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Two nonnegative solutions for two-dimensional nonlinear wave equations

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ABSTRACT

We study a class of initial value problems for two-dimensional nonlinear wave equations. A new topological approach is applied to prove the existence of at least two nonnegative classical solutions. The arguments are based upon a recent theoretical result.

RESUMEN

Estudiamos una clase de problemas de valor inicial para ecuaciones de onda no lineales en dimensión dos. Se aplica un nuevo enfoque topológico para probar la existencia de al menos dos soluciones clásicas no negativas. Los argumentos se basan en un resultado teórico reciente.

Keywords and Phrases: Hyperbolic Equations, positive solution, fixed point, cone, sum of operators.

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1 Introduction

Global existence for nonlinear wave equations is an important mathematical topic. Mathematicians, including F. John, S. Kleinerman, L. Hörmander, etc., have made investigations to this subject. The first non-trivial long-time existence result was established by F. John and S. Kleinerman in [19], where it is proved the almost global existence for a class $3D$ quasilinear scalar wave equations. Global existence for $3D$ quasilinear wave equations was established firstly by S. Kleinerman in [20] and by D. Christodoulou, independently by S. Kleinerman, in [5]. The problem in $2D$ case is quite delicate. Introducing the ghost weight, in [1] was proved the global well-posedness for a class $2D$ nonlinear wave equations. Using a class Hardy-type inequality depending on the compact support of the initial data, in [21] was proved almost global existence for $2D$ case. Here we propose a new approach for investigations for classical solutions of a class $2D$ nonlinear wave equations. We investigate for existence of at least two positive solutions for the following IVP

$$\begin{aligned} u_{tt} - \Delta u &= f(t, x, u, u_t, u_x), \quad t > 0, && x = (x_1, x_2) \in \mathbb{R}^2, \\ u(0, x) &= u_0(x), && x = (x_1, x_2) \in \mathbb{R}^2, \\ u_t(0, x) &= u_1(x), && x = (x_1, x_2) \in \mathbb{R}^2, \end{aligned} \tag{1.1}$$

where $\Delta u = u_{x_1 x_1} + u_{x_2 x_2}$, $u_x = (u_{x_1}, u_{x_2})$.

The initial value problem (1.1) has attracted considerable attention in the mathematical community and the well-posedness theory in the Sobolev spaces for polynomial type nonlinearities has been extensively studied. The case of exponential nonlinearity was recently investigated (see [18] and references therein). In particular, if the nonlinearity f and the initial data u_0, u_1 are smooth then the Cauchy problem (1.1) has a classical local (in time) solution. This follows from Duhamel's formula via the usual fixed point argument in the space $H^s_{loc} \times H^{s-1}_{loc}$, $s > 2$. Such an s guarantee that $u, u_t, \nabla u$ are in L^∞. Note that $u \in H^s_{loc}$ means that the H^s norm over a ball centered at x_0 and with radius 1 is uniformly bounded by a constant independent of x_0. We refer the reader to [23] and references therein for more properties and information on nonlinear wave equations. In [17] is proved existence and uniqueness of generalized solutions of the first initial boundary value problem for strongly hyperbolic systems in bounded domains. In the case when

$$f(t, x, u, u_t, u_x) = f(u(x)), \quad t > 0, \quad x \in \mathbb{R}^2,$$

and

$$u_0(x) = u_1(x) = 0, \quad x \in \mathbb{R}^2,$$

the problem (1.1) is investigated in [14] where the authors prove existence of at least one nontrivial classical solution of the problem (1.1).

We make the following assumptions on the non-linearity and initial data trough the paper.

(H1) $u_0, u_1 \in \mathcal{C}^2(\mathbb{R}^2)$,

$$\begin{aligned} 0 &\leq u_0, |u_{0x_1}|, |u_{0x_1x_1}|, |u_{0x_2}|, |u_{0x_2x_2}| \leq r, \\ 0 &\leq u_1, |u_{1x_1}|, |u_{1x_1x_1}|, |u_{1x_2}|, |u_{1x_2x_2}| \leq r \quad \text{on} \quad \mathbb{R}^2, \end{aligned}$$

where $r > 0$ is a given constant.

(H2) $f \in \mathcal{C}([0,\infty) \times \mathbb{R}^6)$,

$$\begin{aligned} 0 &\leq f(t, x, w_1, w_2, w_3, w_4) \\ &\leq \sum_{j=1}^{l} \left(a_j(t,x)|w_1|^{p_j} + b_j(t,x)|w_2|^{p_j} + c_j(t,x)|w_3|^{p_j} + d_j(t,x)|w_4|^{p_j}\right), \end{aligned}$$

$(t,x) \in [0,\infty) \times \mathbb{R}^2$, where $a_j, b_j, c_j, d_j \in \mathcal{C}([0,\infty) \times \mathbb{R}^2)$,

$$0 \leq a_j, b_j, c_j, d_j \leq a, \quad p_j > 0, \quad j \in \{1, \ldots, l\},$$

where $a > 0$ and $l \in \mathbb{N}$ are given constants.

Our main result reads as follows.

Theorem 1.1. *Suppose* **(H1)** *and* **(H2)***. Then the IVP* (1.1) *has at least two nonnegative classical solutions.*

To prove our main result we use a new topological approach. This approach can be used for investigations for existence of at least one and at least two classical solutions for initial value problems, boundary value problems and initial boundary value problems for some classes ordinary differential equations, partial differential equations and fractional differential equations (see [2, 3, 4, 7, 10, 12, 13, 15, 16] and references therein). So far, for the authors they are not known investigations for existence of multiple solutions for the IVP (1.1).

