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Einstein warped product spaces on Lie groups

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ABSTRACT

We consider a compact Lie group with bi-invariant metric, coming from the Killing form. In this paper, we study Einstein warped product space, $M = M_1 \times_{f_1} M_2$ for the cases, (i) M_1 is a Lie group (ii) M_2 is a Lie group and (iii) both M_1 and M_2 are Lie groups. Moreover, we obtain the conditions for an Einstein warped product of Lie groups to become a simple product manifold. Then, we characterize the warping function for generalized Robertson-Walker spacetime, $(M = I \times_{f_1} G_2, -dt^2 + f_1^2 g_2)$ whose fiber G_2, being semi-simple compact Lie group of $\dim G_2 > 2$, having bi-invariant metric, coming from the Killing form.

RESUMEN

Consideramos un grupo de Lie compacto con métrica bi-invariante, que proviene de la forma de Killing. En este artículo estudiamos espacios productos alabeados de Einstein, $M = M_1 \times_{f_1} M_2$ para los casos (i) M_1 es un grupo de Lie (ii) M_2 es un grupo de Lie y (iii) ambos M_1 y M_2 son grupos de Lie. Más aún, obtenemos condiciones para que un producto alabeado de Einstein de grupos de Lie sea una variedad producto simple. Luego, caracterizamos la función de alabeo para el espacio-tiempo generalizado de Robertson-Walker, $(M = I \times_{f_1} G_2, -dt^2 + f_1^2 g_2)$ cuya fibra G_2 es un grupo de Lie compacto semi-simple de $\dim G_2 > 2$ con una métrica bi-invariante, que proviene de la forma de Killing.

Keywords and Phrases: Einstein space, warped product, Lie group, bi-invariant metric, Killing form.

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1 Introduction

R. L. Bishop and B. O'Neill [3], introduced the notion of warped product space to study the examples of complete Riemannian manifolds of negative sectional curvature. Authors proved that the completeness of warped space is followed by the completeness of base and fiber spaces. Further, the results for isometrically immersed warped product manifold into some Riemannian manifold were considered in [5, 6, 7]. In [9, 19], authors studied the conditions for the warping function to become a constant by using the relation between the scalar curvatures of a warped manifold with its base and fiber spaces.

The concept of the warped product has been generalized to the twisted warped product [11, 28], the doubly warped product and the multiply warped product [25, 32, 33]. A multiply warped product is a product manifold $M = B \times M_1 \times M_2 \times \cdots \times M_k$, equipped with the metric

$$g = \pi^*(g_B) + (f_1 \circ \pi_1)^2 \pi_2^*(g_1) + (f_1 \circ \pi_1)^2 \pi_2^*(g_2) + \cdots + (f_1 \circ \pi_1)^2 \pi_2^*(g_k),$$

where (B, g_B) and (M_i, g_i), $i \in \{1, \ldots, k\}$, are pseudo-Riemannian manifolds, f_i are smooth functions on (M_i, g_i) and π_i are projections from M to M_i. In particular, if $B = (a, b)$, $k = 1$ and $g_B = -dt^2$, then M is known as a generalized Robertson-Walker spacetime [1, 10, 31]. A generalized Robertson-Walker spacetime with a fiber of constant scalar curvature is known as a Robertson-Walker spacetime. The simplest example for Robertson-Walker spacetime is an Einstein static universe. The product manifold $M = M_1 \times M_2$ with metric $g = (f_2 \circ \pi_2)^2 \pi_1^*(g_1) + (f_1 \circ \pi_1)^2 \pi_2^*(g_2)$ is known as a doubly warped product space.

A pseudo-Riemannian manifold M with metric g is an Einstein manifold provided $Ric = cg$, where Ric is a Ricci curvature and c is some real constant. The Einstein metric g is of much interest, both in geometry and physics. A warped product with a constant warping function is considered as simply Riemannian product. In [2, p. 265], A. L. Besse proposed the question, "Does there exist a compact Einstein warped product with non-constant warping function?". Some answers to the question were given in [16, 30]. If M is an Einstein warped product space of nonpositive scalar curvature with a compact base manifold, then the warped product space is reduced to a simply Riemannian product [16]. In [24, 26], authors studied Einstein warped product space by using quarter and semi symmetric connections. The triviality results for Einstein warped product space with non-compact base manifold were studied in [30].

