^n→R is a separable function, can be expressed in the separable form. Similarly the function

$$F(x) = f[L_1(x),\ L_2(x), \ldots,\ L_k(x)] \tag{1.5}$$

can be simplified. Using the substitution

$$y_i = L_i(x) \qquad i = 1, \ldots, k \tag{1.6}$$

we obtain

$$F(x) = f(y_1, y_2, \ldots, y_k). \tag{1.7}$$

This substitution can be exceedingly effective if L_i are linear and f is separable. If (1.5) arises in a programming problem, it can be replaced by (1.7) subject to (1.6). If $k < n$ the number of variables which arise in the function F is reduced but the number of variables and constraints in the model increase, Such a substitution is useful especially if the programming problem can in this way be replaced or approximated by a liner mixed integer programming problem. The large-scale mixed integer programming problem can be simply solved [8].

In part 2 an example for linear model of a business process is given. In part 3 this model is extended to linear mixed integer model. The programming problem

maximize f (x) subject to $x \in B$,

where f (x) is a linear function and B a nonconvex polyhedron, using the expressions given in part 4, can be transformed into a linear mixed integer programming problem. Using this result, many goals can be considered in the model if at least one of them must be achieved.

Some types of continuous piecewise linear functions which can be considered as simple examples of nonsmooth functions, can be explicitly expressed as linear functions using zero-one variables subject to additional linear constraints. In part 5 some expressions for other nonsmooth functions are given. As a special case the absolute value function is considered. This function is very important in optimization models. Using these results, some nonsmooth programming problems [2] containing continuous functions of few variables can either be transformed into liner mixed integer programming problems or approximated by them.

2. MOTIVATIONS EXAMPLE

For illustration we take an example from the woodworking industry. We divide the business process into production activities and market activities. In some cases a division of the business process with the following properties can be successful. For any production activity the consumption of the production elements and the revenue of the products are proportional to the quantity of connected production activity. The income and the revenue are proportional to the quantities sold and bought respctively. A linear model for optimization of the business process can then be used [4].

Let us suppose that in our case the linear model is allowed. Data for construction of the model can be arranged in two tables. For our example they are given in Tables 1 and 2. In Table 1 data for sources and consumers are given. In this table production elements, semi-products and final products are considered. For E1, E3 and E4 we have sources. E3 can be bought and we have two suppliers. The first one can supply a maximum of 100 units and the corresponding purchase prices is 15 monetary units per unit. For E1 we have a source from our own capacity. Instead of the purchase price here we take the variable driving cost per unit of capacity, which is not taken into account by consumption of elements considered in Table 1 or by the cost considered in Table 2. For semi-product E6 we have neither sources nor consumers. Elements E2, E5, E7, E8 and E9 can be sold.

In Table 2 consumption and production of elements and the marginal cost, which is not connected with the consumption of elements considered in the Table 1, are given for each production activity. Per unit of production activity X8 for example 3 units of E5, 2 units of E6 and 12 units of E7 are consumed and one unit of E8 is produced. If consumptions of E5, E6 and E7 are not considered, the variable production cost per unit of X8 is 10 monetary units.

Table 1. *Data about sources and consumers*

Element		Sources and consumers			
Symbol	Name	Symbol	Price	Minimal	Maximal
E1	Production means I	Y1	60		10
E2	Plates 200×100	Z2	15	20	30
E3	Plates 160×100	Y31	15		100
		Y32	16		
E4	Production means II	Y4	2		100
E5	Plates 200×60	Z5	11		
E6	Plates 160×60				
E7	Plates 55×40	Z7	2	100	150
E8	Product A	Z81	100		10
		Z82	95	10	
E9	Product B	Z91	60		
		Z92	65		50

