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HYBRIDIZATIONS OF ARCHIMEDEAN COPULA AND GENERALIZED MSM OPERATORS AND THEIR APPLICATIONS IN INTERACTIVE DECISION-MAKING WITH Q-RUNG PROBABILISTIC DUAL HESITANT FUZZY ENVIRONMENT

Gogineni Anusha[1*], Paladugu Venkata Ramana[2] and Rupak Sarkar[3]

[1] Department of Engg. Mathematics, College of Engg., Koneru Lakshmaiah Education Foundation, Vaddeswaram-522032, Andhra Pradesh, India
[2] Department of Basic Science & Humanities, Vignan's Lara Institute of Technology and Sciences, Vadlamudi-522213, Andhra Pradesh, India
[3] Department of Mathematics, Women's Polytechnic, Hapania, Tripura, India

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Abstract: *The q-rung probabilistic dual hesitant fuzzy sets (qRPDHFSs), which outperform dual hesitant fuzzy sets, probabilistic dual hesitant fuzzy sets, and probabilistic dual hesitant Pythagorean fuzzy sets, are used in this research to develop an interactive group decision-making approach. We first suggest the Archimedean Copula-based operations on q-rung probabilistic dual hesitant fuzzy (qRPDHF) components and investigate their key features before constructing the approach. We then create some new aggregation operators (AOs) in light of these operations, including the qRPDHF generalized Maclaurin symmetric mean (MSM) operator, qRPDHF geometric generalized MSM operator, qRPDHF weighted generalized MSM operator, and qRPDHF weighted generalized geometric generalized MSM operator. These aggregation operators are better than current operators on qRPDHF because they can take into account the interactions between a large number of criteria and probability distributions. The evaluation findings are distorted since the present methodologies do not take expert involvement into account in order to achieve the requisite consistency level. We employ the idea of interaction, consistency, resemblance, and consensus-building among the decision-makers in our method to get around this. We create an optimization model based on the cross-entropy of the qRPDHF components to estimate the weights of the criterion. We provide a contextual research on the choice of open-source software LMS in order to demonstrate the relevance of the*

* Corresponding author.
E-mail addresses: anushagogineni76@kluniversity.in (G. Anusha), paladuguvenkataramana@gmail.com (P.V. Ramana), rs4magt@gmail.com (R. Sarkar)

recommended AOs. Likewise, we ran a sensitivity test on the weights of the criterion to make sure that our model is consistent. The comparison investigation has demonstrated that the suggested approach can overcome the challenges of previous works.

Key words: q-Rung probabilistic dual hesitant fuzzy set, Archimedean Copula, generalized Maclaurin symmetric mean operators, group decision-making.

1. Introduction

Finding the best option(s) from a pool of readily available possibilities based on several features, both quantitative and qualitative, is the main goal of the multiple criteria decision-making (MCDM) technique. Uncovering the numerical values of qualities can occasionally be a difficult process for an expert. In light of recent scientific and technological developments, uncertainty now dominates decision-making (DM) analyses. Zadeh (1965) proposed the idea of fuzzy sets (FSs) to deal with the data's ambiguity. The hesitant fuzzy set (HFS) (Torra, 2010) is an extension of FSs that permits membership degrees (MDs) to assume a limited number of likely entries as opposed to only one. By include hesitant non-membership degrees (NMDs) together with MDs in the inquiry, Zhu et al. (2012) created the concept of dual HFSs (DHFSs) and named its fundamental component dual hesitant fuzzy elements (DHFEs). The case about the presence of the probabilities of the components in the DHFSs and HFSs is not resolved even if DHFSs and HFSs are successfully used to many DM situations. Let's look at an illustration to better comprehend this: Consider a professional who indicates his propensity for anything negative as a hesitant fuzzy element (HFE) of 0.4, 0.6, or 0.7. He mentioned throughout the tests that the comfort level associated to 0.6 is the greatest when compared to others, while the comfort level related to 0.7 is the worst. Thus, under such a situation, the HFE {0.4, 0.6, 0.7} isn't reasonable to depict the data. Additionally, think about a rating of an expert to assess the quality of an item as the DHFE <{0.4, 0.6, 0.7}, {0.2, 0.3, 0.4}>. During the appraisals, the expert accepts that his solace level toward the item evaluating 0.4, 0.7 is twofold than 0.6 in MDs, while triple toward the 0.4 in the NMDs concerning the others. Hence, the DHFE <{0.4, 0.6, 0.7}, {0.2, 0.3, 0.4}> isn't reasonable to portray the information. To handle such issues, the idea of the probabilistic HFS (PHFS) and probabilistic HFEs (PHFEs) were presented by Xu and Zhou (2017) and were extended to probabilistic DHFS (PDHFS) and probabilistic DHFEs (PDHFEs) respectively by Hao et al. (2017). The PDHFE gives a more exact depiction compare to PHFE, HFE, and DHFE and can effectively portray the data in the above-expressed example. The TODIM approach with PDHFSs was utilised by Ren et al. (2017) for enterprise strategy evaluation. New correlation coefficients were put up by Garg and Kaur (2018) and used to solve problems with PDHFSs and multi-criteria decision-making (MCDM). A strategy selection problem was handled by Ren et al. (2019) utilising an integrated VIKOR and AHP technique using PDHF information. PDHFSs is constrained in that the total grade for both membership and non-membership should not be more than 1. Ji et al. (2021) developed the idea of probabilistic dual-hesitant Pythagorean fuzzy sets (PDHPFSs), which adhere to the requirement that the sum of the squares of MD and NMD should not be more than 1. The q-rung probabilistic dual hesitant fuzzy sets (qRPDHFSs) sets, introduced by Li et al. (2020) hold the constraint that the addition of q^{th} power of the MD and the NMD must accomplish the value in [0, 1]. The qRPDHFS reduces to PDHFS when q =1 and PDHPFS when q =2, which means that the qRPDHFSs are extended versions of PDHFSs and PDHPFSs. Thus qRPDHFSs are more powerful than PDHFSs and PDHPFSs.

In the past few decades, interactive technology has made a series of developments. Sakawa took the lead in considering the interaction between group decision makers (experts) to resolve inconsistencies (Sakawa & Yano, 1985). Some studies have shown that interactive DM gradually and dynamically learns about the personal preference structure under the continuous communication and interaction between experts, and finally obtains the most satisfactory results (Bashiri & Badri, 2011; Reverberi & Talamo, 1999; Shi & Xia, 1997). Watson et al. (1991) believed that the interactive mode within the group is a key variable that affects the rationality of the DM results. Xu and Chen (2007) believe that experts modify their preference information through interaction during the DM process, which can make the decision result more reasonable, and they use a hybrid weighted average operator to aggregate decision information in a fuzzy environment. Cheng et al. (2018) considered the consistency of evaluation results and attribute weights through the interaction between venture capital providers and between venture capital providers and entrepreneurs. Gou el al. (2019) introduced a consistency index to judge the linguistic preference relation of acceptable consistency. Thus, in the literature, there is a significant gap regarding the consideration of interactive DM problems with q-rung probabilistic dual hesitant fuzzy (qRPDHF) information.

In any MCDM method, the primary concern is how to fuse the assessment data of various criterias' for alternatives, and afterward to get the fittest one. Two different ways are there to pick the most suitable alternative. One is the conventional assessment tools, and the other is the information aggregation operators (AOs). The conventional assessment tools can only generate the preference order of alternatives, while information AOs not only generate the preference order of alternatives effectively yet additionally provides comprehensive assessment value of each alternative. As a result, the information AOs can tackle MCDM issues in a more feasible way compares to the conventional assessment tools. Recently, the study of PDHF aggregation operators and their extensions has drawn significant attention to researchers. The PDHF weighted averaging (PDHFWA) operator was created by Hao et al. (2017) and used for risk assessment. To address the problem of decision-making, Garg and Kaur (2018) designed various PDHFS-based Einstein AOs with certain information metrics. The PDHF fuzzy power weighted Hamy mean (PDHPFPWHM) operator was created by Ji et. al. (2021) and utilised to address the MCDM issues. For the purpose of resolving MCDM issues, Li et al. (2020) suggested the q-rung PDHF power weighted Muirhead mean (q-RPDHFPWMM) operator.

a) *Objectives of research*

Real-world multi-criteria group decision-making (MCGDM) situations allow for the observation of the relationships between a variety of factors. In this situation, it is crucial to take into account how the various criteria interact in order to arrive at a more logical conclusion. To date, the PDHF Einstein weighted averaging operator (Garg & Kaur, 2018) and the PDHF weighted averaging AO (Hao et al., 2017) have been used to average data incorporating PDHF information. Additionally, the PDHPFPWHM operator is capable of recording the relationship between characteristics. However, they miss out on the linkages between several input criteria. The q-RPDHFPWMM operator (Li et al., 2020) can manage multi-input dependence between criteria, but it is unable to handle probability distributions, leading to information loss during the aggregation phase. The q-RPDHFPWMM operator (Li et al., 2020) was also solely used to address MCDM issues with variable weightings of criteria. In order to prevent information loss during aggregation, an AO that addresses the link between multiple input attributes and probability distributions in the context of an MCGDM setup with qRPDHF information is required.

b) Research gaps and motivations

A PDHFS reduces to a PHFS if the NMDs alongside their associated probabilities are ignored. Also, a PDHFS turns into a DHFS if the hesitant MDs and NMDs are equally probable. Hence, PDHFSs are generalized versions of the PHFSs and DHFSs. But, PDHFSs cannot fully express the real decision information because sum of MD and NMD must not exceed 1. Extending this restriction to their square sums, we get PDHPFSs. But PDHFSs and PDHPFSs are special cases of qRPDHFSs, since they require that the sum of the q^{th} power of MD and q^{th} power of NMD should not surpass 1.Thus, qRPDHFSs can express the criteria values with higher flexibility. Since Hao et al.'s method (2017) and Garg and Kaur's method (2018) are based on PDHFSs and Ji et's method (2021) is based on PDHPFSs, so Li et al's (Li et al., 2020) method based on qRPDHFSs is more effective compare to Hao et al.'s method (2017), and Garg and Kaur's method (2018)and Ji et's method (2021) for solving real decision-making problems. But Li et al's method has certain drawbacks too.

To analyze the shortcomings of Li et al.'s method (2020), we consider the following two counter examples:

Example 1: Suppose, an Institute is interested to choose an OSS-LMS package among three OSS-LMS packages, namely- Sakai (A_1), eFront (A_2), and Moodle (A_3). These OSS-LMSs are to be assessed by three experts (E_1, E_2 and E_3) depending on three attributes. The details of these options and attributes are presented in Table 1 and Table 2, respectively. To choose the best alternative among these five OSS-LMS, a team is formed involving of three experts. Their initial evaluation results are presented in terms of PDHFEs.

Table 1. Description of the OSS-LMS alternatives

OSS-LMS	Description
Sakai (A_1)	Sakai is an OSS-LMS scheme that offers a flexible and versatile context for teaching, training, analysis, and other associations. Sakai constantly grows based on the requirements of the faculty, learners, and corporations (https://sakaiproject.org/).
eFront (A_2)	eFront LMS extends the finest open source resolutions through the most useful of e-learning. The structure is adaptable, commanding, efficient, and completely functional (http://www.efrontlearning.net/).
MOODLE (A_3)	This is the most prevalent open-source LMS to provide teachers, administrators, and students with one robust, secure, and combined system for training atmospheres (Moodle.org).

Table 2. Criteria details

Criteria	Description
Functionality (C1)	Functionality is the strength of the software to accommodate functions that match the user's specifications when utilizing the software under particular conditions. Functionality is utilized to estimate the level in which an LMS meets the functional specifications of an establishment.
Reliability (C2)	Reliability is the capacity of the software package to work consistently without falling under specific situations. Reliability is practiced to evaluate the level of fault tolerance of the software packages.
Security and privacy (C3)	Security and privacy standards are required to authenticate the efficacy of a structure to safeguard private data and safeguard information from attacks and exposure on a user's computer.

The initial assessment matrices are:
$$M_d = \left[\mathfrak{I}^{(ijd)}(P)\right]_{m\times n} = \left[< \bigcup_a \{\Delta_{ijd}^{(a)}(P^{(a)})\}, \bigcup_b \{\nabla_{ijd}^{(b)}(P^{(b)})\} > \right]_{3\times 3} \quad (d = 1(1)3, j = 1(1)3, i = 1(1)3)$$
are given in the form of Table 3. Suppose that the consensus coefficient among experts should be above 0.97 (i.e; $\Omega^* \geq 0.97$).

