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Bayesian Optimal Portfolio Selection in the MSF-SBEKK Model[†]

A b s t r a c t. The aim of this paper is to investigate the predictive properties of the MSF-Scalar BEKK(1,1) model in context of portfolio optimization. The MSF-SBEKK model has been proposed as a feasible tool for analyzing multidimensional financial data (large *n*), but this research examines forecasting abilities of this model for *n* = 2, since for bivariate data we can obtain and compare predictive distributions of the portfolio in many other multivariate SV specifications. Also, approximate posterior results in the MSF-SBEKK model (based on preliminary estimates of nuisance matrix parameters) are compared with the exact ones.

K e y w o r d s: portfolio analysis, MSV models, MSF-SBEKK model, forecasting.

Introduction

It is well known that in portfolio selection (computing the weights of the assets in the portfolio) correlations among the assets are essential. The weights of the minimum variance portfolio depend on the conditional covariance matrix (see Aguilar, West, 2000, Pajor, 2009). Thus for active portfolio management multivariate time series forecasts should be applied.

The aim of the paper is to examine the predictive properties of the MSF-SBEKK model (being the hybrid of the Multiplicative Stochastic Factor and scalar BEKK specifications; hence the model is called MSF-SBEKK; see Osiewalski, Pajor, 2009) in context of the optimal portfolio selection problem. The multi-period minimum conditional variance portfolio is considered (as in Pajor, 2009). In the optimization process we use the predictive distributions of future returns and the predictive conditional covariance matrices obtained from the Bayesian MSF-SBEKK and other multivariate stochastic volatility (MSV) models. In order to compare predictive results in the MSF-SBEKK model with

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those obtained in other MSV specifications, we consider only bivariate portfolios. The bivariate stochastic volatility models are used to describe the daily exchange rate of the euro against the Polish zloty and the daily exchange rate of the US dollar against the Polish zloty. Based on these two currencies we consider the Bayesian portfolio selection problem. In the next section we briefly present the MSF-SBEKK model. Section 3 is devoted to the optimal portfolio construction. In section 4 we present and discuss the empirical results. Some concluding remarks are presented in the last section.

1. Bayesian MSF-SBEKK Model

Let $x_{j,t}$ denote the price of asset j (or the exchange rate as in our application) at time t for j = 1,2, ..., n and t = 1, 2, ..., $T+s$. The vector of growth rates $\mathbf{y}_t=(y_{1,t}, y_{2,t}, ..., y_{n,t})'$, where $y_{j,t}$ = 100 ln $(x_{j,t}/x_{j,t-1})$, is modelled using the basic VAR(1) framework:

$$\mathbf{y}_t = \boldsymbol{\delta} + \mathbf{R}\mathbf{y}_{t-1} + \boldsymbol{\xi}_t,\ t = 1, 2, ... ,T, T+1, ..., T+s, \tag{1}$$

where $\{\boldsymbol{\xi}_t\}$ is a process with time-varying volatility, T denotes the number of the observations used in estimation, and s is the forecast horizon, $\boldsymbol{\delta}$ is a n-dimensional vector, $\mathbf{R}$ is a $n\times n$ matrix of parameters.

Following Osiewalski and Pajor (2009), for $\boldsymbol{\xi}_t$ we assume the so-called type I MSF-SBEKK(1,1) hybrid specification:

$$\boldsymbol{\xi}_t = \sqrt{g_t}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t, \tag{2}$$

$$\ln g_t = \phi \ln g_{t-1} + \sigma_g \eta_t, \quad \{(\boldsymbol{\varepsilon}_t', \eta_t)'\} \sim iiN(\mathbf{0}_{[(n+1)\times 1]}, \mathbf{I}_{n+1}),^{1} \tag{3}$$

$$\mathbf{H}_t = (1-\beta-\gamma)\mathbf{A} + \beta(\boldsymbol{\xi}_{t-1}\boldsymbol{\xi}_{t-1}') + \gamma\mathbf{H}_{t-1}. \tag{4}$$

That is, $\boldsymbol{\xi}_t$ is conditionally normal with mean vector $\mathbf{0}$ and covariance matrix $g_t\mathbf{H}_t$, where g_t is a latent process and $\mathbf{H}_t$ is a square matrix of order n that has the scalar BEKK(1,1) structure . Thus, the conditional distribution of $\mathbf{y}_t$, given its past and latent variables, is normal with mean $\mathbf{y}_t = \boldsymbol{\delta} + \mathbf{R}\mathbf{y}_{t-1}$ and covariance matrix $g_t\mathbf{H}_t$. The model defined by (2)-(4) includes as special cases two simple basic structures. When $\sigma_g \to 0$ and $\phi = 0$ we have the scalar BEKK(1,1) model, while $\beta = 0$ and $\gamma = 0$ lead to the MSF model (see Osiewalski, Pajor, 2009). Note that the model has one latent process which helps in explaining outlying observations, and time-varying conditional correlations as in the scalar BEKK(1,1) structure.

[1] $\{(\boldsymbol{\varepsilon}_t', \eta_t)'\}$ is a sequence of independent and identically distributed normal random vectors with mean vector zero and covariance matrix $\mathbf{I}_{n+1}$.

