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Estimates for the polar derivative of a constrained polynomial on a disk

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ABSTRACT

This work is a part of a recent wave of studies on inequalities which relate the uniform-norm of a univariate complex coefficient polynomial to its derivative on the unit disk in the plane. When there is a limit on the zeros of a polynomial, we develop some additional inequalities that relate the uniform-norm of the polynomial to its polar derivative. The obtained results support some recently established Erdős-Lax and Turán-type inequalities for constrained polynomials, as well as produce a number of inequalities that are sharper than those previously known in a very large literature on this subject.

RESUMEN

Este trabajo es parte de una reciente ola de estudios sobre desigualdades que relacionan la norma uniforme de un polinomio univariado con coeficientes complejos con su derivada en el disco unitario en el plano. Cuando existe un límite sobre los ceros de un polinomio, desarrollamos algunas desigualdades adicionales que relacionan la norma uniforme del polinomio con su derivada polar. Los resultados obtenidos satisfacen desigualdades de tipo Erdős-Lax y Turán para polinomios restringidos recientemente establecidas, y también producen desigualdades que son más estrictas que aquellas conocidas previamente en la larga literatura dedicada a este tema.

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1 Introduction

Experimental data is converted into mathematical notation and mathematical models in scientific inquiries. In order to solve these, it may be necessary to know how large or small the maximum modulus of the derivative of an algebraic equation can be in terms of maximum modulus of the polynomial. In practise, setting boundaries for these circumstances is crucial. The only information available in the literature is in the form of approximations, and there are no closed formulae for calculating these limitations precisely. These approximate boundaries are quite accurate when computed effectively adequate for the demands of investigators and scientists. As a result, there is a constant desire to find boundaries that are superior to those described in the literature. We were inspired to write this note because there is a need for updated and more precise bounds. The inequalities for polynomials and their derivatives, which generalise the classical inequalities for different norms and with different constraints on utilising various methods of geometric function theory, are a fertile topic in analysis. In the literature, for proving the inverse theorems in approximation theory, many inequalities in both directions relating the norm of the derivative and the polynomial itself play a significant role and, of course, have their own intrinsic appeal. As shown by various recent studies, numerous research papers have been published on these inequalities for constrained polynomials (for example, see [11, 13, 17, 19, 20, 21]). We begin with the well-known Bernstein inequality [4] for the uniform norm on the unit disk in the plane: namely, if $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

If we only consider polynomials without zeros in $|z| < 1$, the above inequality (1.1) can then be emphasised. In fact, Erdős conjectured and later Lax [14] proved that, if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

The inequality (1.2) is sharp and equality holds if $P(z)$ has all of its zeros on $|z| = 1$.

When there is a restriction on the polynomial's zeros, Turán's classical inequality [25] offers a lower bound estimate for the size of the derivative of the polynomial on the unit circle in relation to the size of the polynomial. It states that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Aziz and Dawood [2] improved inequality (1.3) to take the form

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{2}\left\{\max_{|z|=1}|P(z)| + \min_{|z|=1}|P(z)|\right\}. \tag{1.4}$$

Any polynomial that has all of its zeros on $|z| = 1$ holds true for (1.3) and (1.4).

The inequalities (1.3) and (1.4) have been generalised and expanded in a number of ways over time. For a polynomial $P(z)$ of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, Govil [8], proved that

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{1+k^n}\max_{|z|=1}|P(z)|. \tag{1.5}$$

As is easy to see that (1.5) becomes an equality if $P(z) = z^n + k^n$, one would expect that if we exclude the class of polynomials having all zeros on $|z| = k$, then it may be possible to improve the bound in (1.5). In this direction, it was shown by Govil [10] that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{1+k^n}\left\{\max_{|z|=1}|P(z)| + \min_{|z|=k}|P(z)|\right\}. \tag{1.6}$$

As an extension of (1.2), Malik [15] proved that, if $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{1+k}\max_{|z|=1}|P(z)|. \tag{1.7}$$

The result is sharp and equality in (1.7) holds for $P(z) = (z+k)^n$.