Table 3. Initial assessment matrix

		C_1	C_2	C_3
E_1	A_1	<{0.1(0.3), 0.3(0.2), 0.4(0.5)}, {0.6(0.4), 0.2(0.3), 0.3(0.3)}>	<{0.6(0.4), 0.8(0.2), 0.3(0.3), 0.2(0.1)}, {0.3(0.4), 0.2(0.2), 0.7(0.3), 0.9(0.1)}>	<{0.4(0.2), 0.1(0.4), 0.5(0.1), 0.7(0.3)}, {0.1(0.1), 0.5(0.3), 0.4(0.3), 0.3(0.3)}>
	A_2	<{0.3(0.1), 0.7(0.5), 0.8(0.2), 0.5(0.2)}, {0.9(0.5), 0.6(0.1), 0.5(0.4)}>	<{0.2(0.5), 0.5(0.2), 0.6(0.1), 0.3(0.2)}, {0.8(0.4), 0.6(0.2), 0.4(0.4)}>	<{0.5(0.1), 0.8(0.6), 0.4(0.3)}, {0.5(0.2), 0.3(0.4), 0.1(0.4)}>
	A_3	<{0.6(0.5), 0.2(0.2), 0.3(0.3)}, {0.1(0.3), 0.3(0.2), 0.4(0.5)}>	<{0.3(0.2), 0.2(0.1), 0.7(0.1), 0.9(0.6)}, {0.6(0.5), 0.8(0.1), 0.3(0.2), 0.2(0.2)}>	<{0.1(0.2), 0.5(0.4), 0.4(0.3), 0.3(0.1)}, {0.4(0.1), 0.1(0.3), 0.5(0.5), 0.7(0.1)}>
E_2	A_1	<{0.4(0.3), 0.6(0.2), 0.5(0.2), 0.3(0.3)}, {0.3(0.3), 0.7(0.1), 0.8(0.3), 0.5(0.3)}>	<{0.7(0.4), 0.8(0.2), 0.6(0.3), 0.4(0.1)}, {0.2(0.4), 0.5(0.2), 0.6(0.3), 0.3(0.1)}>	<{0.5(0.2), 0.3(0.4), 0.1(0.1), 0.4(0.3)}, {0.5(0.1), 0.8(0.6), 0.4(0.3)}>
	A_2	<{0.1(0.1), 0.3(0.5), 0.4(0.4)}, {0.6(0.6), 0.2(0.2), 0.3(0.2)}>	<{0.6(0.5), 0.8(0.2), 0.3(0.1), 0.2(0.2)}, {0.3(0.3), 0.2(0.1), 0.7(0.2), 0.9(0.4)}>	<{0.4(0.1), 0.1(0.1), 0.5(0.5), 0.7(0.3)}, {0.1(0.1), 0.5(0.4), 0.4(0.4), 0.3(0.1)}>
	A_3	<{0.9(0.4), 0.6(0.1), 0.5(0.5)}, {0.9(0.3), 0.6(0.2), 0.5(0.5)}>	<{0.8(0.3), 00.6(0.1), 0.4(0.6)}, {0.8(0.6), 0.6(0.2), 0.4(0.2)}>	<{0.5(0.3), 0.3(0.4), 0.1(0.3)}, {0.3(0.3), 0.1(0.5), 0.5(0.2)}>
E_3	A_1	<{0.6(0.5), 0.2(0.2), 0.3(0.3)}, {0.1(0.3), 0.3(0.1), 0.4(0.6)}>	<{0.3(0.4), 0.2(0.2), 0.7(0.3), 0.9(0.1)}, {0.6(0.4), 0.8(0.2), 0.3(0.3), 0.2(0.1)}>	<{0.1(0.2), 0.5(0.4), 0.4(0.1), 0.3(0.3)}, {0.4(0.1), 0.1(0.3), 0.5(0.3), 0.7(0.3)}>
	A_2	<{0.9(0.1), 0.6(0.5), 0.5(0.4)}, {0.3(0.5), 0.7(0.1), 0.8(0.2), 0.5(0.2)}>	<{0.8(0.7), 0.6(0.1), 0.4(0.2)}, {0.2(0.3), 0.5(0.1), 0.6(0.2), 0.3(0.4)}>	<{0.5(0.4), 0.3(0.1), 0.1(0.5)}, {0.5(0.1), 0.8(0.8), 0.4(0.1)}>

	C_1	C_2	C_3
A_3	<{0.3(0.4), 0.7(0.1), 0.8(0.2), 0.5(0.3)}, {0.6(0.5), 0.2(0.1), 0.3(0.4)}>	<{0.2(0.2), 0.5(0.1), 0.6(0.1), 0.3(0.6)}, {0.3(0.5), 0.2(0.1), 0.7(0.2), 0.9(0.2)}>	<{0.5(0.2), 0.8(0.7), 0.4(0.1)}, {0.1(0.1), 0.5(0.3), 0.4(0.5), 0.3(0.1)}>

Li et al.'s method (Li et al., 2020) has the limitation that it fails to generate any ranking order in the MCGDM problem described above.

Example 2: (Li et al., 2020) "After preliminary analysis, four possible investment alternatives are taken into account; they are denoted by A_1, A_2, A_3, A_4. In this paper, we consider three commonly used attributes in investment evaluation decision: (1) G_1 the quality of product and service; (2) G_2 social and environmental impacts; (3) G_3 economic benefits. The weight vector of the attributes is $w = (0.3, 0.2, 0.5)^T$. The experts need to assess the four alternatives' performance from three aspects respectively". The initial assessment matrix is presented in Table 4".

Table 4. Initial assessment matrix

	C_1	C_2	C_3
A_1	<{0.7(0.2),0.6(0.2), 0.5(0.6)},{0.2(1)}>	<{0.7(1)},{0.2(1)}>	<{0.2(1)},{0.2(1)}>
A_2	<{0.1(1)},{0.4(1)}>	<{0.3(1)},{0.7(1)}>	<{0.7(1)},{0.3(0.5), 0.2(0.5)}>
A_3	<{0.3(1)},{0.5(1)}>	<{0.6(1)},{0.2(1)}>	<{0.1(1)},{0.7(1)}>
A_4	<{0.05(0.7),0.2(0.3)}, {0.5(1)}>	<{0.3(1)}, {0.6(0.5), 0.4(0.5)}>	<{0.8(1)},{0.1(1)}>

Using Li et al.'s method (Li et al., 2020) with qRPDHFPWMM operator, the scores of the alternatives are respectively 0.2574, 0.1535, -0.2095 and 0.3497. Thus, the ranking order is $A_4 \succ A_1 \succ A_2 \succ A_3$.

As we can see from Example 2, Li et al's method (Li et al., 2020) is capable of producing the ranking order for any MCDM problem, but it has certain drawbacks mentioned below:

1. In Li et al's method (Li et al., 2020), experts' assessments were carried out separately, making it challenging to draw a consistent conclusion. Specifically, it fails to depict the ambiguity of articulating information with the collaboration among experts. So, the assessment outcomes get distorted.
2. Sometimes, the information related to criteria weights is not known or partially unknown due to lack of data, and the expert's limited proficiency. These criteria weights can be determined by experts' personnel inclinations. In the DM technique (Li et al., 2020), due to arbitrary assignment of weights of criteria' for the final aggregation procedure, the preference ranking obtained gets affected. Moreover, the method (Li et al., 2020) leads to information loss as it doesn't not consider any information measure.
3. The q-RPDHFPWMM operator (Li et al., 2020) is capable of capturing the dependency among multiple criteria, but it cannot deal with probability distributions during the aggregation process. As a result, the ranking order obtained is not reasonable.

c) Contributions

The following contributions are included in this paper:

1. We have put out a paradigm for aggregation based on interactions and qRPDHF data. The consistency harmonious weight index (CHWI) and expert assessment

similarity ideas have been used in this framework to examine the expert's subjective and objective weights. The final expert weights are then calculated by taking into account a combination of these factors. Finally, until the required consensus value is obtained, the coefficient of consensus is computed again with expert participation.

2. The cross-entropy measure takes into account the weight of each criterion to address the amount of unclear information. Taking use of this, an optimization model is developed in this work to determine the weights of the criterion.
3. Copulas are functions that link several marginal distributions, which can indicate the correlation among variables and also prevent information loss during aggregation, according to a number of scholars (Bacigal et al., 2015; Beliakov et al., 2007; Grabisch et al., 2011; Han et al., 2020; Nather, 2010; Nelsen, 2013; Tao et al., 2018). The generalised Maclaurin Symmetric mean (GMSM) (Wang et al., 2018) operator generalises the Bonferroni mean, Hamy mean, and Maclaurin symmetric mean operators by changing the parameter values. The GMSM operator takes into account the connections between several criteria. Therefore, the GMSM operator has been expanded to include qRPDHF-GMSM operators with their weighted forms employing Archimedean Copula operations on qRPDHF elements (qRPDHFEs).

In section 2, we concisely discuss some essential concepts namely qRPDHFs, GMSM operator and Archimedean Copula. Section 3 investigates the shortcomings of Li et. al's Method (2017). In section 4, we present the Archimedean Copula based operations between qRPDHFEs and the associated GMSM operators. This section also puts forward the qRPDHFGMSM, qRPDHFGGMSM, qRPDHFWGMSM and qRPDHFWGGMSM operators along their characteristics. In section 5, we provide a group DM methodology using the developed AOs. A case study of open-source software LMS assessment is considered in section 6 to express the applicability of the developed approach. Section 7 deals with impact of parameters, and comparison study. The last section is the conclusions.

2. Preliminaries

Some basic notions are presented here that are relevant to our research.

2.1 q-Rung probabilistic dual hesitant fuzzy set (qRPDHFS)

Definition 1 (Li et al., 2020): A qRPDHFS $\Im(P)$ on a universe set $U=\{u_1,u_2,\ldots,u_n\}$ is described as:

$$\Im(P)=\{<u_i,\mu_{\Im}(P)(u_i),\gamma_{\Im}(P)(u_i)>: u_i\in U\}$$

where

$$\mu_{\Im}(P)(u_i)=\bigcup_a\{\Delta_{u_i}^{(a)}(P^{(a)})\},\ \text{and}\ \ \gamma_{\Im}(P)(u_i)=\bigcup_b\{\nabla_{u_i}^{(b)}(P^{(b)})\}$$

$$(0\le P^{(a)},P^{(b)}\le 1;\sum_a P^{(a)}\le 1;\sum_b P^{(b)}\le 1,0\le(\Delta_{u_i}^{(a)})^q+(\nabla_{u_i}^{(b)})^q\le 1)$$

(for each a and b where $q\ge 1$) express the membership and non-membership degrees, respectively of $u_i\in U$ and the related probabilities are $P^{(a)}$ and $P^{(b)}$ respectively. $\Im(P)$ transforms into a qRPDHF element (qRPDHFE) if it is singleton. It is expressed as:

$$\Im(P) = <\bigcup_a \{\Delta^{(a)}(P^{(a)})\},\ \bigcup_b \{\nabla^{(b)}(P^{(b)})\}>$$

where $0 \le P^{(a)}, P^{(b)} \le 1; \sum_a P^{(a)} \le 1; \sum_b P^{(b)} \le 1$ and $0 \le (\Delta^{(a)})^q + (\nabla^{(b)})^q \le 1$ for each *a* and *b*).

Motivated by the score value, deviation degree and ranking rules of PDHFEs (Hao et al., 2017), we define the followings:

Definition 2: The score value of $\Im(P) = <\bigcup_a \{\Delta^{(a)}(P^{(a)})\},\ \bigcup_b \{\nabla^{(b)}(P^{(b)})\}>$ is given by

$$Sc(\Im(P)) = \sum_a ((\Delta^{(a)})^q \times P^{(a)}) - \sum_b ((\nabla^{(b)})^q \times P^{(b)}) .$$

Sometimes qRPDHFs cannot be compared if their score values become identical. To address this issue, their deviation degrees can be used.

Definition 3: The deviation degree of $\Im(P) = <\bigcup_a \{\Delta^{(a)}(P^{(a)})\},\ \bigcup_b \{\nabla^{(b)}(P^{(b)})\}>$ is given by:

$$D(\Im(P)) = \sqrt{\sum_a ((\Delta^{(a)})^q - (Sc(\Im(P)))^2 \times P^{(a)}) + \sum_b ((\nabla^{(b)})^q - (Sc(\Im(P)))^2 \times P^{(b)})}$$

Thus, deviation degree of a qRPDHFE reflects describes the distance from the average value.

Definition 4: For the qRPDHFEs $\Im^{(1)}(P)$ and $\Im^{(2)}(P)$, a ranking rule is defined as:

A. If $Sc(\Im^{(1)}(P)) > Sc(\Im^{(2)}(P))$, then $\Im^{(1)}(P) \succ \Im^{(2)}(P)$

B. If $Sc(\Im^{(1)}(P)) = Sc(\Im^{(2)}(P))$, then

(a) If $D(\Im^{(1)}(P)) > D(\Im^{(2)}(P))$, then $\Im^{(1)}(P) \succ \Im^{(2)}(P)$

(b) If $D(\Im^{(1)}(P)) = D(\Im^{(2)}(P))$, then $\Im^{(1)}(P) = \Im^{(2)}(P)$

2.2 Generalized Maclaurin symmetric mean operator (GMSM)

Definition 5 (Wang et al., 2018): The GMSM operator is defined by:

$$GMSM^{q;t_1,t_2,\ldots\ldots,t_q}(\xi_1,\xi_2,\ldots,\xi_n) = \left(\frac{1}{{}^nC_q}\sum_{1\le p_1<p_2<\ldots<p_q\le n}\left(\prod_{j=1}^{q}(\xi_{p_j})^{t_j}\right)\right)^{\frac{1}{t_1+t_2+\ldots\ldots+t_q}}$$

where $t_1, t_2, \ldots\ldots, t_q \ge 0$, q is a parameter, and $(p_1, p_2, \ldots, p_q)$ denotes a *q*-tuple combination of $(1, 2, \ldots, n)$.

A few specific cases of the GMSM operator are as follows:

Case-I: When *q*=2 and $t_1 = t_2 = t$, we get the Bonferroni mean (BM) operator (Bonferroni, 1950) given below:

$$BM(\xi_1,\xi_2,\ldots,\xi_n) = \left(\frac{1}{{}^nC_2}\sum_{1\le p_1<p_2\le n}\left(\prod_{j=1}^{2}(\xi_{p_j})^{t_j}\right)\right)^{\frac{1}{2t}} = \left(\frac{1}{n(n-1)}\sum_{i,j=1(i\ne j)}^{n}(\xi_i^{\,t}\,\xi_j^{\,t})\right)^{\frac{1}{2t}}$$

Case-II: When $t_1 = t_2 = \ldots\ldots = t_q = 1$, we get the MSM operator (Maclaurin, 1729) given below:

$$MSM(\xi_1,\xi_2,...,\xi_n)=\left(\frac{1}{{}^nC_q}\sum_{1\le p_1<p_2<...<p_q\le n}\left(\prod_{j=1}^{q}\xi_{p_j}\right)\right)^{\frac{1}{1+1+.....+1}}=\left(\frac{1}{{}^nC_q}\sum_{1\le p_1<p_2<...<p_q\le n}\left(\prod_{j=1}^{q}\xi_{p_j}\right)\right)^{\frac{1}{q}}$$

Case-III: When $t_1=t_2=.....=t_q=\frac{1}{q}$, we get the Hamy mean (HM) operator (Hara et al., 1998) given below:

$$HM(\xi_1,\xi_2,...,\xi_n)=\left(\frac{1}{{}^nC_q}\sum_{1\le p_1<p_2<...<p_q\le n}\left(\prod_{j=1}^{q}(\xi_{p_j})^{\frac{1}{q}}\right)\right)^{\frac{1}{\frac{1}{q}+\frac{1}{q}+.....+\frac{1}{q}}}=\frac{1}{{}^nC_q}\sum_{1\le p_1<p_2<...<p_q\le n}\left(\prod_{j=1}^{q}\xi_{p_j}\right)^{\frac{1}{q}}$$

2.3 Archimedean Copula

Definition 6 (Sklar, 1959): A function $g:[0,1]\times[0,1]\to[0,1]$ is termed as a copula if:

(i) $g(u,0)=g(0,u)=0, g(u,1)=g(1,u)=u\ \forall u\in[0,1]$

(ii) $g(u_1,r_1)+g(u_2,r_2)-g(u_2,r_1)-g(u_1,r_2)\ge 0$, for $u_1,r_1,u_2,r_2\in[0,1]$ with $u_1\le u_2$ and $r_1\le r_2$.