Table 2. *Technological data*

Production activity		Consumption		Production	
Symbol	Name	Symbol	Quantity	Symbol	Quantity
X1	Production 200×100	E1	0.1	E2	1
		Cost	10		
X2	Production 160×100	E1	0.08	E3	1
		Cost	9		
X3	Cut E2 1+0+3	E2	1	E5	1
		E4	0.4	E7	3
X4	Cut E2 0+1+4	E2	1	E6	1
		E4	0.5	E7	4
X5	Cut E2 0+0+8	E2	1	E7	8
		E4	0.5		
X6	Cut E3 0+1+2	E3	1	E6	1
		E4	0.3	E7	2
X7	Cut E3 0+0+6	E3	1	E7	6
		E4	0.4		
X8	Production A	E5	3	E8	1
		E6	2		
		E7	12		
		Cost	10		
X9	Production B	E6	3	E9	1
		E7	6		
		Cost	6		

Denote by x_k, y_k and z_k the decision variables connected to the activities Xk, Yk and Zk. Considering the given data we obtain an optimization model in the form:

maximize $15z_2 + 11z_5 + 2z_7 + 100z_{81} + 95z_{82} + 60z_{91} + 65z_{92} - 60y_1 - 15y_{31} - 16y_{32} - 2y_4 - 10x_1 - 9x_2 - 10x_8 - 6x_9$

subject to

$$y_1 - 0.1x_1 - 0.08x_2 \geq 0 \tag{2.1}$$

$$y_1 \leq 10$$

$$x_1 - x_3 - x_4 - x_5 - z_2 \geq 0$$

$$20 \leq z_2 \leq 30$$

$$y_{31} + y_{32} + x_2 - x_6 - x_7 \geq 0$$

$$y_{31} \leq 100$$

$$y_4 - 0.4x_3 - 0.5x_4 - 0.5x_5 - 0.3x_6 - 0.4x_7 \geq 0 \tag{2.2}$$

$$y_4 \leq 100$$

$$x_3 - 3x_8 - z_5 \geq 0$$

$$x_4 + x_6 - 2x_8 - 3x_9 \geq 0$$

$$3x_3 + 4x_4 + 8x_5 + 2x_6 + 6x_7 - 12x_8 - 6x_9 - z_7 \geq 0$$

$$100 \leq z_7 \leq 150$$

$$x_8 - z_{81} - z_{82} \geq 0$$

$$z_{81} \leq 10 \qquad z_{82} \geq 10$$

$$x_9 - z_{91} - z_{92} \geq 0$$

$$z_{92} \leq 50 \tag{2.3}$$

Table 3. *Result for production quantities*

Production activity		Optimal production quantity	Production cost too high for:
Symbol	Name		
X1	Production 200x100	95.000	
X2	Production 160x100	6.250	
X3	Cut E2 1+0+3	30.000	
X4	Cut E2 0+1+4	45.000	
X5	Cut E2 0+0+8		5.750
X6	Cut E3 0+1+2	218.333	
X7	Cut E3 0+0+6		6.417
X8	Production A	10.000	
X9	Production B	81.111	

Using the computer program LOMP we obtain the result given in Tables 3 and 4. The optimal objective function value is 880.417. The reduced cost is negative if the upper bound for corresponding variable is active. By means of the computer program LOMP calculations using shadow prices can be made. In this way opportunity costs are included and the fixed cost is excluded [4].

Table 4. *Result for sources and consumers*

Element		Shadow price	Sources and consumers		
Symbol	Name		Symbol	Quantity	Red. cost
E1	Production means I	87.500	Y1	10.000	—27.500
E2	Plates 200x100	18.750	Z2	20.000	3.750
E3	Plates 160x100	16.000	Y31	100.000	— 1.000
			Y32	112.083	
E4	Production means II	6.667	Y4	100.000	— 4.667
E5	Plates 200x60	15.292	Z5		4.292
E6	Plates 160x60	13.917			
E7	Plates 55x40	2.042	Z7	100.000	0.042
E8	Product A	108.208	Z81		8.208
			Z82	10.000	13.208
E9	Product B	60.000	Z91	31.111	
			Z92	50.000	— 5.000

Table 5. *Calculation for X5 by shadow prices*

Element		Consumed quantity per unit	Production cost per unit	Sum of cost per unit
Symbol	Name			
E2	Plates 200x100	1	18.750	
E4	Production means II	0.5	3.333	22.083

Calculations for X5 and X8 are given in Tables 5 and 6. The shadow price of E8 equals the sum of cost per unit X8 and the production activity X8 is optimal. The revenue per unit of X5 is

$$8*2.04167 = 16.33336$$

and the production cost of X5 is too high for 5.750 monetary units per unit of production activity.