On the other hand, if $P(z) \neq 0$ in $|z| < k$, $k \leq 1$, the precise estimate of maximum of $|P'(z)|$ on $|z| = 1$ does not seem to be known in general, and this problem is still open. However, some special cases in this direction have been considered by many people where some partial extensions of (1.2) are established. In 1980, it was shown by Govil [9] that if $P(z)$ is a polynomial of degree n and $P(z) \neq 0$ in $|z| < k$, $k \leq 1$, then

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{1+k^n}\max_{|z|=1}|P(z)|, \tag{1.8}$$

provided $|P'(z)|$ and $|Q'(z)|$ attain their maximum at the same point on $|z| = 1$, where $Q(z) = z^n\overline{P(1/\overline{z})}$. Under the same hypothesis as in (1.8), Aziz and Ahmad [1] established an improved inequality in the form

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{1+k^n}\left\{\max_{|z|=1}|P(z)| - \min_{|z|=k}|P(z)|\right\}. \tag{1.9}$$

In the literature, more generalised variations of Bernstein and Turán inequalities have emerged, in which the underlying polynomial is replaced with more general classes of functions. One such generalisation is moving from the domain of ordinary derivatives of polynomials to the domain of their polar derivatives. Before drawing any more conclusions, let us first discuss the idea of the polar derivative. For a polynomial $P(z)$ of degree n, we define

$$D_\beta P(z) := nP(z) + (\beta - z)P'(z),$$

the polar derivative of $P(z)$ with respect to the point β. The polynomial $D_\beta P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\beta\to\infty} \left\{ \frac{D_\beta P(z)}{\beta} \right\} = P'(z),$$

uniformly with respect to z for $|z| \leq R,\ R > 0$.

The comprehensive books by Marden [16], Milovanović *et al.* [18], Rahman and Schmeisser [23], and the most recent one by Gardner *et al.* [7] all provide access to the extensive literature on the polar derivative of polynomials.

In 1998, Aziz and Rather [3] established the polar derivative analogue of (1.5) by proving that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \geq k$,

$$\max_{|z|=1} |D_\beta P(z)| \geq n \left(\frac{|\beta| - k}{1 + k^n} \right) \max_{|z|=1} |P(z)|. \tag{1.10}$$

In the same publication, Aziz and Rather extended the inequality (1.4) to the polar derivative of a polynomial. In fact, they proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for any complex number β with $|\beta| \geq 1$,

$$\max_{|z|=1} |D_\beta P(z)| \geq \frac{n}{2} \left\{ (|\beta| - 1) \max_{|z|=1} |P(z)| + (|\beta| + 1) \min_{|z|=1} |P(z)| \right\}. \tag{1.11}$$

The corresponding polar derivative analogue of (1.6) and a refinement of (1.10) was given by Dewan *et al.* [5]. They proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then for any complex number β with $|\beta| \geq k$,

$$\max_{|z|=1} |D_\beta P(z)| \geq \frac{n}{1 + k^n} \left\{ (|\beta| - k) \max_{|z|=1} |P(z)| + \left(|\beta| + \frac{1}{k^{n-1}} \right) \min_{|z|=k} |P(z)| \right\}. \tag{1.12}$$

Singh and Chanam [24] most recently developed the following generalisation and strengthening of (1.10).

Theorem A. *Let $P(z) = z^s \sum_{\nu=0}^{n-s} a_\nu z^\nu$, $0 \le s \le n$, be a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for every complex number β with $|\beta| \ge k$,*

$$\max_{|z|=1} |D_\beta P(z)| \ge (|\beta| - k)\left\{\frac{n+s}{1+k^n} + \frac{k^{(n-s)/2}\sqrt{|a_{n-s}|} - \sqrt{|a_0|}}{(1+k^n)k^{(n-s)/2}\sqrt{|a_{n-s}|}}\right\} \max_{|z|=1} |P(z)|. \tag{1.13}$$

The improvement of inequality (1.8) as a result of Govil [9] was demonstrated by Singh and Chanam in the same paper in the form of the subsequent outcome.