The paper is organized as follows. In the next section, we give some auxiliary results. In Section 3, we prove our main result. In Section 4, we give an example.

2 Auxiliary Results

Let X be a real Banach space.

Definition 2.1. *A mapping $K : X \to X$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

The concept for k-set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness.

Definition 2.2. *Let Ω_X be the class of all bounded sets of X. The Kuratowski measure of noncompactness $\alpha : \Omega_X \to [0, \infty)$ is defined by*

$$\alpha(Y) = \inf \left\{ \delta > 0 : Y = \bigcup_{j=1}^{m} Y_j \quad \text{and} \quad \text{diam}(Y_j) \le \delta, \quad j \in \{1, \dots, m\} \right\},$$

where $\text{diam}(Y_j) = \sup\{\|x - y\|_X : x, y \in Y_j\}$ is the diameter of Y_j, $j \in \{1, \dots, m\}$.

For the main properties of measure of noncompactness we refer the reader to [6].

Definition 2.3. *For a given number $k \ge 0$, a map $K : X \to X$ is said to be k-set contraction if it is continuous, bounded and*

$$\alpha(K(Y)) \le k\alpha(Y)$$

for any bounded set $Y \subset X$.

Obviously, if $K : X \to X$ is a completely continuous mapping, then K is 0-set contraction.

Definition 2.4. *Let X and Y be real Banach spaces. A mapping $K : X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that*

$$\|Kx - Ky\|_Y \ge h\|x - y\|_X$$

for any $x, y \in X$.

Definition 2.5. *A closed, convex set $\mathcal{P}$ in X is said to be a cone if*

(1) $\alpha x \in \mathcal{P}$ for any $\alpha \ge 0$ and for any $x \in \mathcal{P}$,

(2) $x, -x \in \mathcal{P}$ implies $x = 0$.

Let $\mathcal{P} \subset X$ be a cone and define

$$\begin{aligned}
\mathcal{P}^* &= \mathcal{P}\backslash\{0\},\\
\mathcal{P}_{r_1} &= \Big\{u \in \mathcal{P} : \|u\| \le r_1\Big\},\\
\mathcal{P}_{r_1,r_2} &= \Big\{u \in \mathcal{P} : r_1 \le \|u\| \le r_2\Big\}
\end{aligned}$$

for positive constants r_1, r_2 such that $0 < r_1 \le r_2$. The following result will be used to prove Theorem 1.1 . We refer the reader to [8] and [11] for more details.

Theorem 2.6. *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping with constant $h > 1$, $S : \overline{U}_3 \to E$ is a k-set contraction with $0 \leqslant k < h - 1$ and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) *$Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,*

(ii) *there exists $\epsilon > 0$ such that $Sx \neq (I - T)(\lambda x)$, for all $\lambda \ge 1 + \epsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$,*

(iii) *$Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.*

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

Note that (see [9]) the function

$$G(t, x, \tau, \xi) = -\frac{1}{2\pi} \frac{H(t - \tau - |x - \xi|)}{\sqrt{(t - \tau)^2 - |x - \xi|^2}}, \quad t, \tau > 0, \quad x, \xi \in \mathbb{R}^2,$$

where $|x - \xi| = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}$, is the Green function for the two-dimensional wave equation

$$\begin{aligned}
&u_{tt} - \Delta u = h(t, x), \qquad t > 0, \quad x = (x_1, x_2) \in \mathbb{R}^2,\\
&u(0, x) = u_t(0, x) = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2,
\end{aligned}$$

where $H(\cdot)$ denotes the Heaviside function. Observe that

$$G(t, x, \tau, \xi) \le 0, \quad t, \tau > 0, \quad x, \xi \in \mathbb{R}^2.$$

A key lemma in our proof is the following.

Lemma 2.7. *For $h_1, h_2, p > 0$, we have*

$$\left|\int_{\mathbb{R}^2}\int_0^\infty (h_1+h_2\tau)^p G(t,x,\tau,\xi)d\tau d\xi\right| \le (h_1+h_2t)^p I(t), \quad (t,x)\in(0,\infty)\times\mathbb{R}^2, \tag{2.1}$$

where $I(t) = t^3 + t^2(1+|\log t|)$.

Proof. Let $h_1, h_2, p > 0$ and $t > 0$. One has

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}\int_0^\infty (h_1+h_2\tau)^p G(t,x,\tau,\xi)d\tau d\xi\right| \le \frac{1}{2\pi}\int_{|x-\xi|\le t}\int_0^{t-|x-\xi|}\frac{(h_1+h_2\tau)^p}{\sqrt{(t-\tau)^2-|x-\xi|^2}}d\tau d\xi\\
&\le \frac{(h_1+h_2t)^p}{2\pi}\int_{|x-\xi|\le t}\left(\log(t+\sqrt{t^2-|x-\xi|^2})-\log|x-\xi|\right)d\xi\\
&= \frac{(h_1+h_2t)^p}{2\pi}\left(\int_{|x-\xi|\le t}\log(t+\sqrt{t^2-|x-\xi|^2})d\xi-\int_{|x-\xi|\le t}\log|x-\xi|d\xi\right)\\
&\le \frac{(h_1+h_2t)^p}{2\pi}\left(\log(2t)\int_{|x-\xi|\le t}d\xi-2\pi\int_0^t r_1\log r_1 dr_1\right)\\
&= \frac{(h_1+h_2t)^p}{2\pi}\left(\pi t^2\log(2t)-\pi\left(t^2\log t-\frac{t^2}{2}\right)\right)\\
&\le \frac{(h_1+h_2t)^p}{2}\left(t^2\log(1+2t)+t^2|\log t|+\frac{t^2}{2}\right)\\
&\le \frac{(h_1+h_2t)^p}{2}\left(2t^3+t^2|\log t|+\frac{t^2}{2}\right)\\
&\le (h_1+h_2t)^p\left(t^3+t^2(1+|\log t|)\right).
\end{aligned}$$