Table 6. *Calculation for X8 by shadow prices*

Element		Consumed quantity per unit	Production cost per unit	Sum of cost per unit
Symbol	Name			
E5	Plates 200x60	3	45.875	
E6	Plates 160x60	2	27.833	
E7	Plates 55x40	12	24.500	
	Other marginal cost		10.000	108.208

3. PIECEWISE LINEAR MODEL

Consider the business process if the cost, the revenue and the consumption of production elements are piecewise linear functions. Then a linear mixed integer model is obtained [5]. Let us take the example given in part two subject to additional conditions.

Before starting a new cutting plan the production equipment must be made ready. Therefore we need 0.5 units of E4 and the additional cost of 2 monetary units arises. Then instead of (2.2) we obtain

$$y_4 - 0.5t_3 - 0.4x_3 - 0.5t_4 - 0.5x_4 - 0.5t_5 - 0.5x_5 - 0.5t_6 - 0.3x_6 - \\ - 0.5t_7 - 0.4x_7 \geq 0$$

subject to additional constraints

$$400t_j - x_j \geq 0 \qquad t_j = 0 \text{ or } 1 \qquad j = 3, \ldots, 7$$

where 400 is a chosen large enough constant. In the objective function we obtain additional terms

$$-2t_3 - 2t_4 - 2t_5 - 2t_6 - 2t_7.$$

Similarly the cost for the market research can be considered. If 10 monetary units must be spent before the second product can be sold to the second consumer, instead of (2.3) we obtain the constraint

$$50^* t_9 - z_{92} \geq 0 \qquad t_9 = 0 \text{ or } 1$$

and in the objective function the additional term $-10t_9$

Instead of the marginal cost of 6 monetary units connected with the production activity x_9 there arises the marginal cost of 3 monetary units and fixed cost of 120 monetary units at start of x_9, 80 monetary units if x_9 surplus 38 units and 80 monetary units if x_9 surplus 76 units. We must take the substitution

$$x_9 = x_{91} + x_{92} + x_{93}$$

subject to nonnegative variables x_{9j} and

$$\begin{array}{lll} 38u_j - x_{9j} \geq 0 & u_j = 0 \text{ or } 1 & j = 1, 2, 3 \\ x_{91} - 38u_2 \geq 0 & x_{92} - 38u_3 \geq 0. & \end{array}$$

Instead of $-6x_9$ in the objective function we obtain

$$-120u_1 - 80u_2 - 80u_3 - 3x_{91} - 3x_{92} - 3x_{93}.$$

At the beginning of the production of plates 200x100 for the first 3 units we need 0.11 time units and at the beginning of the production of plates 160x100 0.09 time units per unit of production activity. Therefore we must take the substitution

$$x_j = x_{j1} + x_{j2} \qquad j = 1, 2$$

subject to nonnegative variables x_{j1} and x_{j2}. Instead of (2.1) we obtain

$$y_1 - 0.11x_{11} - 0.1x_{12} - 0.09x_{21} - 0.08x_{22} \geq 0$$

and additional constraints

$$\begin{array}{lll} x_{j1} - 3t_j \geq 0 & t_j = 0 \text{ or } 1 & j = 1, 2 \\ 100t_1 - x_{12} \geq 0 & 125t_2 - x_{22} \geq 0. & \end{array}$$

Constants are determined considering (2.1) and $0 \leq y_1 \leq 10$.