In (4) $\mathbf{A}$ is a free symmetric positive definite matrix of order n; for $\mathbf{A}^{-1}$ we assume the Wishart prior with n degrees of freedom and mean I_n; β and γ are free scalar parameters, jointly uniformly distributed over the unit simplex. As regards initial conditions for $\mathbf{H}_t$, we take $\mathbf{H}_0 = h_0\mathbf{I}_n$ and treat $h_0 > 0$ as an additional parameter, *a priori* exponentially distributed with mean 1. For the parameters of the latent process we use the same priors as Osiewalski, Pajor (2009); for ϕ : normal with mean 0 and variance 100, truncated to (-1, 1), for σ_g^{-2} : exponential with mean 200; g_0 is equal 1. The $n(n+1)$ elements of $\boldsymbol{\delta}_0 = (\boldsymbol{\delta}\ vec(\mathbf{R})')'$ are assumed to be *a priori* independent of remaining parameters, with the $N(\mathbf{0}, \mathbf{I}_{n(n+1)})$ prior truncated by the restriction that all eigenvalues of $\mathbf{R}$ lie inside the unit circle.

In this paper we also want to check how the approximation proposed and explained in Osiewalski, Pajor (2009) influences the predictive distribution of future logarithmic returns and, in consequence, the optimal portfolio composition. Therefore we apply this approximation. That is, we use Ordinary Least Squares (OLS) for the VAR(1) parameters and replace $\mathbf{A}$ by the empirical covariance matrix of the OLS residuals from the VAR(1) part. The Bayesian analysis for the remaining parameters and future return rates is based on the conditional posterior and predictive distributions given the particular values of vector $\boldsymbol{\delta}_0$ and matrix $\mathbf{A}$.

All distributions are sampled using the Gibbs scheme with Metropolis-Hastings steps, as shown in detail in Osiewalski, Pajor (2009).

2. Portfolio Selection Problem in the MSF-SBEKK Model

We denote by $\boldsymbol{\Theta}_t$ the latent variable vector at time t, by $\boldsymbol{\theta}$ the parameter vector, and we assume that:

a) $\boldsymbol{\xi}_t = \boldsymbol{\Sigma}_t^{1/2}\boldsymbol{\varepsilon}_t$, where $\{\boldsymbol{\varepsilon}_t\} \sim iiN(\mathbf{0}, \mathbf{I}_n)$,

b) $\boldsymbol{\Sigma}_t$ is a function of the latent variables $\boldsymbol{\Theta}_\tau$ for $\tau \le t$, and of the past of $\boldsymbol{\xi}_t$, i.e. $\boldsymbol{\Sigma}_t = \boldsymbol{\Sigma}(\boldsymbol{\Theta}_\tau, \boldsymbol{\xi}_{\tau-1}; \tau \le t)$,

c) the vector $\boldsymbol{\xi}_t$, conditional on $\sigma(\boldsymbol{\Theta}_\tau, \boldsymbol{\xi}_{\tau-1}; \tau \le t)$, is independent of $\sigma(\boldsymbol{\Theta}_\tau; \tau > t)$.

In Pajor (2009) it was assumed that $\boldsymbol{\Sigma}_t = \boldsymbol{\Sigma}(\boldsymbol{\Theta}_\tau; \tau \le t)$. Now we relax the assumption, allowing $\boldsymbol{\Sigma}_t$ to depend on the past of $\boldsymbol{\xi}_t$ as in the MSF-SBEKK model. The s-period portfolio at time T is defined by a vector $\mathbf{w}_{T+s|T} = (w_{1,T+s|T}, w_{2,T+s|T}, \ldots, w_{n,T+s|T})'$, where $w_{i,T+s|T}$ is the fraction of wealth invested in asset i $(1 \le i \le n)$. The return on the portfolio that places weight $w_{i,T+s|T}$ on asset i at time T is approximately a weighted average of the returns on the individual assets. The weight applied to each return is the fraction of the portfolio invested in that asset:

$$R_{w,T+s|T} \approx \sum_{i=1}^{n} w_{i,T+s|T} z_{i,T+s|T} = \tilde{R}_{w,T+s|T}\ , \tag{5}$$

where $z_{i,T+s|T}$ is the rate of return on the asset i from the period T to $T+s$, i.e. $z_{i,T+s|T} = \sum_{t=T+1}^{T+s} y_{i,t}$ (i = 1, ..., n). If $\mathbf{\Sigma}_{T+s|T}$ is the matrix of conditional covariances of $\mathbf{z}_{T+s|T} = (z_{1,T+s|T}, z_{2,T+s|T}, ..., z_{n,T+s|T})'$, then the conditional variance of return on the portfolio is

$$Var(\tilde{R}_{w,T+s|T} \mid \psi_T, \mathbf{\Theta}_T, ..., \mathbf{\Theta}_{T+s}) = V^2_{T+s|T} = \mathbf{w}_{T+s|T}{}'\mathbf{\Sigma}_{T+s|T}\mathbf{w}_{T+s|T}\ , \tag{6}$$

where ψ_T is the σ-algebra generated by $\boldsymbol{\varepsilon}_\tau$ and $\mathbf{\Theta}_\tau$ for $\tau \le T$, i.e. $\psi_T = \sigma(\boldsymbol{\varepsilon}_\tau, \mathbf{\Theta}_\tau; \tau \le T)$.