This gives (2.1) as desired. □

We make the change $u = v + u_0 + tu_1$. Then, we get the IVP

$$\begin{aligned}
v_{tt}-\Delta v &= f(t,x,v+u_0+tu_1, v_t+u_1, v_x+u_{0x}+tu_{1x})+\Delta u_0+t\Delta u_1\\
&= f_1(t,x,v,v_t,v_x), \quad t>0, \quad x\in\mathbb{R}^2,\\
v(0,x) &= v_t(0,x)=0, \quad x\in\mathbb{R}^2.
\end{aligned} \tag{2.2}$$

Lemma 2.8. *Suppose* **(H2)**. *If $w_k\in\mathbb{R}$, $|w_k|\le b$, $k\in\{1,\dots,4\}$, for some positive b, then*

$$f(t,x,w_1,w_2,w_3,w_4) \le 4a\sum_{j=1}^{l} b^{p_j}.$$

Proof. We have

$$
\begin{aligned}
0 &\le f(t,x,w_1,w_2,w_3,w_4)\\
&\le \sum_{j=1}^{l} \left(a_j(t,x)|w_1|^{p_j} + b_j(t,x)|w_2|^{p_j} + c_j(t,x)|w_3|^{p_j} + d_j(t,x)|w_4|^{p_j}\right)\\
&\le \sum_{j=1}^{l} \left(ab^{p_j} + ab^{p_j} + ab^{p_j} + ab^{p_j}\right)\\
&= 4a\sum_{j=1}^{l} b^{p_j}, \quad (t,x,w_1,w_2,w_3,w_4) \in [0,\infty)\times\mathbb{R}^6.
\end{aligned}
$$

This completes the proof. □

Let $E = \mathcal{C}^2([0,\infty)\times\mathbb{R}^2)$ and for any $u \in E$, denote

$$
\|u\| = \max\left\{\|u\|_\infty, \quad \|u_t\|_\infty, \quad \|u_{tt}\|_\infty \|u_{x_j}\|_\infty, \quad \|u_{x_j x_j}\|_\infty, \quad j \in \{1,2\}\right\},
$$

provided that it is finite, where

$$
\|v\|_\infty = \sup_{(t,x)\in[0,\infty)\times\mathbb{R}^2} |v(t,x)|.
$$

Lemma 2.9. *Suppose* **(H1)** *and* **(H2)***. Let $v \in E$, $\|v\| \le b$, for some positive b. Then*

$$
f(t,x,v+u_0+tu_1, v_t+u_1, v_x+u_{0x}+tu_{1x}) \le 4a\sum_{j=1}^{l}(b+r(1+t))^{p_j}, \quad (t,x)\in[0,\infty)\times\mathbb{R}^2.
$$

Proof. Let

$$
\begin{aligned}
w_1 &= v + u_0 + tu_1,\\
w_2 &= v_t + u_0 + tu_1,\\
w_3 &= u_{x_1} + u_{0x_1} + tu_{1x_1},\\
w_4 &= v_{x_2} + u_{0x_2} + tu_{1x_2}.
\end{aligned}
$$

Then

$$
|w_j| \le b + r(1+t), \quad j \in \{1,\dots,4\}, \quad t \ge 0.
$$

Hence and Lemma 2.8, we get the desired result. This completes the proof. □

Lemma 2.10. *Suppose* **(H1)** *and* **(H2)**. *Let* $v \in E$, $\|v\| \leq b$, *for some positive* b. *Then*

$$|f_1(t,x,v,v_t,v_x)| \leq 4a\sum_{j=1}^{l}(b+r(1+t))^{p_j} + 2r(1+t), \quad (t,x) \in [0,\infty)\times\mathbb{R}^2.$$

Proof. By **(H1)**, we get

$$|\Delta u_0| \leq 2r, \quad |\Delta u_1| \leq 2r \quad \text{on} \quad \mathbb{R}^2.$$

Using Lemma 2.9, we obtain

$$\begin{aligned}|f_1(t,x,v,v_t,v_x)| &\leq f(t,x,v+u_0+tu_1, v_t+u_1, v_x+u_{0x}+tu_{1x}) + |\Delta u_0| + t|\Delta u_1|\\ &\leq 4a\sum_{j=1}^{l}(b+r(1+t))^{p_j} + 2r(1+t), \quad (t,x)\in[0,\infty)\times\mathbb{R}^2.\end{aligned}$$

This completes the proof. □

Now, applying Lemma 2.10 and (2.1), we obtain the following result.

Lemma 2.11. *Suppose* **(H1)** *and* **(H2)**. *Then*

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\int_0^\infty G(t,x,\tau,\xi)f_1(\tau,\xi,v(\tau,\xi),v_t(\tau,\xi),v_x(\tau,\xi))d\tau d\xi\right|\\ &\leq \left(4a\sum_{j=1}^{l}(b+r(1+t))^{p_j}+2r(1+t)\right)I(t)\\ &\leq \left(4a\sum_{j=1}^{l}(2(b+r))^{p_j}+4a\sum_{j=1}^{l}(2r)^{p_j}t^{p_j}+2r(1+t)\right)I(t), \quad (t,x)\in[0,\infty)\times\mathbb{R}^2.\end{aligned}$$

Take a nonnegative function $g \in \mathcal{C}([0,\infty)\times\mathbb{R}^2)$. Suppose that $v \in E$ is a solution to the integral equation.