The model can be written in the form

$$\begin{array}{l} \textit{maximize } 15z_2 + 11z_5 + 2z_7 + 100z_{81} + 95z_{82} + 60z_{91} + 65z_{92} - \\ - 60y_1 - 15y_{31} - 16y_{32} - 2y_4 - 10x_{11} - 10x_{12} - 9x_{21} - \\ - 9x_{22} - 10x_8 - 3x_{91} - 3x_{92} - 3x_{93} - 2t_3 - 2t_4 - 2t_5 - 2t_6 - 2t_7 - \\ - 10t_9 - 120u_1 - 80u_2 - 80u_3 \end{array}$$

subject to nonnegative variables and

$$y_1 - 0.11x_{11} - 0.1x_{12} - 0.09x_{21} - 0.08x_{22} \geq 0$$
$$y_1 \leq 10 \tag{E1}$$

$$x_{11} + x_{12} - x_3 - x_4 - x_5 - z_2 \geq 0$$
$$20 \leq z_2 \leq 30 \tag{E2}$$

$$y_{31} + y_{32} + x_{21} + x_{22} - x_6 - x_7 \geq 0$$
$$y_{31} \leq 100 \tag{E3}$$

$$y_4 - 0.5t_3 - 0.4x_3 - 0.5t_4 - 0.5x_4 - 0.5t_5 - 0.5.x_5 -$$
$$- 0.5t_6 - 0.3x_6 - 0.5t_7 - 0.4x_7 \geq 0$$
$$400t_j - x_j \geq 0 \qquad t_j = 0 \text{ or } 1 \qquad j = 3, \ldots, 7$$
$$y_4 \leq 100 \tag{E4}$$

$$x_3 - 3x_8 - z_5 \geq 0 \tag{E5}$$

$$x_4 + x_6 - 2x_8 - 3x_{91} - 3x_{92} - 3x_{93} \geq 0 \tag{E6}$$

$$3x_3 + 4x_4 + 8x_5 + 2x_6 + 6x_7$$
$$- 12x_8 - 6x_{91} - 6x_{92} - 6x_{93} - z_7 \geq 0$$
$$100 \leq z_7 \leq 150 \tag{E7}$$

$$x_8 - z_{81} - z_{82} \geq 0$$
$$z_{81} \leq 10 \qquad z_{82} \geq 10 \tag{E8}$$

$$x_{91} + x_{92} + x_{93} - z_{91} - z_{92} \geq 0$$
$$50t_9 - z_{92} \geq 0 \qquad t_9 = 0 \text{ or } 1 \tag{E9}$$

$$x_{11} - 3t_1 \geq 0 \qquad 100t_1 - x_{12} \geq 0 \qquad t_1 = 0 \text{ or } 1 \tag{X1}$$

$$x_{21} - 3t_2 \geq 0 \qquad 125t_2 - x_{22} \geq 0 \qquad t_2 = 0 \text{ or } 1 \tag{X2}$$

$$38u_j - x_{9j} \geq 0 \qquad u_j = 0 \text{ or } 1 \qquad j = 1, 2, 3$$

$$x_{91} - 38u_2 \geq 0 \qquad x_{92} - 38u_3 \geq 0 \tag{X9}$$

Table 7. *Result for sources and consumers*

Element		Shadow price	Sources and consumers		
Symbol	Name		Symbol	Quantity	Red. cost
E1	Production means I	96.000	Y1	10.000	—36.000
E2	Plates 200x100	19.600	Z2	20.000	4.600
E3	Plates 160x100	16.000	Y31	100.000	— 1.000
			Y32	98.300	
E4	Production means II	2.000	Y4	97.840	
E5	Plates 200x60	14.400	Z5		3.400
E6	Plates 160x60	12.600			
E7	Plates 55x40	2.000	Z7	109.400	
E8	Product A	102.400	Z81		2.400
			Z82	10.000	7.400
E9	Product B	60.000	Z91	26.000	
			Z92	50.000	— 5.000

Table 8. *Result for production quantities*

Production activity		Optimal production quantity	Production cost too high for:
Symbol	Name		
X11	Production 200x100	3.000	0.960
X12		96.700	
X21	Production 160x100		1.640
X22			0.680
X3	Cut E2 1+0+3	30.000	
X4	Cut E2 0+1+4	49.700	
X5	Cut E2 0+0+8		4.600
X6	Cut E3 0+1+2	198.300	
X7	Cut E3 0+0+6		4.800
X8	Production A	10.000	
X91	Production B	38.000	
X92		38.000	
X93			7.200

The result obtained by PC is given in Tables 7 and 8. Zero-one variables t_2, t_5, t_7 and u_3 equal zero, other zero-one variables equal one, The optimal objective function value is 869.320.

