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Fixed point results of (ϕ, ψ)-weak contractions in ordered b-metric spaces

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ABSTRACT

The purpose of this paper is to prove some results on fixed point, coincidence point, coupled coincidence point and coupled common fixed point for the mappings satisfying generalized (ϕ, ψ)-contraction conditions in complete partially ordered b-metric spaces. Our results generalize, extend and unify most of the fundamental metrical fixed point theorems in the existing literature. A few examples are illustrated to support our findings.

RESUMEN

El propósito de este artículo es demostrar algunos resultados sobre puntos fijos, puntos de coincidencia, puntos de coincidencia acoplados y puntos de coincidencia acoplados comunes para aplicaciones que satisfacen condiciones de (ϕ, ψ)-contracción generalizadas en b-espacios métricos completos parcialmente ordenados. Nuestros resultados generalizan, extienden y unifican la mayoría de los teoremas de punto fijo métricos fundamentales en la literatura existente. Se ilustran algunos ejemplos para apoyar nuestros resultados.

Keywords and Phrases: Fixed point, coupled coincidence point, coupled common fixed point, partially ordered b-metric space, compatible, mixed f-monotone.

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1 Introduction

The usual metric space has been generalized and enhanced in many different directions, one of such generalizations is a b-metric space which was first coined by Czerwik in [16] and is also known as metric type space (Khamsi and Hussain [35] used recently the term "metric type space")'. Indeed, in some papers it is considered that this concept has been introduced by Bourbaki [14] in 1974, or that it has been introduced by Bakhtin [12] in 1989, or by Czerwik [16] in 1993 or even by Czerwik [17] in 1998. After extensive searches in zbMATH and Mathematical Reviews, it appears that the first fixed point theorem in a quasimetric space (b-metric spaces) has been established in 1981 by Vulpe *et al.* [55], who transposed the Picard-Banach contraction mapping principle from metric spaces to the framework of a quasimetric space. Some important information on the introduction of a b-metric spaces can be found from the article "The early developments in fixed point theory on b-metric spaces: a brief survey and some important related aspects" by Berinde and Pacurar [13]. Later, a series of papers have been dedicated to the improvement of fixed point results for single valued and multi-valued operators on b-metric spaces by following various topological properties, some of such are from [1, 3, 6, 5, 9, 20, 22, 28, 29, 30, 32, 34, 36, 39, 40, 41, 43, 53].

The concept of coupled fixed points for certain mappings in ordered spaces was first introduced by Bhaskar *et al.* [23] and applied their results to study the existence and uniqueness of the solutions for boundary valued problems. While the concept of coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings with monotone property in complete partially ordered metric spaces was first introduced by Lakshmikantham *et al.* [37]. Since then, several authors have carried out further generalizations and improvements in various spaces (see [10, 18, 21, 24, 44, 48]). Aghajani *et al.* [2] proved some coupled coincidence and coupled fixed point results for mappings satisfying generalized (ψ, ϕ, θ)-contractive conditions in partially ordered complete b-metric spaces. Later, the results of [2] have been improved and generalized by Huaping Huang *et al.* [27] in the same space. More works on coupled coincidence and coupled fixed point results for generalized contraction mappings in ordered spaces can be seen from [4, 7, 8, 11, 15, 19, 25, 26, 31, 38, 42, 45, 46, 47, 49, 50, 51, 52]. Recently, some results on fixed point, coincidence point and coupled coincidence points for the mappings satisfying generalized weak contraction contractions in partially ordered b-metric spaces have been discussed by Belay Mituku *et al.* [39], Seshagiri Rao *et al.* [53, 54] and Kalyani *et al.* [33].

The aim of this work is to provide some results on fixed point and coincidence point, coupled coincidence point for the mappings satisfying generalized (ϕ, ψ)-contractive conditions in an ordered b-metric space. Our results are the variations and the generalizations of the results of [25, 26, 31, 38, 42, 45, 52] and several comparable results in the existing literature. A few numerical examples are illustrated to support the findings.

2 Mathematical Preliminaries

The following definitions and results will be needed in what follows.

Definition 2.1 ([39, 53]). *A mapping $d : P \times P \to [0, +\infty)$, where P is a non-empty set is said to be a b-metric, if it satisfies the properties given below for any $\upsilon, \xi, \mu \in P$ and for some real number $s \geq 1$,*

(a) $d(\upsilon, \xi) = 0$ if and only if $\upsilon = \xi$,

(b) $d(\upsilon, \xi) = d(\xi, \upsilon)$,

(c) $d(\upsilon, \xi) \leq s\,(d(\upsilon, \mu) + d(\mu, \xi))$.

Then (P, d, s) is known as a b-metric space. If $(P, \preceq)$ is still a partially ordered set, then $(P, d, s, \preceq)$ is called a partially ordered b-metric space.

Definition 2.2 ([39, 53]). *Let (P, d, s) be a b-metric space. Then*

(1) a sequence $\{\upsilon_n\}$ is said to converge to υ, if $\lim_{n\to+\infty} d(\upsilon_n, \upsilon) = 0$ and written as $\lim_{n\to+\infty} \upsilon_n = \upsilon$.

(2) $\{\upsilon_n\}$ is said to be a Cauchy sequence in P, if $\lim_{n,m\to+\infty} d(\upsilon_n, \upsilon_m) = 0$.

(3) (P, d) is said to be complete, if every Cauchy sequence in it is convergent.

Definition 2.3. *If the metric d is complete then $(P, d, s, \preceq)$ is called complete partially ordered b-metric space.*

Definition 2.4 ([39]). *Let $(P, \preceq)$ be a partially ordered set and let $\not{f}, \not{g} : P \to P$ be two mappings. Then*

(1) $\not{g}$ is called monotone non-decreasing, if $\not{g}\upsilon \preceq \not{g}\xi$ for all $\upsilon, \xi \in P$ with $\upsilon \preceq \xi$.

(2) an element $\upsilon \in P$ is called a coincidence (common fixed) point of $\not{f}$ and $\not{g}$, if $\not{f}\upsilon = \not{g}\upsilon$ ($\not{f}\upsilon = \not{g}\upsilon = \upsilon$).

(3) $\not{f}$ and $\not{g}$ are called commuting, if $\not{f}\not{g}\upsilon = \not{g}\not{f}\upsilon$, for all $\upsilon \in P$.

(4) $\not{f}$ and $\not{g}$ are called compatible, if any sequence $\{\upsilon_n\}$ with $\lim_{n\to+\infty} \not{f}\upsilon_n = \lim_{n\to+\infty} \not{g}\upsilon_n = \mu$, for $\mu \in P$ then $\lim_{n\to+\infty} d(\not{g}\not{f}\upsilon_n, \not{f}\not{g}\upsilon_n) = 0$.

(5) a pair of self maps $(\not{f}, \not{g})$ is called weakly compatible, if $\not{f}\not{g}\upsilon = \not{g}\not{f}\upsilon$, when $\not{g}\upsilon = \not{f}\upsilon$ for some $\upsilon \in P$.

(6) $\not{g}$ is called monotone $\not{f}$-non-decreasing, if

$$\not{f}\upsilon \preceq \not{f}\xi \text{ implies } \not{g}\upsilon \preceq \not{g}\xi, \text{ for any } \upsilon, \xi \in P.$$

(7) a non empty set P is called well ordered set, if every two elements of it are comparable i.e., $\upsilon \preceq \xi$ or $\xi \preceq \upsilon$, for $\upsilon, \xi \in P$.

Definition 2.5 ([2, 37])**.** *Let $(P, \preceq)$ be a partially ordered set and, let $\hbar : P \times P \to P$ and $\ell : P \to P$ be two mappings. Then*

(1) $\hbar$ has the mixed ℓ-monotone property, if $\hbar$ is non-decreasing ℓ-monotone in its first argument and is non-increasing ℓ-monotone in its second argument, that is for any $\upsilon, \xi \in P$

$$\upsilon_1, \upsilon_2 \in P, \quad \ell\upsilon_1 \preceq \ell\upsilon_2 \ \textit{ implies } \ \hbar(\upsilon_1, \xi) \preceq \hbar(\upsilon_2, \xi) \textit{ and}$$
$$\xi_1, \xi_2 \in P, \quad \ell\xi_1 \preceq \ell\xi_2 \ \textit{ implies } \ \hbar(\upsilon, \xi_1) \succeq \hbar(\upsilon, \xi_2).$$

Suppose, if ℓ is the identity mapping then $\hbar$ is said to have the mixed monotone property.

(2) an element $(\upsilon, \xi) \in P \times P$ is called a coupled coincidence point of $\hbar$ and ℓ, if $\hbar(\upsilon, \xi) = \ell\upsilon$ and $\hbar(\xi, \upsilon) = \ell\xi$. Note that, if ℓ is the identity mapping then (υ, ξ) is said to be a coupled fixed point of $\hbar$.

(3) an element $\upsilon \in P$ is called a common fixed point of $\hbar$ and ℓ, if $\hbar(\upsilon, \upsilon) = \ell\upsilon = \upsilon$.

(4) $\hbar$ and ℓ are commutative, if for all $\upsilon, \xi \in P$, $\hbar(\ell\upsilon, \ell\xi) = \ell(\hbar\upsilon, \hbar\xi)$.

(5) $\hbar$ and ℓ are said to be compatible, if

$$\lim_{n\to+\infty} d(\ell(\hbar(\upsilon_n, \xi_n)), \hbar(\ell\upsilon_n, \ell\xi_n)) = 0 \textit{ and } \lim_{n\to+\infty} d(\ell(\hbar(\xi_n, \upsilon_n)), \hbar(\ell\xi_n, \ell\upsilon_n)) = 0,$$

whenever $\{\upsilon_n\}$ and $\{\xi_n\}$ are any two sequences in P such that $\lim_{n\to+\infty} \hbar(\upsilon_n, \xi_n) = \lim_{n\to+\infty} \ell\upsilon_n = \upsilon$ and $\lim_{n\to+\infty} \hbar(\xi_n, \upsilon_n) = \lim_{n\to+\infty} \ell\xi_n = \xi$, for any $\upsilon, \xi \in P$.