The vector of the rates of return at time $T+k$ ($k > 0$, $k \le s$) satisfies:

$$\mathbf{y}_{T+k} = \sum_{j=0}^{k-1} \mathbf{R}^j \boldsymbol{\delta} + \mathbf{R}^k \mathbf{y}_T + \sum_{j=1}^{k} \mathbf{R}^{k-j} \boldsymbol{\xi}_{T+j}. \tag{7}$$

Based on equation (7) we have:

$$\mathbf{z}_{T+s|T} = \sum_{k=1}^{s} \sum_{j=0}^{k-1} \mathbf{R}^j\ \boldsymbol{\delta} + \sum_{k=1}^{s} \mathbf{R}^k \mathbf{y}_T + \sum_{j=1}^{s} \sum_{i=0}^{s-j} \mathbf{R}^i\ \boldsymbol{\xi}_{T+j}, \tag{8}$$

Since $E(\boldsymbol{\xi}_{T+i}\boldsymbol{\xi}_{T+j}{}' \mid \psi_T, \mathbf{\Theta}_T, ..., \mathbf{\Theta}_{T+s}) = \mathbf{0}$ for $i \ne j$, the conditional covariance matrix of $\mathbf{z}_{T+s|T}$ in the MSF-SBEKK(1,1) model becomes:

$$\mathbf{\Sigma}_{T+s|T} = \sum_{j=1}^{s} (\sum_{i=0}^{s-j} \mathbf{R}^i) \mathbf{\Sigma}^*_{T+j} (\sum_{i=0}^{s-j} \mathbf{R}^i)', \tag{9}$$

where

$$\mathbf{\Sigma}^*_{T+j} = E(\boldsymbol{\xi}_{T+j}\boldsymbol{\xi}_{T+j}{}' \mid \psi_T, \mathbf{\Theta}_T, ..., \mathbf{\Theta}_{T+s}).$$

Finally, the conditional variance of return on the portfolio is:

$$Var(\tilde{R}_{w,T+s|T} \mid \psi_T, \mathbf{\Theta}_T, ..., \mathbf{\Theta}_{T+s}) = \mathbf{w}_{T+s|T}{}' \sum_{j=1}^{s} (\sum_{i=0}^{s-j} \mathbf{R}^i) \mathbf{\Sigma}^*_{T+j} (\sum_{i=0}^{s-j} \mathbf{R}^i)' \mathbf{w}_{T+s|T}.$$

It is easy to show that in the MSF-SBEKK(1,1) model:

$$E(\boldsymbol{\xi}_{T+1}\boldsymbol{\xi}_{T+1}{}' \mid \psi_T, \mathbf{\Theta}_T, ..., \mathbf{\Theta}_{T+s}) = g_{T+1}\mathbf{H}_{T+1},$$

$$\mathbf{H}_{T+1} = E(\mathbf{H}_{T+1} \mid \psi_T, \mathbf{\Theta}_T, ..., \mathbf{\Theta}_{T+s}) = (1-\gamma-\beta)\mathbf{A} + \beta\boldsymbol{\xi}_T\boldsymbol{\xi}_T{}' + \gamma\mathbf{H}_T,$$

and for $2 < k \le s$:

$$E(\boldsymbol{\xi}_{T+k}\boldsymbol{\xi}_{T+k}'|\psi_T,\boldsymbol{\Theta}_T,...,\boldsymbol{\Theta}_{T+s}) =$$
$$= g_{T+k}[(1-\gamma-\beta)\mathbf{A}+(\beta g_{T+k-1}+\gamma)E(\mathbf{H}_{T+k-1}\,|\,\psi_T,\boldsymbol{\Theta}_T,...,\boldsymbol{\Theta}_{T+s})],$$
$$E(\mathbf{H}_{T+k}\,|\,\psi_T,\boldsymbol{\Theta}_T,...,\boldsymbol{\Theta}_{T+s}) =$$
$$= (1-\gamma-\beta)\mathbf{A}+(\beta g_{T+k-1}+\gamma)E(\mathbf{H}_{T+k-1}\,|\,\psi_T,\boldsymbol{\Theta}_T,...,\boldsymbol{\Theta}_{T+s}).$$

Consequently[2]:

$$E(\boldsymbol{\xi}_{T+k}\boldsymbol{\xi}_{T+k}'|\psi_T,\boldsymbol{\Theta}_T,...,\boldsymbol{\Theta}_{T+s}) =$$
$$= g_{T+k}\left[1+\sum_{i=1}^{k-2}\prod_{j=1}^{i}\beta g_{T+k-j}+\gamma\right](1-\gamma-\beta)\mathbf{A}+g_{T+k}\ \mathbf{H}_{T+1}\prod_{j=1}^{k-1}(\beta g_{T+k-j}+\gamma).$$

The most popular approach assumes that the investor selects the portfolio with minimum variance (see Markowitz, 1959, Elton, Gruber, 1991). Here we assume that the conditional variance of the portfolio is minimized and that short sales are allowed ($w_{i,T+s|T} < 0$ reflects a short selling). Then the problem for the investor reduces to solving the quadratic programming problem:

$$\min_{\mathbf{w}_{T+s|T}} \mathbf{w}_{T+s|T}'\boldsymbol{\Sigma}_{T+s|T}\mathbf{w}_{T+s|T} \quad \text{subject to } w_{1,T+s|T}+w_{2,T+s|T}+...+w_{n,T+s|T}=1.$$