4. DIFFERENT EXPRESSIONS OF NONCONVEX POLYHEDRONS

Let f_{ij}: $R^n \to R$ be linear functions. Then

$$B = \bigcup_{j=1}^{k} \{x \in R^n | f_{ij}(x) = b_{ij} \text{ for } i \in E_j, f_{ij}(x) \leq b_{ij} \text{ for } i \in L_j\}$$

is a polyhedron which can be nonconvex. If B is bounded it can be expressed in the form

$$B = \{x \in R^n | d(u_j - 1) \leq f_{ij}(x) - b_{ij} \leq d(1 - u_j) \text{ for } i \in E_j,$$
$$f_{ij}(x) - b_{ij} \leq d(1 - u_j) \text{ for } i \in L_j,$$
$$u_j = 0 \text{ or } 1 \text{ for } j = 1, \ldots, k, \sum_{j=1}^{k} u_j \geq 1\}$$

where d is a suitable large enough constant. This can be proved using the implication

$$u_j = 1 \to x \in B \text{ for any } j \text{ subject to } 1 \leq j \leq k.$$

The nonconvex polyhedron can also be defined in the form

$$B = \{x \in B_c | x \notin \bigcup_{j=1}^{k} P_j\}$$

where B_c is a convex polyhedron,

$$P_j = \{x \in R^n | f_{ij}(x) < b_{ij} \text{ for } i = 1, \ldots, m_j\} \quad j = , \ldots, k$$

and f_{ij} are linear functions. If B is bounded it can be expressed in the form

$$B = \{x \in B_c | b_{ij} - f_{ij}(x) \leq d u_{ij} \text{ for } i = 1, \ldots, m_j,$$
$$\sum_{i=1}^{mj} u_{ij} \leq m_j - 1 \quad \text{for} \quad j = 1, \ldots, k\}.$$

The constant d is needed also in this case. For each inequation we can take another constant.

Let us consider an example of a linear mixed integer model of a business process where m goals are defined and at least one of them must be achieved. Let the goals be expressed in the form

$$f_i(x_1, \ldots, x_n, u_1, \ldots, u_k) \geq 0 \qquad i = 1, \ldots, m \tag{4.1}$$

where x_j are decision variables and u_j are zero-one variables contained in the model. This condition can be expressed by following constraints:

$$f_i(x_1, \ldots, x_n, u_1, \ldots, u_k) \geq -dv_i \qquad i = 1, \ldots, m \qquad (4.2)$$
$$v_1 + v_2 + \ldots + v_m \leq m-1 \qquad (4.3)$$
$$v_i = 0 \text{ or } 1 \text{ for } i = 1, \ldots, m$$

where d is a positive large enough constant. If the solution does not satisfy the condition (4.1) then the i-th goal is not achieved. In this case since (4.2) $v_i = 1$. Therefore at least one of goals must be achieved for any solution which satisfies (4.2) and (4.3). If m goals exist and n of them must be achieved whre $n < m$, similar expression can be used. These results can be extended and the general assignment problem can be included in the optimization model.

Consider the example given in part 3 where at least one of the following three conditions must be satisfied:

$$95z_{82} + 60z_{91} \geq 4000$$
$$y_{31} = 0$$
$$y_4 \leq 90.$$

From (4.2) and (4.3) it follows

$$95z_{82} + 60z_{91} + 4000v_1 \geq 4000 \qquad (4.4)$$
$$-y_{31} + 100v_2 \geq 0 \qquad (4.5)$$
$$y_4 - 10v_3 \leq 90 \qquad (4.6)$$
$$v_1 + v_2 + v_3 \leq 2 \qquad v_i = 0 \text{ or } 1 \qquad i = 1,2,3. \qquad (4.7)$$

The terms with integer variables in (4.4 — (4.6) are transposed on the left-hand side. Instead of d in (4.4) — (4.6) suitable different constants are taken. We obtain the same objective function as in part 3 subject to constraints given in part 3 and additional constraints (4.4) — (4.7). The constraint $y_4 \leq 100$ can since (4.6) be omited and the constraint $y_{31} \leq 100$ can since (4.5) be omited.