We know that a b-metric is not continuous and then we use frequently the following lemma in the proof of our results for the convergence of sequences in b-metric spaces.

Lemma 2.6 ([2])**.** *Let $(P, d, s, \preceq)$ be a b-metric space with $s > 1$ and suppose that $\{\upsilon_n\}$ and $\{\xi_n\}$ are b-convergent to υ and ξ respectively. Then we have*

$$\frac{1}{s^2} d(\upsilon, \xi) \leq \lim_{n\to+\infty} \inf d(\upsilon_n, \xi_n) \leq \lim_{n\to+\infty} \sup d(\upsilon_n, \xi_n) \leq s^2 d(\upsilon, \xi).$$

In particular, if $\upsilon = \xi$, then $\lim_{n\to+\infty} d(\upsilon_n, \xi_n) = 0$. Moreover, for each $\tau \in P$, we have

$$\frac{1}{s} d(\upsilon, \tau) \leq \lim_{n\to+\infty} \inf d(\upsilon_n, \tau) \leq \lim_{n\to+\infty} \sup d(\upsilon_n, \tau) \leq s d(\upsilon, \tau).$$

3 Main Results

The following distance functions are used throughout the paper.

A self mapping ϕ defined on $[0,+\infty)$ is said to be an altering distance function, if it satisfies the following conditions:

(i) ϕ is non-decreasing and continuous function,

(iii) $\phi(t)=0$ if and only if $t=0$.

Let us denote the set of all altering distance functions on $[0,+\infty)$ by Φ.

Similarly, Ψ denotes the set of all functions $\psi:[0,+\infty)\to[0,+\infty)$ satisfying the following conditions:

(i) ψ is lower semi-continuous,

(ii) $\psi(t)=0$ if and only if $t=0$.

Let $(P,d,s,\preceq)$ be a partially ordered b-metric space with parameter $s>1$ and, let $g:P\to P$ be a mapping. Set

$$M(\upsilon,\xi)=\max\left\{\frac{d(\xi,g\xi)\,[1+d(\upsilon,g\upsilon)]}{1+d(\upsilon,\xi)},\frac{d(\upsilon,g\upsilon)\;d(\upsilon,g\xi)}{1+d(\upsilon,g\xi)+d(\xi,g\upsilon)},d(\upsilon,\xi)\right\}, \tag{3.1}$$

and

$$N(\upsilon,\xi)=\max\left\{\frac{d(\xi,g\xi)\,[1+d(\upsilon,g\upsilon)]}{1+d(\upsilon,\xi)},d(\upsilon,\xi)\right\}. \tag{3.2}$$

Let $\phi\in\Phi$ and $\psi\in\Psi$. The mapping g is a generalized (ϕ,ψ)-contraction mapping if it satisfies the following condition

$$\phi(sd(g\upsilon,g\xi))\le\phi(M(\upsilon,\xi))-\psi(N(\upsilon,\xi)), \tag{3.3}$$

for any $\upsilon,\xi\in P$ with $\upsilon\preceq\xi$ and M,N are same as above.

Now, we prove some results for the existence of fixed point, coincidence point, coupled coincidence point and coupled common fixed point of the mappings satisfying a generalized (ϕ,ψ)-contraction condition in the context of partially ordered b-metric space. We begin with the following fixed point theorem in this paper.

Theorem 3.1. *Suppose that $(P,d,s,\preceq)$ is a complete partially ordered b-metric space with parameter $s>1$. Let $g:P\to P$ be a generalized (ϕ,ψ)-contractive mapping, and be continuous, non-decreasing mapping with respect to $\preceq$. If there exists $\upsilon_0\in P$ with $\upsilon_0\preceq g\upsilon_0$, then g has a fixed point in P.*

Proof. For some $\upsilon_0 \in P$ such that $g\upsilon_0 = \upsilon_0$, then we have the result. Assume that $\upsilon_0 \prec g\upsilon_0$, then construct a sequence $\{\upsilon_n\} \subset P$ by $\upsilon_{n+1} = g\upsilon_n$, for $n \geq 0$. Since g is non-decreasing, then by induction we obtain that

$$\upsilon_0 \prec g\upsilon_0 = \upsilon_1 \preceq \cdots \preceq \upsilon_n \preceq g\upsilon_n = \upsilon_{n+1} \preceq \cdots . \tag{3.4}$$

If for some $n_0 \in \mathbb{N}$ such that $\upsilon_{n_0} = \upsilon_{n_0+1}$ then from (3.4), υ_{n_0} is a fixed point of g and we have nothing to prove. Suppose that $\upsilon_n \neq \upsilon_{n+1}$, for all $n \geq 1$. Since $\upsilon_n > \upsilon_{n-1}$ for all $n \geq 1$ and then by condition (3.3), we have

$$\begin{aligned} \phi(d(\upsilon_n, \upsilon_{n+1})) = \phi(d(g\upsilon_{n-1}, g\upsilon_n)) &\leq \phi(sd(g\upsilon_{n-1}, g\upsilon_n)) \\ &\leq \phi(M(\upsilon_{n-1}, \upsilon_n)) - \psi(N(\upsilon_{n-1}, \upsilon_n)). \end{aligned} \tag{3.5}$$

From (3.5), we get

$$d(\upsilon_n, \upsilon_{n+1}) = d(g\upsilon_{n-1}, g\upsilon_n) \leq \frac{1}{s} M(\upsilon_{n-1}, \upsilon_n), \tag{3.6}$$

where

$$\begin{aligned} M(\upsilon_{n-1}, \upsilon_n) &= \max\left\{ \frac{d(\upsilon_n, g\upsilon_n)\,[1 + d(\upsilon_{n-1}, g\upsilon_{n-1})]}{1 + d(\upsilon_{n-1}, \upsilon_n)}, \frac{d(\upsilon_{n-1}, g\upsilon_{n-1})\; d(\upsilon_{n-1}, g\upsilon_n)}{1 + d(\upsilon_{n-1}, g\upsilon_n) + d(\upsilon_n, g\upsilon_{n-1})}, \right. \\ &\qquad \left. d(\upsilon_{n-1}, \upsilon_n) \right\} \\ &= \max\left\{ d(\upsilon_n, \upsilon_{n+1}), \frac{d(\upsilon_{n-1}, \upsilon_n)\; d(\upsilon_{n-1}, \upsilon_{n+1})}{1 + d(\upsilon_{n-1}, \upsilon_{n+1})}, d(\upsilon_{n-1}, \upsilon_n) \right\} \\ &\leq \max\{d(\upsilon_n, \upsilon_{n+1}), d(\upsilon_{n-1}, \upsilon_n)\}. \end{aligned} \tag{3.7}$$

If $\max\{d(\upsilon_n, \upsilon_{n+1}), d(\upsilon_{n-1}, \upsilon_n)\} = d(\upsilon_n, \upsilon_{n+1})$ for some $n \geq 1$, then from (3.6) follows

$$d(\upsilon_n, \upsilon_{n+1}) \leq \frac{1}{s} d(\upsilon_n, \upsilon_{n+1}), \tag{3.8}$$

which is a contradiction. This means that $\max\{d(\upsilon_n, \upsilon_{n+1}), d(\upsilon_{n-1}, \upsilon_n)\} = d(\upsilon_{n-1}, \upsilon_n)$ for $n \geq 1$. Hence, we obtain from (3.6) that

$$d(\upsilon_n, \upsilon_{n+1}) \leq \frac{1}{s} d(\upsilon_{n-1}, \upsilon_n). \tag{3.9}$$

Since, $\frac{1}{s} \in (0, 1)$ then the sequence $\{\upsilon_n\}$ is a Cauchy sequence by [1, 6, 41, 22]. But P is complete, then there exists $\mu \in P$ such that $\upsilon_n \to \mu$.

Also, the continuity of g implies that

$$g\mu = g(\lim_{n\to+\infty} \upsilon_n) = \lim_{n\to+\infty} g\upsilon_n = \lim_{n\to+\infty} \upsilon_{n+1} = \mu. \tag{3.10}$$

Therefore, μ is a fixed point of g in P. □

Last result is still valid for g not necessarily continuous, assuming an additional hypothesis on P.

Theorem 3.2. *In Theorem 3.1 assume that P satisfies,*

$$\textit{if a non-decreasing sequence } \{\upsilon_n\} \to \mu \textit{ in } P, \textit{ then } \upsilon_n \preceq \mu \textit{ for all } n \in \mathbb{N}, \textit{i.e., } \mu = \sup \upsilon_n.$$

Then a non-decreasing mapping g has a fixed point in P.

Proof. From Theorem 3.1, we take the same sequence $\{\upsilon_n\}$ in P such that $\upsilon_0 \preceq \upsilon_1 \preceq \cdots \preceq \upsilon_n \preceq \upsilon_{n+1} \preceq \cdots$, that is, $\{\upsilon_n\}$ is non-decreasing and converges to some $\mu \in P$. Thus from the hypotheses, we have $\upsilon_n \preceq \mu$, for any $n \in \mathbb{N}$, implies that $\mu = \sup \upsilon_n$.