In this way we obtain so-called the minimum conditional variance portfolio (the portfolio that has the lowest risk of any feasible portfolio):

$$\mathbf{w}_{MV,T+s|T} = \frac{\boldsymbol{\Sigma}_{T+s|T}^{-1}\boldsymbol{\iota}}{\boldsymbol{\iota}'\boldsymbol{\Sigma}_{T+s|T}^{-1}\boldsymbol{\iota}}, \tag{10}$$

which has a return:

$$R_{MV,T+s|T} = \frac{\boldsymbol{\iota}'\boldsymbol{\Sigma}_{T+s|T}^{-1}\mathbf{z}_{T+s|T}}{\boldsymbol{\iota}'\boldsymbol{\Sigma}_{T+s|T}^{-1}\boldsymbol{\iota}}, \tag{11}$$

and the conditional variance at time T:

$$Var(\mathbf{w}_{MV,T+s|T}'\mathbf{y}_{T+s}\,|\,\psi_T,\boldsymbol{\Theta}_T,...,\boldsymbol{\Theta}_{T+s}) = V_{MV,T+s|T}^2 = \frac{1}{\boldsymbol{\iota}'\boldsymbol{\Sigma}_{T+s|T}^{-1}\boldsymbol{\iota}}, \tag{12}$$

where $\boldsymbol{\iota}$ is an $n\times 1$ vector of ones.

Next we consider a s-period portfolio selection problem where the investor minimizes the conditional variance of the portfolio with a given level of return $\tilde{R}_{w,T+s|T} \geq R^*_{p,T+s|T}$. This problem reduces to solving the quadratic programming problem:

[2] A very similar result was obtained by Piotr De Silva in his unpublished master's dissertation.

$$\min_{\mathbf{w}_{T+s|T}} \mathbf{w}_{T+s|T}'\mathbf{\Sigma}_{T+s|T}\mathbf{w}_{T+s|T} \text{ subject to } \begin{cases} \mathbf{w}_{T+s|T}'\boldsymbol{\iota} = 1, \\ \mathbf{w}_{T+s|T}'\mathbf{z}_{T+s|T} \geq R^{*}_{p,T|T+s}. \end{cases}$$

When $\tilde{R}_{w,T+s|T} = R^{*}_{p,T+s|T}$, the solution for the s-period portfolio is:

$$\mathbf{w}_{MVR_p^*,T+s|T} = \frac{(\mathbf{\Sigma}^{-1}_{T+s|T}\mathbf{z}_{T+s|T}\boldsymbol{\iota}'\mathbf{\Sigma}^{-1}_{T+s|T} - \mathbf{\Sigma}^{-1}_{T+s|T}\boldsymbol{\iota}\mathbf{z}_{T+s|T}'\mathbf{\Sigma}^{-1}_{T+s|T})(\boldsymbol{\iota}R^{*}_{p,T+s|T} - \mathbf{z}_{T+s|T})}{(\boldsymbol{\iota}'\mathbf{\Sigma}^{-1}_{T+s|T}\boldsymbol{\iota})(\mathbf{z}_{T+s|T}'\mathbf{\Sigma}^{-1}_{T+s|T}\mathbf{z}_{T+s|T}) - (\boldsymbol{\iota}'\mathbf{\Sigma}^{-1}_{T+s|T}\mathbf{z}_{T+s|T})^2}. \quad (13)$$

It is important to stress that the classic portfolio choice scheme assumes the covariance matrix and expected returns at time T to be known. In our Bayesian models the minimum conditional variance portfolio ($\mathbf{w}_{MV,T+s|T}$), and the minimum conditional variance portfolio with a given level of return, ($\mathbf{w}_{MVR_p^*,T+s|T}$) are random vectors as measurable functions of $\mathbf{z}_{T+s|T}$, and $\mathbf{\Sigma}_{T+s|T}$. Hence, the predictive distributions of $\mathbf{w}_{MV,T+s|T}$, and $\mathbf{w}_{MVR_p^*,T+s|T}$ (also, of $V_{MV,T+s|T}$, and $V_{MVR_p^*,T+s|T}$) are induced by the distribution of $\mathbf{z}_{T+s|T}$, and $\mathbf{\Sigma}_{T+s|T}$. In practice, to compute the weights of the assets in the portfolio we must use some characteristic of these predictive distributions. As the predictive mean (for $\mathbf{w}_{MV,T+s|T}$ or $\mathbf{w}_{MVR_p^*,T+s|T}$) may not exist, we consider the predictive medians of $\mathbf{w}_{MV,i,T+s|T}$ and $\mathbf{w}_{MVR_p^*,i,T+s|T}$, denoted by $\mathbf{w}^{op}_{MV,T+s|T} = (w^{op}_{MV,1,T+s|T},\ldots,w^{op}_{MV,n,T+s|T})'$ and $\mathbf{w}^{op}_{MVR_p^*,T+s|T} = (w^{op}_{MVR_p^*,T+s|T},\ldots,w^{op}_{MVR_p^*,n,T+s|T})'$, and defined respectively by conditions:

$\Pr\{w_{MV,i,T+s|T} \geq w^{op}_{MV,i,T+s|T} \mid y\} \leq 0.5$ and $\Pr\{w_{MV,i,T+s|T} \leq w^{op}_{MV,i,T+s|T} \mid y\} \geq 0.5$,