In 1976, Milnor investigated the curvature properties of left-invariant metrics in Lie groups [20]. Most of the Lie groups carry the more than one left-invariant metric, because in [18], authors showed that for a non-Abelian Lie group with a unique left-invariant metric up to homothety, the group is either the hyperbolic space H^n, or $R^{n-3} \times H_3$, where H_3 is a Heisenberg group. The Heisenberg group H_3 has a unique Riemannian metric up to homothety, whereas it has three metrics in the Lorentzian case [29]. Classifications for four-dimensional nilpotent Lie groups were considered in [4, 17]. The class of Lie groups obtaining a bi-invariant metric is smaller than that of Lie groups with a left-invariant metric. In [14, 15], authors study the warped product Einstein metrics on spaces of constant scalar curvature and homogeneous spaces. The classifications of warped product Einstein metric were studied in [13]. In [8, 22], the authors study the general helices and slant helices in three dimensional Lie group equipped with a bi-invariant metric.

In our paper, we discuss the few possible answers to the question "Does there exist a compact Einstein warped product with non-constant warping function?" for a compact Einstein warped product of Lie groups. We know that every compact Lie group has a bi-invariant metric and bi-invariant metric is much easier to handle than the left invariant metric. That is why, we use the bi-invariant metric in our paper. Now the results of the left invariant metric are still open to study. Section 2, of this paper includes some of the basic results. The central part of our paper is section 3, where we prove our main results for a warped product having either base manifold or fiber manifold is a compact Lie group with bi-invariant metric, coming from the Killing form. We show that an Einstein warped product space of nonnegative scalar curvature with a one-dimensional base manifold (Riemannian manifold) and fiber being a compact Lie group with bi-invariant metric, coming from the Killing form does not exist. Also, the characteristic of warping function in generalized Robertson-Walker spacetime is studied in Theorem 3.9. Finally, we give examples of warped products, obtained using a semi-simple compact Lie group taking bi-invariant metric from the Killing form.

2 Preliminaries

A Lie group G_1 is a smooth manifold with a group structure such that the multiplicative and inverse maps are smooth. To study the geometry of G_1, it becomes necessary to associate a left invariant metric with it. A metric in which left multiplication behaves as an isometry is known as a left invariant metric, and for a metric in which right multiplication behaves as an isometry is known as a right invariant metric. Left multiplication and right multiplication on G_1, are defined as $L_{a_1} : G_1 \mapsto G_1$, $L_{a_1}x_1 = a_1x_1$ and $R_{a_1} : G_1 \mapsto G_1$, $R_{a_1}x_1 = x_1a_1$, for all $a_1, x_1 \in G_1$. Let $\mathfrak{g}_1$ be the Lie algebra of G_1, then an adjoint representation, $Ad : G_1 \mapsto \mathfrak{g}_1$, of a Lie group G_1 is a map such that $Ad_{a_1} : \mathfrak{g}_1 \mapsto \mathfrak{g}_1$ is linear isomorphism given by $Ad_{a_1} = d(R_{a_1^{-1}} \circ L_{a_1})_{e_1}$ for all $a_1 \in G_1$. An inner product g_1 on $\mathfrak{g}_1$ is said to be Ad-invariant if

$$g_1(Ad_{a_1}X_1, Ad_{a_1}Y_1) = g_1(X_1, Y_1),$$

for all $a_1 \in G_1$ and $X_1, Y_1 \in \mathfrak{g}_1$.

A metric g_1, which is both left invariant and right invariant is said to be a bi-invariant metric.

The metric g_1 is bi-invariant if and only if

$$g_1([S_1, K_1], T_1) = g_1(K_1, [T_1, S_1]) = g_1(S_1, [K_1, T_1]),$$

for all $S_1, K_1, T_1 \in \mathfrak{g_1}$. Also, using the Koszul formula and above equation, we obtain

$$\nabla_{S_1} K_1 = \frac{1}{2}[S_1, K_1], \ \ \forall \ \ S_1, K_1 \in \mathfrak{g_1}.$$

Corresponding to bi-invariant metric g_1 on m_1- dimensional Lie group G_1, the Riemann curvature tensor R, and the Ricci tensor Ric, are given by

$$R(X_1, Y_1)Z_1 = \frac{1}{4}\big[[X_1, Y_1], Z_1\big],$$

$$Ric(X_1, Y_1) = \frac{1}{4} g_1\big([X_1, E_i], [Y_1, E_i]\big),$$

where $\{E_1, \ldots, E_{m_1}\}$, is an orthonormal frame for $\mathfrak{g_1}$. From [12, p. 622], we get the existence of bi-invariant metric on Lie group.