Definition 7 (Sklar, 1959): An Archimedean copula is a mapping $g:[0,1]\times[0,1]\to[0,1]$ given by $g(u,r)=\psi(\Theta(u)+\Theta(r))$ where Θ is a strictly decreasing function and $\psi:[0,\infty)\to[0,1]$ is given as: $\psi(t)=\begin{cases}\Theta^{-1}(t), & t\in[0,\eta(0)]\\ 0, & t\in[\Theta(0),\infty]\end{cases}$

Further, if g is strictly increasing and ψ coincides with Θ, then g is called strict Archimedean Copula and we write: $g(u,r)=\Theta^{-1}(\Theta(u)+\Theta(r))$.

3. Operations between qRPDHFEs and associated GMSM operators

Let us take two qRPDHFEs $\Im^{(1)}(P)$ and $\Im^{(2)}(P)$ and suppose that the probabilities and the fuzzy values presented in $\Im^{(1)}(P)$ and $\Im^{(2)}(P)$ are different. Then, multiplying the fuzzy values with their corresponding probabilities, we may get some unreasonable results. To avoid this, the probabilities can be adjusted in the following manner.

Example 3: Suppose $\Im^{(1)}(P)=<\{0.4(0.8),0.3(0.2)\},\{0.7(0.5),0.8(0.5)\}>$ and $\Im^{(2)}(P)=<\{0.5(1)\},\{0.6(0.4),\ 0.8(0.6)\}>$. Then their corresponding adjusted qRPDHFEs are: $\Im^{(1)}(P)=<\{0.4(0.8),0.3(0.2)\},\ \{0.7(0.4),0.7(0.1),0.8(0.5)\}>$ and $\Im^{(2)}(P)=<\{0.5(0.8),0.5(0.2)\},\{0.6(0.4),0.8(0.1),0.8(0.5)\}>$.

Now, based on the adjusted qRPDHFEs, we propose Archimedean Copula (AC) operations and develop the corresponding GMSM operators.

3.1 Operations between qRPDHFEs based on Archimedean Copula

Definition 8: Let $\Im^{(j)}(P)=<\bigcup_a\{\Delta_j^{(a)}(P^{(a)})\},\bigcup_b\{\nabla_j^{(b)}(P^{(b)})\}>(j=1,2)$ be two adjusted qRPDHFEs. Then, for any $\lambda,\lambda_1,\lambda_2>0$, we define:

1. $\Im^{(1)}(P)\oplus\Im^{(2)}(P)=\left\langle \begin{array}{c} \bigcup_{a}\{\sqrt[q]{1-\Theta^{-1}(\Theta(1-(\Delta_1^{(a)})^q)+\Theta(1-(\Delta_2^{(a)})^q))}(P^{(a)})\}, \\ \bigcup_{b}\{\sqrt[q]{\Theta^{-1}(\Theta((\nabla_1^{(b)})^q)+\Theta((\nabla_2^{(b)})^q))}(P^{(b)})\} \end{array} \right\rangle$

2. $\Im^{(1)}(\Delta)\otimes\Im^{(2)}(\Delta)=\left\langle \begin{array}{c} \bigcup_{a}\{\sqrt[q]{\Theta^{-1}(\Theta((\Delta_1^{(a)})^q)+\Theta((\Delta_2^{(a)})^q))}(P^{(a)})\}, \\ \bigcup_{b}\{\sqrt[q]{1-\Theta^{-1}(\Theta(1-(\nabla_1^{(b)})^q)+\Theta(1-(\nabla_2^{(b)})^q))}(P^{(b)})\} \end{array} \right\rangle$

3. $\lambda\Im^{(1)}(P)=\left\langle \bigcup_{a}\{\sqrt[q]{1-\Theta^{-1}(\lambda\Theta(1-(\Delta_1^{(a)})^q))}\,(P^{(a)})\},\bigcup_{b}\{\sqrt[q]{\Theta^{-1}(\lambda\Theta((\nabla_1^{(b)})^q))}(P^{(b)})\} \right\rangle$

4. $(\Im^{(1)}(P))^{\lambda}=\left\langle \bigcup_{a}\{\sqrt[q]{\Theta^{-1}(\lambda\Theta((\Delta_1^{(a)})^q))}(P^{(a)})\},\bigcup_{b}\{\sqrt[q]{1-\Theta^{-1}(\lambda\Theta(1-(\nabla_1^{(b)})^q))}(P^{(b)})\} \right\rangle$

Theorem 1: Let $\Im^{(j)}(P)=<\bigcup_{a}\{\Delta_j^{(a)}(P^{(a)})\},\bigcup_{b}\{\Delta_j^{(b)}(P^{(b)})\}>(j=1,2)$ be two adjusted qRPDHFEs. Then for $\lambda,\lambda_1,\lambda_2>0$, we have,

1. $\Im^{(1)}(P)\oplus\Im^{(2)}(P)=\Im^{(2)}(P)\oplus\Im^{(1)}(P)$
2. $\Im^{(1)}(P)\otimes\Im^{(2)}(P)=\Im^{(2)}(P)\otimes\Im^{(1)}(P)$
3. $\lambda(\Im^{(1)}(P)\oplus\Im^{(2)}(P))=\lambda\Im^{(1)}(P)\oplus\lambda\Im^{(2)}(P)$
4. $(\Im^{(1)}(P)\otimes\Im^{(2)}(P))^{\lambda}=(\Im^{(1)}(P))^{\lambda}\,\tilde{\otimes}\,(\Im^{(2)}(P))^{\lambda}$
5. $(\lambda_1+\lambda_2)\Im^{(1)}(P)=\lambda_1\Im^{(1)}(P)\oplus\lambda_2\Im^{(1)}(P)$
6. $\Im^{(1)}(P)^{\lambda_1+\lambda_2}=(\Im^{(1)}(P))^{\lambda_1}\otimes(\Im^{(1)}(P))^{\lambda_2}$

Proof: (1)-(2) Straight forward.

3. We have, $\Im^{(1)}(P)\oplus\Im^{(2)}(P)$

$$=\left\langle \begin{array}{c} \bigcup_{a}\{\sqrt[q]{1-\Theta^{-1}(\Theta(1-(\Delta_1^{(a)})^q)+\Theta(1-(\Delta_2^{(a)})^q))}(P^{(a)})\}, \\ \bigcup_{b}\{\sqrt[q]{\Theta^{-1}(\Theta((\nabla_1^{(b)})^q)+\Theta((\nabla_2^{(b)})^q))}(P^{(b)})\} \end{array} \right\rangle$$

Therefore, $\lambda(\Im^{(1)}(P)\oplus\Im^{(2)}(P))$

$$=\left\langle \begin{array}{c} \bigcup_{a}\{\sqrt[q]{1-\Theta^{-1}(\lambda\Theta(1-(\Delta_1^{(a)})^q)+\lambda\Theta(1-(\Delta_2^{(a)})^q))}(P^{(a)})\}, \\ \bigcup_{b}\{\sqrt[q]{\Theta^{-1}(\lambda\Theta((\nabla_1^{(b)})^q)+\lambda\Theta((\nabla_2^{(b)})^q))}(P^{(b)})\} \end{array} \right\rangle$$

On the other hand, $\lambda\Im^{(1)}(P)\oplus\lambda\Im^{(2)}(P)$

$$= \left\langle \bigcup_a \{\sqrt[q]{1-\Theta^{-1}(\lambda\Theta(1-(\Delta_1^{(a)}))^q)}\,(P^{(a)})\}, \bigcup_b \{\sqrt[q]{\Theta^{-1}(\lambda\Theta((\nabla_1^{(b)}))^q)}(P^{(b)})\} \right\rangle$$

$$\oplus \left\langle \bigcup_a \{\sqrt[q]{1-\Theta^{-1}(\lambda\Theta(1-(\Delta_2^{(a)}))^q)}\,(P^{(a)})\}, \bigcup_b \{\sqrt[q]{\Theta^{-1}(\lambda\Theta((\nabla_2^{(b)})^q))}(P^{(b)})\} \right\rangle$$

$$= \left\langle \begin{array}{l} \bigcup_a \{\sqrt[q]{1-\Theta^{-1}(\lambda\Theta(1-(\Delta_1^{(a)})^q)+\lambda\Theta(1-(\Delta_2^{(a)}))^q)}(P^{(a)})\}, \\ \qquad \bigcup_b \{\sqrt[q]{\Theta^{-1}(\lambda\Theta((\nabla_1^{(b)})^q)+\lambda\Theta((\nabla_2^{(b)})^q))}(P^{(b)})\} \end{array} \right\rangle$$

Hence, $\lambda(\Im^{(1)}(P)\oplus\Im^{(2)}(P)) = \lambda\Im^{(1)}(P)\oplus\lambda\Im^{(2)}(P)$.

(4)-(6) Proof is similar to (3).

3.2 Archimedean Copula based GMSM operators

Based on the AC operational laws for qRPDHFEs, we firstly propose qRPDHFGMSM operator and study it's properties.

Definition 9: Let $\Im^{(j)}(P) = < \bigcup_a \{\Delta_j^{(a)}(P^{(a)})\}, \bigcup_b \{\Delta_j^{(b)}(P^{(b)})\} >$ $(j = 1(1)n)$ be a collection of adjusted qRPDHFEs. Then the AC based GMSM operator on qRPDHFEs is denoted by $qRPDHFGMSM$ and is defined by:

$$qRPDHFGMSM^{r;t_1,t_2,\ldots\ldots,t_r}(\Im^{(1)}(P),\Im^{(2)}(P),\ldots,\Im^{(n)}(P))$$
$$= \left(\frac{1}{{}^nC_r} \bigoplus_{1\le p_1<p_2<\ldots<p_r\le n} \left(\bigotimes_{j=1}^r (\Im^{(p_j)}(P))^{t_j} \right) \right)^{\frac{1}{t_1+t_2+\ldots\ldots+t_r}}$$

where $t_1,t_2,\ldots\ldots,t_r \ge 0$, r is a parameter, and $(p_1,p_2,\ldots,p_r)$ denotes a r-tuple combination of $(1,2,\ldots,n)$.

Theorem 2: $qRPDHFGMSM^{r;t_1,t_2,\ldots\ldots,t_r}(\Im^{(1)}(P),\Im^{(2)}(P),\ldots,\Im^{(n)}(P))$ is also a qRPDHFE and

$$qRPDHFGMSM^{r;t_1,t_2,\ldots\ldots,t_r}(\Im^{(1)}(P),\Im^{(2)}(P),\ldots,\Im^{(n)}(P))$$
$$= \left\langle \bigcup_a \left\{ \sqrt[q]{\Theta^{-1}\left(\frac{1}{t_1+t_2+\ldots\ldots+t_r}\Theta\left(1-\Theta^{-1}\left(\frac{1}{{}^nC_r}\sum_{1\le p_1<p_2<\ldots<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^r t_j\Theta((\Delta_{p_j}^{(a)})^q)\right)\right)\right)\right)\right)}(P^{(a)})\right\} \right.$$
$$\left. \bigcup_b \left\{ \sqrt[q]{1-\Theta^{-1}\left(\frac{1}{t_1+t_2+\ldots\ldots+t_r}\Theta\left(1-\Theta^{-1}\left(\frac{1}{{}^nC_r}\sum_{1\le p_1<p_2<\ldots<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^r t_j\Theta(1-(\nabla_{p_j}^{(b)})^q)\right)\right)\right)\right)\right)}(P^{(b)})\right\} \right\rangle$$

Proof: We have,

$$(\Im^{(p_j)}(P))^{t_j} = \left\langle \bigcup_a \left\{ \sqrt[q]{\Theta^{-1}(t_j\Theta((\Delta_{p_j}^{(a)})^q))}(P^{(a)}) \right\}, \bigcup_b \left\{ \sqrt[q]{1-\Theta^{-1}(t_j\Theta(1-(\nabla_{p_j}^{(b)})^q))}(P^{(b)}) \right\} \right\rangle.$$

Therefore, $\bigotimes_{j=1}^r (\Im^{(p_j)}(P))^{t_j}$

$$= \left\langle \bigcup_a \left\{ \sqrt[q]{\Theta^{-1}\left(\sum_{j=1}^r t_j\Theta((\Delta_{p_j}^{(a)})^q)\right)}(P^{(a)}) \right\}, \bigcup_b \left\{ \sqrt[q]{1-\Theta^{-1}\left(\sum_{j=1}^r t_j\Theta(1-(\nabla_{p_j}^{(b)})^q)\right)}(P^{(b)}) \right\} \right\rangle$$

Then, $\underset{1\le p_1<p_2<\ldots<p_r\le n}{\oplus}\left(\overset{r}{\underset{j=1}{\otimes}}(\mathfrak{I}^{(p_j)}(P))^{t_j}\right)$

$$=\left\langle \bigcup_a\left\{\sqrt[q]{1-\Theta^{-1}\left(\sum_{1\le p_1<p_2<\ldots<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^{r}t_j\Theta((\Delta_{p_j}^{(a)})^q)\right)\right)\right)}(P^{(a)})\right\},\right.$$

$$\left.\bigcup_b\left\{\sqrt[q]{\Theta^{-1}\left(\sum_{1\le p_1<p_2<\ldots<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^{r}t_j\Theta(1-(\nabla_{p_j}^{(b)})^q)\right)\right)\right)}(P^{(b)})\right\}\right\rangle$$

Now $\frac{1}{{}^nC_r}\underset{1\le p_1<p_2<\ldots<p_r\le n}{\oplus}\left(\overset{r}{\underset{j=1}{\otimes}}(\mathfrak{I}^{(p_j)}(P))^{t_j}\right)$

$$=\left\langle \bigcup_a\left\{\sqrt[q]{1-\Theta^{-1}\left(\frac{1}{{}^nC_r}\sum_{1\le p_1<p_2<\ldots<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^{r}t_j\Theta((\Delta_{p_j}^{(a)})^q)\right)\right)\right)}(P^{(a)})\right\},\right.$$

$$\left.\bigcup_b\left\{\sqrt[q]{\Theta^{-1}\left(\frac{1}{{}^nC_r}\sum_{1\le p_1<p_2<\ldots<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^{r}t_j\Theta(1-(\nabla_{p_j}^{(b)})^q)\right)\right)\right)}(P^{(b)})\right\}\right\rangle$$

Hence, $qRPDHFGMSM^{r;t_1,t_2,\ldots\ldots,t_r}(\mathfrak{I}^{(1)}(P),\mathfrak{I}^{(2)}(P),\ldots,\mathfrak{I}^{(n)}(P))$

$$=\left(\frac{1}{{}^nC_r}\underset{1\le p_1<p_2<\ldots<p_r\le n}{\oplus}\left(\overset{r}{\underset{j=1}{\otimes}}(\mathfrak{I}^{(p_j)}(P))^{t_j}\right)\right)^{\frac{1}{t_1+t_2+\ldots..+t_r}}$$

$$=\left\langle \bigcup_a\left\{\sqrt[q]{\Theta^{-1}\left(\frac{1}{t_1+t_2+\ldots..+t_r}\Theta\left(1-\Theta^{-1}\left(\frac{1}{{}^nC_r}\sum_{1\le p_1<p_2<\ldots<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^{r}t_j\Theta((\Delta_{p_j}^{(a)})^q)\right)\right)\right)\right)\right)}\,(P^{(a)})\right\}\right.$$

$$\left.\bigcup_b\left\{\sqrt[q]{1-\Theta^{-1}\left(\frac{1}{t_1+t_2+\ldots..+t_r}\Theta\left(1-\Theta^{-1}\left(\frac{1}{{}^nC_r}\sum_{1\le p_1<p_2<\ldots<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^{r}t_j\Theta(1-(\nabla_{p_j}^{(b)})^q)\right)\right)\right)\right)\right)}(P^{(b)})\right\}\right\rangle$$

In the following, some vital properties of the $qRPDHFGMSM$ operator are presented.