$\Pr\{w_{MVR_p^*,i,T+s|T} \geq w^{op}_{MVR_p^*,i,T+s|T} \mid y\} \leq 0.5$ and $\Pr\{w_{MVR_p^*,i,T+s|T} \leq w^{op}_{MVR_p^*,i,T+s|T} \mid y\} \geq 0.5$,

for $i = 1, \ldots, n$-1, and

$$w^{op}_{MV,n,T+s|T} = 1 - \sum_{i=1}^{n-1} w^{op}_{MV,i,T+s|T}, \quad w^{op}_{MVR_p^*,n,T+s|T} = 1 - \sum_{i=1}^{n-1} w^{op}_{MVR_p^*,i,T+s|T}.$$

In multivariate stochastic variance models there is no analytical solution for the optimal portfolio selection problem even for $n = 2$ assets. To evaluate the quantiles of the predictive distributions of $\mathbf{w}_{MV,T+s|T}$ and $\mathbf{w}_{MVR_p^*,T+s|T}$, and then find the portfolio, we use Markov chain Monte Carlo methods – the Gibbs sampler with the Metropolis-Hastings algorithm.

3. Empirical Results

As the dataset we use the same daily exchange rates as in Pajor (2009). Thus, we consider the daily exchange rate of the euro against the Polish zloty and the daily exchange rate of the US dollar against the Polish zloty from January 2, 2002 to June 29, 2007. The data were downloaded from the website of the National Bank of Poland. The dataset of the percentage daily logarithmic growth (return) rates, $\mathbf{y}_t$, consists of 1388 observations (for each series). As the first growth rates are used as initial conditions, $T = 1387$ remaining observations on $\mathbf{y}_t$ are modelled.

3.1. Bayesian Model Comparison

In Table 1 we rank the models by the increasing value of the decimal logarithm of the Bayes factor of VAR(1)-SJSV against the alternative models. We see that for our dataset the models with three latent processes describe the time-varying conditional covariance matrix much better than the models with one or two latent processes. The VAR(1)-SJSV model wins our model comparison, being about 8.5 orders of magnitude better than the VAR(1)-$\text{TSV}_{\text{EUR\_USD}}$ model. The decimal log of the Bayes factor of the VAR(1)-MSF-SBEKK model relative to the VAR(1)-SJSV model is 27.32. The presence of more latent processes improves fit enormously, but seems infeasible for highly dimensional time series. Assuming equal prior model probabilities, the VAR(1)-MSF-SBEKK model is about 20.73 orders of magnitude more probable a posterior than the VAR(1)-MSF model (with the constant conditional correlations), and about 32 orders of magnitude better than the VAR(1)-SBEKK model. Note that the VAR(1)-MSF-SBEKK model is about 6.6 orders of magnitude better than another hybrid model – the VAR(1)-MSF-DCC model, proposed by Osiewalski, Pajor (2007).

Table 1. Logs of Bayes factors in favour of VAR(1)-SJSV model

Model	Number of latent processes	Number of parameters	Log_{10} ($B_{\text{SJSV},i}$)	Rank
VAR(1)-SJSV	3	18	0	1
VAR(1)-$\text{TSV}_{\text{EUR\_USD}}$	3	18	8.51	2
VAR(1)-$\text{TSV}_{\text{USD\_EUR}}$	3	18	11.10	3
VAR(1)-JSV	2	15	19.60	4
VAR(1)-MSF-SBEKK	1	14	27.32	5
VAR(1)-MSF-SBEKK with the approximation	1	14	29.30	6
VAR(1)-MSF-IDCC	1	18	32.00	7
VAR(1)-MSF-DCC	1	20	33.88	8
VAR(1)-MSF(SDF)	1	12	48.05	9
VAR(1)-SBEKK	0	12	59.70	10
VAR(1)-BMSV	2	14	158.51	11

Note: the decimal logarithm of the Bayes factors were calculated using the Newton and Raftery method (see Newton, Raftery, 1994). Only the results for the VAR(1)-MSF-SBEKK, VAR(1)-MSF and VAR(1)-SBEKK models are new; the remaining ones were obtained by Pajor (2009).

In the bivariate case considered here it is possible to compare exact and approximate Bayesian results relate to estimation of the VAR(1)-MSF-SBEKK model. Thus, in Tables 1 we present the decimal logarithm of the Bayes factor for both cases. Using the approximate Bayesian approach proposed by Osiewalski, Pajor (2009) leads to smaller values of the data density, but it seems that the fit does not significantly change.

Of course, our model comparison relies on the prior distributions for the parameters of the models, but these prior distributions are not very informative.

3.1. Predictive Properties of the MSF-SBEKK Models in Portfolio Selection

It is important to investigate the predictive properties of the MSF-SBEKK model in portfolio selection. In addition, we can examine how the exact and approximate posterior results may differ. Thus, in this section we report the results of building the optimal portfolios using the MSF-SBEKK model. We consider the hypothetical portfolios, which consist of two currencies: the US dollar and euro. We assume that there are no transaction costs and that we may reallocate zloty to long as well as to short positions across the currencies. Allocation decisions are made at time T based on the predictive distribution for $\mathbf{y}_{T+k}$ and $\mathbf{\Sigma}_{T+k}$ for $k = 1, ..., 60$.