$$\begin{aligned}0 &= \frac{1}{8}\int_0^t\int_0^{x_1}\int_0^{x_2}(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2 g(t_1,s_1,s_2)v(t_1,s_1,s_2)ds_2ds_1dt_1\\ &-\frac{1}{16\pi}\int_0^t\int_0^{x_1}\int_0^{x_2}(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2 g(t_1,s_1,s_2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}G(t_1,s_1,s_2,t_2,\xi_1,\xi_2)\\ &\times f_1(t_2,\xi_1,\xi_2,v(t_2,\xi_1,\xi_2),v_t(t_2,\xi_1,\xi_2),v_x(t_2,\xi_1,\xi_2))dt_2d\xi_2d\xi_1ds_2ds_1dt_1,\end{aligned} \tag{2.3}$$

$t \geq 0$, $(x_1,x_2)\in\mathbb{R}^2$. We differentiate three times in t, three times in x_1 and three times in x_2 the equation (2.3) and we obtain

$$0 = g(t,x)v(t,x) - \frac{1}{2\pi}g(t,x)\int_{\mathbb{R}^2}\int_0^\infty G(t,x,\tau,\xi)f_1(\tau,\xi,v(\tau,\xi),v_t(\tau,\xi),v_x(\tau,\xi))d\tau d\xi,$$

$t \geq 0$, $x \in \mathbb{R}^2$, whereupon

$$0 = v(t,x) - \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_0^\infty G(t,x,\tau,\xi) f_1(\tau,\xi,v(\tau,\xi),v_t(\tau,\xi),v_x(\tau,\xi))d\tau d\xi,$$

$t \geq 0$, $x \in \mathbb{R}^2$. Hence, using the Green function, we conclude that v is a solution of the IVP (2.2). Thus, any solution $v \in E$ of the integral equation (2.3) is a solution to the IVP (2.2).

(H3) Let $m > 0$ be large enough and A, r_1, L_1, R_1 be positive constants that satisfy the following conditions

$$r_1 < L_1 < R_1, \quad r_1 < r, \quad R_1 > \left(\frac{2}{5m}+1\right)L_1,$$

$$A\left(R_1 + 4a\sum_{j=1}^{l}(2(R_1+r))^{p_j} + 4a\sum_{j=1}^{l}(2r)^{p_j} + 2r\right) < \frac{L_1}{20}.$$

(H4) There exists a nonnegative function $g \in \mathcal{C}([0,\infty)\times\mathbb{R}^2)$ such that

$$\begin{aligned} q(t,x_1,x_2) = &\int_0^t\int_0^{x_1}\int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2) g(t_1,s_1,s_2) \\ &\times \left(1+|x_1-s_1|+(x_1-s_1)^2\right)\left(1+|x_2-s_2|+(x_2-s_2)^2\right) \\ &\times \left(1+(t-t_1)+(t-t_1)^2\right)\left(1+\left(1+t_1+\sum_{j=1}^{l} t_1^{p_j}\right)I(t_1)\right) ds_2 ds_1 dt_1 \\ &\leq A, \quad (t,x_1,x_2) \in [0,\infty)\times\mathbb{R}^2. \end{aligned}$$

In the last section we will give an example for the constants m, A, r, L_1, R_1 and R and for a function g that satisfy **(H3)** and **(H4)**. For $v \in E$, define the operator

$$\begin{aligned} Fv(t,x_1,x_2) = &\frac{1}{8}\int_0^t\int_0^{x_1}\int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2 g(t_1,s_1,s_2) \\ &\times v(t_1,s_1,s_2) ds_2 ds_1 dt_1 \\ &- \frac{1}{16\pi}\int_0^t\int_0^{x_1}\int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2 g(t_1,s_1,s_2) \\ &\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^\infty G(t_1,s_1,s_2,t_2,\xi_1,\xi_2) \\ &\times f_1(t_2,\xi_1,\xi_2,v(t_2,\xi_1,\xi_2),v_t(t_2,\xi_1,\xi_2),v_x(t_2,\xi_1,\xi_2)) dt_2 d\xi_2 d\xi_1 ds_2 ds_1 dt_1, \end{aligned}$$

$(t,x_1,x_2) \in [0,\infty)\times\mathbb{R}^2$.

Lemma 2.12. *Suppose* **(H1)**–**(H3)**. *Then, for $v \in E$, $\|v\| \leq b$, for some positive b, we have*

$$\|Fv\| \leq A\left(b + 4a\sum_{j=1}^{l}(2(b+r))^{p_j} + 4a\sum_{j=1}^{l}(2r)^{p_j} + 2r\right).$$

Proof. Using Lemma 2.11 and **(H3)**, we get

$$\begin{aligned}|Fv(t,x_1,x_2)| &\leq \frac{1}{8}\int_0^t\int_0^{x_1}\int_0^{x_2}\operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2 g(t_1,s_1,s_2)\\ &\quad\times|v(t_1,s_1,s_2)|ds_2ds_1dt_1\\ &\quad+\frac{1}{16\pi}\int_0^t\int_0^{x_1}\int_0^{x_2}\operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2 g(t_1,s_1,s_2)\\ &\quad\times\bigg|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}G(t_1,s_1,s_2,t_2,\xi_1,\xi_2)\\ &\quad\times f_1(t_2,\xi_1,\xi_2,v(t_2,\xi_1,\xi_2),v_t(t_2,\xi_1,\xi_2),v_x(t_2,\xi_1,\xi_2))dt_2d\xi_2d\xi_1\bigg|ds_2ds_1dt_1\\ &\leq bA+4a\sum_{j=1}^{l}(2(b+r))^{p_j}\int_0^t\int_0^{x_1}\int_0^{x_2}\operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2\\ &\quad\times g(t_1,s_1,s_2)I(t_1)ds_2ds_1dt_1\\ &\quad+4a\sum_{j=1}^{l}(2r)^{p_j}\int_0^t\int_0^{x_1}\int_0^{x_2}\operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2\\ &\quad\times g(t_1,s_1,s_2)t_1^{p_j}I(t_1)ds_2ds_1dt_1\\ &\quad+2r\int_0^t\int_0^{x_1}\int_0^{x_2}\operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)^2\\ &\quad\times g(t_1,s_1,s_2)(1+t_1)I(t_1)ds_2ds_1dt_1\\ &\leq A\left(b+4a\sum_{j=1}^{l}(2(b+r))^{p_j}+4a\sum_{j=1}^{l}(2r)^{p_j}+2r\right),\end{aligned}$$