Proposition 2.1. *Let G_1 be a Lie group with Lie algebra $\mathfrak{g_1}$ and metric g_1, then g_1 induces a bi-invariant metric if and only if $\overline{Ad(G_1)}$ is compact. In other words, every compact Lie group has a bi-invariant metric.*

Also, for a connected Lie group G_1, the metric g_1 induce a bi-invariant metric if and only if $Ad_{a_1} : \mathfrak{g_1} \mapsto \mathfrak{g_1}$, is skew adjoint for all $a_1 \in G_1$, which means

$$g_1(Ad_{a_1} X_1, Y_1) = -g_1(X_1, Ad_{a_1} Y_1), \qquad \forall \ X_1, Y_1 \in \mathfrak{g_1}.$$

Definition 2.2 ([2, 23])**.** *The Killing form $B : \mathfrak{g} \times \mathfrak{g} \mapsto \mathbb{R}$ is a symmetric $B(X_1, Y_1) = B(Y_1, X_1)$, $Ad(G_1)$-invariant $B([X_1, Y_1], Z_1) = B(X_1, [Y_1, Z_1])$ and bilinear form, defined by*

$$B(X_1, Y_1) = tr(ad(X_1) \circ ad(Y_1)),$$

where $ad(X_1) : \mathfrak{g_1} \mapsto \mathfrak{g_1}$ is a map, sending each Z_1 to $[X_1, Z_1]$, for all $X_1, Y_1, Z_1 \in \mathfrak{g_1}$.

A Killing form on a Lie group G_1 is nondegenerate if and only if G_1 is semisimple. In case of compact semisimple Lie group, the Killing form is always negative definite. From [23, p. 304–306], we have

Corollary 2.3. *Let G_1 be a semisimple compact Lie group with bi-invariant metric g_1, then*

$(a.1)$ *For nondegenerate plane spanned by S and K in $\mathfrak{g}_1$, the sectional curvature is given by*

$$\mathcal{K} = \frac{1}{4}\Big(\frac{g_1([S,K],[S,K])}{g_1(S,S)g_1(K,K) - g_1(S,K)g_1(S,K)}\Big).$$

$(a.2)$ *If the metric g_1 is induced from the Killing form, then G_1 is an Einstein ($Ric_1 = -\frac{1}{4}g_1$) and the scalar curvature (τ), is given by*

$$\tau = \frac{1}{4}\dim(G).$$

It is clear from $(a.1)$, that if g_1 is a Riemannian metric then $\mathcal{K} \geq 0$ and $\mathcal{K} = 0$, if G_1 is an Abelian group.

Let (M_1, g_1) and (M_2, g_2) be two pseudo-Riemannian manifolds of dimensions m_1, m_2 and f_1 be a positive smooth function on M_1. Then for natural projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$, the warped product $(M = M_1 \times_{f_1} M_2, g)$ is a product manifold $M_1 \times M_2$ with the metric

$$g = \pi_1^*(g_1) + (f_1 \circ \pi_1)^2 \pi_2^*(g_2),$$

where * representing the pull-back operator and f_1 is a warping function on M. Whereas M_1 and M_2 are known as the base, and the fiber of (M, g), respectively. Let Ric, Ric_1 and Ric_2 are Ricci tensors on M, M_1 and M_2, respectively. Then from [23, p. 211], we have

Proposition 2.4. *Let $M = M_1 \times_{f_1} M_2$ be a warped product space, then Ricci tensors on M, M_1 and M_2, satisfies*

$$Ric = Ric_1 - \frac{m_2}{f_1}H^{f_1} + Ric_2 - f^\sharp g_2, \tag{2.1}$$

where $f^\sharp = -f_1\Delta f_1 + (m_2-1)g_1(grad\ f_1, grad\ f_1)$. Here $grad\ f_1$, H^{f_1} and Δf_1 denote the gradient of f_1, the Hessian of f_1 and the Laplacian of f_1, defined as $\Delta f_1 = -trH^{f_1}$.

Corollary 2.5. *The warped product $M = M_1 \times_{f_1} M_2$ is an Einstein with $Ric = \lambda g$ if and only if*

$(a.3)$ $Ric_1 = \lambda g_1 + \dfrac{m_2}{f_1}H^{f_1}$,

$(a.4)$ *(M_2, g_2) is an Einstein, such that $Ric_2 = \nu g_2$, where $\nu = f^\sharp + \lambda f_1^2$.*

3 Main results

Proposition 3.1. *Let (M_2, g_2) be a pseudo-Riemannian manifold and (G_1, g_1) be a semi-simple compact Lie group whose bi-invariant metric coming from the Killing form. Then warped product* *manifold* $(M = G_1 \times_{f_1} M_2, g)$*, is an Einstein manifold* $(Ric = \lambda g)$ *if and only if*