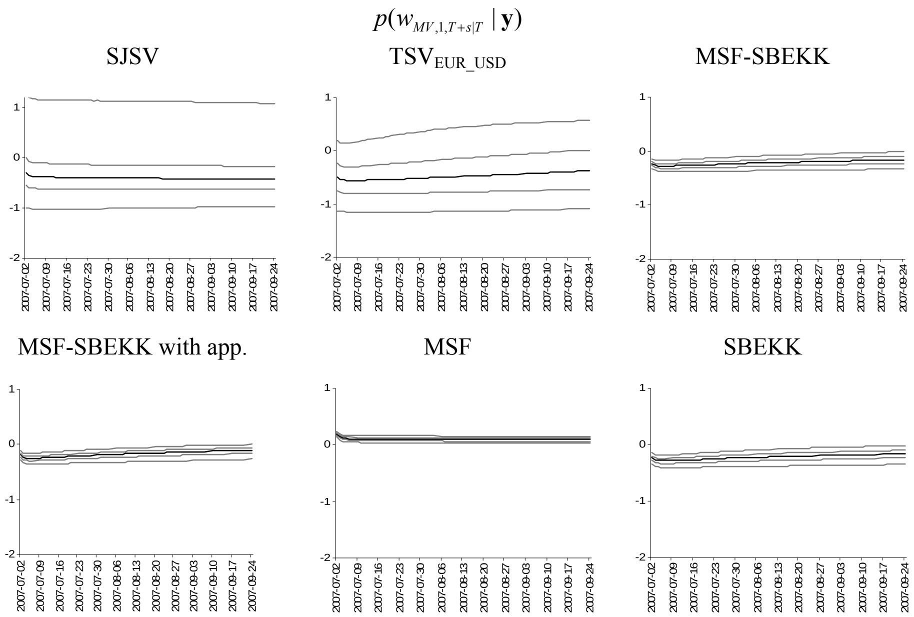

Figure 1. Quantiles of the predictive distributions of the minimum conditional variance portfolios (the fractions of wealth invested in the US dollar). The central black lines represent the medians, and the grey lines represent the quantiles of order 0.05, 0.25, 0.75, 0.95, respectively

In Figure 1 we show the quantiles of the predictive distributions of the minimum conditional variance portfolio $w_{MV,1,T|T+s}$ (the fraction of wealth invested in the US dollar). If the medians of the marginal predictive distributions are treated as point forecasts, in model with time-varying conditional correlation coefficient the optimum weights to invest in the USD/PLN are negative, indicating the short sale of the US dollar (the median of the marginal predictive distribution of $w_{MV,1,T|T+s}$ is equal to about -0.4 in the most probable a posterior model, and about -0.22 in the VAR(1)-MSF-SBEKK model). The short position on the US dollar is connected with corresponding long position on the euro. We see that in VAR(1)-MSV models with more than one latent process the predictive distributions are very widely dispersed and fat-tailed, thus leaving us with considerable uncertainty about the future returns of these portfolios. Surprisingly, in the VAR(1)-MSV models with one latent process or in the VAR(1)-SBEKK model the minimum conditional variance portfolios are estimated more precisely – the inter-quartile ranges are relatively small. It seams that the VAR(1)-MSF-SBEKK and VAR(1)-SBEKK models produce portfolios with lowest risk measured by the conditional variance (see Figure 2). Note that the predictive distributions of $w_{MV,1,T|T+s}$ for $s = 1, ..., 60$ produced by the VAR(1)-MSF-SBEKK model are located in areas of high predictive densities obtained in the best model (i.e. VAR(1)-SJSV).

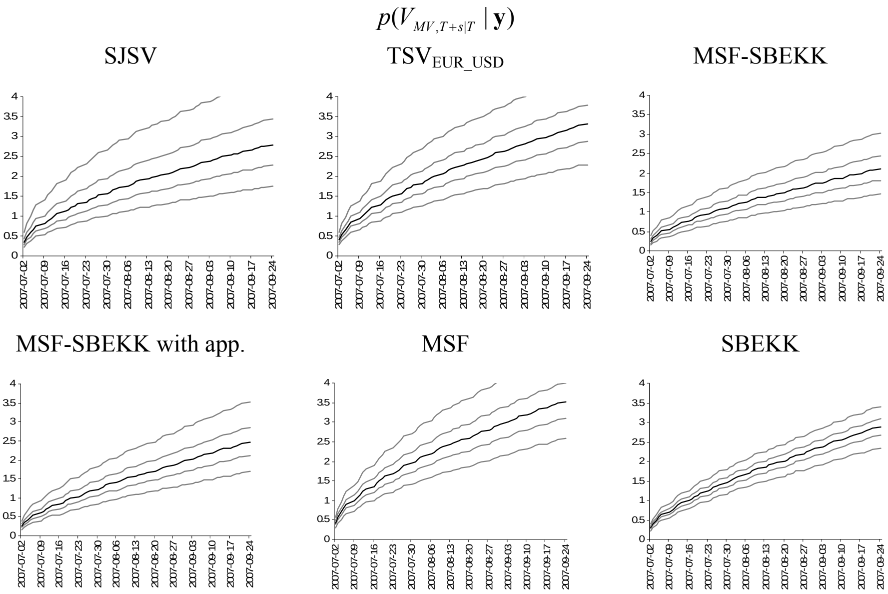

Figure 2. Quantiles of the predictive distributions of the conditional standard deviation of the minimum conditional variance portfolios. The central black lines represent the medians, and the grey lines represent the quantiles of order 0.05, 0.25, 0.75, 0.95, respectively

As in Pajor (2009) we can see that the predictive distributions related to the portfolio with bound on return are more diffuse – the inter-quartile ranges are higher (see Figure 3 and 4). Comparing the minimum conditional variance portfolio and the minimum conditional variance portfolio with the return equal to at least 5%, we can see that the distributions of the forecasted value of $w_{MVR_p^*,T|T+s}$ and $V_{MVR_p^*,T|T+s}$ are more dispersed and have very thick tails. Thus uncertainty connected with the optimal portfolio with return at least 5% on annual base is huge. In all models the quantiles of the conditional standard deviation of the optimal portfolios (see Figure 4) indicate increasing volatility with the forecast horizon.