Table 9. *Result for sources and consumers*

Element		Shadow price	Sources and consumers		
Symbol	Name		Symbol	Quantity	Red. cost
E1	Production means I	87.500	Y1	10.000	—27.500
E2	Plantes 200×100	18.750	Z2	20.000	3.750
E3	Plates 160×100	16.000	Y31	100.000	—1.000
			Y32	74.500	
E4	Production means II	10.000	Y4	90.000	—8.000
E5	Plates 200×60	15.625	Z5		4.625
E6	Plates 160×60	14.250			
E7	Plates 55×40	2.375	Z7	100.000	0.375
E8	Product A	113.875	Z81		13.875
			Z82	10.000	18.875
E9	Product B	60.000	Z91	18.333	
			Z92	50.000	—5.000

The optimal solution is given by Tables 9. and 10. The reduced cost for v_3 is 80, zero-one variables v_3, t_5, t_7 and u_3 equal zero, other zero-one variables equal one, the optimal objective function value is 807.500.

Table 10. *Result for production activities*

Production		Optimal production quantity	Production cost too high for:
Symbol	Name		
X11	Production 200×100	3.000	0.875
X12		92.000	
X21	Production 160×100	3.000	0.875
X22		2.500	
X3	Cut E2 1 + 0 +3	30.000	
X4	Cut E2 0 + 1 + 4	45.000	
X5	Cut E2 0 + 0 + 8		4.750
X6	Cut E3 0 + 1 + 2	180.000	
X7	Cut E3 0 + 0 +6		5.750
X8	Production A	10.000	
X91	Production B	38.000	
X92		30.333	
X93			0.000

If the choice of the third goal expressed by (4.6) is omitted, we obtain the same result for production activities and for shadow prices as in part 3. The difference appears only in the market activities Y31 and Y32 and in the objective function.

$$y_{31} = 0 \qquad y_{32} = 198.300 \qquad z = 769.320.$$

If goals are differently favorable and the favorableness is measurable, then they can be considered in the objective function by the sum

$$c_1(1 - v_1) + c_2(1 - v_2) + \ldots + c_k(1 - v_k)$$

where c_j are prices of goals expressed in the same unit as the objective function.

Since determination of the constant d in some cases is not trivial, the expression of unbounded nonconvex polyhedron can be used. The convex hull of the polyhedron

$$B = \bigcup_{j=1}^{k} B_j$$

where B_j are convex polyhedrons, can be expressed in the form

$$C = \{x \in R^n | x = \sum_{j=1}^{k} u_j y_j , \sum_{j=1}^{k} u_j = 1 , y_j \in B_j \text{ for } j = 1, \ldots, k\}$$

where u_j are nonnegative. If u_j are zero-one variables, then C = B, where B is a nonconvex polyhedron. If the constant d can be determined simply this expression is not advantageous since more additional variables must be defined.

5. EXPRESSIONS OF NONSMOOTH FUNCTIONS

Consider the function

$$f(x) = \begin{cases} f_1(x) & \text{for } g(x) \le 0 \\ f_2(x) & \text{for } g(x) > 0 \end{cases} \tag{5.1}$$

where $f_1(x)$, $f_2(x)$ and $g(x)$ are continuous real funtions for $x \in R^n$ and $g(x)$ let be bounded. If

$$f_1(x) = f_2(x) \quad \text{for} \quad x \in G = \{x \in R^n | g(x) = 0\}, \tag{5.2}$$

then f(x) is continuous for $x \in G$. The function (5.1) can be expressed in the form