$(t,x_1,x_2)\in[0,\infty)\times\mathbb{R}^2$, and

$$\begin{aligned}\left|\frac{\partial}{\partial t}Fv(t,x_1,x_2)\right| &\leq \frac{1}{4}\int_0^t\int_0^{x_1}\int_0^{x_2}\operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)g(t_1,s_1,s_2)\\ &\quad\times|v(t_1,s_1,s_2)|ds_2ds_1dt_1\\ &\quad+\frac{1}{8\pi}\int_0^t\int_0^{x_1}\int_0^{x_2}\operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)g(t_1,s_1,s_2)\\ &\quad\times\bigg|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}G(t_1,s_1,s_2,t_2,\xi_1,\xi_2)\\ &\quad\times f_1(t_2,\xi_1,\xi_2,v(t_2,\xi_1,\xi_2),v_t(t_2,\xi_1,\xi_2),v_x(t_2,\xi_1,\xi_2))dt_2d\xi_2d\xi_1\bigg|ds_2ds_1dt_1\\ &\leq bA\\ &\quad+4a\sum_{j=1}^{l}(2(b+r))^{p_j}\int_0^t\int_0^{x_1}\int_0^{x_2}\operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)\\ &\quad\times g(t_1,s_1,s_2)I(t_1)ds_2ds_1dt_1\end{aligned}$$

$$+ 4a\sum_{j=1}^{l}(2r)^{p_j}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)$$
$$\times g(t_1,s_1,s_2)t_1^{p_j}I(t_1)ds_2ds_1dt_1$$
$$+ 2r\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2(t-t_1)$$
$$\times g(t_1,s_1,s_2)(1+t_1)I(t_1)ds_2ds_1dt_1$$
$$\leq A\left(b+4a\sum_{j=1}^{l}(2(b+r))^{p_j}+4a\sum_{j=1}^{l}(2r)^{p_j}+2r\right),$$

$(t,x_1,x_2)\in[0,\infty)\times\mathbb{R}^2$, and

$$\left|\frac{\partial^2}{\partial t^2}Fv(t,x_1,x_2)\right| \leq \frac{1}{4}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2g(t_1,s_1,s_2)$$
$$\times|v(t_1,s_1,s_2)|ds_2ds_1dt_1$$
$$+\frac{1}{8\pi}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2g(t_1,s_1,s_2)$$
$$\times\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}G(t_1,s_1,s_2,t_2,\xi_1,\xi_2)\right.$$
$$\left.\times f_1(t_2,\xi_1,\xi_2,v(t_2,\xi_1,\xi_2),v_t(t_2,\xi_1,\xi_2),v_x(t_2,\xi_1,\xi_2))dt_2d\xi_2d\xi_1\right|ds_2ds_1dt_1$$
$$\leq bA+4a\sum_{j=1}^{l}(2(b+r))^{p_j}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2$$
$$\times g(t_1,s_1,s_2)I(t_1)ds_2ds_1dt_1$$
$$+4a\sum_{j=1}^{l}(2r)^{p_j}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2$$
$$\times g(t_1,s_1,s_2)t_1^{p_j}I(t_1)ds_2ds_1dt_1$$
$$+2r\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)(x_1-s_1)^2(x_2-s_2)^2$$
$$\times g(t_1,s_1,s_2)(1+t_1)I(t_1)ds_2ds_1dt_1$$
$$\leq A\left(b+4a\sum_{j=1}^{l}(2(b+r))^{p_j}+4a\sum_{j=1}^{l}(2r)^{p_j}+2r\right),$$

$(t,x_1,x_2)\in[0,\infty)\times\mathbb{R}^2$, and

$$\left|\frac{\partial}{\partial x_1}Fv(t,x_1,x_2)\right| \leq \frac{1}{4}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)|x_1-s_1|(x_2-s_2)^2(t-t_1)^2g(t_1,s_1,s_2)$$
$$\times|v(t_1,s_1,s_2)|ds_2ds_1dt_1$$
$$+\frac{1}{8\pi}\int_0^t\int_0^{x_1}\int_0^{x_2}\mathrm{sign}(x_1)\mathrm{sign}(x_2)|x_1-s_1|(x_2-s_2)^2(t-t_1)^2g(t_1,s_1,s_2)$$