Theorem B. *Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n having no zeros in $|z| < k$, $k \le 1$, and let $Q(z) = z^n \overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain their maximum at the same point on $|z| = 1$,*

$$\max_{|z|=1} |P'(z)| \le \left\{\frac{n}{1+k^n} - \frac{\left(\sqrt{|a_0|} - k^{n/2}\sqrt{|a_n|}\right)k^n}{(1+k^n)\sqrt{|a_0|}}\right\} \max_{|z|=1} |P(z)|. \tag{1.14}$$

The result is sharp and equality holds in (1.14) for $P(z) = z^n + k^n$.

The study of these inequalities for a certain class of polynomials having a zero of order $s \ge 0$ at the origin is continued in this paper, and we set some new upper and lower bounds for the derivative and polar derivative of a polynomial on the unit disk while taking into account the location of the zeros and extremal coefficients of the underlying polynomial.

2 Main results

We begin this section by proving the following Turán-type inequality giving generalisations and refinements of (1.10)–(1.13) and related inequalities.

Theorem 2.1. *Let $P(z) = z^s \sum_{\nu=0}^{n-s} a_\nu z^\nu$, $0 \le s \le n$, be a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for every complex number β with $|\beta| \ge k$,*

$$\begin{aligned}\max_{|z|=1} |D_\beta P(z)| \ge{}& \frac{n}{1+k^n}\left\{(|\beta| - k)\max_{|z|=1} |P(z)| + \left(|\beta| + \frac{1}{k^{n-1}}\right) m_k\right\} \\ &+ \left(\frac{|\beta| - k}{1+k^n}\right)\left\{s + \frac{\sqrt{k^{n-s}|a_{n-s}| - m_k} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - m_k}}\right\}\left(\max_{|z|=1} |P(z)| - \frac{m_k}{k^n}\right),\end{aligned} \tag{2.1}$$

where $m_k = \min_{|z|=k} |P(z)|$.

Setting $s = 0$ in (2.1), we get the following refinement of (1.12) and hence of (1.10) and (1.11) as well.

Corollary 2.2. *Let $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ be a polynomial of degree n having all its zeros in $|z|\le k$, $k\ge 1$, then for every complex number β with $|\beta|\ge k$,*

$$\begin{aligned}\max_{|z|=1}|D_\beta P(z)|\ge&\frac{n}{1+k^n}\left\{(|\beta|-k)\max_{|z|=1}|P(z)|+\left(|\beta|+\frac{1}{k^{n-1}}\right)m_k\right\}\\&+\left(\frac{|\beta|-k}{1+k^n}\right)\left\{\frac{\sqrt{k^n|a_n|-m_k}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-m_k}}\right\}\left(\max_{|z|=1}|P(z)|-\frac{m_k}{k^n}\right),\end{aligned}\tag{2.2}$$

where m_k is as defined in Theorem 2.1.

By taking $k=1$ in (2.2), we easily get a refinement of (1.11). If we divide both sides of (2.1) and (2.2) by $|\beta|$ and let $|\beta|\to\infty$, we get the following results:

Corollary 2.3. *Let $P(z)=z^s\sum_{\nu=0}^{n-s}a_\nu z^\nu$, $0\le s\le n$, be a polynomial of degree n having all its zeros in $|z|\le k,\ k\ge 1$, then*

$$\begin{aligned}\max_{|z|=1}|P'(z)|\ge&\frac{n}{1+k^n}\left(\max_{|z|=1}|P(z)|+m_k\right)\\&+\left\{\frac{s}{1+k^n}+\frac{\sqrt{k^{n-s}|a_{n-s}|-m_k}-\sqrt{|a_0|}}{(1+k^n)\sqrt{k^{n-s}|a_{n-s}|-m_k}}\right\}\left(\max_{|z|=1}|P(z)|-\frac{m_k}{k^n}\right),\end{aligned}\tag{2.3}$$

where m_k is as defined in Theorem 2.1. Equality in (2.3) holds for $P(z)=z^n+k^n$.