$$f(x) = h(x,u) = uf_1(x) + (1 - u)f_2(x) \tag{5.3}$$

subject to zero-one variable u and

$$-du < g(x) \le d(1 - u) \tag{5.4}$$

where d is a large enough constant. In (5.3) it is considered that from (5.4) for zero-one variable u it follows

$$u = u(x).$$

If $f_1(x)$ and $f_2(x)$ satisfy the condition (5.2), then (5.4) can be replaced by

$$-du \le g(x) \le d(1 - u). \tag{5.5}$$

Considering (1.1) — (1.4) from (5.3) it follows

$$h(x,u) = y_1 + y_2 \tag{5.6}$$

subject to (5.5) and

$$f_1(x) = y_1 + z_1 \tag{5.7}$$
$$f_2(x) = y_2 + z_2 \tag{5.8}$$

$$-du \le y_1 + z_2 \le du \tag{5.9}$$

$$d(u-1) \le z_1 + y_2 \le d(1-u). \tag{5.10}$$

The function (5.1) can be replaced by (5.6) subject to (5.4) and additional constraints (5.7) — (5.10). If g(x), $f_1(x)$ and $f_2(x)$ are linear, then this substitution except (5.4) can be used in linear mixed integer programming. If $f_1(x)$ and $f_2(x)$ satisfy (5.2) then (5.5) can be used.

Let $f_j(x)$ and $g_{ij}(x)$ be real function for $x \in R^n$. Consider the function

$$f(x) = \begin{cases} f_j(x) \quad for \quad x \in G_j & j = 1, \ldots, k \\ undefined \ otherwise \end{cases} \tag{5.11}$$

where

$$G_j = \{x \in R^n | g_{ij}(x) \le 0 \quad for \quad i = 1, \ldots, m_j\} \qquad j = 1, \ldots, k.$$

If $f_i(x) = f_j(x)$ for $x \in G_i \cap G_j$, $i \ne j$, then f(x) is a single valued function. If G_i and G_j have not internal joint points for $i \ne j$ and G_j are bounded, then (5.11) can be written in the form

$$f(x) = \sum_{j=1}^{k} u_j f_j(x)$$

subject to zero-one variables u_j and

$$u_1 + u_2 + \ldots + u_k = 1$$

$$g_{ij}(x) \le d(1-u_j) \qquad i = 1, \ldots, m_j \qquad j = 1, \ldots, k.$$

Consider the function (5.3) subject to (5.5) if $y \in G$ exists for which (5.2) is not true. In this case the function (5.3) is not single valued for y. The single valued function can be defined by (5.3) subject to (5.4). Since (5.4) is not convenient in the mathematical programming, instead of (5.3) in some practical cases the function

$$h(x,p) = (1-p)f_1(x) + pf_2(x) \tag{5.12}$$

subject to zero-one variables u_1 and u_2 and

$$-du_1 \le g(x) \le du_2 \qquad u_2 \le p \le 1 - u_1 \tag{5.13}$$

can be used. If $f_1(x)$ and $f_2(x)$ satify the condition (5.2), then

$$h(x,p) = f_1(x) = f_2(x) \quad for \quad x \in G.$$

If (5.2) is not true, then for $x \in G$ it follows

$$min(f_1(x), \ f_2(x)) \le h(x,p) \le max(f_1(x), \ f_2(x)). \tag{5.14}$$

If the function

$$F(x) = \begin{cases} f(x) \text{ defined by (5.1)} & \text{for } g(x) \neq 0 \\ \text{satisfies the condition (5.14)} & \text{for } g(x) = 0 \end{cases}$$

arises in a programming problem, then it can be replaced by (5.12) subject to (5.13). If F(x) arises in the objective function or in one inequation only, then for g(x) = 0 the most favorable value for F(x) is either $f_1(x)$ or $f_2(x)$. In this case F(x) can be replaced by (5.3) subject to (5.5), and (5.3) can be simplified using (1.1) — (1.4). Similar analysis can be made for (5.11).