$$\times \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} G(t_1, s_1, s_2, t_2, \xi_1, \xi_2) \right.$$
$$\left. \times f_1(t_2, \xi_1, \xi_2, v(t_2, \xi_1, \xi_2), v_t(t_2, \xi_1, \xi_2), v_x(t_2, \xi_1, \xi_2)) dt_2 d\xi_2 d\xi_1 \right| ds_2 ds_1 dt_1$$
$$\leq bA$$
$$+ 4a \sum_{j=1}^{l} (2(b+r))^{p_j} \int_0^t \int_0^{x_1} \int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2)|x_1 - s_1|(x_2 - s_2)^2 (t - t_1)^2$$
$$\times g(t_1, s_1, s_2) I(t_1) ds_2 ds_1 dt_1$$
$$+ 4a \sum_{j=1}^{l} (2r)^{p_j} \int_0^t \int_0^{x_1} \int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2)|x_1 - s_1|(x_2 - s_2)^2 (t - t_1)^2$$
$$\times g(t_1, s_1, s_2) t_1^{p_j} I(t_1) ds_2 ds_1 dt_1$$
$$+ 2r \int_0^t \int_0^{x_1} \int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2)|x_1 - s_1|(x_2 - s_2)^2 (t - t_1)^2$$
$$\times g(t_1, s_1, s_2)(1 + t_1) I(t_1) ds_2 ds_1 dt_1$$
$$\leq A \left(b + 4a \sum_{j=1}^{l} (2(b+r))^{p_j} + 4a \sum_{j=1}^{l} (2r)^{p_j} + 2r \right),$$

$(t, x_1, x_2) \in [0, \infty) \times \mathbb{R}^2$, and

$$\left| \frac{\partial^2}{\partial x_1^2} Fv(t, x_1, x_2) \right| \leq \frac{1}{4} \int_0^t \int_0^{x_1} \int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_2 - s_2)^2 (t - t_1)^2 g(t_1, s_1, s_2)$$
$$\times |v(t_1, s_1, s_2)| ds_2 ds_1 dt_1$$
$$+ \frac{1}{8\pi} \int_0^t \int_0^{x_1} \int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_2 - s_2)^2 (t - t_1)^2 g(t_1, s_1, s_2)$$
$$\times \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} G(t_1, s_1, s_2, t_2, \xi_1, \xi_2) \right.$$
$$\left. \times f_1(t_2, \xi_1, \xi_2, v(t_2, \xi_1, \xi_2), v_t(t_2, \xi_1, \xi_2), v_x(t_2, \xi_1, \xi_2)) dt_2 d\xi_2 d\xi_1 \right| ds_2 ds_1 dt_1$$
$$\leq bA$$
$$+ 4a \sum_{j=1}^{l} (2(b+r))^{p_j} \int_0^t \int_0^{x_1} \int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_2 - s_2)^2 (t - t_1)^2$$
$$\times g(t_1, s_1, s_2) I(t_1) ds_2 ds_1 dt_1$$
$$+ 4a \sum_{j=1}^{l} (2r)^{p_j} \int_0^t \int_0^{x_1} \int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_2 - s_2)^2 (t - t_1)^2$$
$$\times g(t_1, s_1, s_2) t_1^{p_j} I(t_1) ds_2 ds_1 dt_1$$
$$+ 2r \int_0^t \int_0^{x_1} \int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2)(x_2 - s_2)^2 (t - t_1)^2$$
$$\times g(t_1, s_1, s_2)(1 + t_1) I(t_1) ds_2 ds_1 dt_1$$

$$\leq A\left(b+4a\sum_{j=1}^{l}(2(b+r))^{p_j}+4a\sum_{j=1}^{l}(2r)^{p_j}+2r\right),$$

$(t,x_1,x_2)\in[0,\infty)\times\mathbb{R}^2$. As above, one can obtain

$$\left|\frac{\partial}{\partial x_2}Fv(t,x_1,x_2)\right|\leq A\left(b+4a\sum_{j=1}^{l}(2(b+r))^{p_j}+4a\sum_{j=1}^{l}(2r)^{p_j}+2r\right),$$

$(t,x_1,x_2)\in[0,\infty)\times\mathbb{R}^2$, and

$$\left|\frac{\partial^2}{\partial x_2^2}Fv(t,x_1,x_2)\right|\leq A\left(b+4a\sum_{j=1}^{l}(2(b+r))^{p_j}+4a\sum_{j=1}^{l}(2r)^{p_j}+2r\right),$$

$(t,x_1,x_2)\in[0,\infty)\times\mathbb{R}^2$. Consequently

$$\|Fv\|\leq A\left(b+4a\sum_{j=1}^{l}(2(b+r))^{p_j}+4a\sum_{j=1}^{l}(2r)^{p_j}+2r\right).$$

This completes the proof. □

3 Proof of the Main Result

Let

$$\widetilde{\mathcal{P}}=\{u\in E: u\geq 0\quad\text{on}\quad[0,\infty)\times\mathbb{R}^2\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{\mathcal{P}}$. Note that $Fv\geq 0$ for any $v\in\mathcal{P}$. Let $\epsilon>0$. For $v\in E$, define the operators

$$Tv(t,x)=(1+m\epsilon)v(t,x)-\epsilon\frac{L_1}{10},$$
$$Sv(t,x)=-\epsilon Fv(t,x)-m\epsilon v(t,x)-\epsilon\frac{L_1}{10},$$

$(t,x)\in[0,\infty)\times\mathbb{R}^2$. Note that any fixed point $v\in E$ of the operator $T+S$ is a solution to the IVP (2.2). Define

$$U_1=\mathcal{P}_{r_1}=\{v\in\mathcal{P}:\|v\|<r_1\},$$
$$U_2=\mathcal{P}_{L_1}=\{v\in\mathcal{P}:\|v\|<L_1\},$$
$$U_3=\mathcal{P}_{R_1}=\{v\in\mathcal{P}:\|v\|<R_1\},$$

$$R_2 = R_1 + \frac{A}{m}\left(R_1 + 4a\sum_{j=1}^{l}(2(R_1+r))^{p_j} + 4a\sum_{j=1}^{l}(2r)^{p_j} + 2r\right) + \frac{L_1}{5},$$
$$\Omega = \overline{\mathcal{P}_{R_2}} = \{v \in \mathcal{P} : \|v\| \le R_2\}.$$

1. For $v_1, v_2 \in \Omega$, we have

$$\|Tv_1 - Tv_2\| = (1+m\epsilon)\|v_1 - v_2\|,$$

whereupon $T : \Omega \to E$ is an expansive operator with a constant $1 + m\epsilon > 1$.