Next, we prove that μ is a fixed point of g in P, that is $g\mu = \mu$. Suppose that $g\mu \neq \mu$. Let

$$M(\upsilon_n, \mu) = \max\left\{\frac{d(\mu, g\mu)\,[1 + d(\upsilon_n, g\upsilon_n)]}{1 + d(\upsilon_n, \mu)}, \frac{d(\upsilon_n, g\upsilon_n)\ d(\upsilon_n, g\mu)}{1 + d(\upsilon_n, g\mu) + d(\mu, g\upsilon_n)}, d(\upsilon_n, \mu)\right\}, \tag{3.11}$$

and

$$N(\upsilon_n, \mu) = \max\left\{\frac{d(\mu, g\mu)\,[1 + d(\upsilon_n, g\upsilon_n)]}{1 + d(\upsilon_n, \mu)}, d(\upsilon_n, \mu)\right\}. \tag{3.12}$$

Letting $n \to +\infty$ and from the fact that $\lim_{n\to+\infty} \upsilon_n = \mu$, we get

$$\lim_{n\to+\infty} M(\upsilon_n, \mu) = \max\{d(\mu, g\mu), 0, 0\} = d(\mu, g\mu), \tag{3.13}$$

and

$$\lim_{n\to+\infty} N(\upsilon_n, \mu) = \max\{d(\mu, g\mu), 0\} = d(\mu, g\mu). \tag{3.14}$$

We know that $\upsilon_n \preceq \mu$ for all n, then from contraction condition (3.3), we get

$$\phi(d(\upsilon_{n+1}, g\mu)) = \phi(d(g\upsilon_n, g\mu) \leq \phi(sd(g\upsilon_n, g\mu) \leq \phi(M(\upsilon_n, \mu)) - \psi(N(\upsilon_n, \mu)). \tag{3.15}$$

Letting $n \to +\infty$ and use of (3.13) and (3.14), we get

$$\phi(d(\mu, g\mu)) \leq \phi(d(\mu, g\mu)) - \psi(d(\mu, g\mu)) < \phi(d(\mu, g\mu)), \tag{3.16}$$

which is a contradiction under (3.16). Thus, $g\mu = \mu$, that is g has a fixed point μ in P. □

Now we give a sufficient condition for the uniqueness of the fixed point that exists in Theorem 3.1 and Theorem 3.2.

$$\text{every pair of elements has a lower bound or an upper bound.} \tag{3.17}$$

This condition is equivalent to,

$$\text{for every } \upsilon, \xi \in P, \text{ there exists } w \in P \text{ which is comparable to } \upsilon \text{ and } \xi.$$

Theorem 3.3. *In addition to the hypotheses of Theorem 3.1 (or Theorem 3.2), condition* (3.17) *provides the uniqueness of a fixed point of* g *in* P.

Proof. From Theorem 3.1 (or Theorem 3.2), we conclude that g has a nonempty set of fixed points. Suppose that υ^* and ξ^* be two fixed points of g then, we claim that $\upsilon^* = \xi^*$. Suppose that $\upsilon^* \neq \xi^*$, then from the hypotheses we have

$$\phi(d(g\upsilon^*, g\xi^*)) \leq \phi(sd(g\upsilon^*, g\xi^*)) \leq \phi(M(\upsilon^*, \xi^*)) - \psi(N(\upsilon^*, \xi^*)). \tag{3.18}$$

Consequently, we get

$$d(\upsilon^*, \xi^*) = d(g\upsilon^*, g\xi^*) \leq \frac{1}{s} M(\upsilon^*, \xi^*), \tag{3.19}$$

where

$$\begin{aligned} M(\upsilon^*, \xi^*) &= \max\left\{ \frac{d(\xi^*, g\xi^*)\,[1 + d(\upsilon^*, g\upsilon^*)]}{1 + d(\upsilon^*, \xi^*)}, \frac{d(\upsilon^*, g\upsilon^*)\; d(\upsilon^*, g\xi^*)}{1 + d(\upsilon^*, g\xi^*) + d(\xi^*, g\upsilon^*)}, d(g\upsilon^*, g\xi^*) \right\} \\ &= \max\left\{ \frac{d(\xi^*, \xi^*)\,[1 + d(\upsilon^*, \upsilon^*)]}{1 + d(\upsilon^*, \xi^*)}, \frac{d(\upsilon^*, \upsilon^*)\; d(\upsilon^*, \xi^*)}{1 + d(\upsilon^*, \xi^*) + d(\xi^*, \upsilon^*)}, d(\upsilon^*, \xi^*) \right\} \\ &= \max\{0, 0, d(\upsilon^*, \xi^*)\} \\ &= d(\upsilon^*, \xi^*). \end{aligned} \tag{3.20}$$

From (3.19), we obtain that

$$d(\upsilon^*, \xi^*) \leq \frac{1}{s} d(\upsilon^*, \xi^*) < d(\upsilon^*, \xi^*), \tag{3.21}$$

which is a contradiction. Hence, $\upsilon^* = \xi^*$. This completes the proof. □

Let $(P, d, s, \preceq)$ be a partially ordered b-metric space with parameter $s > 1$, and let $g, f : P \to P$ be two mappings. Set

$$M_f(\upsilon, \xi) = \max\left\{ \frac{d(f\xi, g\xi)\,[1 + d(f\upsilon, g\upsilon)]}{1 + d(f\upsilon, f\xi)}, \frac{d(f\upsilon, g\upsilon)\; d(f\upsilon, g\xi)}{1 + d(f\upsilon, g\xi) + d(f\xi, g\upsilon)}, d(f\upsilon, f\xi) \right\}, \tag{3.22}$$

and

$$N_f(\upsilon, \xi) = \max\left\{ \frac{d(f\xi, g\xi)\,[1 + d(f\upsilon, g\upsilon)]}{1 + d(f\upsilon, f\xi)}, d(f\upsilon, f\xi) \right\}. \tag{3.23}$$

Now, we introduce the following definition.

Definition 3.4. *Let $(P, d, s, \preceq)$ be a partially ordered b-metric space with $s > 1$. The mapping $g : P \to P$ is called a generalized (ϕ,ψ)-contraction mapping with respect to $f : P \to P$ for some $\phi \in \Phi$ and $\psi \in \Psi$, if*

$$\phi(sd(gv, g\xi)) \leq \phi(M_f(v,\xi)) - \psi(N_f(v,\xi)), \tag{3.24}$$

for any $v, \xi \in P$ with $fv \preceq f\xi$, where $M_f(v,\xi)$ and $N_f(v,\xi)$ are given by (3.22) and (3.23) respectively.

Theorem 3.5. *Suppose that $(P, d, s, \preceq)$ is a complete partially ordered b-metric space with $s > 1$. Let $g : P \to P$ be a generalized (ϕ,ψ)-contractive mapping with respect to $f : P \to P$ and, g and f are continuous such that g is a monotone f-non-decreasing mapping, compatible with f and $gP \subseteq fP$. If for some $v_0 \in P$ such that $fv_0 \preceq gv_0$, then g and f have a coincidence point in P.*

Proof. By following the proof of Theorem 2.2 in [8], we construct two sequences $\{v_n\}$ and $\{\xi_n\}$ in P such that

$$\xi_n = gv_n = fv_{n+1} \quad \text{for all } n \geq 0, \tag{3.25}$$

for which

$$fv_0 \preceq fv_1 \preceq \cdots \preceq fv_n \preceq fv_{n+1} \preceq \cdots. \tag{3.26}$$

Again from [8], we have to show that

$$d(\xi_n, \xi_{n+1}) \leq \lambda d(\xi_{n-1}, \xi_n), \tag{3.27}$$

for all $n \geq 1$ and where $\lambda \in [0, \frac{1}{s})$. Now from (3.24) and using (3.25) and (3.26), we get

$$\begin{aligned}\phi(sd(\xi_n, \xi_{n+1})) &= \phi(sd(gv_n, gv_{n+1})) \\ &\leq \phi(M_f(v_n, v_{n+1})) - \psi(N_f(v_n, v_{n+1})),\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}M_f(v_n, v_{n+1}) = \max\Bigg\{&\frac{d(fv_{n+1}, gv_{n+1})\,[1 + d(fv_n, gv_n)]}{1 + d(fv_n, fv_{n+1})}, \frac{d(fv_n, gv_n)\ d(fv_n, gv_{n+1})}{1 + d(fv_n, gv_{n+1}) + d(fv_{n+1}, gv_n)}, \\ &d(fv_n, fv_{n+1})\Bigg\} \\ = \max\Bigg\{&\frac{d(\xi_n, \xi_{n+1})\,[1 + d(\xi_{n-1}, \xi_n)]}{1 + d(\xi_{n-1}, \xi_n)}, \frac{d(\xi_{n-1}, \xi_n)\ d(\xi_{n-1}, \xi_{n+1})}{1 + d(\xi_{n-1}, \xi_{n+1}) + d(\xi_n, \xi_n)}, d(\xi_{n-1}, \xi_n)\Bigg\} \\ = \max\{&d(\xi_{n-1}, \xi_n), d(\xi_n, \xi_{n+1})\}\end{aligned}$$

and

$$\begin{aligned}N_{\mathcal{f}}(\upsilon_n,\upsilon_{n+1}) &= \max\left\{\frac{d(\mathcal{f}\upsilon_{n+1},\mathcal{g}\upsilon_{n+1})\,[1+d(\mathcal{f}\upsilon_n,\mathcal{g}\upsilon_n)]}{1+d(\mathcal{f}\upsilon_n,\mathcal{f}\upsilon_{n+1})}, d(\mathcal{f}\upsilon_n,\mathcal{f}\upsilon_{n+1})\right\}\\ &= \max\left\{\frac{d(\xi_n,\xi_{n+1})\,[1+d(\xi_{n-1},\xi_n)]}{1+d(\xi_{n-1},\xi_n)}, d(\xi_{n-1},\xi_n)\right\}\\ &= \max\{d(\xi_{n-1},\xi_n), d(\xi_n,\xi_{n+1})\}.\end{aligned}$$

Therefore from equation (3.28), we get

$$\phi(sd(\xi_n,\xi_{n+1})) \le \phi(\max\{d(\xi_{n-1},\xi_n), d(\xi_n,\xi_{n+1})\}) - \psi(\max\{d(\xi_{n-1},\xi_n), d(\xi_n,\xi_{n+1})\}). \quad (3.29)$$