Corollary 2.4. *Let $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ be a polynomial of degree n having all its zeros in $|z|\le k$, $k\ge 1$, then*

$$\begin{aligned}\max_{|z|=1}|P'(z)|\ge&\frac{n}{1+k^n}\left(\max_{|z|=1}|P(z)|+m_k\right)\\&+\frac{\sqrt{k^n|a_n|-m_k}-\sqrt{|a_0|}}{(1+k^n)\sqrt{k^n|a_n|-m_k}}\left(\max_{|z|=1}|P(z)|-\frac{m_k}{k^n}\right),\end{aligned}\tag{2.4}$$

where m_k is as defined in Theorem 2.1. Equality in (2.4) holds for $P(z)=z^n+k^n$.

Remark 2.5. *It is clear that, in general for any polynomial of degree n of the form $P(z)=z^s(a_0+a_1z+\cdots+a_{n-s}z^{n-s}),\ 0\le s\le n$, having all its zeros in $|z|\le k,\ k\ge 1$, the inequality (2.1) improves the inequality (1.13) considerably, excepting the case when $P(z)$ has a zero on $|z|=k$. For the class of polynomials having a zero on $|z|=k$, the inequality (2.2) will give bounds that are sharper than the bound obtained from the inequality (1.12). One can also observe that the inequality (2.4) improves inequality (1.6) considerably when $\sqrt{k^n|a_n|-m_k}-\sqrt{|a_0|}\ne 0$.*

As an application of Corollary 2.4, we prove the following result for the class of polynomials having no zeros in $|z|<k,\ k\le 1$, which in turn provides a generalization and refinement to Theorem B.

Theorem 2.6. *Let $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ be a polynomial of degree n having no zeros in $|z|<k,\ k\le 1$, and let $Q(z)=z^n\overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain their maximum at the same point on $|z|=1$, then for every complex number β with $|\beta|\ge 1$,*

$$\begin{aligned}\max_{|z|=1}|D_\beta P(z)|\le &\frac{n(|\beta|+k^n)}{1+k^n}\max_{|z|=1}|P(z)|-\frac{nm_k(|\beta|-1)}{1+k^n}\\&-\frac{(|\beta|-1)\left(\sqrt{|a_0|-m_k}-k^{n/2}\sqrt{|a_n|}\right)k^n}{(1+k^n)\sqrt{|a_0|-m_k}}\left\{\max_{|z|=1}|P(z)|-m_k\right\},\end{aligned}\tag{2.5}$$

where m_k is as defined in Theorem 2.1. Equality in (2.5) holds for $P(z)=z^n+k^n$, with real $\beta\ge 1$.

If we divide both sides of inequality (2.5) by $|\beta|$ and let $|\beta|\to\infty$, we get the following result.

Corollary 2.7. *Let $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ be a polynomial of degree n having no zeros in $|z|<k,\ k\le 1$, and let $Q(z)=z^n\overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain their maximum at the same point on $|z|=1$, then*

$$\begin{aligned}\max_{|z|=1}|P'(z)|\le &\frac{n}{1+k^n}\max_{|z|=1}|P(z)|-\frac{nm_k}{1+k^n}\\&-\frac{\left(\sqrt{|a_0|-m_k}-k^{n/2}\sqrt{|a_n|}\right)k^n}{(1+k^n)\sqrt{|a_0|-m_k}}\left\{\max_{|z|=1}|P(z)|-m_k\right\},\end{aligned}\tag{2.6}$$

where m_k is as defined in Theorem 2.1. Equality in (2.6) holds for $P(z)=z^n+k^n$.

Remark 2.8. It may be remarked here that, in general for any polynomial of degree n of the form $P(z)=a_0+a_1z+a_2z^2+\cdots+a_nz^n$, having no zeros in $|z|<k$, $k\le 1$, the inequality (2.6) improves the inequality (1.14), excepting the case when $P(z)$ has a zero on $|z|=k$. For the class of polynomials having a zero on $|z|=k$, the inequality (2.5) sharpens a result of Mir and Breaz [20, Corollary 2] considerably.

3 Lemmas

In order to prove our results, we need the following lemmas. The first lemma is a simple deduction from the Maximum Modulus Principle (see [22]).