In case f(x) = |g(x)| from (5.3) it follows

$$f_1(x) = -g(x)$$
$$f_2(x) = g(x)$$
$$|g(x)| = uf_1(x) + (1-u)f_2(x) = (1-2u)g(x).$$

Considering (1.1) — (1.4) for bouded g(x) we obtain

$$|g(x)| = y + z \tag{5.15}$$

subject to (5.5), suitable constant d, zero-one variable u and

$$g(x) = z - y \qquad 0 \le y \le du \qquad 0 \le z \le d(1-u). \tag{5.16}$$

Consider the statistical problem

$$\text{minimize} \sum_{j=1}^{n} (x_j - \sum_{i=1}^{m} p_i f_i(t_j))^2. \tag{5.17}$$

Here p_i are unknown parameters, x_j and t_j are given values, $f_i : R \to R$ are chosen functions and m < n. Instead of (5.17) we can take

$$\text{minimize} \sum_{j=1}^{n} |x_j - \sum_{i=1}^{m} p^i f_i(t_j)|.$$

This problem can be transformed into linear mixed integer programming problem. From (5.15) and (5.16) it follows

$$\text{minimize} \sum_{j=1}^{n} (y_j + z_j)$$

subject to

$$x_j - \sum_{i=1}^{m} p_i f_i(t_j) = z_j - y_j$$

$$-du_j \leq x_j - \sum_{i=1}^{m} p_i f_i(t_j) \leq d(1-u_j)$$

$$0 \leq y_j \leq du_j \qquad 0 \leq z_j \leq d(1-u_j) \qquad j = 1, \ldots, n.$$

Consider the programming problem [1]

$$maximize \sum_{j=1}^{n} c_j x_j + \sum_{l=1}^{k} \alpha_l r_l + \sum_{t=1}^{p} |s_t|$$

subject to $r_l = -1$ or 1 and

$$\sum_{j=1}^{n} a_{ij} x_j + \sum_{l=1}^{k} h_{il} r_l = q_i \qquad i = 1, \ldots, m$$

$$\sum_{j=1}^{n} b_{tj} x_j + s_t = \beta_t \qquad t = 1, \ldots, p$$

where a_{ij}, b_{tj}, h_{il}, β_t, c_j, q_i and α_l are given parameters. It can be transformed into linear mixed integer programming problem.

$$maximize \sum_{j=1}^{n} c_j x_j + \sum_{l=1}^{k} \alpha_l (1-2v_l) + \sum_{t=1}^{p} (y_t + z_t)$$

subject to zero-one variables u_t and v_l, suitable constand d and

$$\sum_{j=1}^{n} a_{ij} x_j + \sum_{l=1}^{k} h_{il}(1-2v_l) = q_i \qquad i = 1, \ldots, m$$

$$\beta_t - \sum_{j=1}^{n} b_{tj} x_j = z_t - y_t$$

$$-du_t \leq \beta_t - \sum_{j=1}^{n} b_{tj} x_j \leq d(1-u_t)$$

$$0 \leq y_t \leq du_t \qquad 0 \leq z_t \leq d(1-u_t) \qquad t = 1, \ldots, p.$$

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NEGLADKI IN NEKONVEKSNI MODELI POSLOVNEGA PROCESA

Rezime

V nekaterih primerih je pri optimiranju poslovanja primerno uporabiti linearni model. V tem članku je s primerom pokazano, da je z linearnim mešanim celoštevilskim modelom mogoče poslovni proces natančneje opisati. S pomočjo linearnega mešanega celoštevilskega modela, ki ga je mogoče obdelati z obstoječimi računalniškimi programi, lahko izrazimo tudi optimizacijske naloge, pri katerih je množica možnih rešitev nekonveksni polieder. Taka naloga se pojavi pri optimiranju poslovanja, kadar imamo na voljo več ciljev, in želimo doseči vsaj enega od njih, če je te cilje mogoče izraziti kot linearne funkcije odločitvenih spremenljivk. Podobne izražave obstajajo tudi za nekatere negladke funkcije. Tako lahko izražamo absolutne vrednosti funkcij, kar je uporabljeno na primeru iz statistike.