If $0 < d(\xi_{n-1},\xi_n) \le d(\xi_n,\xi_{n+1})$ for some $n \in \mathbb{N}$, then from (3.29) we get

$$\phi(sd(\xi_n,\xi_{n+1})) \le \phi(d(\xi_n,\xi_{n+1})) - \psi(d(\xi_n,\xi_{n+1})) < \phi(d(\xi_n,\xi_{n+1})), \quad (3.30)$$

or equivalently

$$sd(\xi_n,\xi_{n+1}) \le d(\xi_n,\xi_{n+1}). \quad (3.31)$$

This is a contradiction. Hence from (3.29) we obtain that

$$sd(\xi_n,\xi_{n+1}) \le d(\xi_{n-1},\xi_n). \quad (3.32)$$

Thus equation (3.27) holds, where $\lambda \in [0, \frac{1}{s})$. Therefore from (3.27) and Lemma 3.1 of [32], we conclude that $\{\xi_n\} = \{\mathcal{g}\upsilon_n\} = \{\mathcal{f}\upsilon_{n+1}\}$ is a Cauchy sequence in P and then converges to some $\mu \in P$ as P is complete such that

$$\lim_{n\to+\infty} \mathcal{g}\upsilon_n = \lim_{n\to+\infty} \mathcal{f}\upsilon_{n+1} = \mu.$$

Thus by the compatibility of $\mathcal{g}$ and $\mathcal{f}$, we obtain that

$$\lim_{n\to+\infty} d(\mathcal{f}(\mathcal{g}\upsilon_n), \mathcal{g}(\mathcal{f}\upsilon_n)) = 0, \quad (3.33)$$

and from the continuity of $\mathcal{g}$ and $\mathcal{f}$, we have

$$\lim_{n\to+\infty} \mathcal{f}(\mathcal{g}\upsilon_n) = \mathcal{f}\mu, \qquad \lim_{n\to+\infty} \mathcal{g}(\mathcal{f}\upsilon_n) = \mathcal{g}\mu. \quad (3.34)$$

Further, from the triangular inequality of a b-metric and, from equations (3.33) and (3.34) , we get

$$\frac{1}{s}d(\mathcal{g}\mu,\mathcal{f}\mu) \le d(\mathcal{g}\mu,\mathcal{g}(\mathcal{f}\upsilon_n)) + sd(\mathcal{g}(\mathcal{f}\upsilon_n),\mathcal{f}(\mathcal{g}\upsilon_n)) + sd(\mathcal{f}(\mathcal{g}\upsilon_n),\mathcal{f}\mu). \quad (3.35)$$

Finally, we arrive at $d(\mathcal{g}v,\mathcal{f}v) = 0$ as $n \to +\infty$ in (3.35). Therefore, v is a coincidence point of $\mathcal{g}$

and $\mathcal{f}$ in P. $\square$

Relaxing the continuity of the mappings $\mathcal{f}$ and $\mathcal{g}$ in Theorem 3.5, we obtain the following result.

Theorem 3.6. *In Theorem 3.5, assume that P satisfies*

> *for any non-decreasing sequence $\{\mathcal{f}\upsilon_n\} \subset P$ with $\lim_{n\to+\infty} \mathcal{f}\upsilon_n = \mathcal{f}\upsilon$ in $\mathcal{f}P$, where $\mathcal{f}P$ is a closed subset of P implies that $\mathcal{f}\upsilon_n \preceq \mathcal{f}\upsilon, \mathcal{f}\upsilon \preceq \mathcal{f}(\mathcal{f}\upsilon)$ for $n \in \mathbb{N}$.*

If there exists $\upsilon_0 \in P$ such that $\mathcal{f}\upsilon_0 \preceq \mathcal{g}\upsilon_0$, then the weakly compatible mappings $\mathcal{g}$ and $\mathcal{f}$ have a coincidence point in P. Furthermore, $\mathcal{g}$ and $\mathcal{f}$ have a common fixed point, if $\mathcal{g}$ and $\mathcal{f}$ commute at their coincidence points.

Proof. The sequence, $\{\xi_n\} = \{\mathcal{g}\upsilon_n\} = \{\mathcal{f}\upsilon_{n+1}\}$ is a Cauchy sequence from the proof of Theorem 3.5. Since $\mathcal{f}P$ is closed, then there is some $\mu \in P$ such that

$$\lim_{n\to+\infty} \mathcal{g}\upsilon_n = \lim_{n\to+\infty} \mathcal{f}\upsilon_{n+1} = \mathcal{f}\mu.$$

Thus from the hypotheses, we have $\mathcal{f}\upsilon_n \preceq \mathcal{f}\mu$ for all $n \in \mathbb{N}$. Now, we have to prove that μ is a coincidence point of $\mathcal{g}$ and $\mathcal{f}$.

From equation (3.24), we have

$$\phi(sd(\mathcal{g}\upsilon_n, \mathcal{g}\upsilon)) \leq \phi(M_{\mathcal{f}}(\upsilon_n, \upsilon)) - \psi(N_{\mathcal{f}}(\upsilon_n, \upsilon)), \tag{3.36}$$

where

$$\begin{aligned} M_{\mathcal{f}}(\upsilon_n, \mu) &= \max\left\{\frac{d(\mathcal{f}\mu, \mathcal{g}\mu)\,[1 + d(\mathcal{f}\upsilon_n, \mathcal{g}\upsilon_n)]}{1 + d(\mathcal{f}\upsilon_n, \mathcal{f}\mu)}, \frac{d(\mathcal{f}\upsilon_n, \mathcal{g}\upsilon_n)\ d(\mathcal{f}\upsilon_n, \mathcal{g}\mu)}{1 + d(\mathcal{f}\upsilon_n, \mathcal{g}\mu) + d(\mathcal{f}\mu, \mathcal{g}\upsilon_n)}, d(\mathcal{f}\upsilon_n, \mathcal{f}\mu)\right\} \\ &\to \max\{d(\mathcal{f}\mu, \mathcal{g}\mu), 0, 0\} \\ &= d(\mathcal{f}\mu, \mathcal{g}\mu) \quad \text{as } n \to +\infty, \end{aligned}$$

and

$$\begin{aligned} N_{\mathcal{f}}(\upsilon_n, \mu) &= \max\left\{\frac{d(\mathcal{f}\mu, \mathcal{g}\mu)\,[1 + d(\mathcal{f}\upsilon_n, \mathcal{g}\upsilon_n)]}{1 + d(\mathcal{f}\upsilon_n, \mathcal{f}\mu)}, d(\mathcal{f}\upsilon_n, \mathcal{f}\mu)\right\} \\ &\to \max\{d(\mathcal{f}\mu, \mathcal{g}\mu), 0\} \\ &= d(\mathcal{f}\mu, \mathcal{g}\mu) \quad \text{as } n \to +\infty. \end{aligned}$$

Therefore equation (3.36) becomes

$$\phi(s \lim_{n\to+\infty} d(\mathcal{g}\upsilon_n, \mathcal{g}\upsilon)) \leq \phi(d(\mathcal{f}\mu, \mathcal{g}\mu)) - \psi(d(\mathcal{f}\mu, \mathcal{g}\mu)) < \phi(d(\mathcal{f}\mu, \mathcal{g}\mu)). \tag{3.37}$$

Consequently, we get

$$\lim_{n\to+\infty} d(\mathscr{g}\upsilon_n, \mathscr{g}\upsilon) < \frac{1}{s} d(\mathscr{f}\mu, \mathscr{g}\mu). \tag{3.38}$$

Further by triangular inequality, we have

$$\frac{1}{s} d(\mathscr{f}\mu, \mathscr{g}\mu) \leq d(\mathscr{f}\mu, \mathscr{g}\upsilon_n) + d(\mathscr{g}\upsilon_n, \mathscr{g}\mu), \tag{3.39}$$

then (3.38) and (3.39) lead to contradiction, if $\mathscr{f}\mu \neq \mathscr{g}\mu$. Hence, $\mathscr{f}\mu = \mathscr{g}\mu$. Let $\mathscr{f}\mu = \mathscr{g}\mu = \rho$, that is $\mathscr{g}$ and $\mathscr{f}$ commute at ρ, then $\mathscr{g}\rho = \mathscr{g}(\mathscr{f}\mu) = \mathscr{f}(\mathscr{g}\mu) = \mathscr{f}\rho$. Since $\mathscr{f}\mu = \mathscr{f}(\mathscr{f}\mu) = \mathscr{f}\rho$, then by equation (3.36) with $\mathscr{f}\mu = \mathscr{g}\mu$ and $\mathscr{f}\rho = \mathscr{g}\rho$, we get

$$\phi(sd(\mathscr{g}\mu, \mathscr{g}\rho)) \leq \phi(M_{\mathscr{f}}(\mu, \rho)) - \psi(N_{\mathscr{f}}(\mu, \rho)) < \phi(d(\mathscr{g}\mu, \mathscr{g}\rho)), \tag{3.40}$$

or equivalently,

$$sd(\mathscr{g}\mu, \mathscr{g}\rho) \leq d(\mathscr{g}\mu, \mathscr{g}\rho),$$

which is a contradiction, if $\mathscr{g}\mu \neq \mathscr{g}\rho$. Thus, $\mathscr{g}\mu = \mathscr{g}\rho = \rho$. Hence, $\mathscr{g}\mu = \mathscr{f}\rho = \rho$, that is ρ is a common fixed point of $\mathscr{g}$ and $\mathscr{f}$. □