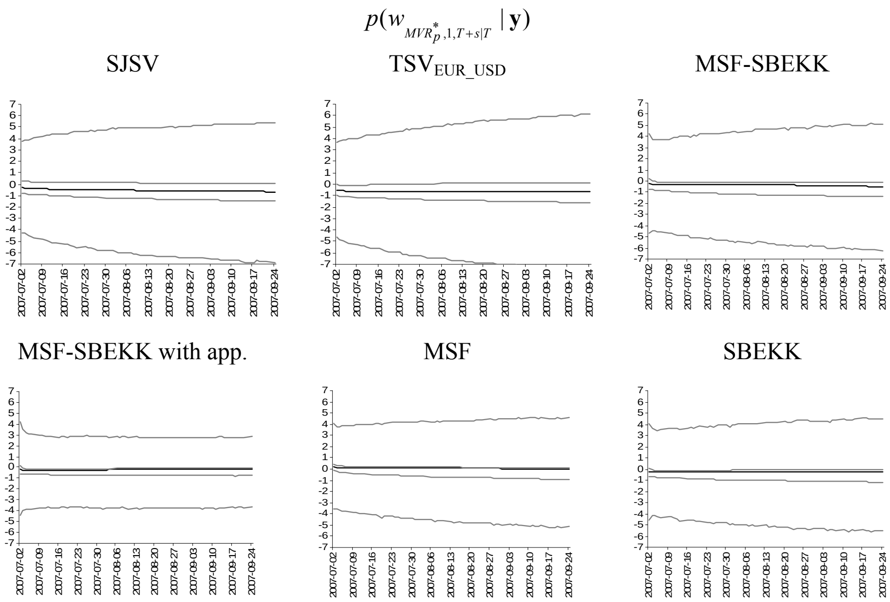

Figure 3. Quantiles of the predictive distributions of the minimum conditional variance portfolios with the return equal to at least 5% on annual base (the fraction of wealth invested in the US dollar). The central black lines represent the medians, and the grey lines represent the quantiles of order 0.05, 0.25, 0.75, 0.95, respectively

Finally, as in Pajor (2009) we use the medians of $w_{MVR_p^*,1,T+s|T}$ to construct hypothetical portfolios for s = 1, 2, .., 60. Let W_T = 10000 PLN be the initial wealth of the investor at time T (on June 29, 2007). If we assume that there are no transaction costs and the investor uses the median of the predictive distribution of $\mathbf{w}_{MVR_p^*,T+s|T}$ (denoted by $w^{op}_{MVR_p^*,1,T+s|T}$) to construct optimal portfolio, then the investor's wealth at time T+s is given by:

$$W_{MVR_p^*,T+s|T} = W_T[w^{op}_{MVR_p^*,1,T+s|T}(x_{1,T+s}/x_{1,T}) + w^{op}_{MVR_p^*,2,T+s|T}(x_{2,T+s}/x_{2,T})],$$

$s = 1, 2, .., 60.$

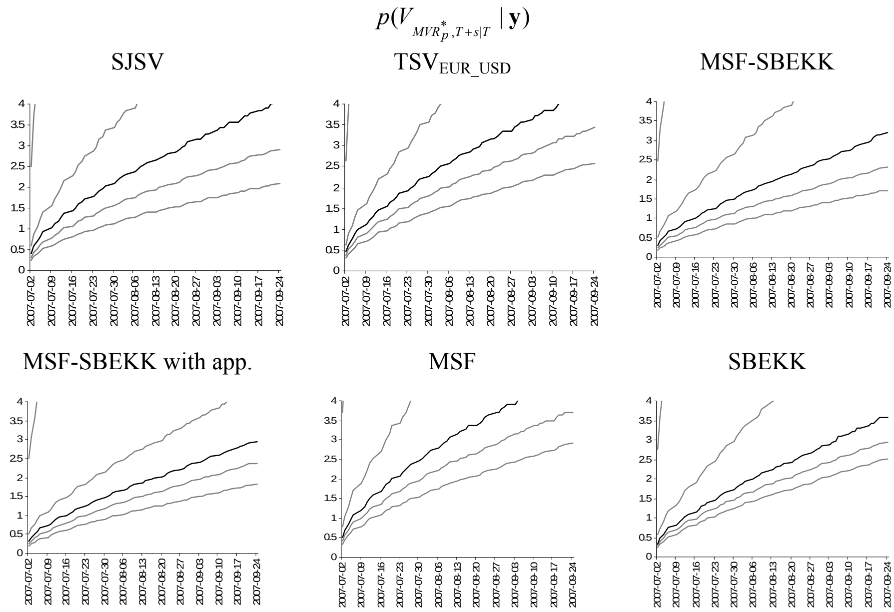

Figure 4. Quantiles of the predictive distributions of the conditional standard deviation of the minimum conditional variance portfolio with the return equal to at least 5% on annual base. The central black lines represent the medians, and the grey lines represent the quantiles of order 0.05, 0.25, 0.75, 0.95, respectively