$(a.5)$ $H^{f_1} = -\dfrac{(1+4\lambda)f_1}{4m_2} g_1,$

$(a.6)$ (M_2, g_2) *is an Einstein with* $Ric_2 = \nu g_2$*, where*

$$\nu = -f_1 \Delta f_1 + (m_2 - 1) g_1(grad\ f_1, grad\ f_1) + \lambda f_1^2.$$

Proof. Let $(M = G_1 \times_{f_1} M_2, g)$ be an Einstein manifold $(Ric = \lambda g)$, where (M_2, g_2) is a pseudo-Riemannian manifold and (G_1, g_1) is a semi-simple compact Lie group taking bi-invariant metric from the Killing form. Then from (2.1), we have

$$\lambda g_1 + f_1^2 \lambda g_2 = Ric_1 - \frac{m_2}{f_1} H^{f_1} + Ric_2 - f^\sharp g_2, \tag{3.1}$$

where λ is some constant and $f^\sharp = -f_1 \Delta f_1 + (m_2 - 1) g_1(grad\ f_1, grad\ f_1)$. Now, by restricting the argument (horizontal and vertical vectors) on G_1, M_2, and taking $Ric_1 = -\frac{1}{4} g_1$ in (3.1), we get

$$\begin{cases} \lambda g_1 & = -\dfrac{1}{4} g_1 - \dfrac{m_2}{f_1} H^{f_1}, \\ f_1^2 \lambda g_2 & = Ric_2 - f^\sharp g_2. \end{cases} \tag{3.2}$$

Conversely, assume that $(M = G_1 \times_{f_1} M_2, g)$ be a warped product with conditions $(a.5)$ and $(a.6)$. Then from (2.1), we get

$$Ric = \lambda g_1 + \frac{m_2}{f_1} H^{f_1} - \frac{m_2}{f_1} H^{f_1} + \nu g_2 - f^\sharp g_2. \tag{3.3}$$

Since $\nu = -f_1 \Delta f_1 + (m_2 - 1) g_1(grad\ f_1, grad\ f_1) + \lambda f_1^2$, so from (3.3), we have

$$Ric = \lambda(g_1 + f_1^2 g_2) = \lambda g. \tag{3.4}$$

□

Proposition 3.2. *Let* (M_1, g_1) *be a pseudo-Riemannian manifold and* (G_2, g_2) *be a semi-simple compact Lie group whose bi-invariant metric coming from the Killing form. Then warped product manifold* $(M = M_1 \times_{f_1} G_2, g)$*, is an Einstein manifold* $(Ric = \lambda g)$ *if and only if*

$(a.7)$ $Ric_1 = \lambda g_1 + \dfrac{m_2}{f_1} H^{f_1},$

$(a.8)$ (M_2, g_2) *is an Einstein with* $Ric_2 = \nu g_2$*, where*

$$\nu = -\frac{1}{4} = -f_1 \Delta f_1 + (m_2 - 1) g_1(grad\ f_1, grad\ f_1) + \lambda f_1^2.$$

Proof. Since (G_2, g_2) is a semi-simple compact Lie group taking bi-invariant metric from the Killing form, so using $Ric_2 = -\frac{1}{4}g_2$ in $(a.6)$, we have

$$Ric_2 = -\frac{1}{4}g_2 = \nu g_2 = (f^\sharp + \lambda f_1^2)g_2. \quad \square$$

Lemma 3.3 ([16])**.** *Let f_1 be a smooth function on semi-Riemannian manifold M_1, then the divergence of Hessian tensor satisfies*

$$div(H^{f_1})(X_1) = Ric_1(grad\ f_1, X_1) - d(\Delta f_1)(X_1), \tag{3.5}$$

for all $X_1 \in \Gamma TM_1$.

Theorem 3.4. *Let (G_1, g_1) be a semi-simple compact Lie group of dimension $m_1 > 2$ and whose bi-invariant metric coming from the Killing form. If $4m_2H^{f_1} + (1+4\lambda)f_1g_1 = 0$, where $\lambda \in \mathbb{R}$, $m_2 \in \mathbb{N}$ and f_1 is a non constant smooth function on G_1, then f_1 satisfy the condition*

$$\nu = -f_1\Delta f_1 + (m_2 - 1)g_1(grad\ f_1, grad\ f_1) + \lambda f_1^2,$$

where $\nu \in \mathbb{R}$.