Lemma 3.1. *If $P(z)$ is a polynomial of degree at most n, then for $R\ge 1$,*

$$\max_{|z|=R}|P(z)|\le R^n\max_{|z|=1}|P(z)|.$$

The following lemma is due to Dewan and Upadhye [6].

Lemma 3.2. *If $P(z)$ is a polynomial of degree n having all zeros in $|z| \le k,\ k \ge 1$, then*

$$\max_{|z|=k}|P(z)| \ge \frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)| + \frac{k^n-1}{k^n+1}\min_{|z|=k}|P(z)|.$$

Lemma 3.3. *If $P(z) = z^s\sum_{\nu=0}^{n-s} a_\nu z^\nu,\ 0 \le s \le n$, is a polynomial of degree n having all zeros in $|z| \le 1$, then for any complex number β with $|\beta| \ge 1$ and $|z| = 1$,*

$$|D_\beta P(z)| \ge (|\beta|-1)\left\{\frac{n+s}{2} + \frac{\sqrt{|a_{n-s}|}-\sqrt{|a_0|}}{2\sqrt{|a_{n-s}|}}\right\}|P(z)|.$$

The above lemma is due to Singh and Chanam [24].

Lemma 3.4. *If $P(z) = z^s\sum_{\nu=0}^{n-s} a_\nu z^\nu$, $0 \le s \le n$, is a polynomial of degree n having all zeros in $|z| \le 1$, then for any complex number β with $|\beta| \ge 1$ and $|z| = 1$,*

$$\begin{aligned}|D_\beta P(z)| \ge& \frac{n}{2}\Big((|\beta|-1)|P(z)| + (|\beta|+1)m_1\Big)\\ &+\left(\frac{|\beta|-1}{2}\right)\left\{s + \frac{\sqrt{|a_{n-s}|-m_1}-\sqrt{|a_0|}}{\sqrt{|a_{n-s}|-m_1}}\right\}(|P(z)|-m_1),\end{aligned}$$

where $m_1 = \min_{|z|=1}|P(z)|$.

Proof. By hypothesis $P(z) = z^s\sum_{\nu=0}^{n-s} a_\nu z^\nu$, $0 \le s \le n$, has all its zeros in $|z| \le 1$. If the polynomial $h(z) = \sum_{\nu=0}^{n-s} a_\nu z^\nu$ has a zero on $|z| = 1$, then $m_1 = \min_{|z|=1}|P(z)| = 0$ and the result follows by Lemma 3.3 in this case. Henceforth, we assume that all the zeros of $P(z) = z^s h(z)$ lie in $|z| < 1$, so that $m_1 > 0$. Therefore, we have $m_1 \le |P(z)|$ for $|z| = 1$. This implies for any complex number μ with $|\mu| < 1$, that

$$m_1|\mu z^n| < |P(z)| \quad \text{for} \quad |z| = 1.$$

Since all the zeros of $P(z)$ lie in $|z| < 1$, it follows by Rouché's theorem that all the zeros of

$$P(z) - \mu m_1 z^n = z^s\Big(a_0 + a_1 z + \cdots + (a_{n-s} - \mu m_1)z^{n-s}\Big)$$

also lie in $|z| < 1$. Hence, by Lemma 3.3, we get for $|\beta| \ge 1$ and $|z| = 1$,

$$|D_\beta(P(z) - \mu m_1 z^n)| \ge (|\beta|-1)\left\{\frac{n+s}{2} + \frac{\sqrt{|a_{n-s}-\mu m_1|}-\sqrt{|a_0|}}{2\sqrt{|a_{n-s}-\mu m_1|}}\right\}|P(z) - \mu m_1 z^n|. \tag{3.1}$$