Theorem 3: (Idempotency) If $\mathfrak{I}^{(j)}(P)=\mathfrak{I}^{(L)}(P)\ \forall j$ (L being a fixed natural number), then $qRPDHFGMSM^{r;t_1,t_2,\ldots\ldots,t_r}(\mathfrak{I}^{(1)}(P),\mathfrak{I}^{(2)}(P),\ldots,\mathfrak{I}^{(n)}(P))=\mathfrak{I}^{(L)}(P)$.

Theorem 4: (Monotonicity) Let $\mathfrak{I}'^{(j)}(P)=<\bigcup_a\{\Delta_j'^{(a)}(P^{(a)})\},\bigcup_b\{\Delta_j'^{(b)}(P^{(b)})\}>$ ($j=1(1)n$) be another collection of adjusted qRPDHFEs such that $\forall j$, $\Delta_j^{(a)}\le\Delta_j'^{(a)}$ and $\nabla_j^{(b)}\ge\nabla_j'^{(b)}$. Then,

$qRPDHFGMSM^{r;t_1,t_2,\ldots,t_r}(\mathfrak{I}^{(1)}(P),\mathfrak{I}^{(2)}(P),\ldots,\mathfrak{I}^{(n)}(P))\prec qRPDHFGMSM^{r;t_1,t_2,\ldots,t_r}\mathfrak{I}^{(n)}(P))\prec$
$qRPDHFGMSM^{r;t_1,t_2,\ldots\ldots,t_r}(\mathfrak{I}'^{(1)}(P),\mathfrak{I}'^{(2)}(P),\ldots,\mathfrak{I}'^{(n)}(P))$.

Theorem 5: (Boundedness) If $\Im^{(j-)}(P) =< \{\min_a \Delta_j^{(a)}(P^{(a)})\}, \{\max_b \nabla_j^{(b)}(P^{(b)})\} >$ and $\Im^{(j+)}(P) =< \{\max_a \Delta_j^{(a)}(P^{(a)})\}, \{\min_b \nabla_j^{(b)}(P^{(b)})\} >$, then $\Im^{(j-)}(P) \prec qRPDHFGMSM^{r;t_1,t_2,......,t_r}(\Im^{(1)}(P), \Im^{(2)}(P), \ldots, \Im^{(n)}(P)) \prec \Im_{\wp}^{(j+)}(P)$.

Next, based on the Archimedean Copula operational laws for qRPDHFEs, we propose qRPDHF geometric GMSM operator and study it's properties.

Definition 10: The AC based geometric GMSM operator on qRPDHFEs is denoted by $qRPDHFGGMSM$ and is defined by:

$$qRPDHFGGMSM^{r;t_1,t_2,...,t_r}(\Im^{(1)}(P), \Im^{(2)}(P), \ldots, \Im^{(n)}(P))$$

$$= \frac{1}{t_1 + t_2 + ... + t_r}\left(\bigotimes_{1 \le p_1 < p_2 < \ldots < p_r \le n}\left(\bigoplus_{j=1}^{r}\left(t_j \Im^{(p_j)}(P)\right)\right)\right)^{\frac{1}{^nC_r}}$$

where $t_1, t_2, ..., t_r \ge 0$, r is a parameter, and $(p_1, p_2, \ldots, p_r)$ denotes a r-tuple combination of $(1, 2, \ldots, n)$.

Theorem 6: $qRPDHFGGMSM^{r;t_1,t_2,......,t_r}(\Im^{(1)}(P), \Im^{(2)}(P), \ldots, \Im^{(n)}(P))$ is also a qRPDHFE and

$$qRPDHFGGMSM^{r;t_1,t_2,......,t_r}(\Im^{(1)}(P), \Im^{(2)}(P), \ldots, \Im^{(n)}(P))$$

$$= \left\langle \bigcup_a \left\{ \sqrt[q]{1 - \Theta^{-1}\left(\frac{1}{\sum_r t_r}\Theta\left(1 - \Theta^{-1}\left(\frac{1}{^nC_r}\sum_{1 \le p_1 < p_2 < \ldots < p_r \le n}\Theta\left(1 - \Theta^{-1}\left(\sum_{j=1}^{r} t_j \Theta(1 - (\Delta_{p_j}^{(a)})^q)\right)\right)\right)\right)\right)}(P^{(a)}) \right\},\right.$$

$$\left. \bigcup_b \left\{ \sqrt[q]{\Theta^{-1}\left(\frac{1}{\sum_r t_r}\Theta\left(1 - \Theta^{-1}\left(\frac{1}{^nC_r}\sum_{1 \le p_1 < p_2 < \ldots < p_r \le n}\Theta\left(1 - \Theta^{-1}\left(\sum_{j=1}^{r} t_j \Theta((\nabla_{p_j}^{(b)})^q)\right)\right)\right)\right)\right)}(P^{(b)}) \right\} \right\rangle$$

Proof: Similar to Theorem 2.

In the following, some vital properties of the $qRPDHFGGMSM$ operator are presented.

Theorem 7: (Idempotency) If $\Im^{(j)}(P) = \Im^{(L)}(P)\ \forall j$ (L being a fixed natural number), then $qRPDHFGGMSM^{r;t_1,t_2,......,t_r}(\Im^{(1)}(P), \Im^{(2)}(P), \ldots, \Im^{(n)}(P)) = \Im^{(L)}(P)$.

Theorem 8: (Monotonicity) Let $\Im'^{(j)}(P) =< \bigcup_a \{\Delta_j'^{(a)}(P^{(a)})\}, \bigcup_b \{\Delta_j'^{(b)}(P^{(b)})\} > (j = 1(1)n)$ be another collection of adjusted qRPDHFEs such that $\forall j$, $\Delta_j^{(a)} \le \Delta_j'^{(a)}$ and $\nabla_j^{(b)} \ge \nabla_j'^{(b)}$. Then,

$qRPDHFGGMSM^{r;t_1,t_2,....,t_r}(\Im^{(1)}(P), \Im^{(2)}(P), \ldots, \Im^{(n)}(P)) \prec qRPDHFGGMSM^{r;t_1,t_2,....,t_r}(\Im'^{(1)}(P), \Im'^{(2)}(P), \ldots, \Im'^{(n)}(P))$.

Theorem 9: (Boundedness) If $\Im^{(j-)}(P) =< \{\min_a \Delta_j^{(a)}(P^{(a)})\}, \{\max_b \nabla_j^{(b)}(P^{(b)})\} >$ and $\Im^{(j+)}(P) =< \{\max_a \Delta_j^{(a)}(P^{(a)})\}, \{\min_b \nabla_j^{(b)}(P^{(b)})\} >$, then $\Im^{(j-)}(P) \prec qRPDHFGGMSM^{r;t_1,t_2,......,t_r}(\Im^{(1)}(P), \Im^{(2)}(P), \ldots, \Im^{(n)}(P)) \prec \Im_{\wp}^{(j+)}(P)$.

3.3 Archimedean copula based weighted GMSM operator

Although the qRPDHFGMSM operator can tackle the interrelationship among multiple input criteria, it does not take into account the self importance of the qRPDHFEs. To overcome this problem, we propose qRPDHF weighted GMSM operator (qRPDHFWGMSM operator) based on Archimedean Copula.

Definition 11: The AC based qRPDHFWGMSM operator on qRPDHFEs is defined by:

$$qRPDHFWGMSM^{r;t_1,t_2,\ldots\ldots,t_r}(\Im^{(1)}(P),\Im^{(2)}(P),\ldots,\Im^{(n)}(P))$$

$$=\left(\frac{1}{{}^{n}C_r}\underset{1\le p_1<p_2<\ldots<p_r\le n}{\oplus}\left(\overset{r}{\underset{j=1}{\otimes}}(w_{p_j}\Im^{(p_j)}(P))^{t_j}\right)\right)^{\frac{1}{t_1+t_2+\ldots..+t_r}}$$

where $t_1,t_2,\ldots\ldots,t_r\ge 0$, r is a parameter, $(p_1,p_2,\ldots,p_r)$ denotes a r-tuple combination of $(1,2,\ldots,n)$ and w_j denotes the weight of $\Im^{(j)}(P)$ with $w_j\ge 0$ and $\sum_j w_j=1$.

Theorem 10: $qRPDHFWGMSM^{r;t_1,t_2,\ldots\ldots,t_r}(\Im^{(1)}(P),\Im^{(2)}(P),\ldots,\Im^{(n)}(P))$ is also a qRPDHFE and

$$qRPDHFWGMSM^{r;t_1,t_2,\ldots\ldots,t_r}(\Im^{(1)}(P),\Im^{(2)}(P),\ldots,\Im^{(n)}(P))$$

$$=\left\langle \bigcup_a\left\{\sqrt[q]{\Theta^{-1}\left(\frac{1}{\sum_r t_r}\Theta\left(1-\Theta^{-1}\left(\frac{1}{{}^{n}C_r}\sum_{1\le p_1<p_2<\ldots<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^{r}t_j\Theta(1-\Theta^{-1}(w_{p_j}\Theta(1-(\Delta_{p_j}^{(a)})^q)))\right)\right)\right)\right)\right)}(P^{(a)})\right\},\right.$$

$$\left.\bigcup_b\left\{\sqrt[q]{1-\Theta^{-1}\left(\frac{1}{\sum_r t_r}\Theta\left(1-\Theta^{-1}\left(\frac{1}{{}^{n}C_r}\sum_{1\le p_1<p_2<\ldots<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^{r}t_j\Theta(1-\Theta^{-1}(w_{p_j}\Theta((\nabla_{p_j}^{(b)})^q)))\right)\right)\right)\right)\right)}(P^{(b)})\right\}\right\rangle$$

Proof: We have, $w_{p_j}\Im^{(p_j)}(P)$

$$=\left\langle \bigcup_a\left\{\sqrt[q]{1-\Theta^{-1}(w_{p_j}\Theta(1-(\Delta_{p_j}^{(a)})^q))}(P^{(a)})\right\},\bigcup_b\left\{\sqrt[q]{\Theta^{-1}(w_{p_j}\Theta((\nabla_{p_j}^{(b)})^q))}(P^{(b)})\right\}\right\rangle.$$

Then, $(w_{p_j}\Im^{(p_j)}(P))^{t_j}$

$$=\left\langle \bigcup_a\left\{\sqrt[q]{\Theta^{-1}(t_j\Theta(1-\Theta^{-1}(w_{p_j}\Theta(1-(\Delta_{p_j}^{(a)})^q))))}(P^{(a)})\right\},\right.$$

$$\left.\left\{\bigcup_b\left\{\sqrt[q]{1-\Theta^{-1}(t_j\Theta(1-\Theta^{-1}(w_{p_j}\Theta((\nabla_{p_j}^{(b)})^q))))}(P^{(b)})\right\}\right\}\right\rangle$$

So, $\overset{r}{\underset{j=1}{\otimes}}(w_{p_j}\Im^{(p_j)}(P))^{t_j}$

$$=\left\langle \begin{array}{l}\bigcup_a\left\{\sqrt[q]{\Theta^{-1}\left(\sum_{j=1}^{r}t_j\Theta(1-\Theta^{-1}(w_{p_j}\Theta(1-(\Delta_{p_j}^{(a)})^q)))\right)}(P^{(a)})\right\},\\ \bigcup_b\left\{\sqrt[q]{1-\Theta^{-1}\left(\sum_{j=1}^{r}t_j\Theta(1-\Theta^{-1}(w_{p_j}\Theta((\nabla_{p_j}^{(b)})^q)))\right)}(P^{(b)})\right\}\end{array}\right\rangle$$

Therefore, $\underset{1\le p_1<p_2<...<p_r\le n}{\oplus}\left(\overset{r}{\underset{j=1}{\otimes}}(w_{p_j}\mathfrak{I}^{(p_j)}(P))^{t_j}\right)$

$$=\left\langle \bigcup_a\left\{\sqrt[q]{1-\Theta^{-1}\left(\sum_{1\le p_1<p_2<...<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^{q}t_j\Theta(1-\Theta^{-1}(w_{p_j}\Theta(1-(\Delta_{p_j}^{(a)})^q)))\right)\right)\right)}(P^{(a)})\right\},\right.$$

$$\left.\bigcup_b\left\{\sqrt[q]{\Theta^{-1}\left(\sum_{1\le p_1<p_2<...<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^{q}t_j\Theta(1-\Theta^{-1}(w_{p_j}\Theta((\nabla_{p_j}^{(b)})^q)))\right)\right)\right)}\ (P^{(b)})\right\}\right\rangle$$