2. For $v \in \overline{\mathcal{P}_{R_1}}$, we get

$$\begin{aligned}\|Sv\| &\le \epsilon\|Fv\| + m\epsilon\|v\| + \frac{L_1}{10}\\ &\le \epsilon\left(A\left(R_1 + 4a\sum_{j=1}^{l}(2(R_1+r))^{p_j} + 4a\sum_{j=1}^{l}(2r)^{p_j} + 2r\right) + mR_1 + \frac{L_1}{10}\right).\end{aligned}$$

Therefore $S(\overline{\mathcal{P}_{R_1}})$ is uniformly bounded. Since $S : \overline{\mathcal{P}_{R_1}} \to E$ is continuous, we have that $S(\overline{\mathcal{P}_{R_1}})$ is equi-continuous. Consequently $S : \overline{\mathcal{P}_{R_1}} \to E$ is a 0-set contraction.

3. Let $v_1 \in \overline{\mathcal{P}_{R_1}}$. Set
$$v_2 = v_1 + \frac{1}{m}Fv_1 + \frac{L_1}{5m}.$$
Note that by the second inequality of $(H3)$ and by Lemma 2.12, it follows that $\epsilon Fv + \epsilon\frac{L_1}{5} \ge 0$ on $[0,\infty)\times\mathbb{R}^2$. We have $v_2 \ge 0$ on $[0,\infty)\times\mathbb{R}^2$ and

$$\begin{aligned}\|v_2\| &\le \|v_1\| + \frac{1}{m}\|Fv_1\| + \frac{L_1}{5m}\\ &\le R_1 + \frac{A}{m}\left(R_1 + 4a\sum_{j=1}^{l}(2(R_1+r))^{p_j} + 4a\sum_{j=1}^{l}(2r)^{p_j} + 2r\right) + \frac{L_1}{5}\\ &= R_2.\end{aligned}$$

Therefore $v_2 \in \Omega$ and
$$-\epsilon m v_2 = -\epsilon m v_1 - \epsilon F v_1 - \epsilon\frac{L_1}{10} - \epsilon\frac{L_1}{10}$$
or
$$(I-T)v_2 = -\epsilon m v_2 + \epsilon\frac{L_1}{10} = Sv_1.$$
Consequently $S(\overline{\mathcal{P}_{R_1}}) \subset (I-T)(\Omega)$.

4. Suppose that there exists an $v_0 \in \mathcal{P}^*$ such that $T(v-\lambda v_0) \in \mathcal{P}$, $v \in \partial\mathcal{P}_{r_1}$, $v \in \partial\mathcal{P}_{r_1}\bigcap(\Omega+\lambda u_0)$

and $Sv = v - \lambda v_0$ for some $\lambda \geq 0$. Then

$$\begin{aligned} r_1 &= \|v - \lambda v_0\| \\ &= \|Sv\| \\ &\geq -Sv(t,x) \\ &= \epsilon Fv(t,x) + \epsilon m v(t,x) + \epsilon \frac{L_1}{10} \\ &\geq \epsilon \frac{L_1}{20}, \quad (t,x) \in [0,\infty) \times \mathbb{R}^2, \end{aligned}$$

because by the second inequality of **(H3)** and by Lemma 2.12, it follows that $\epsilon Fv + \epsilon \frac{L_1}{20} \geq 0$ on $[0,\infty) \times \mathbb{R}^2$.

5. Suppose that for any $\epsilon_1 > 0$ small enough there exist a $u \in \partial \mathcal{P}_L$ and $\lambda_1 \geq 1 + \epsilon_1$ such that $\lambda_1 u \in \overline{\mathcal{P}_{R_1}}$ and

$$Su = (I - T)(\lambda_1 u). \tag{3.1}$$

In particular, for $\epsilon_1 > \frac{2}{5m}$, we have $u \in \partial \mathcal{P}_L$, $\lambda_1 u \in \overline{\mathcal{P}_{R_1}}$, $\lambda_1 \geq 1 + \epsilon_1$ and (3.1) holds. Since $u \in \partial \mathcal{P}_L$ and $\lambda_1 u \in \overline{\mathcal{P}_{R_1}}$, it follows that

$$\left(\frac{2}{5m} + 1 \right) L < \lambda_1 L = \lambda_1 \|u\| \leq R_1.$$

Moreover,

$$-\epsilon Fu - m\epsilon u - \epsilon \frac{L}{10} = -\lambda_1 m \epsilon u + \epsilon \frac{L}{10},$$

or

$$Fu + \frac{L}{5} = (\lambda_1 - 1) m u.$$

From here,

$$2\frac{L}{5} \geq \left\| Fu + \frac{L}{5} \right\| = (\lambda_1 - 1) m \|u\| = (\lambda_1 - 1) m L$$

and

$$\frac{2}{5m} + 1 \geq \lambda_1,$$

which is a contradiction.

Therefore all conditions of Theorem 2.6 hold. Hence, the IVP (2.2) has at least two solutions v_1 and v_2 so that

$$r_1 < \|v_1\| < L_1 < \|v_2\| < R_1,$$

and

$$u = v_1 + u_0 + t u_1, \quad w = v_2 + u_0 + t u_1$$

are two different positive solutions of the IVP (1.1). This completes the proof.