Proof. The trace of $(a.5)$, provide us

$$\frac{m_2}{f_1}\Delta f_1 + \frac{(1+4\lambda)m_1}{4} = 0. \tag{3.6}$$

On differentiating (3.6), we get

$$\frac{m_2}{f_1^2}\left(\Delta f_1 df_1 - f_1 d(\Delta f_1)\right) = 0. \tag{3.7}$$

By the definition of divergence and Hessian for any vector field X_1 and g_1-orthonormal frame $\{E_1, \ldots, E_{m_1}\}$ on G_1, we have

$$\begin{aligned} div\left(\frac{1}{f_1}H^{f_1}\right)(X_1) &= \sum_i \epsilon_i \Big(D_{E_i}(\frac{1}{f_1}H^{f_1})\Big)(E_i, X_1) \\ &= -\frac{1}{f_1^2}H^{f_1}(grad\ f_1, X_1) + \frac{1}{f_1}div(H^{f_1})(X_1), \end{aligned} \tag{3.8}$$

where $\epsilon_i = g_1(E_i, E_i)$. Using the fact that $Ric_1 = -\frac{1}{4}g_1$, in equation (3.5), the divergence of Hessian becomes

$$div(H^{f_1})(X_1) = -\frac{1}{4}g_1(grad\ f_1, X_1) - d(\Delta f_1)(X_1). \tag{3.9}$$

Also, from $(a.5)$ and $H^{f_1}(grad\ f_1, X_1) = (D_{X_1} df_1)(grad\ f_1) = \frac{1}{2} d(g_1(grad\ f_1, grad\ f_1))$, we have

$$-\frac{1}{4} g_1(grad\ f_1, X_1) = \frac{m_2}{2f_1} d(g_1(grad\ f_1, grad\ f_1))(X_1) + \lambda df_1(X_1). \tag{3.10}$$

In view of equations (3.9) and (3.10), the equation (3.8) becomes

$$div\left(\frac{1}{f_1} H^{f_1}\right) = \frac{1}{2f_1^2}\Big((m_2 - 1) d(g_1(grad\ f_1, grad\ f_1)) + 2\lambda_1 f_1 df_1(X_1) - 2f_1 d(\Delta f_1)\Big). \tag{3.11}$$

But the divergence of $(a.5)$, implies that $div\left(\frac{1}{f_1} H^{f_1}\right) = 0$. Hence from (3.11), we get

$$(m_2 - 1) d(g_1(grad\ f_1, grad\ f_1)) + 2\lambda_1 f_1 df_1 - 2f_1 d(\Delta f_1) = 0. \tag{3.12}$$

Therefore from equations (3.7) and (3.12), we obtain

$$d\Big((m_2 - 1)(g_1(grad\ f_1, grad\ f_1)) + \lambda_1 f_1^2 - f_1(\Delta f_1)\Big) = d(\nu) = 0. \tag{3.13}$$

Hence from equation (3.13), we can conclude that for a compact Einstein manifold (M_2, g_2) with dimension m_2 and $Ric_2 = \nu g_2$, the construction of an Einstein warped manifold $M = G_1 \times_{f_1} M_2$ is possible. □

Corollary 3.5. *Let $M = G_1 \times_{f_1} M_2$ be an Einstein warped product space with semi-simple compact Lie group G_1 of dimension $m_1 > 2$ and whose bi-invariant metric coming from the Killing form. Then M reduces to a simply Riemannian product.*

Proof. Rearranging the equation (3.6), we have

$$\Delta f_1 = \frac{(1 + 4\lambda) m_1}{4m_2} f_1. \tag{3.14}$$

As λ is a constant, so for $\lambda \leq -\frac{1}{4}$, equation (3.14) implies that $\Delta f_1 \leq 0$, hence f_1 is constant. Similarly if $\lambda \geq -\frac{1}{4}$, then $\Delta f_1 \geq 0$. Since according to the weak maximum principle, if f_1 is subharmonic or superharmonic *i.e.* ($\Delta f_1 \geq 0$ or $\Delta f_1 \leq 0$), then f_1 is constant [27, p. 75]. Hence M is a simply Riemannian product. □

In our next result, we prove that if fiber space of warped space is also a semi-simple compact Lie group of dimension $m_2 > 2$ and inherits the bi-invariant metric from the Killing form, then the only possible values for f_1 are ± 1.

Corollary 3.6. *Let G_1 and G_2 be semi-simple compact Lie groups of dimensions $m_1, m_2 > 2$ and bi-invariant metric tensors coming from their respective Killing forms. Then $M = G_1 \times_{f_1} G_2$ is an Einstein if and only if $f_1 = \pm 1$.*

Proof. Let $M = G_1 \times_{f_1} G_2$ be an Einstein, then from Proposition 3.1, Corollary 3.5 and using the fact that $Ric_2 = -\frac{1}{4}g_2$, we obtain, $\nu = \lambda = -\frac{1}{4}$. Therefore $f_1^2 = 1$.