For every $\mu \in \mathbb{C}$, we have

$$|a_{n-s} - \mu m_1| \geq |a_{n-s}| - |\mu| m_1,$$

and since the function $\psi(x) = \frac{\left(\sqrt{x} - \sqrt{|a_0|}\right)}{\sqrt{x}}$, $x > 0$, is a non-decreasing function of x, it follows from (3.1) that for every μ with $|\mu| < 1$ and $|z| = 1$,

$$|D_\beta(P(z) - \mu m_1 z^n)| \geq (|\beta| - 1)\left\{\frac{n+s}{2} + \frac{\sqrt{|a_{n-s}| - |\mu| m_1} - \sqrt{|a_0|}}{2\sqrt{|a_{n-s}| - |\mu| m_1}}\right\}|P(z) - \mu m_1 z^n|. \tag{3.2}$$

It is a simple deduction of Laguerre theorem (see [16, p. 52]) on the polar derivative of a polynomial that for any β with $|\beta| \geq 1$, the polynomial

$$D_\beta(P(z) - \mu m_1 z^n) = D_\beta P(z) - \mu \beta n m_1 z^{n-1}$$

has all its zeros in $|z| < 1$. This implies that

$$|D_\beta P(z)| \geq m_1 n |\beta| |z|^{n-1} \quad \text{for} \quad |z| \geq 1. \tag{3.3}$$

Now choosing the argument of μ suitably on the left hand side of (3.2) such that

$$\left|D_\beta P(z) - \mu \beta n m_1 z^{n-1}\right| = |D_\beta P(z)| - |\mu||\beta| n m_1 \quad \text{for} \quad |z| = 1,$$

which is possible by (3.3), we get for $|z| = 1$

$$|D_\beta P(z)| - m_1 n |\mu||\beta| \geq (|\beta| - 1)\left\{\frac{n+s}{2} + \frac{\sqrt{|a_{n-s}| - |\mu| m_1} - \sqrt{|a_0|}}{2\sqrt{|a_{n-s}| - |\mu| m_1}}\right\}\big(|P(z)| - |\mu| m_1\big). \tag{3.4}$$

If in (3.4), we make $|\mu| \to 1$, we easily get for $|z| = 1$,

$$\begin{aligned} |D_\beta P(z)| \geq {} & \frac{n}{2}\Big((|\beta| - 1)|P(z)| + (|\beta| + 1) m_1\Big) \\ & + \left(\frac{|\beta| - 1}{2}\right)\left\{s + \frac{\sqrt{|a_{n-s}| - m_1} - \sqrt{|a_0|}}{\sqrt{|a_{n-s}| - m_1}}\right\}(|P(z)| - m_1). \end{aligned}$$

This completes the proof of Lemma 3.4. □

Lemma 3.5. *If $P(z)$ is a polynomial of degree n and, $Q(z) = z^n \overline{P(1/\overline{z})}$, then on $|z| = 1$,*

$$|P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)|.$$

The above lemma is due to Govil and Rahman [12].

4 Proofs of the main results

Proof of Theorem 2.1. Recall that $P(z)$ has all its zeros in $|z| \leq k,\ k \geq 1$, therefore, all the zeros of the polynomial $E(z) = P(kz)$ lie in $|z| \leq 1$. Applying Lemma 3.4 to the polynomial $E(z)$ and noting that $|\beta|/k \geq 1$, we get

$$\begin{aligned} \max_{|z|=1} \left|D_{\beta/k}E(z)\right| \geq \frac{n}{2}\bigg\{ \left(\frac{|\beta|}{k} - 1\right) \max_{|z|=1} |E(z)| + \left(\frac{|\beta|}{k} + 1\right) m^* \bigg\} \\ + \left(\frac{|\beta|}{k} - 1\right) \left\{ \frac{s}{2} + \frac{\sqrt{k^{n-s}|a_{n-s}| - m^*} - \sqrt{|a_0|}}{2\sqrt{k^{n-s}|a_{n-s}| - m^*}} \right\} \left(\max_{|z|=1} |E(z)| - m^* \right), \end{aligned} \tag{4.1}$$

where $m^* = \min_{|z|=1} |E(z)| = \min_{|z|=1} |P(kz)| = \min_{|z|=k} |P(z)| = m_k$.