Then, $\frac{1}{{}^nC_r}\underset{1\le p_1<p_2<...<p_r\le n}{\oplus}\left(\overset{r}{\underset{j=1}{\otimes}}(w_{p_j}\mathfrak{I}^{(p_j)}(P))^{t_j}\right)$

$$=\left\langle \bigcup_a\left\{\sqrt[q]{1-\Theta^{-1}\left(\frac{1}{{}^nC_r}\sum_{1\le p_1<p_2<...<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^{r}t_j\Theta(1-\Theta^{-1}(w_{p_j}\Theta(1-(\Delta_{p_j}^{(a)})^q)))\right)\right)\right)}(P^{(a)})\right\},\right.$$

$$\left.\bigcup_b\left\{\sqrt[q]{\Theta^{-1}\left(\frac{1}{{}^nC_r}\sum_{1\le p_1<p_2<...<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^{r}t_j\Theta(1-\Theta^{-1}(w_{p_j}\Theta((\nabla_{p_j}^{(b)})^q)))\right)\right)\right)}\ (P^{(b)})\right\}\right\rangle$$

Hence, $qRPDHFWGMSM^{r;t_1,t_2,......,t_r}(\mathfrak{I}^{(1)}(P),\mathfrak{I}^{(2)}(P),...,\mathfrak{I}^{(n)}(P))$

$$=\left\langle \bigcup_a\left\{\sqrt[q]{\Theta^{-1}\left(\frac{1}{\sum_r t_r}\Theta\left(1-\Theta^{-1}\left(\frac{1}{{}^nC_r}\sum_{1\le p_1<p_2<...<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^{r}t_j\Theta(1-\Theta^{-1}(w_{p_j}\Theta(1-(\Delta_{p_j}^{(a)})^q)))\right)\right)\right)\right)\right)}(P^{(a)})\right\},\right.$$

I

$$\left.\bigcup_b\left\{\sqrt[q]{1-\Theta^{-1}\left(\frac{1}{\sum_r t_r}\Theta\left(1-\Theta^{-1}\left(\frac{1}{{}^nC_r}\sum_{1\le p_1<p_2<...<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^{r}t_j\Theta(1-\Theta^{-1}(w_{p_j}\Theta((\nabla_{p_j}^{(b)})^q)))\right)\right)\right)\right)\right)}\ (P^{(b)})\right\}\right\rangle$$

n the following, some vital properties of the $qRPDHFWGMSM$ operator are presented.

Theorem 11: (Idempotency) If $\mathfrak{I}^{(j)}(P)=\mathfrak{I}^{(L)}(P)\ \forall j$ (L being a fixed natural number), then $qRPDHFWGMSM^{r;t_1,t_2,......,t_r}(\mathfrak{I}^{(1)}(P),\mathfrak{I}^{(2)}(P),...,\mathfrak{I}^{(n)}(P))=\mathfrak{I}^{(L)}(P)$.

Theorem 12: (Monotonicity) Let $\mathfrak{I}'^{(j)}(P)=<\bigcup_a\{\Delta_j'^{(a)}(P^{(a)})\},\bigcup_b\{\Delta_j'^{(b)}(P^{(b)})\}>$ $(j=1(1)n)$ be another collection of adjusted qRPDHFEs such that $\forall j$, $\Delta_j^{(a)}\le\Delta_j'^{(a)}$ and $\nabla_j^{(b)}\ge\nabla_j'^{(b)}$. Then $qRPDHFWGMSM^{r;t_1,t_2,....,t_r}(\mathfrak{I}^{(1)}(P),\mathfrak{I}^{(2)}(P),...,\mathfrak{I}^{(n)}(P))\prec qRPDHFWGMSM^{r;t_1,t_2,....,t_r}(\mathfrak{I}'^{(1)}(P),$ $\mathfrak{I}'^{(2)}(P),...,\mathfrak{I}'^{(n)}(P))$.

Theorem 13: (Boundedness) If $\mathfrak{I}^{(j-)}(P)=<\{\min_a\Delta_j^{(a)}(P^{(a)})\},\{\max_b\nabla_j^{(b)}(P^{(b)})\}>$ and $\mathfrak{I}^{(j+)}(P)=<\{\max_a\Delta_j^{(a)}(P^{(a)})\},$ $\{\min_b\nabla_j^{(b)}(P^{(b)})\}>$, then $\mathfrak{I}^{(j-)}(P)\prec qRPDHFWGMSM^{r;t_1,t_2,......,t_r}(\mathfrak{I}^{(1)}(P),\mathfrak{I}^{(2)}(P),\ ...,\mathfrak{I}^{(n)}(P))\prec\mathfrak{I}_{\wp}^{(j+)}(P)$.

The qRPDHFGMSM operator considers the interrelationship among multiple criteria. But, it does not deal with the self priority of the qRPDHFEs. To overcome this problem, we propose qRPDHF weighted geometric GMSM operator (qRPDHFWGGMSM operator) based on Archimedean Copula.

Definition 12: The AC based qRPDHFWGGMSM operator on qRPDHFEs is defined by:

$$qRPDHFWGGMSM^{r;t_1,t_2,\ldots\ldots,t_r}(\Im^{(1)}(P),\Im^{(2)}(P),\ldots,\Im^{(n)}(P))$$

$$=\frac{1}{\sum_r t_r}\left(\underset{1\le p_1<p_2<\ldots<p_r\le n}{\otimes}\left(\overset{r}{\underset{j=1}{\oplus}}\left(t_j\left(\Im^{(p_j)}(P)\right)^{w_{p_j}}\right)\right)\right)^{\frac{1}{^nc_r}}$$

where $t_1,t_2,\ldots..,t_r\ge 0$, r is a parameter, $(p_1,p_2,\ldots,p_r)$ denotes a r-tuple combination of $(1,2,\ldots,n)$ and w_j denotes the weight of $\Im^{(j)}(P)$ with $w_j\ge 0$ and $\sum_j w_j=1$.

Theorem 14: $qRPDHFWGGMSM^{r;t_1,t_2,\ldots\ldots,t_r}(\Im^{(1)}(P),\Im^{(2)}(P),\ldots,\Im^{(n)}(P))$ is also a qRPDHFE and

$$qRPDHFWGGMSM^{r;t_1,t_2,\ldots\ldots,t_r}(\Im^{(1)}(P),\Im^{(2)}(P),\ldots,\Im^{(n)}(P))$$

$$=\left\langle\bigcup_a\left\{\sqrt[q]{1-\Theta^{-1}\left(\frac{1}{\sum_r t_r}\Theta\left(1-\Theta^{-1}\left(\frac{1}{^nC_r}\sum_{1\le p_1<p_2<\ldots<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^{r}t_j\Theta(1-\Theta^{-1}(w_{p_j}\Theta((\Delta_{p_j}^{(a)})^q)))\right)\right)\right)\right)\right)}(P^{(a)})\right\},\right.$$ P

$$\left.\bigcup_b\left\{\Theta^{-1}\left(\sqrt[q]{\frac{1}{\sum_r t_r}\Theta\left(1-\Theta^{-1}\left(\frac{1}{^nC_r}\sum_{1\le p_1<p_2<\ldots<p_r\le n}\Theta\left(1-\Theta^{-1}\left(\sum_{j=1}^{r}t_j\Theta(1-\Theta^{-1}(w_{p_j}\Theta(1-(\nabla_{p_j}^{(b)})^q)))\right)\right)\right)\right)}\right)(P^{(b)})\right\}\right\rangle$$

roof: Similar to Theorem 10.

A few crucial properties of the $qRPDHFWGGMSM$ operator are demonstrated below.

Theorem 15: (Idempotency) If $\Im^{(j)}(P)=\Im^{(L)}(P)\ \forall j$ (L being a fixed natural number), then $qRPDHFWGGMSM^{r;t_1,t_2,\ldots\ldots,t_r}(\Im^{(1)}(P),\Im^{(2)}(P),\ldots,\Im^{(n)}(P))=\Im^{(L)}(P)$.

Theorem 16: (Monotonicity) Let $\Im'^{(j)}(P)=<\bigcup_a\{\Delta_j'^{(a)}(P^{(a)})\},\bigcup_b\{\Delta_j'^{(b)}(P^{(b)})\}>(j=1(1)n)$ be another collection of adjusted qRPDHFEs such that $\forall j$, $\Delta_j^{(a)}\le\Delta_j'^{(a)}$ and $\nabla_j^{(b)}\ge\nabla_j'^{(b)}$. Then, $qRPDHFWGGMSM^{r;t_1,t_2,\ldots\ldots,t_r}(\Im^{(1)}(P),\Im^{(2)}(P),\ldots,\Im^{(n)}(P))\prec qRPDHFWGGMSM^{r;t_1,t_2,\ldots\ldots,t_r}$ $(\Im'^{(1)}(P),\Im'^{(2)}(P),\ldots,\Im'^{(n)}(P))$.

Theorem 17: (Boundedness) For $\Im^{(j-)}(P)=<\{\min_a\Delta_j^{(a)}(P^{(a)})\},\{\max_b\nabla_j^{(b)}(P^{(b)})\}>$ and $\Im^{(j+)}(P)=<\{\max_a\Delta_j^{(a)}(P^{(a)})\},\ \{\min_b\nabla_j^{(b)}(P^{(b)})\}>$, $\Im^{(j-)}(P)\prec qRPDHFWGGMSM^{r;t_1,t_2,\ldots\ldots,t_r}$ $(\Im^{(1)}(P),\Im^{(2)}(P),\ldots,\Im^{(n)}(P))\prec\Im_{\wp}^{(j+)}(P)$.

4. Group decision-making methodology with interaction of experts

Suppose there are *m* number of options A_i $(i=1,2,\ldots,m)$ and *n* number of criteria C_j $(j=1,2,\ldots,n)$ connected with a decision-making issue with '*d*' number of experts

$E_d\ (d=1,2,\ldots,l)$ under qRPDHF setting. Consider the qRPDHF matrices $M_d=\left[\Im^{(ijd)}(P)\right]_{m\times n}=\left[<\bigcup_a\{\Delta_{ijd}^{(a)}(P^{(a)})\},\bigcup_b\{\nabla_{ijd}^{(b)}(P^{(b)})\}>\right]_{m\times n}$ as initial assessments of experts. Then the developed method has the steps mentioned below:

Step 1: Derive the adjusted qRPDHF matrices $\tilde{M}_d=\left[\tilde{\Im}^{(ijd)}(P)\right]_{m\times n}=\left[<\bigcup_a\{\tilde{\Delta}_{ijd}^{(a)}(P^{(a)})\},\bigcup_b\{\tilde{\nabla}_{ijd}^{(b)}(P^{(b)})\}>\right]_{m\times n}$ $(d=1,2,\ldots,l)$ where each $\tilde{\Im}^{(ijd)}(P)$ represents an adjusted qRPDHFE.

Step 2: Calculate the weights of the experts.

Suppose ξ_d = weight of the expert E_d,

ξ_d' = subjective weight of the expert E_d,

ξ_d'' = objective weight of the expert E_d.

As pointed out by Cheng et al. (2018), ξ_d' gives the CHWI (consistency harmonious weight index defined by

$$CHWI(E_d)=\sum_{j=1}^{n}n\Bigg/\left(\sum_{i=1}^{m}\sum_{t=1}^{m}\hbar(\tilde{\Im}^{(ijd)}(P))\times\hbar(\tilde{\Im}^{(tji)}(P))\right)\quad(d=1,2,\ldots,l)\tag{1}$$

$$\text{where }\hbar(\tilde{\Im}^{(ijd)}(P))=\sum_a\tilde{\Delta}_{ijd}^{(a)}\times P^{(a)}+\sum_b\tilde{\nabla}_{ijd}^{(b)}\times P^{(b)}.$$

If $CHWI(E_d)=1$, $\tilde{M}_d$ is consistent (Han & Li, 1994). The subjective weight ξ_d' is calculated as:

$$\xi_d'=CHWI(E_d)\Bigg/\sum_{d=1}^{l}CHWI(E_d)\quad(d=1(1)l)\tag{2}$$

In the event that the closeness between $\tilde{M}_d$ and $\tilde{M}_s\ (d\neq s)$ is high, the impact of $\tilde{M}_d$ is more important.

Let θ_{ds} be the angle between $\tilde{M}_d$ and $\tilde{M}_s\ (d\neq s)$. Then θ_{ds} can be computed by

$$\theta_{ds}=\frac{\left\langle V(\tilde{M}_d),V(\tilde{M}_s)\right\rangle}{\left\|V(\tilde{M}_d)\right\|\times\left\|V(\tilde{M}_s)\right\|}\qquad(d,s=1(1)l)\tag{3}$$

where $V(\tilde{M}_d)$ =derived vector of $\tilde{M}_d$

$$=\Big(\tilde{\Im}^{(11d)}(P),\tilde{\Im}^{(21d)}(P),\ldots,\tilde{\Im}^{(m1d)}(P),\tilde{\Im}^{(12d)}(P),\tilde{\Im}^{(22d)}(P),\ldots,\tilde{\Im}^{(m2d)}(P),\ldots,\tilde{\Im}^{(1nd)}(P),$$
$$\tilde{\Im}^{(2nd)}(P),\ldots,\tilde{\Im}^{(mnd)}(P)\Big)$$

and $\left\langle V(\tilde{M}_d),V(\tilde{M}_s)\right\rangle=\sum_{i=1}^{m}\sum_{j=1}^{n}(\hbar(\tilde{\Im}^{(ijd)}(P))+\hbar(\tilde{\Im}^{(ijs)}(P)))$.

Obviously, $0\le\theta_{ds}\le1\ (d,s=1(1)l)$.

$$\text{Suppose}\quad\theta_d=\sum_{s=1,s\neq d}^{l}\theta_{ds}\tag{4}$$

Then θ_d expresses the closeness between $\tilde{M}_d$ and $\tilde{M}_s\ (d\neq s)$. We normalize θ_d using Eq. (5) to obtain the objective weight of each criterion.

$$\xi''_d = \frac{\theta_d}{\sum_{d=1}^{l}\theta_d} \quad (d = 1(1)l) \tag{5}$$

Utilizing a mix of $\xi'_d\ (d = 1(1)l)$ and $\xi''_d\ (d = 1(1)l)$, experts' final weights can be calculated by:

$$\xi_d = \varpi\xi'_d + (1-\varpi)\xi''_d\ (d = 1(1)l) \tag{6}$$

In Eq. (6), ϖ sorts out which weight is dominating. The experts lean towards the subjective weights if ϖ is high; and in case if ϖ is low, experts favor the objective weights. The parameter $\varpi \in [0,\ 1]$ is termed as the risk attitude of experts. The greater the value of ϖ, the more inclination of the expert towards risk. Also, he/she turns into a risk adverse person who regards the trustworthiness of the evaluation group.