Now conversely assume that $f_1 = \pm 1$, then $Ric = Ric_1 + Ric_2 = -\frac{1}{4}(g_1 + g_2) = -\frac{1}{4}g$, hence $M = G_1 \times G_2$ is an Einstein. □

Next, we consider those warped product spaces whose base is any pseudo-Riemannian manifold and fiber space is a semi-simple compact Lie group of dimension $m_2 > 2$, taking bi-invariant metric from the Killing form.

Theorem 3.7. *Let $M = M_1 \times_{f_1} G_2$ be an Einstein warped product space with fiber G_2 as a semi-simple compact Lie group of dimension $m_2 > 2$ and having bi-invariant metric coming from the Killing form. If M has negative scalar curvature, then the warped product becomes a simply Riemannian product.*

Proof. Let $M = M_1 \times_{f_1} G_2$ be an Einstein warped product space with fiber G_2 as a semi-simple compact Lie group of dimension $m_2 > 2$, having bi-invariant metric is coming from the Killing form. Then from $(a.4)$, we can say that

$$-f_1\Delta f_1 + (m_2 - 1)g_1(grad\ f_1, grad\ f_1) + \lambda f_1^2 = -\frac{1}{4}. \tag{3.15}$$

Since M is an Einstein, therefore the trace of $Ric = \lambda g$, implies that

$$\tau = \lambda(m_1 + m_2), \tag{3.16}$$

where τ is a scalar curvature of M. Now assume that p_1 and p_2 are maximum and minimum points of f_1 on M_1. Therefore $grad\ f_1(p_1) = grad\ f_1(p_2) = 0$, $\Delta f_1(p_1) \geq 0$ and $\Delta f_1(p_2) \leq 0$. From (3.16) it is clear that $\tau \leq 0$, implies $\lambda \leq 0$, therefore

$$f_1(p_1)^2 \geq f_1(p_2)^2 \implies \lambda f_1(p_1)^2 \leq \lambda f_1(p_2)^2 \implies \lambda f_1(p_1)^2 + \frac{1}{4} \leq \lambda f_1(p_2)^2 + \frac{1}{4}, \tag{3.17}$$

where ν is some constant. Since $\Delta f_1(p_2)f(p_2) \leq 0$ and $\Delta f_1(p_1)f(p_1) \geq 0$, therefore from (3.15), $\lambda f_1(p_2)^2 + \frac{1}{4} \leq 0$ and $\lambda f_1(p_1)^2 + \frac{1}{4} \geq 0$, gives us

$$\lambda f_1(p_2)^2 + \frac{1}{4} \leq \lambda f_1(p_1)^2 + \frac{1}{4}. \tag{3.18}$$

Comparing equations (3.17) and (3.18), we have $f_1(p_1) = f_1(p_2)$ for $\lambda < 0$. □

Theorem 3.8. *Let $M = I \times_{f_1} G_2$ be an Einstein warped product space with the metric $g = dt^2 + f_1^2(t)g_2$, where I is an open interval in $\mathbb{R}$ and G_2 is a semi-simple compact Lie group of dimension $m_2 > 2$ and having bi-invariant metric coming from the Killing form. If M has non* *negative scalar curvature, then there does not exist any such f_1, so that $M = I \times_{f_1} G_2$ is an Einstein warped product space.*

Proof. Let $M = I \times_{f_1} G_2$ have positive scalar curvature ($\lambda > 0$). Then taking $f_1 = e^{\frac{u}{2}}$, the Hessian of f_1,

$$H^{f_1} = \frac{u''}{2}e^{\frac{u}{2}} + \frac{(u')^2}{4}e^{\frac{u}{2}}.$$

Using the above equation in $(a.7)$, we have

$$\frac{u''}{2} + \frac{(u')^2}{4} = -\frac{\lambda}{m_2}. \tag{3.19}$$

Also, from $(a.8)$, we get

$$\Big(\frac{u''}{2} + \frac{(u')^2}{4}\Big) + (m_2-1)\frac{(u')^2}{4} + \lambda = -\frac{1}{4}e^{-u}. \tag{3.20}$$