Definition 3.7. *Let $(P, d, s, \preceq)$ be a complete partially ordered b-metric space with $s > 1$, $\phi \in \Phi$ and $\psi \in \Psi$. A mapping $\hbar : P \times P \to P$ is said to be a generalized (ϕ, ψ)-contractive mapping with respect to $\mathscr{f} : P \to P$ such that*

$$\phi(s^k d(\hbar(\upsilon, \xi), \hbar(\rho, \tau))) \leq \phi(M_{\mathscr{f}}(\upsilon, \xi, \rho, \tau)) - \psi(N_{\mathscr{f}}(\upsilon, \xi, \rho, \tau)), \tag{3.41}$$

for all $\upsilon, \xi, \rho, \tau \in P$ with $\mathscr{f}\upsilon \preceq \mathscr{f}\rho$ and $\mathscr{f}\xi \succeq \mathscr{f}\tau$, $k > 2$ where

$$M_{\mathscr{f}}(\upsilon, \xi, \rho, \tau) = \max\left\{ \frac{d(\mathscr{f}\rho, \hbar(\rho, \tau))\,[1 + d(\mathscr{f}\upsilon, \hbar(\upsilon, \xi))]}{1 + d(\mathscr{f}\upsilon, \mathscr{f}\rho)}, \frac{d(\mathscr{f}\upsilon, \hbar(\upsilon, \xi))\ d(\mathscr{f}\upsilon, \hbar(\rho, \tau))}{1 + d(\mathscr{f}\upsilon, \hbar(\rho, \tau)) + d(\mathscr{f}\rho, \hbar(\upsilon, \xi))}, d(\mathscr{f}\upsilon, \mathscr{f}\rho) \right\},$$

and

$$N_{\mathscr{f}}(\upsilon, \xi, \rho, \tau) = \max\left\{ \frac{d(\mathscr{f}\rho, \hbar(\rho, \tau))\,[1 + d(\mathscr{f}\upsilon, \hbar(\upsilon, \xi))]}{1 + d(\mathscr{f}\upsilon, \mathscr{f}\rho)}, d(\mathscr{f}\upsilon, \mathscr{f}\rho) \right\}.$$

Theorem 3.8. *Let $(P, d, s, \preceq)$ be a complete partially ordered b-metric space with $s > 1$. Suppose that $\hbar : P \times P \to P$ be a generalized (ϕ, ψ)- contractive mapping with respect to $\mathscr{f} : P \to P$ and, $\hbar$ and $\mathscr{f}$ are continuous functions such that $\hbar$ has the mixed $\mathscr{f}$-monotone property and commutes with $\mathscr{f}$. Also assume that $\hbar(P \times P) \subseteq \mathscr{f}(P)$. Then $\hbar$ and $\mathscr{f}$ have a coupled coincidence point in P, if there exists $(\upsilon_0, \xi_0) \in P \times P$ such that $\mathscr{f}\upsilon_0 \preceq \hbar(\upsilon_0, \xi_0)$ and $\mathscr{f}\xi_0 \succeq \hbar(\xi_0, \upsilon_0)$.*

Proof. From the hypotheses and following the proof of Theorem 2.2 of [8], we construct two sequences $\{\upsilon_n\}$ and $\{\xi_n\}$ in P such that

$$\ell\upsilon_{n+1} = \hbar(\upsilon_n,\xi_n), \quad \ell\xi_{n+1} = \hbar(\xi_n,\upsilon_n), \quad \text{for all } n \geq 0.$$

In particular, $\{\ell\upsilon_n\}$ is non-decreasing and $\{\ell\xi_n\}$ is non-increasing sequences in P. Now from (3.41) by replacing $\upsilon=\upsilon_n, \xi=\xi_n, \rho=\upsilon_{n+1}, \tau=\xi_{n+1}$, we get

$$\begin{aligned}\phi(s^k d(\ell\upsilon_{n+1},\ell\upsilon_{n+2})) &= \phi(s^k d(\hbar(\upsilon_n,\xi_n),\hbar(\upsilon_{n+1},\xi_{n+1})))\\ &\leq \phi(M_\ell(\upsilon_n,\xi_n,\upsilon_{n+1},\xi_{n+1})) - \psi(N_\ell(\upsilon_n,\xi_n,\upsilon_{n+1},\xi_{n+1})),\end{aligned} \tag{3.42}$$

where

$$M_\ell(\upsilon_n,\xi_n,\upsilon_{n+1},\xi_{n+1}) \leq \max\{d(\ell\upsilon_n,\ell\upsilon_{n+1}), d(\ell\upsilon_{n+1},\ell\upsilon_{n+2})\} \tag{3.43}$$

and

$$N_\ell(\upsilon_n,\xi_n,\upsilon_{n+1},\xi_{n+1}) = \max\{d(\ell\upsilon_n,\ell\upsilon_{n+1}), d(\ell\upsilon_{n+1},\ell\upsilon_{n+2})\}. \tag{3.44}$$

Therefore from (3.42), we have

$$\begin{aligned}\phi(s^k d(\ell\upsilon_{n+1},\ell\upsilon_{n+2})) &\leq \phi(\max\{d(\ell\upsilon_n,\ell\upsilon_{n+1}), d(\ell\upsilon_{n+1},\ell\upsilon_{n+2})\})\\ &\quad - \psi(\max\{d(\ell\upsilon_n,\ell\upsilon_{n+1}), d(\ell\upsilon_{n+1},\ell\upsilon_{n+2})\}).\end{aligned} \tag{3.45}$$

Similarly by taking $\upsilon=\xi_{n+1}, \xi=\upsilon_{n+1}, \rho=\upsilon_n, \tau=\upsilon_n$ in (3.41), we get

$$\begin{aligned}\phi(s^k d(\ell\xi_{n+1},\ell\xi_{n+2})) &\leq \phi(\max\{d(\ell\xi_n,\ell\xi_{n+1}), d(\ell\xi_{n+1},\ell\xi_{n+2})\})\\ &\quad - \psi(\max\{d(\ell\xi_n,\ell\xi_{n+1}), d(\ell\xi_{n+1},\ell\xi_{n+2})\}).\end{aligned} \tag{3.46}$$

From the fact that $\max\{\phi(c),\phi(d)\} = \phi\{\max\{c,d\}\}$ for all $c,d \in [0,+\infty)$. Then combining (3.45) and (3.46), we get

$$\begin{aligned}\phi(s^k\delta_n) &\leq \phi(\max\{d(\ell\upsilon_n,\ell\upsilon_{n+1}), d(\ell\upsilon_{n+1},\ell\upsilon_{n+2}), d(\ell\xi_n,\ell\xi_{n+1}), d(\ell\xi_{n+1},\ell\xi_{n+2})\})\\ &\quad - \psi(\max\{d(\ell\upsilon_n,\ell\upsilon_{n+1}), d(\ell\upsilon_{n+1},\ell\upsilon_{n+2}), d(\ell\xi_n,\ell\xi_{n+1}), d(\ell\xi_{n+1},\ell\xi_{n+2})\}),\end{aligned} \tag{3.47}$$

where

$$\delta_n = \max\{d(\ell\upsilon_{n+1},\ell\upsilon_{n+2}), d(\ell\xi_{n+1},\ell\xi_{n+2})\}. \tag{3.48}$$

Let us denote,

$$\Delta_n = \max\{d(\ell\upsilon_n,\ell\upsilon_{n+1}), d(\ell\upsilon_{n+1},\ell\upsilon_{n+2}), d(\ell\xi_n,\ell\xi_{n+1}), d(\ell\xi_{n+1},\ell\xi_{n+2})\}. \tag{3.49}$$

Hence from equations (3.45)-(3.48), we obtain

$$s^k \delta_n \leq \Delta_n. \tag{3.50}$$

Next, we prove that

$$\delta_n \leq \lambda \delta_{n-1}, \tag{3.51}$$

for all $n \geq 1$ and where $\lambda = \frac{1}{s^k} \in [0, 1)$.

Suppose that if $\Delta_n = \delta_n$ then from (3.50), we get $s^k \delta_n \leq \delta_n$ which leads to $\delta_n = 0$ as $s > 1$ and hence (3.51) holds. If $\Delta_n = \max\{d(\ell \upsilon_n, \ell \upsilon_{n+1}), d(\ell \xi_n, \ell \xi_{n+1})\}$, *i.e.*, $\Delta_n = \delta_{n-1}$ then (3.50) follows (3.51).

Now from (3.50), we obtain that $\delta_n \leq \lambda^n \delta_0$ and hence,

$$d(\ell \upsilon_{n+1}, \ell \upsilon_{n+2}) \leq \lambda^n \delta_0 \ \text{ and } \ d(\ell \xi_{n+1}, \ell \xi_{n+2}) \leq \lambda^n \delta_0. \tag{3.52}$$

Therefore from Lemma 3.1 of [32], the sequences $\{\ell \upsilon_n\}$ and $\{\ell \xi_n\}$ are Cauchy sequences in P. Hence, by following the remaining proof of Theorem 2.2 of [2], we can show that $\hbar$ and ℓ have a coincidence point in P. □

Corollary 3.9. *Let $(P, d, s, \preceq)$ be a complete partially ordered b-metric space with $s > 1$, and $\hbar : P \times P \to P$ be a continuous mapping such that $\hbar$ has a mixed monotone property. Suppose there exists $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\phi(s^k d(\hbar(\upsilon, \xi), \hbar(\rho, \tau))) \leq \phi(M_\ell(\upsilon, \xi, \rho, \tau)) - \psi(N_\ell(\upsilon, \xi, \rho, \tau)),$$

for all $\upsilon, \xi, \rho, \tau \in P$ with $\upsilon \preceq \rho$ and $\xi \succeq \tau$, $k > 2$ where

$$M_\ell(\upsilon, \xi, \rho, \tau) = \max\left\{ \frac{d(\rho, \hbar(\rho, \tau))\,[1 + d(\upsilon, \hbar(\upsilon, \xi))]}{1 + d(\upsilon, \rho)}, \frac{d(\upsilon, \hbar(\upsilon, \xi))\ d(\upsilon, \hbar(\rho, \tau))}{1 + d(\upsilon, \hbar(\rho, \tau)) + d(\rho, \hbar(\upsilon, \xi))}, d(\upsilon, \rho) \right\},$$

and

$$N_\ell(\upsilon, \xi, \rho, \tau) = \max\left\{ \frac{d(\rho, \hbar(\rho, \tau))\,[1 + d(\upsilon, \hbar(\upsilon, \xi))]}{1 + d(\upsilon, \rho)}, d(\upsilon, \rho) \right\}.$$

Then $\hbar$ has a coupled fixed point in P, if there exists $(\upsilon_0, \xi_0) \in P \times P$ such that $\upsilon_0 \preceq \hbar(\upsilon_0, \xi_0)$ and $\xi_0 \succeq \hbar(\xi_0, \upsilon_0)$.