The above inequality (4.1) is equivalent to

$$\begin{aligned} \max_{|z|=1} \left| nP(kz) + \left(\frac{\beta}{k} - z\right) kP'(kz) \right| \geq \frac{n}{2}\bigg\{ \left(\frac{|\beta| - k}{k}\right) \max_{|z|=1} |P(kz)| + \left(\frac{|\beta|}{k} + 1\right) m_k \bigg\} \\ + \left(\frac{|\beta| - k}{k}\right) \left\{ \frac{s}{2} + \frac{\sqrt{k^{n-s}|a_{n-s}| - m_k} - \sqrt{|a_0|}}{2\sqrt{k^{n-s}|a_{n-s}| - m_k}} \right\} \\ \times \left(\max_{|z|=1} |P(kz)| - m_k \right). \end{aligned}$$

The last inequality yields

$$\begin{aligned} \max_{|z|=k} |D_\beta P(z)| \geq \frac{n}{2}\bigg\{ \left(\frac{|\beta| - k}{k}\right) \max_{|z|=k} |P(z)| + \left(\frac{|\beta|}{k} + 1\right) m_k \bigg\} \\ + \left(\frac{|\beta| - k}{k}\right) \left\{ \frac{s}{2} + \frac{\sqrt{k^{n-s}|a_{n-s}| - m_k} - \sqrt{|a_0|}}{2\sqrt{k^{n-s}|a_{n-s}| - m_k}} \right\} \left(\max_{|z|=k} |P(z)| - m_k \right). \end{aligned} \tag{4.2}$$

Since $D_\beta P(z)$ is a polynomial of degree at most $n-1$, we have by Lemma 3.1 for $R = k \geq 1$,

$$\max_{|z|=k} |D_\beta P(z)| \leq k^{n-1} \max_{|z|=1} |D_\beta P(z)|.$$

On using this and Lemma 3.2, the above inequality (4.2) clearly gives

$$\begin{aligned} k^{n-1} \max_{|z|=1} |D_\beta P(z)| \geq \frac{n}{2}\bigg\{ \left(\frac{|\beta| - k}{k}\right) \left(\frac{2k^n}{1+k^n} \max_{|z|=1} |P(z)| + \left(\frac{k^n - 1}{k^n + 1}\right) m_k \right) + \left(\frac{|\beta|}{k} + 1\right) m_k \bigg\} \\ + \left(\frac{|\beta| - k}{k}\right) \left\{ \frac{s}{2} + \frac{\sqrt{k^{n-s}|a_{n-s}| - m_k} - \sqrt{|a_0|}}{2\sqrt{k^{n-s}|a_{n-s}| - m_k}} \right\} \\ \times \left\{ \frac{2k^n}{1+k^n} \max_{|z|=1} |P(z)| + \left(\frac{k^n - 1}{k^n + 1}\right) m_k - m_k \right\}. \end{aligned}$$

After rearranging the terms, we get

$$\max_{|z|=1}|D_\beta P(z)| \geq \frac{n}{1+k^n}\left\{(|\beta|-k)\max_{|z|=1}|P(z)| + \left(|\beta|+\frac{1}{k^{n-1}}\right)m_k\right\} + \left(\frac{|\beta|-k}{1+k^n}\right)\left\{s+\frac{\sqrt{k^{n-s}|a_{n-s}|-m_k}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|-m_k}}\right\}\left(\max_{|z|=1}|P(z)|-\frac{m_k}{k^n}\right),$$

which is exactly (2.1). This completes the proof of Theorem 2.1. □

Proof of Theorem 2.6. Let $Q(z) = z^n\overline{P(1/\overline{z})}$. Since $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ in $|z| < k$, $k \leq 1$, the polynomial $Q(z)$ of degree n has all its zeros in $|z| \leqslant 1/k$, $1/k \geq 1$. On applying inequality (2.4) of Corollary 2.4 to $Q(z)$, we get

$$\max_{|z|=1}|Q'(z)| \geq \frac{n}{1+\frac{1}{k^n}}\left(\max_{|z|=1}|Q(z)|+m'_k\right) + \frac{\sqrt{\frac{1}{k^n}|a_0|-m'_k}-\sqrt{|a_n|}}{(1+\frac{1}{k^n})\sqrt{\frac{1}{k^n}|a_0|-m'_k}}\left\{\max_{|z|=1}|Q(z)|-k^n m'_k\right\}. \tag{4.3}$$