Thus from (3.19) and (3.20), we obtain

$$(u')^2 = -\Big(\frac{1}{m_2-1}e^{-u} + \frac{4}{m_2}\lambda\Big). \tag{3.21}$$

The possible solutions for (3.21) (with the help of Maple), are

$$\begin{cases} u = -\ln\Big(-\frac{4\lambda(m_2-1)}{m_2}\Big), \\ u = -\ln\Big(-\frac{4(m_2-1)}{m_2}\lambda\Big(1+\tan^2\big(-\sqrt{\frac{\lambda}{m_2}}t + c\sqrt{\frac{\lambda}{m_2}}\big)\Big)\Big), \end{cases} \tag{3.22}$$

where c is some constant. It is clear from (3.22) that the function u is not well defined. Furthermore, as u is a real valued function, therefore $(u')^2 \geq 0$ and $-\Big(e^{-u}\frac{1}{m_2-1} + \frac{4}{m_2}\lambda\Big) < 0$, for any point on I. Therefore from equation (3.21), we can conclude that there does not exist any real solution for the equation.

For $\lambda = 0$, $(a.7)$ and $(a.8)$, imply that $f_1'' = 0$ and $f_1 f_1'' + (m_2-1)(f_1')^2 = -\frac{1}{4}$, respectively. Hence

$$f_1 = at + b \implies (m_2-1)(a)^2 = -\frac{1}{4}, \tag{3.23}$$

where a and b are some real constants. Thus from (3.22) and (3.23), we can say that there does not exist any such f_1 such that $M = I \times_{f_1} G_2$ be an Einstein warped product space of non negative scalar curvature. □

Next, we find the characteristic of warping function in generalized Robertson-Walker spacetime, whose fiber is semi-simple and compact Lie group of dimension $m_2 > 2$.

Theorem 3.9. *Let $M = I \times_{f_1} G_2$ be an Einstein warped product space with the metric $g = -dt^2 + f_1^2(t)g_2$, where I is an open interval in $\mathbb{R}$ and G_2 is a semi-simple compact Lie group of dimension $m_2 > 2$ and having bi-invariant metric coming from the Killing form. Then*

(*i*) *If M is Ricci flat, then there exists a non-constant function f_1 on I such that $f_1 = \frac{1}{2\sqrt{m_2-1}}t + b$, where b is some constant.*

(*ii*) *If M has positive scalar curvature ($\tau > 0$) or negative scalar curvature ($\tau < 0$), then there does not exist any such f_1, so that $M = I \times_{f_1} G_2$ be an Einstein warped product space.*

Proof. Let $M = I \times_{f_1} G_2$ be an Einstein warped product space with the metric $g = -dt^2 + f_1^2(t)g_2$, then from Proposition 3.2, we get

$$f_1'' = \frac{\lambda f_1}{m_2}, \quad \text{and} \quad f_1 f_1'' - (m_2 - 1)(f_1')^2 + \lambda f_1^2 = -\frac{1}{4}. \tag{3.24}$$

From these two differential equations, we obtain

$$(f_1')^2 - \frac{\lambda(1 + m_2)}{m_2(m_2 - 1)} f_1^2 = \frac{1}{4(m_2 - 1)}. \tag{3.25}$$

As λ is constant, therefore to obtain the solutions for differential equation (3.25), we have to consider all possible values of λ.

(*i*) If $\lambda = 0$, then from (3.25), we obtain

$$f_1 = \frac{1}{2\sqrt{m_2 - 1}} t + b, \tag{3.26}$$

where b is some constant. Since f_1 is also satisfying (3.24), hence in the Ricci flat manifold case, it is possible to find a non-constant function on I.

(*ii*) (*a*) Let M be an Einstein manifold with positive scalar curvature $\lambda > 0$, then from (3.25), the possible solutions are

$$\begin{cases} f_1 = \pm\sqrt{\frac{-m_2}{4\lambda(1+m_2)}}, \\ f_1 = \frac{\sqrt{m_2(m_2-1)}}{2\sqrt{\lambda(1+m_2)}}\Big(- \frac{1}{4(m_2-1)} e^{\sqrt{\frac{\lambda(1+m_2)}{m_2(m_2-1)}}(c_1-t)} + e^{\sqrt{\frac{\lambda(1+m_2)}{m_2(m_2-1)}}(t-c_1)}\Big), \\ f_1 = \frac{\sqrt{m_2(m_2-1)}}{2\sqrt{\lambda(1+m_2)}}\Big(- \frac{1}{4(m_2-1)} e^{\sqrt{\frac{\lambda(1+m_2)}{m_2(m_2-1)}}(t-c_1)} + e^{\sqrt{\frac{\lambda(1+m_2)}{m_2(m_2-1)}}(c_1-t)}\Big), \end{cases} \tag{3.27}$$

where c_1 is some constant. As $m_2 > 2$, so $f_1 = \pm\sqrt{\frac{-m_2}{4\lambda(1+m_2)}} \notin \mathbb{R}$, hence constant solution of f_1 is not possible. From second and third part of (3.27), we have

$$f_1'' = \frac{\lambda(1 + m_2)}{m_2(m_2 - 1)} f_1. \tag{3.28}$$

Equation (3.28), showing that second and third part of (3.27), is not satisfying (3.24). Hence there does not exist such type of f_1 which satisfies the equation (3.24) for $\lambda > 0$.