Proof. Set $\ell = I_P$ in Theorem 3.8. □

Corollary 3.10. *Let $(P, d, s, \preceq)$ be a complete partially ordered b-metric space with $s > 1$, and $\hbar : P \times P \to P$ be a continuous mapping such that $\hbar$ has a mixed monotone property. Suppose* *there exists* $\psi \in \Psi$ *such that*

$$d(\hbar(\upsilon, \xi), \hbar(\rho, \tau)) \leq \frac{1}{s^k} M_{\ell}(\upsilon, \xi, \rho, \tau) - \frac{1}{s^k} \psi(N_{\ell}(\upsilon, \xi, \rho, \tau)),$$

for all $\upsilon, \xi, \rho, \tau \in P$ *with* $\upsilon \preceq \rho$ *and* $\xi \succeq \tau$, $k > 2$ *where*

$$M_{\ell}(\upsilon, \xi, \rho, \tau) = \max\left\{\frac{d(\rho, \hbar(\rho, \tau))\,[1 + d(\upsilon, \hbar(\upsilon, \xi))]}{1 + d(\upsilon, \rho)}, \frac{d(\upsilon, \hbar(\upsilon, \xi))\; d(\upsilon, \hbar(\rho, \tau))}{1 + d(\upsilon, \hbar(\rho, \tau)) + d(\rho, \hbar(\upsilon, \xi))}, d(\upsilon, \rho)\right\},$$

and

$$N_{\ell}(\upsilon, \xi, \rho, \tau) = \max\left\{\frac{d(\rho, \hbar(\rho, \tau))\,[1 + d(\upsilon, \hbar(\upsilon, \xi))]}{1 + d(\upsilon, \rho)}, d(\upsilon, \rho)\right\}.$$

If there exists $(\upsilon_0, \xi_0) \in P \times P$ *such that* $\upsilon_0 \preceq \hbar(\upsilon_0, \xi_0)$ *and* $\xi_0 \succeq \hbar(\xi_0, \upsilon_0)$, *then* $\hbar$ *has a coupled fixed point in* P.

Theorem 3.11. *In addition to Theorem 3.8, if for all* $(\upsilon, \xi), (r, s) \in P \times P$, *there exists* $(c^*, d^*) \in P \times P$ *such that* $(\hbar(c^*, d^*), \hbar(d^*, c^*))$ *is comparable to* $(\hbar(\upsilon, \xi), \hbar(\xi, \upsilon))$ *and to* $(\hbar(r, s), \hbar(s, r))$, *then* $\hbar$ *and* ℓ *have a unique coupled common fixed point in* $P \times P$.

Proof. From Theorem 3.8, we know that there exists at least one coupled coincidence point in P for $\hbar$ and ℓ. Assume that (υ, ξ) and (r, s) are two coupled coincidence points of $\hbar$ and ℓ, *i.e.*, $\hbar(\upsilon, \xi) = \ell\upsilon$, $\hbar(\xi, \upsilon,) = \ell\xi$ and $\hbar(r, s) = \ell r$, $\hbar(s, r) = \ell s$. Now, we have to prove that $\ell\upsilon = \ell r$ and $\ell\xi = \ell s$.

From the hypotheses, there exists $(c^*, d^*) \in P \times P$ such that $(\hbar(c^*, d^*), \hbar(d^*, c^*))$ is comparable to $(\hbar(\upsilon, \xi), \hbar(\xi, \upsilon))$ and to $(\hbar(r, s), \hbar(s, r))$. Suppose that

$$(\hbar(\upsilon, \xi), \hbar(\xi, \upsilon)) \leq (\hbar(c^*, d^*), \hbar(d^*, c^*)) \text{ and } (\hbar(r, s), \hbar(s, r)) \leq (\hbar(c^*, d^*), \hbar(d^*, c^*)).$$

Let $c_0^* = c^*$ and $d_0^* = d^*$ and then choose $(c_1^*, d_1^*) \in P \times P$ as

$$\ell c_1^* = \hbar(c_0^*, d_0^*),\ \ell d_1^* = \hbar(d_0^*, c_0^*) \quad (n \geq 1).$$

By repeating the same procedure above, we can obtain two sequences $\{\ell c_n^*\}$ and $\{\ell d_n^*\}$ in P such that

$$\ell c_{n+1}^* = \hbar(c_n^*, d_n^*),\ \ell d_{n+1}^* = \hbar(d_n^*, c_n^*) \quad (n \geq 0).$$

Similarly, define the sequences $\{\ell\upsilon_n\}$, $\{\ell\xi_n\}$ and $\{\ell r_n\}$, $\{\ell s_n\}$ as above in P by setting $\upsilon_0 = \upsilon$, $\xi_0 = \xi$ and $r_0 = r$, $s_0 = s$. Further, we have that

$$\ell\upsilon_n \to \hbar(\upsilon, \xi),\ \ell\xi_n \to \hbar(\xi, \upsilon),\ \ell r_n \to \hbar(r, s),\ \ell s_n \to \hbar(s, r) \quad (n \geq 1). \tag{3.53}$$

Since, $(\hbar(\upsilon, \xi), \hbar(\xi, \upsilon)) = (\ell\upsilon, \ell\xi) = (\ell\upsilon_1, \ell\xi_1)$ is comparable to $(\hbar(c^*, d^*), \hbar(d^*, c^*)) = (\ell c^*, \ell d^*) =$

$(\not{f} c_1^*, \not{f} d_1^*)$ and hence we get $(\not{f} \upsilon_1, \not{f} \xi_1) \leq (\not{f} c_1^*, \not{f} d_1^*)$. Thus, by induction we obtain that

$$(\not{f} \upsilon_n, \not{f} \xi_n) \leq (\not{f} c_n^*, \not{f} d_n^*) \quad (n \geq 0). \tag{3.54}$$

Therefore from (3.41), we have

$$\begin{aligned} \phi(d(\not{f} \upsilon, \not{f} c_{n+1}^*)) \leq \phi(s^k d(\not{f} \upsilon, \not{f} c_{n+1}^*)) &= \phi(s^k d(\hbar(\upsilon, \xi), \hbar(c_n^*, d_n^*))) \\ &\leq \phi(M_{\not{f}}(\upsilon, \xi, c_n^*, d_n^*)) - \psi(N_{\not{f}}(\upsilon, \xi, c_n^*, d_n^*)), \end{aligned} \tag{3.55}$$

where

$$\begin{aligned} M_{\not{f}}(\upsilon, \xi, c_n^*, d_n^*) = \max \bigg\{ & \frac{d(\not{f} c_n^*, \hbar(c_n^*, d_n^*)) \, [1 + d(\not{f} \upsilon, \hbar(\upsilon, \xi))]}{1 + d(\not{f} \upsilon, \not{f} c_n^*)} , \\ & \frac{d(\not{f} \upsilon, \hbar(\upsilon, \xi)) \; d(\not{f} \upsilon, \hbar(c_n^*, d_n^*))}{1 + d(\not{f} \upsilon, \hbar(c_n^*, d_n^*)) + d(\not{f} c_n^*, \hbar(\upsilon, \xi))}, d(\not{f} \upsilon, \not{f} c_n^*) \bigg\} \\ = {} & \max\{0, 0, d(\not{f} \upsilon, \not{f} c_n^*)\} \\ = {} & d(\not{f} \upsilon, \not{f} c_n^*) \end{aligned}$$

and

$$\begin{aligned} N_{\not{f}}(\upsilon, \xi, c_n^*, d_n^*) &= \max \left\{ \frac{d(\not{f} c_n^*, \hbar(c_n^*, d_n^*)) \, [1 + d(\hbar \upsilon, \hbar(\upsilon, \xi))]}{1 + d(\not{f} \upsilon, \not{f} c_n^*)}, d(\not{f} \upsilon, \not{f} c_n^*) \right\} \\ &= d(\not{f} \upsilon, \not{f} c_n^*). \end{aligned}$$

Thus from (3.55),

$$\phi(d(\not{f} \upsilon, \not{f} c_{n+1}^*)) \leq \phi(d(\not{f} \upsilon, \not{f} c_n^*)) - \psi(d(\not{f} \upsilon, \not{f} c_n^*)). \tag{3.56}$$

As by the similar process, we can prove that

$$\phi(d(\not{f} \xi, \not{f} d_{n+1}^*)) \leq \phi(d(\not{f} \xi, \not{f} d_n^*)) - \psi(d(\not{f} \xi, \not{f} d_n^*)). \tag{3.57}$$

From (3.56) and (3.57), we have

$$\begin{aligned} \phi(\max\{d(\not{f} \upsilon, \not{f} c_{n+1}^*), d(\not{f} \xi, \not{f} d_{n+1}^*)\}) &\leq \phi(\max\{d(\not{f} \upsilon, \not{f} c_n^*), d(\not{f} \xi, \not{f} d_n^*)\}) \\ &\quad - \psi(\max\{d(\not{f} \upsilon, \not{f} c_n^*), d(\not{f} \xi, \not{f} d_n^*)\}) \\ &< \phi(\max\{d(\not{f} \upsilon, \not{f} c_n^*), d(\not{f} \xi, \not{f} d_n^*)\}). \end{aligned} \tag{3.58}$$