Step 3: Obtain the consensus coefficient.

In an innovative MCGDM procedure, the weights of experts will change if the information provided changes. To get a more reasonable decision result, experts should interrelate with each other and lastly take a decision on assessing information. We utilize the symbol $\Omega *$ to express the consensus coefficient. The judgment matrix of the dth expert obtained from the δ round interaction is defined by $V(\tilde{M}_d^{\delta})$ = $\left(\tilde{\Im}^{(11d)(\delta)}(P), \tilde{\Im}^{(21d)(\delta)}(P), \ldots, \tilde{\Im}^{(m1d)(\delta)}(P), \tilde{\Im}^{(12d)(\delta)}(P), \tilde{\Im}^{(22d)(\delta)}(P), \ldots, \tilde{\Im}^{(m2d)(\delta)}(P), \ldots, \tilde{\Im}^{(1nd)(\delta)}(P), \tilde{\Im}^{(2nd)(\delta)}(P), \ldots, \tilde{\Im}^{(mnd)(\delta)}(P)\right)\ (d = 1(1)l).$

The consensus coefficient $\Omega^{(\delta)}$ which is computed by the δ th adjustment can be defined by

$$\Omega^{(\delta)} = \frac{\sum_{d=1}^{l}\sum_{s=1}^{l}\theta_{ds}}{l(l-1)} \quad (d \neq s) \tag{7}$$

Especially $\Omega^{(\delta)}$ =1 if and only if $V(\tilde{M}_d^{\delta}) = V(\tilde{M}_s^{\delta})$. In that case the opinions of the experts are fully unified, then $0 \le \Omega^{(\delta)} < 1$. Thus, throughout the DM procedure, one expert should provide the weights to others based on the assessment information and estimate the consensus coefficient in the δ round after he/she has obtained a consensus value $\Omega *$ in advance, and then check whether meets $\Omega^{(\delta)} \ge \Omega *$. If $\Omega^{(\delta)} < \Omega *$, experts need to carry out interactive communication and then recalculate $\Omega^{(\delta)}$ until $\Omega^{(\delta)} \ge \Omega *$.

Step 4: Aggregate the qRPDHF matrices using the proposed AOs.

Here *qRPDHFWGMSM* (or *qRPDHFWGGMSM*) operator is utilized to obtain the aggregated qRPDHF matrix $\left[\tilde{\Im}^{(ij)}(P)\right]_{m\times n} = <\bigcup_a\{\tilde{\Delta}_{ij}^{(a)}(P^{(a)})\}, \bigcup_b\{\tilde{\nabla}_{ij}^{(b)}(P^{(b)})\}>$.

$$\tilde{\Im}^{(ij)}(P) = qRPDHFWGMSM\,(\tilde{\Im}^{(ij1)}(P), \tilde{\Im}^{(ij2)}(P), \ldots, \tilde{\Im}^{(ijl)}(P)) \tag{8}$$

or

$$\tilde{\Im}^{(ij)}(P) = qRPDHFWGGMSM\,(\tilde{\Im}^{(ij1)}(P), \tilde{\Im}^{(ij2)}(P), \ldots, \tilde{\Im}^{(ijl)}(P)) \tag{9}$$

Step 5: Construct the normalized aggregated qRPDHF matrix $\left[\tilde{\Im}^{(ij)*}(P)\right]_{m\times n} (i = 1(1)m,\ j = 1(1)n)$.

Here $\tilde{\Im}^{(ij)*}(P) = <\bigcup_a\{\tilde{\Delta}_{ij}^{(a)}(P^{(a)})\},\bigcup_b\{\tilde{\nabla}_{ij}^{(b)}(P^{(b)})\}>$ (or $<\bigcup_b\{\tilde{\nabla}_{ij}^{(b)}(P^{(b)})\},\bigcup_a\{\tilde{\Delta}_{ij}^{(a)}(P^{(a)})\}>$) if C_j is a benefit criteria (or cost criteria).

Step 6: Calculate the criteria weights.

The weights of criteria' play an important role on the final outcome.

Consider an expert E_d, and qRPDHF data under C_j. Then the following divergence measure can be used to describe how the alternative A_i differs from other alternatives.

$$DIV_{ij}^{d} = \frac{1}{m-1}\sum_{p=1}^{m} C(\tilde{\Im}^{(ijd)}(P),\tilde{\Im}^{(pjd)}(P)) \tag{10}$$

where $C(\tilde{\Im}^{(ijd)}(P),\tilde{\Im}^{(pjd)}(P))$ stands for cross-entropy measure between the qRPDHFEs $\tilde{\Im}^{(ijd)}(P)$ and $\tilde{\Im}^{(pjd)}(P)$. We define it by:

$$C(\tilde{\Im}^{(ijd)}(P),\tilde{\Im}^{(pjd)}(P))$$

$$= \left(\frac{1}{a}\sum_a(\tilde{\Delta}_{ijd}^{(a)}\times P^{(a)})\right)\times\ln\left(\frac{2\sum_a(\tilde{\Delta}_{ijd}^{(a)}\times P^{(a)})}{\sum_a(\tilde{\Delta}_{ijd}^{(a)}\times P^{(a)})+\sum_a(\tilde{\Delta}_{pjd}^{(a)}\times P^{(a)})}\right)$$

$$+\left(\frac{1}{a}\sum_a(\tilde{\Delta}_{pjd}^{(a)}\times P^{(a)})\right)\times\ln\left(\frac{2\sum_a(\tilde{\Delta}_{pjd}^{(a)}\times P^{(a)})}{\sum_a(\tilde{\Delta}_{ijd}^{(a)}\times P^{(a)})+\sum_a(\tilde{\Delta}_{pjd}^{(a)}\times P^{(a)})}\right)$$

$$+\left(1-\frac{1}{a}\sum_a(\tilde{\Delta}_{ijd}^{(a)}\times P^{(a)})\right)\times\ln\left(\frac{1-\frac{1}{a}\sum_a(\tilde{\Delta}_{ijd}^{(a)}\times P^{(a)})}{1-\frac{1}{2a}\left(\sum_a(\tilde{\Delta}_{ijd}^{(a)}\times P^{(a)})+\sum_a(\tilde{\Delta}_{pjd}^{(a)}\times P^{(a)})\right)}\right)$$

$$+\left(1-\frac{1}{a}\sum_a(\tilde{\Delta}_{pjd}^{(a)}\times P^{(a)})\right)\times\ln\left(\frac{1-\frac{1}{a}\sum_a(\tilde{\Delta}_{pjd}^{(a)}\times P^{(a)})}{1-\frac{1}{2a}\left(\sum_a(\tilde{\Delta}_{ijd}^{(a)}\times P^{(a)})+\sum_a(\tilde{\Delta}_{pjd}^{(a)}\times P^{(a)})\right)}\right)$$

$$+\left(\frac{1}{b}\sum_b(\tilde{\nabla}_{ijd}^{(b)}\times P^{(b)})\right)\times\ln\left(\frac{2\sum_b(\tilde{\nabla}_{ijd}^{(b)}\times P^{(b)})}{\sum_b(\tilde{\nabla}_{ijd}^{(b)}\times P^{(b)})+\sum_b(\tilde{\nabla}_{pjd}^{(b)}\times P^{(b)})}\right)$$

$$+\left(\frac{1}{b}\sum_b(\tilde{\nabla}_{pjd}^{(b)}\times P^{(b)})\right)\times\ln\left(\frac{2\sum_b(\tilde{\nabla}_{pjd}^{(b)}\times P^{(b)})}{\sum_b(\tilde{\nabla}_{ijd}^{(b)}\times P^{(b)})+\sum_b(\tilde{\nabla}_{pjd}^{(b)}\times P^{(b)})}\right)$$

$$+\left(1-\frac{1}{b}\sum_{b}(\tilde{\nabla}_{ijd}^{(b)}\times P^{(b)})\right)\times\ln\left(\frac{1-\frac{1}{b}\sum_{b}(\tilde{\nabla}_{ijd}^{(b)}\times P^{(b)})}{1-\frac{1}{2a}\left(\sum_{b}(\tilde{\nabla}_{ijd}^{(b)}\times P^{(b)})+\sum_{b}(\tilde{\nabla}_{pjd}^{(b)}\times P^{(b)})\right)}\right)$$
$$+\left(1-\frac{1}{b}\sum_{b}(\tilde{\nabla}_{pjd}^{(b)}\times P^{(b)})\right)\times\ln\left(\frac{1-\frac{1}{b}\sum_{b}(\tilde{\nabla}_{pjd}^{(b)}\times P^{(b)})}{1-\frac{1}{2a}\left(\sum_{b}(\tilde{\nabla}_{ijd}^{(b)}\times P^{(b)})+\sum_{b}(\tilde{\nabla}_{pjd}^{(b)}\times P^{(b)})\right)}\right) \quad (11)$$

The total divergence for the criterion C_j is:

$$DIV_j^d=\frac{1}{m-1}\sum_{i=1}^{m}\sum_{p=1}^{m}C(\tilde{\mathfrak{I}}^{(ijd)}(P),\tilde{\mathfrak{I}}^{(pjd)}(P)) \quad (12)$$

Considering the importance of all experts, We can determine the overall divergences for each option over the given criterion C_j.

$$DIV_j=\sum_{d=1}^{l}\xi_d DIV_j^d=\sum_{d=1}^{l}\xi_d\frac{1}{m-1}\sum_{i=1}^{m}\sum_{p=1}^{m}C(\tilde{\mathfrak{I}}^{(ijd)}(P),\tilde{\mathfrak{I}}^{(pjd)}(P)) \quad (13)$$

From the above discussions, it is clear that the following optimization model can be used to calculate weights of criteria.

$$Max\ \chi=\sum_{j=1}^{n}w_j\sum_{d=1}^{l}\xi_d\frac{1}{m-1}\sum_{i=1}^{m}\sum_{p=1}^{m}C(\tilde{\mathfrak{I}}^{(ijd)}(P),\tilde{\mathfrak{I}}^{(pjd)}(P))$$

Subject to (14)

$$\begin{cases} w_j\in\Pi,\\ \sum_{j=1}^{n}w_j=1,\\ w_j\geq 0\ \forall\ j\end{cases}$$

where Π is the set of partial information's about criteria weights.

Step 7: Applying the idea of adjusted probabilities, we create the adjusted aggregated qRPDHF matrix:

$$\left[\tilde{\mathfrak{I}}^{(ij)*}(P')\right]_{m\times n}=\left[<\bigcup_{a}\{\tilde{\Delta}_{ij}^{\prime(a)}(P^{\prime(a)})\},\bigcup_{b}\{\tilde{\nabla}_{ij}^{\prime(b)}(P^{\prime(b)})\}>\right]_{m\times n}\ (i=1(1)m,\ j=1(1)n).$$

To develop this final adjusted aggregated qRPDHF matrix $\left[\tilde{\mathfrak{I}}^{(i)*}(P')\right]_{m\times n}$, the *qRPDHFWGMSM* operator (or *qRPDHFWGGMSM* operator) is used.

$$\tilde{\mathfrak{I}}^{(i)*}(P')=qRPDHFWGMSM(\tilde{\mathfrak{I}}^{(i1)*}(P'),\tilde{\mathfrak{I}}^{(i2)*}(P'),........,\tilde{\mathfrak{I}}^{(in)*}(P')) \quad (15)$$

or

$$\tilde{\mathfrak{I}}^{(i)*}(P')=qRPDHFWGGMSM(\tilde{\mathfrak{I}}^{(i1)*}(P'),\tilde{\mathfrak{I}}^{(i2)*}(P'),........,\tilde{\mathfrak{I}}^{(in)*}(P')) \quad (16)$$

Step 8: Generate the preference the options $A_i\ (i=1(1)m)$ using the scores of $\tilde{\mathfrak{I}}^{(i)*}(P')\ (i=1(1)m)$ and select the optimal option.

5. Case Study and solution

The proposed method is employed in Example-1 in order to assess the OSS-LMS alternatives with qRPDHF information.

Step 1: The initial assessments of experts are shown in Table-5.