(b) Let M be an Einstein manifold with negative scalar curvature $\lambda < 0$, then (3.25), reduced to

$$(f_1')^2 + \frac{a(1+m_2)}{m_2(m_2-1)} f_1^2 = \frac{1}{4(m_2-1)}, \tag{3.29}$$

where $\lambda = -a$ and a is some positive real number. The solutions for differential equation (3.29), are

$$\begin{cases} f_1 = \pm\sqrt{\frac{m_2}{4a(1+m_2)}}, \\ f_1 = \pm\sqrt{\frac{m_2}{4a(1+m_2)}} \sin\left(\sqrt{\frac{a(1+m_2)}{m_2(m_2-1)}}(-t+c_1)\right). \end{cases} \tag{3.30}$$

Since solutions obtained in (3.30), are not satisfying the equation (3.24), hence there is no solution for (3.24). $\square$

Examples for warped product of Lie groups

The Lie groups $\mathbf{SU}(n)$, $n \geq 2$, and $SO(n)$, $n \geq 3$ are examples of semi-simple compact Lie groups. The Lie algebra $\mathfrak{su}(n)$ of $\mathbf{SU}(n)$, set of $n \times n$ skew hermitian matrices with zero trace. For $n = 2$, the elements of $\mathfrak{su}(2)$,

$$X_1 = \begin{pmatrix} a_1\iota & a_2 + a_3\iota \\ -a_2 + a_3\iota & -a_1\iota \end{pmatrix}, \qquad a_1, a_2, a_3 \in \mathbb{R}.$$

Similarly, If $Y_1 \in \mathfrak{su}(2)$, then

$$Y_1 = \begin{pmatrix} b_1\iota & b_2 + b_3\iota \\ -b_2 + b_3\iota & -b_1\iota \end{pmatrix}, \qquad b_1, b_2, b_3 \in \mathbb{R}.$$

The basis E_1, E_2 and E_3, for $\mathfrak{su}(2)$, can be chosen as

$$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix}.$$

Hence Ad_{X_1} and Ad_{X_2}, are obtained as

$$Ad_{X_1} = \begin{pmatrix} 0 & -2a_1 & 2a_3 \\ 2a_1 & 0 & -2a_2 \\ -2a_3 & 2a_2 & 0 \end{pmatrix}, \qquad Ad_{X_2} = \begin{pmatrix} 0 & -2b_1 & 2b_3 \\ 2b_1 & 0 & -2b_2 \\ -2b_3 & 2b_2 & 0 \end{pmatrix}.$$

Thus, the Killing form $B(X_1, Y_1)$ on $\mathfrak{su}(2)$, will be

$$B(X_1, Y_1) = tr(Ad_{X_1} \circ Ad_{Y_1}) = -8a_1b_1 - 8a_2b_2 - 8a_3b_3 = 4tr(X_1Y_1).$$

So, we can made the following examples from all the above discussions.

1. The warped product manifold $M = \mathbf{SU}(2) \times_{f_1} M_2$, with metric $g = B + f_1^2 g_2$, where (M_2, g_2) is any pseudo-Riemannian manifold and non constant function f_1 on $\mathbf{SU}(2)$, is not an Einstein.

2. The product manifold $M = \mathbf{SU}(2) \times \mathbf{SO}(2)$, with metric $g = B_1 + B_2$, is an Einstein manifold, where B_1 and B_2 are Killing forms on $\mathfrak{su}(2)$ and $\mathfrak{so}(2)$, respectively.

Conclusion 3.10. *In [21], Mustafa proved that for every compact manifold G_1 there exist a metric on it such that non trivial Einstein warped products with base G_1 cannot be constructed. In our paper, from Corollary 3.5, we can say that bi-invariant metric generated by the Killing form on semi-simple compact Lie group G_1 is one in which we cannot construct non trivial Einstein warped product with base G_1.*

Data availability statement

No new data were created and analyzed in this study.

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