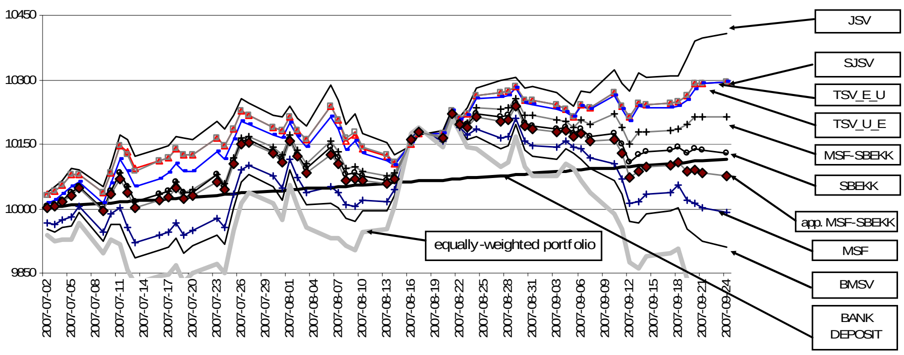

Figure 5. Wealth of the investor at time $T+s$ for $s = 1, ..., 60$ (the optimal portfolio is constructed on the medians of $w_{MVR_p^*,T+s|T}$)

In Figure 5, we present the plot of $W_{MVR_p^*,T+s|T}$ for s =1, 2, ..., 60, and compare them with a bank deposit with the interest rate equal to 4.7% on annual base (the quotation of the 3-month Warsaw Interbank Offered Rate on June, 29 2007). The best results we obtain in the VAR(1)-JSV model – at a 2-month horizon the average return of the optimal portfolios is equal to 0.098%, which represents annual return of 24.58%. In the best model (i.e. VAR(1) – SJSV) the average return of the optimal portfolios is equal to 0.065%, which represents annual return of 16.34%, whereas in the VAR(1)-MSF-SBEKK and VAR(1)-SBEKK models we have 0.048% and 0.044%, respectively (i.e. 12.02% and 11.05% per annum, respectively). It is important to stress that in the VAR(1)-MSF-SBEKK model the returns of the hypothetical investments are higher than those of the bank deposit, indicating good forecasting properties of the model. In the VAR(1)-MSF model (with constant conditional correlation) the average return of the portfolio is negative (we obtained -0.006% i.e. -0.16% per annum). Thus the SBEKK structure is very important in forecasting. In the approximated VAR(1)-MSF-SBEKK model the average return is equal to 0.04% (i.e. 9.43% per annum). Thus using approximation in the VAR(1)-MSF-SBEKK leads to worse predictive results. After two months the return of the optimal portfolio is lower than the interest rate of the bank deposit, but still it is positive. Note that the average return of equally-weighted portfolio is equal to -0.047, i.e. -11.80% per annum.

Conclusions

The paper investigates the predictive abilities of the VAR(1)-MSF-SBEKK model in portfolio selection. The predictive distributions of the optimal portfolios produced by the VAR(1)-MSF-SBEKK model are compared with those obtained in unparsimonious (but more probable a posterior) MSV specifications. The predictive distributions of the weights of the optimal portfolios produced by the VAR(1)-MSF-SBEKK model are located in areas of high predictive densities obtained in the best MSV model (i.e. VAR(1)-SJSV). Unfortunately, in all models the predictive distributions of the optimal portfolio are very spread and have heavy tails. Our main finding is that the VAR(1)-MSF-SBEKK model is useful (but not very impressive) for building the multi-period optimal minimum conditional variance portfolio. It seems that the approximation proposed by Osiewalski, Pajor (2009) results in worse predictive properties of the VAR(1)-MSF-BEKK model, but for large portfolios this approximation is necessary.

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Bayesowska optymalizacja portfela w modelu MSF-SBEKK

Z a r y s t r e ś c i. Celem artykułu jest analiza prognostycznych własności bayesowskiego modelu MSF-SBEKK w kontekście wyboru optymalnego portfela inwestycyjnego. Wykorzystywany w artykule wielowymiarowy proces MSF-SBEKK posiada elementy struktury skalarnego procesu BEKK oraz procesu MSF. Obecność, w jego definicji, odrębnego czynnika losowego pozwala lepiej opisywać zjawisko grubych ogonów, zaś w strukturze SBEKK uzależnia się warunkowe wariancje oraz warunkowe korelacje od przeszłych wartości procesu. Proces MSF-SBEKK posiada zatem nietrywialną strukturę i może być wykorzystany do opisu zależności miedzy stopami zwrotu kilkudziesięciu (a nawet kilkuset) instrumentów finansowych. W artykule dokonane zostało porównanie prognoz uzyskanych w dwuwymiarowym modelu MSF-SBEKK oraz w innych modelach z klasy MSV na przykładzie portfela walutowego, złożonego z kursu dolara amerykańskiego oraz euro. Uzyskane wyniki wskazują na dobre własności prognostyczne modelu MSF-SBEKK, choć uproszczenia w sposobie jego estymacji mogą je pogarszać.

S ł o w a k l u c z o w e: model MSF-SBEKK, modele MSV, analiza portfelowa, prognoza.