Now,

$$m'_k = \min_{|z|=1/k}|Q(z)| = \min_{|z|=1/k}\left|z^n\overline{P\left(\frac{1}{\overline{z}}\right)}\right| = \frac{1}{k^n}\min_{|z|=k}|P(z)| = \frac{m_k}{k^n}$$

and

$$\max_{|z|=1}|Q(z)| = \max_{|z|=1}|P(z)|.$$

Using these observations in (4.3), we get

$$\max_{|z|=1}|Q'(z)| \geq \frac{nk^n}{1+k^n}\left(\max_{|z|=1}|P(z)|+\frac{m_k}{k^n}\right) + \frac{\left(\sqrt{|a_0|-m_k}-k^{n/2}\sqrt{|a_n|}\right)k^n}{(1+k^n)\sqrt{|a_0|-m_k}}\left\{\max_{|z|=1}|P(z)|-m_k\right\}. \tag{4.4}$$

Since $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, we have

$$\max_{|z|=1}(|P'(z)|+|Q'(z)|) = \max_{|z|=1}|P'(z)| + \max_{|z|=1}|Q'(z)|. \tag{4.5}$$

On combining (4.4), (4.5) and Lemma 3.5, we get

$$n\max_{|z|=1}|P(z)| \geq \max_{|z|=1}|P'(z)| + \frac{nk^n}{1+k^n}\left(\max_{|z|=1}|P(z)|+\frac{m_k}{k^n}\right) + \frac{\left(\sqrt{|a_0|-m_k}-k^{n/2}\sqrt{|a_n|}\right)k^n}{(1+k^n)\sqrt{|a_0|-m_k}}\left\{\max_{|z|=1}|P(z)|-m_k\right\},$$

which gives

$$\max_{|z|=1}|P'(z)| \le n\max_{|z|=1}|P(z)| - \frac{nk^n}{1+k^n}\left(\max_{|z|=1}|P(z)| + \frac{m_k}{k^n}\right) - \frac{\left(\sqrt{|a_0|-m_k} - k^{n/2}\sqrt{|a_n|}\right)k^n}{(1+k^n)\sqrt{|a_0|-m_k}}\left\{\max_{|z|=1}|P(z)| - m_k\right\}. \tag{4.6}$$

Also, it is easy to verify that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)|. \tag{4.7}$$

Note that for any complex number β, and $|z| = 1$, we have

$$|D_\beta P(z)| = |nP(z) + (\beta - z)P'(z)| \le |nP(z) - zP'(z)| + |\beta||P'(z)|,$$

which gives by (4.7) and $|\beta| \ge 1$, that

$$\begin{aligned}|D_\beta P(z)| &\le |Q'(z)| + |\beta||P'(z)| = |Q'(z)| + |P'(z)| - |P'(z)| + |\beta||P'(z)| \\ &\le n\max_{|z|=1}|P(z)| + (|\beta|-1)|P'(z)| \qquad \text{(by Lemma 3.5)} \\ &\le n\max_{|z|=1}|P(z)| + (|\beta|-1)\max_{|z|=1}|P'(z)|. \end{aligned} \tag{4.8}$$

Inequality (4.8), in conjunction with (4.6), gives for $|z| = 1$,

$$|D_\beta P(z)| \le n|\beta|\max_{|z|=1}|P(z)| - \frac{nk^n(|\beta|-1)}{1+k^n}\left(\max_{|z|=1}|P(z)| + \frac{m_k}{k^n}\right) - \frac{(|\beta|-1)\left(\sqrt{|a_0|-m_k} - k^{n/2}\sqrt{|a_n|}\right)k^n}{(1+k^n)\sqrt{|a_0|-m_k}}\left\{\max_{|z|=1}|P(z)| - m_k\right\},$$

which is equivalent to (2.5). This completes the proof of Theorem 2.6. □

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