4 An Example

Let

$$l=1,\quad p_1=\frac{3}{5},\quad R_1=r=1,\quad a=200,\quad L_1=\frac{1}{2},\quad r_1=\frac{1}{100},$$
$$m=10^{50},\quad \epsilon=50,\quad A=\frac{1}{10^{10}},\quad R=100.$$

Then

$$R_1>\left(\frac{2}{5m}+1q\right)L_1,\quad r_1<L_1<R_1,\quad r_1<\frac{L_1}{20}.$$

Also,

$$\begin{aligned}A\left(R_1+4a\sum_{j=1}^{l}(2(R_1+r))^{p_j}+4a\sum_{j=1}^{l}(2r)^{p_j}+2r\right)&=\frac{1}{10^{10}}\left(1+800\cdot(4)^2+800\cdot 4+2\right)\\&<\frac{1}{40}=\frac{L_1}{20}.\end{aligned}$$

Consequently **(H3)** holds. Now, we will construction the function g in **(H4)**. Let

$$h(x)=\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}},\quad l(s)=\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}},\quad s\in\mathbb{R}.$$

Then

$$\begin{aligned}h'(s)&=\frac{22\sqrt{2}s^{10}(1-s^{22})}{(1-s^{11}\sqrt{2}+s^{22})(1+s^{11}\sqrt{2}+s^{22})},\\ l'(s)&=\frac{11\sqrt{2}s^{10}(1+s^{20})}{1+s^{40}},\quad s\in\mathbb{R}.\end{aligned}$$

Therefore

$$-\infty<\lim_{s\to\pm\infty}(1+s+s^2)h(s)<\infty,$$
$$-\infty<\lim_{s\to\pm\infty}(1+s+s^2)l(s)<\infty.$$

Hence, there exists a positive constant C_1 so that

$$(1+s+s^2)\left(\frac{1}{44\sqrt{2}}\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}+\frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\right)\leq C_1,\quad s\in\mathbb{R}.$$

Note that by [22, p. 707, Integral 79], we have

$$\int \frac{dz}{1+z^4} = \frac{1}{4\sqrt{2}} \log \frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1+s^{44})(1+s+s^2)^2(1+((1+s+s^2)I(s))^2)}, \quad s \in \mathbb{R},$$

and

$$g_1(t,x_1,x_2) = Q(t)Q(x_1)Q(x_2), \quad t \in [0,\infty), \quad x_1, x_2 \in \mathbb{R}.$$

Then there exists a constant $C_2 > 0$ so that

$$\begin{aligned} C_2 \geq & \int_0^t \int_0^{x_1} \int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2) g_1(t_1,s_1,s_2) \\ & \times \left(1+|x_1-s_1|+(x_1-s_1)^2\right)\left(1+|x_2-s_2|+(x_2-s_2)^2\right) \\ & \times \left(1+(t-t_1)+(t-t_1)^2\right)\left(1+(1+t_1+t_1^2)I(t_1)\right) ds_2 ds_1 dt_1, \quad (t,x_1,x_2) \in [0,\infty) \times \mathbb{R}^2. \end{aligned}$$

Now, we take

$$g(t,x_1,x_2) = \frac{1}{10^{20} C_2} g_1(t,x_1,x_2), \quad (t,x_1,x_2) \in [0,\infty) \times \mathbb{R}^2.$$

Then

$$\begin{aligned} A &= \frac{1}{10^{10}} \\ & \geq \int_0^t \int_0^{x_1} \int_0^{x_2} \operatorname{sign}(x_1)\operatorname{sign}(x_2) g(t_1,s_1,s_2) \\ & \times \left(1+|x_1-s_1|+(x_1-s_1)^2\right)\left(1+|x_2-s_2|+(x_2-s_2)^2\right) \\ & \times \left(1+(t-t_1)+(t-t_1)^2\right)\left(1+(1+t_1+t_1^2)I(t_1)\right) ds_2 ds_1 dt_1, \quad (t,x_1,x_2) \in [0,\infty) \times \mathbb{R}^2. \end{aligned}$$

Now, consider the IVP

$$\begin{aligned} u_{tt} - u_{x_1x_1} - u_{x_2x_2} &= w(t) u^{\frac{3}{5}}, \quad (t,x_1,x_2) \in (0,\infty) \times \mathbb{R}^2, \\ u(0,x) &= u_t(0,x) = 0, \quad (x_1,x_2) \in \mathbb{R}^2, \end{aligned} \tag{4.1}$$

where

$$w(t) = \begin{cases} 10(9t^2 - 9t + 2) & t \in [0,1] \\ \\ 20 & t > 1. \end{cases}$$

Here $l = 1$,

$$a_1(t, x_1, x_2) = |w(t)| \leq a = 200,$$
$$b_1(t, x_1, x_2) = c_1(t, x_1, x_2) = d_1(t, x_1, x_2) = 0,$$

$(t, x_1, x_2) \in [0, \infty) \times \mathbb{R}^2$, and

$$u_0(x) = u_1(x) = 0 \leq 1 = r, \quad (x_1, x_2) \in \mathbb{R}^2.$$

We have that **(H1)** and **(H2)** hold. The IVP (4.1) has two nonnegative solutions $u_1(t, x) = 0$, $(t, x) \in [0, \infty) \times \mathbb{R}^2$, and

$$u_2(t, x) = \begin{cases} (t(1-t))^5 & (t, x) \in [0, 1] \times \mathbb{R}^2 \\ \\ 0 & (t, x) \in (1, \infty) \times \mathbb{R}^2. \end{cases}$$

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