Table 5. Adjusted assessment matrix

		C_1	C_2	C_3
E_1	A_1	<{0.1(0.3), 0.3(0.2), 0.4(0.2), 0.4(0.3)}, {0.6(0.3), 0.6(0.1), 0.2(0.3), 0.3(0.3)}>	<{0.6(0.4), 0.8(0.2), 0.3(0.3), 0.2(0.1)}, {0.3(0.4), 0.2(0.2), 0.7(0.3), 0.9(0.1)}>	<{0.4(0.2), 0.1(0.4), 0.5(0.1), 0.7(0.3)}, {0.1(0.1), 0.5(0.3), 0.4(0.3), 0.3(0.3)}>
	A_2	<{0.3(0.1), 0.7(0.5), 0.8(0.2), 0.5(0.2)}, {0.9(0.5), 0.6(0.1), 0.5(0.2), 0.5(0.2)}>	<{0.2(0.5), 0.5(0.2), 0.6(0.1), 0.3(0.2)}, {0.8(0.3), 0.8(0.1), 0.6(0.2), 0.4(0.4)}>	<{0.5(0.1), 0.8(0.1), 0.8(0.5), 0.4(0.3)}, {0.5(0.1), 0.3(0.4), 0.1(0.4), 0.5(0.1)}>
	A_3	<{0.6(0.4), 0.6(0.1), 0.2(0.2), 0.3(0.3)}, {0.1(0.3), 0.3(0.2), 0.4(0.1), 0.4(0.4)}>	<{0.3(0.2), 0.2(0.1), 0.7(0.1), 0.9(0.6)}, {0.6(0.5), 0.8(0.1), 0.3(0.2), 0.2(0.2)}>	<{0.1(0.2), 0.5(0.4), 0.4(0.3), 0.3(0.1)}, {0.4(0.1), 0.1(0.3), 0.5(0.5), 0.7(0.1)}>
E_2	A_1	<{0.4(0.3), 0.6(0.2), 0.5(0.2), 0.3(0.3)}, {0.3(0.3), 0.7(0.1), 0.8(0.3), 0.5(0.3)}>	<{0.7(0.4), 0.8(0.2), 0.6(0.3), 0.4(0.1)}, {0.2(0.4), 0.5(0.2), 0.6(0.3), 0.3(0.1)}>	<{0.5(0.2), 0.3(0.4), 0.1(0.1), 0.4(0.3)}, {0.5(0.1), 0.8(0.3), 0.8(0.3), 0.4(0.3)}>
	A_2	<{0.1(0.1), 0.3(0.5), 0.4(0.2), 0.4(0.2)}, {0.6(0.5), 0.6(0.1), 0.2(0.2), 0.3(0.2)}>	<{0.6(0.5), 0.8(0.2), 0.3(0.1), 0.2(0.2)}, {0.3(0.3), 0.2(0.1), 0.7(0.2), 0.9(0.4)}>	<{0.4(0.1), 0.1(0.1), 0.5(0.5), 0.7(0.3)}, {0.1(0.1), 0.5(0.4), 0.4(0.4), 0.3(0.1)}>
	A_3	<{0.9(0.4), 0.6(0.1), 0.5(0.2), 0.5(0.3)}, {0.9(0.3), 0.6(0.2), 0.5(0.1), 0.5(0.4)}>	<{0.8(0.2), 0.8(0.1), 0.6(0.1), 0.4(0.6)}, {0.8(0.5), 0.8(0.1), 0.6(0.2), 0.4(0.2)}>	<{0.5(0.2), 0.3(0.4), 0.1(0.3), 0.5(0.1)}, {0.5(0.1), 0.3(0.3), 0.1(0.5), 0.5(0.1)}>
E_3	A_1	<{0.6(0.3), 0.6(0.2), 0.2(0.2), 0.3(0.3)}, {0.1(0.3), 0.3(0.1), 0.4(0.3), 0.4(0.3)}>	<{0.3(0.4), 0.2(0.2), 0.7(0.3), 0.9(0.1)}, {0.6(0.4), 0.8(0.2), 0.3(0.3), 0.2(0.1)}>	<{0.1(0.2), 0.5(0.4), 0.4(0.1), 0.3(0.3)}, {0.4(0.1), 0.1(0.3), 0.5(0.3), 0.7(0.3)}>
	A_2	<{0.9(0.1), 0.6(0.5), 0.5(0.2), 0.5(0.2)}, {0.3(0.5), 0.7(0.1), 0.8(0.2), 0.5(0.2)}>	<{0.8(0.5), 0.8(0.2), 0.6(0.1), 0.4(0.2)}, {0.2(0.3), 0.5(0.1), 0.6(0.2), 0.3(0.4)}>	<{0.5(0.1), 0.3(0.1), 0.1(0.5), 0.5(0.3)}, {0.5(0.1), 0.8(0.4), 0.8(0.4), 0.4(0.1)}>
	A_3	<{0.3(0.4), 0.7(0.1), 0.8(0.2), 0.5(0.3)}, {0.6(0.3), 0.6(0.2), 0.2(0.1), 0.3(0.4)}>	<{0.2(0.2), 0.5(0.1), 0.6(0.1), 0.3(0.6)}, {0.3(0.5), 0.2(0.1), 0.7(0.2), 0.9(0.2)}>	<{0.5(0.2), 0.8(0.4), 0.8(0.3), 0.4(0.1)}, {0.1(0.1), 0.5(0.3), 0.4(0.5), 0.3(0.1)}>

Step 2: We compute the subjective weights of experts with the use of Eq. (1) and Eq. (2). The connections between the experts' opinions are gained using Eq. (3) and Eq.(4) and are shown in Table-6. The objective weights of are calculated using Eq. (5). Their overall weight of experts' are calculated by Eq. (6) (choosing ϖ =0.5) and are given in Table-7.

Table 6. Similarity between experts' judgments

	D_1	D_2	D_3
D_1	1	0.94892766	0.975803776
D_2	0.94892766	1	0.955188852
D_3	0.975803776	0.955188852	1

Table 7. Weights of experts

	Subjective	**Objective**	**Final**
D_1	$\xi_1' = 0.355950765$	$\xi_1'' = 0.334164012$	$\xi_1 = 0.3451$
D_2	$\xi_2' = 0.307840016$	$\xi_2'' = 0.330584933$	$\xi_2 = 0.3192$
D_3	$\xi_3' = 0.336209218$	$\xi_3'' = 0.335251055$	$\xi_3 = 0.3357$

Step 3: The consensus coefficient value is obtained using Eq. (7) and is found to be equal to 0.960680. This means that consensus level is not reached since $\Omega^{(\delta)} = 0.960680 < \Omega^* = 0.97$. According to Table-6, low level of connections is found among the 2nd and other experts. This signifies a biased decision of 2nd expert. Therefore, the evaluation data of 2nd expert should get modified. The revised assessment information of the expert E_2 is presented in Table-8.

Table 8. Revised adjusted evaluation matrix for expert E_2

	C_1	C_2	C_3
A_1	<{0.4(0.3), 0.6(0.2), 0.5(0.2), 0.3(0.3)}, {0.3(0.3), 0.7(0.1), 0.8(0.3), 0.5(0.3)}>	<{0.7(0.4), 0.8(0.2), 0.6(0.3), 0.4(0.1)}, {0.2(0.4), 0.5(0.2), 0.6(0.3), 0.3(0.1)}>	<{0.5(0.2), 0.3(0.4), 0.1(0.1), 0.4(0.3)}, {0.5(0.1), 0.8(0.3), 0.8(0.3), 0.4(0.3)}>
A_2	<{0.9(0.1), 0.3(0.5), 0.4(0.2), 0.4(0.2)}, {0.6(0.5), 0.6(0.1), 0.2(0.2), 0.3(0.2)}>	<{0.6(0.5), 0.8(0.2), 0.3(0.1), 0.3(0.2)}, {0.3(0.3), 0.2(0.1), 0.7(0.2), 0.9(0.4)}>	<{0.4(0.1), 0.9(0.1), 0.5(0.5), 0.7(0.3)}, {0.1(0.1), 0.5(0.4), 0.4(0.4), 0.3(0.1)}>
A_3	<{0.9(0.4), 0.6(0.1), 0.5(0.2), 0.5(0.3)}, {0.9(0.3), 0.6(0.2), 0.5(0.1), 0.5(0.4)}>	<{0.8(0.2), 0.8(0.1), 0.6(0.1), 0.4(0.6)}, {0.8(0.5), 0.8(0.1), 0.6(0.2), 0.4(0.2)}>	<{0.5(0.2), 0.3(0.4), 0.9(0.3), 0.5(0.1)}, {0.5(0.1), 0.3(0.3), 0.1(0.5), 0.5(0.1)}>

The updated similarity between the experts is shown in Table-9. The revised subjective weights, objective weights and final weights of experts are given in Table-10.

Table 9. Similarity between experts' judgments

	D_1	D_2	D_3
D_1	1	0.960562903	0.975803776
D_2	0.960562903	1	0.973687808
D_3	0.975803776	0.973687808	1

Table 10. Weights of experts

	Subjective	**Objective**	**Final**
D_1	$\xi_1' = 0.36673953$	$\xi_1'' = 0.332702822$	$\xi_1 = 0.3497$
D_2	$\xi_2' = 0.286860847$	$\xi_2'' = 0.33233926$	$\xi_2 = 0.3096$
D_3	$\xi_3' = 0.346399623$	$\xi_3'' = 0.334957918$	$\xi_3 = 0.3407$

We again calculate the consensus coefficient and we get $\Omega^{(\delta)} = 0.970018 > \Omega^* = 0.97$. This means that the required consensus level has been achieved.

Step 4: Utilizing Eq. (8) and taking *r*=2, *q*=3, $t_1 = t_2 = 2$, and $\Theta(x) = \frac{1}{x} - 1 \; (x \in (0,1])$, the aggregated qRPDHF matrix is obtained (Table-11).

Table 11. Aggregated qRPDHF matrix

	C_1	C_2	C_3
A_1	<{0.227895306 (0.3), 0.34146202 (0.2), 0.238279964 (0.2), 0.22633804 (0.3)}, {0.463528684 (0.3), 0.743639962 (0.1), 0.611580768 (0.3), 0.548554159 (0.3)}>	<{0.366855654 (0.4), 0.473175705 (0.2), 0.363325668 (0.3), 0.257291879 (0.1)}, {0.309685763 (0.4), 0.62809717 (0.2), 0.741543607 (0.3), 0.487899542 (0.1)}>	<{0.220229689(0.2), 0.174306177 (0.4), 0.221593385 (0.1), 0.282222658 (0.3)}, {0.518936043 (0.1), 0.693934888 (0.3), 0.740074614 (0.3), 0.531694149 (0.3)}>
A_2	<{0.61529503 (0.1), 0.367099559 (0.5), 0.366606178 (0.2), 0.326855285 (0.2)}, {0.805611894 (0.5), 0.743422141 (0.1), 0.328076273 (0.2), 0.455493106 (0.2)}>	<{0.361526112 (0.5), 0.527071257 (0.2), 0.343395786 (0.1), 0.226706449 (0.2)}, {0.48300963 (0.3), 0.333296918 (0.1), 0.783092523 (0.2), 0.65499119 (0.4)}>	<{0.326855285 (0.1), 0.53612698 (0.1), 0.296510226 (0.5), 0.361337545 (0.3)}, {0.168341209 (0.1), 0.63591727 (0.4), 0.517516245 (0.4), 0.451420463 (0.1)}>
A_3	<{0.402114899 (0.4), 0.463087898 (0.1), 0.307250966 (0.2), 0.294264417 (0.3)}, {0.797044611 (0.3), 0.679474704 (0.2), 0.535805538 (0.1), 0.55446316 (0.4)}>	<{0.211311653 (0.2), 0.313720714 (0.1), 0.462607172 (0.1), 0.287719327 (0.6)}, {0.778036281 (0.5), 0.869439288 (0.1), 0.705266928 (0.2), 0.526905632 (0.2)}>	<{0.25019694 (0.2), 0.336742161 (0.4), 0.548149113 (0.3), 0.269852656 (0.1)}, {0.527350985 (0.1), 0.387209963 (0.3), 0.168263919 (0.5), 0.677942932 (0.1)}>

Step 5: Normalization is not required as no cost type criterion is considered.

Step 6: Assume that $\Pi = \{0.10 \le w_1 \le 0.30,\ 0.15 \le w_2 \le 0.40, 0.20 \le w_3 \le 0.35\}$

Then the following optimization model is obtained:

$$Max\, \chi = 0.018622 w_1 + 0.015832 w_2 + 0.025745 w_3$$

Subject to

$$\begin{cases} 0.10 \le w_1 \le 0.30,\ 0.15 \le w_2 \le 0.40, 0.20 \le w_3 \le 0.35, \\ w_1 + w_2 + w_3 = 1, \\ w_1, w_2, w_3 \ge 0 \end{cases}$$

Solving the above optimization model, we get, $w_1 = 0.30, w_2 = 0.35, w_3 = 0.35$ and $Max\, \chi = 0.0201$.

Step 7: Based on Table-11 and the criteria weights ($w_1 = 0.30, w_2 = 0.35, w_3 = 0.35$), the final adjusted aggregated qRPDHF matrix $\left[\Im_{\wp}^{(i)*}(\Delta')\right]_{3\times 1}$ (Table-12) is constructed using on Eq. 15 by taking $\Theta(x) = \frac{1}{x} - 1 \; (x \in (0,1])$.

Table 12. Final adjusted aggregated qRPDHF matrix

	Aggregated qRPDHFEs
A_1	<{ 0.174194042 (0.2), 0.157802072 (0.1), 0.192021822 (0.1), 0.204677709 (0.1), 0.165325088 (0.1), 0.177843433 (0.1), 0.193274562 (0.2), 0.175371303 (0.1)}, {0.607032006 (0.1), 0.673758863 (0.2), 0.808611568 (0.1), 0.821571833 (0.2), 0.850950169 (0.1), 0.781393058 (0.2), 0.695389123 (0.1)}>
A_2	<{0.278236296 (0.1), 0.281989064 (0.1), 0.236699614 (0.3), 0.258450061 (0.1), 0.258286059 (0.1), 0.24977418 (0.1), 0.204800915 (0.2)}, {0.684475187 (0.1), 0.82164947 (0.2), 0.803535681 (0.1), 0.886928017 (0.1), 0.856991069 (0.1), 0.685299996 (0.2), 0.717825489 (0.1), 0.695270867 (0.1)}>
A_3	<{0.183693202 (0.2), 0.242842796 (0.1), 0.275604637 (0.1), 0.243090499 (0.1), 0.215248104 (0.1), 0.235707269 (0.1), 0.231045195 (0.2), 0.197489731 (0.1)}, {0.873194635 (0.1), 0.865508021 (0.2), 0.821881338 (0.1), 0.813389698 (0.1), 0.756819281 (0.1), 0.719803034 (0.1), 0.719803034 (0.1), 0.638226621 (0.1), 0.759939279 (0.1)}>

Step 8: The score values of the aggregated qRPDHFEs are:

$$Sc(\tilde{\mathfrak{I}}^{(1)*}(P')) = -0.431971,$$

$$Sc(\tilde{\mathfrak{I}}^{(2)*}(P')) = -0.447465,$$

$$Sc(\tilde{\mathfrak{I}}^{(3)*}(P')) = -0.481656.$$

Hence, the priority order is: $A_1 \succ A_2 \succ A_3$. Therefore, the optimal option is A_1.

6. Discussion

6.1 Impact of parameters upon priority order

To signify the effects of the parameters t_1 and t_2 (taking *r*=2, *q*=3) upon score values, the operators qRPDHFWGMSM and qRPDHFWGGMSM are used choosing $t_1, t_2 \in \{1, 2, \ldots, 10\}$.

(1) Suppose the value of t_2 is fixed (say, t_2 =2). To assess the parameter's effect t_1 upon ranking order, we employ a range of parameter values (t_1) with qRPDHFWGMSM operator. The related scores of alternatives are depicted in Fig. 1. As demonstrated by Fig.1, the ranking order is $A_2 \succ A_3 \succ A_1$ and $A_1 \succ A_2 \succ A_3$ for $1 \le t_1 < 2$ and $2 \le t_1 < 10$ respectively and thus the best alternative is A_1 or A_2 when qRPDHFWGMSM operator is used.