Hence by the property of ϕ, we get

$$\max\{d(\not{f} \upsilon, \not{f} c_{n+1}^*), d(\not{f} \xi, \not{f} d_{n+1}^*)\} < \max\{d(\not{f} \upsilon, \not{f} c_n^*), d(\not{f} \xi, \not{f} d_n^*)\},$$

which shows that $\max\{d(\not{f} \upsilon, \not{f} c_n^*), d(\not{f} \xi, \not{f} d_n^*)\}$ is a decreasing sequence and by a result there exists

$\gamma \geq 0$ such that

$$\lim_{n\to+\infty} \max\{d(\ell \upsilon, \ell c_n^*), d(\ell \xi, \ell d_n^*)\} = \gamma.$$

From (3.58) taking upper limit as $n \to +\infty$, we get

$$\phi(\gamma) \leq \phi(\gamma) - \psi(\gamma), \tag{3.59}$$

from which we get $\psi(\gamma) = 0$, implies that $\gamma = 0$. Thus,

$$\lim_{n\to+\infty} \max\{d(\ell \upsilon, \ell c_n^*), d(\ell \xi, \ell d_n^*)\} = 0.$$

Consequently, we get

$$\lim_{n\to+\infty} d(\ell \upsilon, \ell c_n^*) = 0 \quad \text{and} \quad \lim_{n\to+\infty} d(\ell \xi, \ell d_n^*) = 0. \tag{3.60}$$

By similar argument, we get

$$\lim_{n\to+\infty} d(\ell r, \ell c_n^*) = 0 \quad \text{and} \quad \lim_{n\to+\infty} d(\ell s, \ell d_n^*) = 0. \tag{3.61}$$

Therefore from (3.60) and (3.61), we get $\ell \upsilon = \ell r$ and $\ell \xi = \ell s$. Since $\ell \upsilon = \hbar(\upsilon, \xi)$ and $\ell \xi = \hbar(\xi, \upsilon)$, then by the commutativity of $\hbar$ and ℓ, we have

$$\ell(\ell \upsilon) = \ell(\hbar(\upsilon,\xi)) = \hbar(\ell \upsilon, \ell \xi) \quad \text{and} \quad \ell(\ell \xi) = \ell(\hbar(\xi,\upsilon)) = \hbar(\ell \xi, \ell \upsilon). \tag{3.62}$$

Let $\ell \upsilon = a^*$ and $\ell \xi = b^*$ then (3.62) becomes

$$\ell(a^*) = \hbar(a^*, b^*) \quad \text{and} \quad \ell(b^*) = \hbar(b^*, a^*), \tag{3.63}$$

which shows that (a^*, b^*) is a coupled coincidence point of $\hbar$ and ℓ. It follows that $\ell(a^*) = \ell r$ and $\ell(b^*) = \ell s$ that is $\ell(a^*) = a^*$ and $\ell(b^*) = b^*$. Thus from (3.63), we get $a^* = \ell(a^*) = \hbar(a^*, b^*)$ and $b^* = \ell(b^*) = \hbar(b^*, a^*)$. Therefore, (a^*, b^*) is a coupled common fixed point of $\hbar$ and ℓ.

For the uniqueness, let (u^*, v^*) be another coupled common fixed point of $\hbar$ and ℓ, then we have $u^* = \ell u^* = \hbar(u^*, v^*)$ and $v^* = \ell v^* = \hbar(v^*, u^*)$. Since (u^*, v^*) is a coupled common fixed point of $\hbar$ and ℓ, then we get $\ell u^* = \ell \upsilon = a^*$ and $\ell v^* = \ell \xi = b^*$. Thus, $u^* = \ell u^* = \ell a^* = a^*$ and $v^* = \ell v^* = \ell b^* = b^*$. Hence the result. □

Theorem 3.12. *In addition to the hypotheses of Theorem 3.11, if $\ell \upsilon_0$ and $\ell \xi_0$ are comparable, then $\hbar$ and ℓ have a unique common fixed point in P.*

Proof. From Theorem 3.11, $\hbar$ and ℓ have a unique coupled common fixed point $(\upsilon, \xi) \in P$. Now, it is enough to prove that $\upsilon = \xi$. From the hypotheses, we have $\ell\upsilon_0$ and $\ell\xi_0$ are comparable then we assume that $\ell\upsilon_0 \preceq \ell\xi_0$. Hence by induction we get $\ell\upsilon_n \preceq \ell\xi_n$ for all $n \geq 0$, where $\{\ell\upsilon_n\}$ and $\{\ell\xi_n\}$ are from Theorem 3.8.

Now by use of Lemma 2.6, we get

$$\begin{aligned}\phi(s^{k-2}d(\upsilon,\xi)) = \phi\left(s^k\frac{1}{s^2}d(\upsilon,\xi)\right) &\leq \lim_{n\to+\infty}\sup\phi(s^kd(\upsilon_{n+1},\xi_{n+1}))\\ &= \lim_{n\to+\infty}\sup\phi(s^kd(\hbar(\upsilon_n,\xi_n),\hbar(\xi_n,\upsilon_n)))\\ &\leq \lim_{n\to+\infty}\sup\phi(M_\ell(\upsilon_n,\xi_n,\xi_n,\upsilon_n)) - \lim_{n\to+\infty}\inf\psi(N_\ell(\upsilon_n,\xi_n,\xi_n,\upsilon_n))\\ &\leq \phi(d(\upsilon,\xi)) - \lim_{n\to+\infty}\inf\psi(N_\ell(\upsilon_n,\xi_n,\xi_n,\upsilon_n))\\ &< \phi(d(\upsilon,\xi)),\end{aligned}$$

which is a contradiction. Thus, $\upsilon = \xi$, *i.e.*, $\hbar$ and ℓ have a common fixed point in P. □

Remark 3.13. *It is well known that b-metric space is a metric space when $s = 1$. So, from the result of Jachymski [31], the condition*

$$\phi(d(\hbar(\upsilon,\xi),\hbar(\rho,\tau))) \leq \phi(\max\{d(\ell\upsilon,\ell\rho),d(\ell\xi,\ell\tau)\}) - \psi(\max\{d(\ell\upsilon,\ell\rho),d(\ell\xi,\ell\tau)\})$$

is equivalent to,

$$d(\hbar(\upsilon,\xi),\hbar(\rho,\tau)) \leq \varphi(\max\{d(\ell\upsilon,\ell\rho),d(\ell\xi,\ell\tau)\}),$$

where $\phi \in \Phi$, $\psi \in \Psi$ and $\varphi : [0,+\infty) \to [0,+\infty)$ is continuous, $\varphi(t) < t$ for all $t > 0$ and $\varphi(t) = 0$ if and only if $t = 0$. So, in view of above our results generalize and extend the results of [15, 23, 25, 31, 37, 38] and several other comparable results.

Corollary 3.14. *Suppose $(P, d, s, \preceq)$ be a complete partially ordered b-metric space with parameter $s > 1$. Let $g : P \to P$ be a continuous, non-decreasing mapping with regards to $\preceq$ such that there exists $\upsilon_0 \in P$ with $\upsilon_0 \preceq g\upsilon_0$. Suppose that*

$$\phi(sd(g\upsilon,g\xi)) \leq \phi(M(\upsilon,\xi)) - \psi(M(\upsilon,\xi)), \tag{3.64}$$

where $M(\upsilon,\xi)$ and the conditions upon ϕ, ψ are same as in Theorem 3.1. Then g has a fixed point in P.

Proof. Set $N(\upsilon,\xi) = M(\upsilon,\xi)$ in a contraction condition (3.3) and apply Theorem 3.1, we have the required proof. □

Note 1. *Similarly by removing the continuity of a non-decreasing mapping g and taking a non-decreasing sequence $\{\upsilon_n\}$ as above in Theorem 3.2, we can obtain a fixed point for g in P. Also one can obtain the uniqueness of a fixed point of g by using condition (3.17) in P as by following the proof of Theorem 3.3.*

Note 2. *By following the proofs of Theorems 3.5 - 3.6, we can find the coincidence point for the mappings g and f in P. Similarly, from Theorem 3.8, Theorem 3.11 and Theorem 3.12, one can obtain a coupled coincidence point and its uniqueness, and a unique common fixed point for the mappings $\hbar$ and f in $P \times P$ and on P satisfying an almost generalized contraction condition (3.64), where $M(\upsilon, \xi)$, $M_f(\upsilon, \xi)$, $M_f(\upsilon, \xi, \rho, \tau)$ and the conditions upon ϕ, ψ are same as above defined.*

Corollary 3.15. *Suppose that $(P, d, s, \preceq)$ be a complete partially ordered b-metric space with $s > 1$. Let $g : P \to P$ be a continuous, non-decreasing mapping with regards to $\preceq$. If there exists $k \in [0, 1)$ and for any $\upsilon, \xi \in P$ with $\upsilon \preceq \xi$ such that*

$$d(g\upsilon, g\xi) \leq \frac{k}{s} \max\left\{\frac{d(\xi, g\xi)\,[1 + d(\upsilon, g\upsilon)]}{1 + d(\upsilon, \xi)}, \frac{d(\upsilon, g\upsilon)\ d(\upsilon, g\xi)}{1 + d(\upsilon, g\xi) + d(\xi, g\upsilon)}, d(\upsilon, \xi)\right\}. \tag{3.65}$$

If there exists $\upsilon_0 \in P$ with $\upsilon_0 \preceq g\upsilon_0$, then g has a fixed point in P.