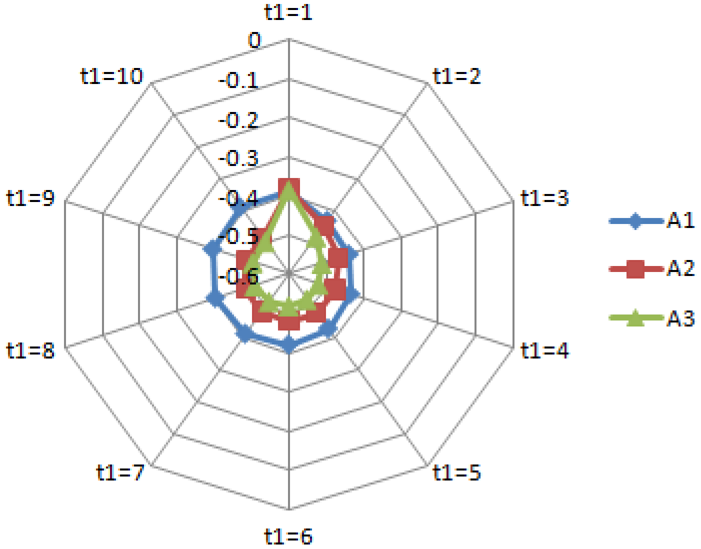

Figure 1. Scores of alternatives when t_1 =1(1)10 using qRPDHFWGMSM operator

(2) Next, suppose the value of t_1 is fixed (say, t_1 =4). We utilize diverse parameter (t_2) values to illustrate its impact upon priority order using the qRPDHFWGMSM operator. The related score values of alternatives are depicted in Fig. 2. As indicated by Fig.2, the ranking order is $A_1 \succ A_2 \succ A_3$ and $A_2 \succ A_3 \succ A_1$ for $1 \le t_1 \le 7$ and $7 < t_1 < 10$ respectively and thus the best alternative is A_1 or A_2 when qRPDHFWGMSM operator is used.

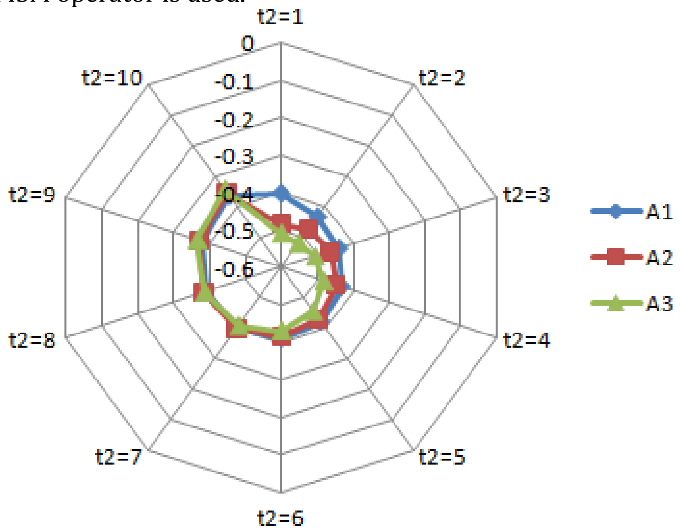

Figure 2. Scores of alternatives when t_2 =1(1)10 using qRPDHFWGMSM operator

(3) To assess the parameter's effect t_1 upon ranking order, we employ a range of parameter values (t_1) with qRPDHFWGGMSM operator. We first take a fixed value of t_2, say, t_2 =2. The related scores of alternatives are depicted in Fig. 3. As

demonstrated by Fig.3, the ranking order is $A_2 \succ A_3 \succ A_1$ and $A_3 \succ A_2 \succ A_1$ for $1 \le t_1 < 6$ and $6 \le t_1 < 10$ respectively and thus the best alternative is A_3 or A_2 when qRPDHFWGGMSM operator is used.

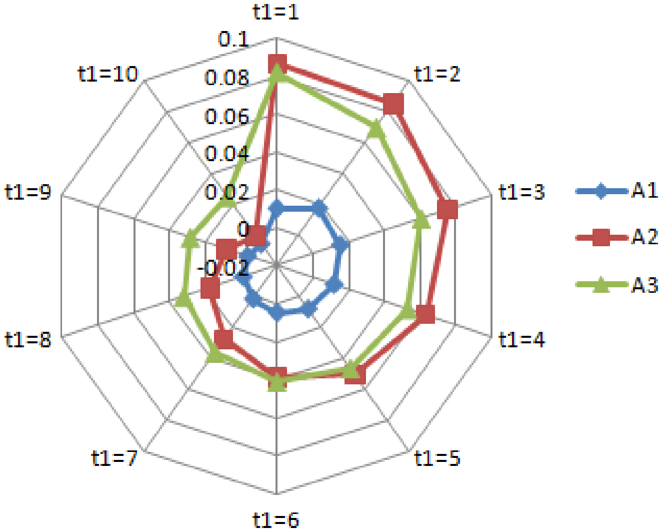

Figure 3. Scores of alternatives when t_1 =1(1)10 using qRPDHFWGGMSM operator

(4) Lastly, suppose the value of t_1 is fixed (say, t_1 =4). We utilize diverse parameter (t_2) values to illustrate its impact upon priority order using the qRPDHFWGGMSM operator. The related score values of alternatives are depicted in Fig. 4. As demonstrated by Fig.4, The priority is $A_3 \succ A_2 \succ A_1$ and $A_2 \succ A_3 \succ A_1$ for $1 \le t_1 \le 2$ and $2 < t_1 \le 10$ respectively and thus the best alternative is A_3 or A_2 when qRPDHFWGGMSM operator is used.

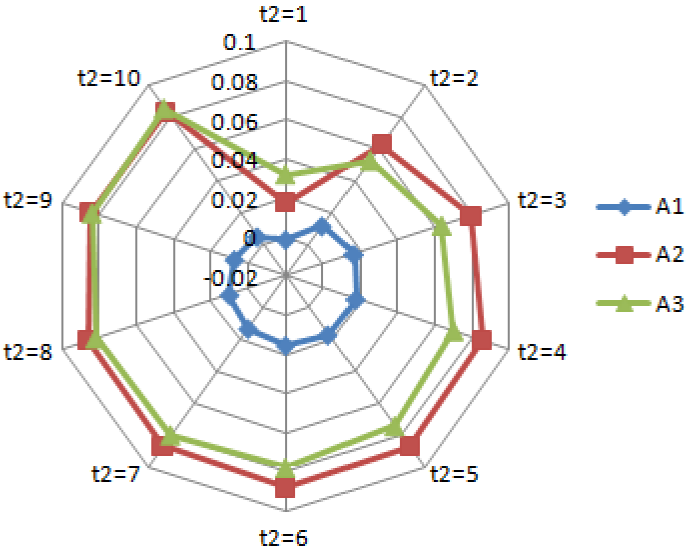

Figure 4. Scores of alternatives when t_2 =1(1)10 using qRPDHFWGGMSM operator

6.2 Comparative study

A research comparing our suggested approach with Li et al.'s method (2020) is offered to assess its efficacy. It is based on the qRPDHFWGMSM and qRPDHFWGGMSM operators. Table 13 lists the characteristics that set them apart from one another. In section 3's Example 1, we use the methodology. Because it is based on several MCDM approaches, Li et al.'s method (2020) fails to produce any preference order of options, as shown in Table 13. . But, our proposed method gives the priority order $A_2 \succ A_1 \succ A_3$ (without interaction) and $A_1 \succ A_2 \succ A_3$ (with interaction). As a result, the approach created using the qRPDHFWGMSM (or qRPDHFWGGMSM) operator is successful.

Table 13. Comparative investigation

Aspects	Proposed	Li et al. (2020)
Information type	qRPDHF	qRPDHF
Decision-making type	*Group DM*	Individual DM
Hesitation in preferences	*Considered*	Considered
Probabilistic information	*Considered*	Considered
Adjustment of probabilities	*Considered*	No
Determination of experts' weights	*Subjective and objective weights with interaction among experts*	Not applicable
Criteria' weights selection	*A cross entropy based optimization model*	Direct
Aggregation operators	*Archimedean copula based weighted generalized Maclaurin symmetric mean AOs*	qRPDHF power weighted Muirhead mean (qRPDHFPWMM) operator
Flexibility of the AOs	*Very high*	High
Whether consider dependency among multi-input criteria	*Yes*	No
Determination of consensus coefficient	*Considered*	Not considered
Ranking of alternatives (without interaction of experts) with qRPDHFWGMSM operator taking *r*=2, *q*=3, and $t_1 = t_2 = 2$	$A_2 \succ A_1 \succ A_3$	Can't be determined
Ranking of alternatives (with interaction of experts) with qRPDHFWGMSM operator taking *r*=2, *q*=3, and $t_1 = t_2 = 2$	$A_1 \succ A_2 \succ A_3$	Can't be determined

In the last sub-section, we have seen that Li et al.'s method (2020) fail to generate any preference order of alternatives as we had taken a group decision-making problem. So, in order to compare our method with the methods developed by Li et al.

(2020), one case study related to MCDM (Example 2 in Section 3) is considered here. In this case, by the developed method, we obtain the scores -0.52571, -0.39996, -0.14094, -0.12571 respectively and the ranking order is $A_4 \succ A_3 \succ A_2 \succ A_1$.Thus, our proposed method and Li et al.'s method (2020) generate different preference order, but the best alternative remains the same in both cases which means that our method is effective.

The advantages of our method in comparison to existing approach are as follows:

1. Li et al.'s technique (2020) is based on the power AO and can therefore lessen the influence of an expert's bias on the results. However, these techniques can't handle MCGDM issues and reduce the accuracy of the results of decisions.However, our approach can handle MCGDM issues. Our approach is therefore far more trustworthy and efficient.
2. Because Li et al.'s technique (2020) is based on the Muirhead mean operator, it can take into account the relationships among several criteria, but it is limited in that it can only link one marginal distribution. But our proposed qRPDHF generalized MSM operators' aggregate qRPDHF data with higher flexibility. It also takes into account the relationships between many input criteria. These operators may link more than one marginal distribution and can thus prevent the information loss that results from the aggregation process since they were designed based on the Archimedean Copula operations on qRPDHFEs.
3. It is clear that approach (Li et al., 2020) falls short in its attempt to reduce the impact of highly inflated attribute values from a few unreliable experts who have different biases. This unspoken problem has a negative impact on how decisions are made in any MCGDM process. By permitting the expert engagement that is lacking in the present study, this issue has been remedied utilising the suggested methodology (Li et al., 2020).
4. The preference ranking produced in the DM approach (Li et al., 2020) is impacted by the random distribution of weights of criteria during the final aggregation step. Additionally, the current approach (Li et al., 2020) loses information because it doesn't take any information measures into account. Our technique computes criteria weights using an optimization model based on the cross entropy measure. By highlighting the importance of each criterion, this optimization approach quantifies the amount of ambiguous data.

7. Conclusion

The qRPDHFSs can effectively portray the dubiousness and uncertainty in reality due to the inclusion of the MDs and NMDs with their corresponding probabilities. The joint occurrence of the stochastic and the non-stochastic ambiguity make the qRPDHFSs more realistic and superior. A comprehensive study on the usefulness of Archimedean Copulas under qRPDHF setting is demonstrated in our study. New operations for qRPDHF elements are formed via Archimedean Copulas. The existing AOs (Hao et al., 2017; Garg & Kaur, 2018) for aggregation PDHF data are limited to algebraic, and Einstein operators. So, they are not capable of considering dependency between multiple attributes. On the other hand, although the AOs (Ji et al., 2021; Li et al., 2020) based on Hamy mean operator and Muirhead mean operator respectively can consider dependency among criteria, cannot connect more than one marginal distribution. These facts motivated us to develop the Archimedean Copula based GMSM operators with their weighted forms under qRPDHF setting. Some pivotal qualities like idempotency, boundedness, and monotonicity, and of proposed AOs are introduced. Subsequently, a MCGDM procedure is exhibited to track down the best option in qRPDHF setting. Here, the weights of criteria are determined using an optimization model and experts weights are figured utilizing the linear combination of objective and subjective weights and interaction among experts. To give a superior comprehension of our technique, we have incorporated a case study including OSS-LMS selection. The robustness of our method has been demonstrated through sensitivity analysis of weights of criteria. The comparative study suggests that the proposed methodology can be adequately utilized in MCGDM issues containing correlated criteria's in the PDHF setting.

The only limitation of the proposed method is that in absence of partial weights information of criteria the proposed method fails. In such a scenario, other objective methods like CRITIC, MEREC, entropy measure, etc can be utilized for determination of criteria weights. In further research, other aggregation operators (Saha et al., 2022a; Saha et al., 2021a; Senapati, 2021; Senapati et al., 2022; Saha et al., 2022b; Saha et al., 2021b) can also be extended to tackle the dependency among attributes with qRPDHF information and the proposed weight determination technique. Our model can be used to provide a realistic solution to well-known problems, such as sustainable supplier selection (Mishra et al., 2022a), warehouse site selection (Saha et al., 2023), bio-energy production technology selection (Hezam et al., 2023), solid waste disposal method selection (Mishra et al., 2022b), renewable energy source selection (Mishra et al., 2022c), low carbon tourism assessment (Mishra et al., 2022d), biomass feedstock selection (Saha et al., 2021c), cloud vendor selection (Krishankumar et al., 2022), and food waste treatment technology selection (Rani et al., 2021) as it can effectively avoid distorting evaluation information and handle the relationships between multiple criteria.

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List of abbreviations

Abbreviation	Meaning
DM	Decision-making
MCDM	Multi-criteria DM
MCGDM	Multi-criteria group DM
AO	Aggregation operator
GMSM	Generalized Maclaurin symmetric mean
qRPDHF	q-Rung probabilistic dual hesitant fuzzy
qRPDHFS	qRPDHF set
qRPDHFE	qRPDHF element
qRPDHFGMSM	qRPDHF Generalized Maclaurin symmetric mean
qRPDHFGGMSM	qRPDHF geometric Generalized Maclaurin symmetric mean
qRPDHFWGMSM	qRPDHF weighted Generalized Maclaurin symmetric mean
qRPDHFWGGMSM	qRPDHF weighted geometric Generalized Maclaurin symmetric mean

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