Proof. Set $\phi(t) = t$ and $\psi(t) = (1 - k)t$, for all $t \in (0, +\infty)$ in Corollary 3.14. □

Note 3. *Relaxing the continuity of a map g in Corollary 3.15, one can obtains a fixed point for g on taking a non-decreasing sequence $\{\upsilon_n\}$ in P by following the proof of Theorem 3.2.*

Example 3.16. *Define a metric $d : P \times P \to P$ as below and $\leq$ is an usual order on P, where $P = \{1, 2, 3, 4, 5, 6\}$*

$$\begin{aligned}
&d(\upsilon, \xi) = d(\xi, \upsilon) = 0, \ \textit{if } \upsilon, \xi = 1, 2, 3, 4, 5, 6 \textit{ and } \upsilon = \xi,\\
&d(\upsilon, \xi) = d(\xi, \upsilon) = 3, \ \textit{if } \upsilon, \xi = 1, 2, 3, 4, 5 \textit{ and } \upsilon \neq \xi,\\
&d(\upsilon, \xi) = d(\xi, \upsilon) = 12, \ \textit{if } \upsilon = 1, 2, 3, 4 \textit{ and } \xi = 6,\\
&d(\upsilon, \xi) = d(\xi, \upsilon) = 20, \ \textit{if } \upsilon = 5 \textit{ and } \xi = 6.
\end{aligned}$$

Define a map $g : P \to P$ by $g1 = g2 = g3 = g4 = g5 = 1, g6 = 2$ and let $\phi(t) = \frac{t}{2}$, $\psi(t) = \frac{t}{4}$ for $t \in [0, +\infty)$. Then g has a fixed point in P.

Proof. It is apparent that, $(P, d, s, \preceq)$ is a complete partially ordered b-metric space for $s = 2$. Consider the possible cases for υ, ξ in P:

Case 1 Suppose $\upsilon, \xi \in \{1, 2, 3, 4, 5\}$, $\upsilon < \xi$ then $d(g\upsilon, g\xi) = d(1, 1) = 0$. Hence,

$$\phi(2d(g\upsilon, g\xi)) = 0 \leq \phi(M(\upsilon, \xi)) - \psi(M(\upsilon, \xi)).$$

Case 2 Suppose that $\upsilon \in \{1,2,3,4,5\}$ and $\xi = 6$, then $d(g\upsilon, g\xi) = d(1,2) = 3$, $M(6,5) = 20$ and $M(\upsilon, 6) = 12$, for $\upsilon \in \{1,2,3,4\}$. Therefore, we have the following inequality,

$$\phi(2d(g\upsilon, g\xi)) \le \frac{M(\upsilon,\xi)}{4} = \phi(M(\upsilon,\xi)) - \psi(M(\upsilon,\xi)).$$

Thus, condition (3.64) of Corollary 3.14 holds. Furthermore, the remaining assumptions in Corollary 3.14 are fulfilled. Hence, g has a fixed point in P as Corollary 3.14 is appropriate to g, ϕ, ψ and $(P, d, s, \preceq)$. □

Example 3.17. *A metric $d : P \times P \to P$, where $P = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \frac{1}{n}, \dots\}$ with usual order $\le$ is defined as follows*

$$d(\upsilon,\xi) = \begin{cases} 0, & \text{if } \upsilon = \xi \\ 1, & \text{if } \upsilon \ne \xi \in \{0,1\} \\ |\upsilon - \xi|, & \text{if } \upsilon, \xi \in \left\{0, \frac{1}{2n}, \frac{1}{2m} : n \ne m \ge 1\right\} \\ 3, & \text{otherwise.} \end{cases}$$

A map $g : P \to P$ be such that $g0 = 0, g\frac{1}{n} = \frac{1}{12n}$ for all $n \ge 1$ and let $\phi(t) = t$, $\psi(t) = \frac{4t}{5}$ for $t \in [0, +\infty)$. Then, g has a fixed point in P.

Proof. It is obvious that for $s = \frac{12}{5}$, $(P, d, s, \preceq)$ is a complete partially ordered b-metric space and also by definition, d is discontinuous b-metric space. Now for $\upsilon, \xi \in P$ with $\upsilon < \xi$, we have the following cases:

Case 1 If $\upsilon = 0$ and $\xi = \frac{1}{n}$, $n \ge 1$, then $d(g\upsilon, g\xi) = d(0, \frac{1}{12n}) = \frac{1}{12n}$ and $M(\upsilon,\xi) = \frac{1}{n}$ or $M(\upsilon,\xi) = \{1, 3\}$. Therefore, we have

$$\phi\left(\frac{12}{5} d(g\upsilon, g\xi)\right) \le \frac{M(\upsilon,\xi)}{5} = \phi(M(\upsilon,\xi)) - \psi(M(\upsilon,\xi)).$$

Case 2 If $\upsilon = \frac{1}{m}$ and $\xi = \frac{1}{n}$ with $m > n \ge 1$, then

$$d(g\upsilon, g\xi) = d\left(\frac{1}{12m}, \frac{1}{12n}\right) \text{ and } M(\upsilon,\xi) \ge \frac{1}{n} - \frac{1}{m} \text{ or } M(\upsilon,\xi) = 3.$$

Therefore,

$$\phi\left(\frac{12}{5} d(g\upsilon, g\xi)\right) \le \frac{M(\upsilon,\xi)}{5} = \phi(M(\upsilon,\xi)) - \psi(M(\upsilon,\xi)).$$

Hence, condition (3.64) of Corollary 3.14 and remaining assumptions are satisfied. Thus, g has a fixed point in P. □

Example 3.18. *Let $P = C[a, b]$ be the set of all continuous functions. Let us define a b-metric d on P by*

$$d(\theta_1, \theta_2) = \sup_{t \in C[a,b]} \{|\theta_1(t) - \theta_2(t)|^2\}$$

for all $\theta_1, \theta_2 \in P$ with partial order $\preceq$ defined by $\theta_1 \preceq \theta_2$ if $a \leq \theta_1(t) \leq \theta_2(t) \leq b$, for all $t \in [a, b]$, $0 \leq a < b$. Let $\mathcal{g} : P \to P$ be a mapping defined by $\mathcal{g}\theta = \frac{\theta}{5}, \theta \in P$ and the two altering distance functions by $\phi(t) = t$, $\psi(t) = \frac{t}{3}$, for any $t \in [0, +\infty]$. Then $\mathcal{g}$ has a unique fixed point in P.

Proof. From the hypotheses, it is clear that $(P, d, s, \preceq)$ is a complete partially ordered b-metric space with parameter $s = 2$ and fulfill all the conditions of Corollary 3.14 and Note 1. Furthermore for any $\theta_1, \theta_2 \in P$, the function $\min(\theta_1, \theta_2)(t) = \min\{\theta_1(t), \theta_2(t)\}$ is also continuous and the conditions of Corollary 3.14 and Note 1 are satisfied. Hence, $\mathcal{g}$ has a unique fixed point $\theta = 0$ in P. $\square$

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References

[1] M. Abbas, B. Ali, B. Bin-Mohsin, N. Dedović, T. Nazir and S. Radenović, "Solutions and Ulam-Hyers stability of differential inclusions involving Suzuki type multivalued mappings on b-metric spaces", Vojnotehnički Glasnik/Military Technical Courier, vol. 68, no. 3, pp. 438–487, 2020.

[2] A. Aghajani and R. Arab, "Fixed points of (ψ, ϕ, θ)-contractive mappings in partially ordered b-metric spaces and applications to quadratic integral equations", Fixed Point Theory Appl., Art. ID 245, 20 pages, 2013.

[3] A. Aghajani, M. Abbas and J. R. Roshan, "Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces", Math. Slovaca, vol. 64, no. 4, pp. 941–960, 2014.

[4] M. Akkouchi, "Common fixed point theorems for two selfmappings of a b-metric space under an implicit relation", Hacet. J. Math. Stat., vol. 40, no. 6, pp. 805–810, 2011.

[5] S. Aleksić, Z. Mitrović and S. Radenović, "On some recent fixed point results for single and multi-valued mappings in b-metric spaces", Fasc. Math., no. 61, pp. 5–16, 2018.

[6] S. Aleksić, H. Huang, Z. D. Mitrović and S. Radenović, "Remarks on some fixed point results in b-metric spaces", J. Fixed Point Theory Appl., vol. 20, no. 4, Paper No. 147, 17 pages, 2018.

[7] S. Aleksić, Z. D. Mitrović and S. Radenović, "Picard sequences in b-metric spaces", Fixed Point Theory, vol. 21, no. 1, pp. 35–45, 2020.

[8] R. Allahyari, R. Arab and A. Shole Haghighi, "A generalization on weak contractions in partially ordered b-metric spaces and its application to quadratic integral equations", J. Inequal. Appl., Art. ID 355, 15 pages, 2014.

[9] A. Amini-Harandi, "Fixed point theory for quasi-contraction maps in b-metric spaces", Fixed Point Theory, vol. 15, no. 2, pp. 351–358, 2014.

[10] H. Aydi, M.-F. Bota, E. Karapinar and S. Moradi, "A common fixed point for weak ϕ-contractions on b-metric spaces", Fixed Point Theory, vol. 13, no. 2, pp. 337–346, 2012.

[11] H. Aydi, A. Felhi and S. Sahmim, "On common fixed points for (α, ψ)-contractions and generalized cyclic contractions in b-metric-like spaces and consequences", J. Nonlinear Sci. Appl., vol. 9, no. 5, pp. 2492–2510, 2016.

[12] I. A. Bakhtin, "The contracting mapping principle in an almost metric space", (Russian) Funkts. Anal., No. 30, pp. 26–37, 1989.

[13] V. Berinde and M. Păcurar, “The early developments in fixed point theory on b-metric spaces: a brief survey and some important related aspects”, Carpathian J. Math., vol. 38, no. 3, pp. 523–538, 2022.

[14] N. Bourbaki, *Éléments de mathématique. Topologie générale.* Chapitres 5 à 10, Paris: Hermann, 1974.

[15] B. S. Choudhury, N. Metiya and A. Kundu, “Coupled coincidence point theorems in ordered metric spaces”, Ann. Univ. Ferrara Sez. VII Sci. Mat., vol. 57, no. 1, pp. 1–16, 2011.

[16] S. Czerwik, “Contraction mappings in b-metric spaces‘”, Acta Math. Inform. Univ. Ostraviensis, vol. 1, pp. 5–11, 1993.

[17] S. Czerwik, “Nonlinear set-valued contraction mappings in b-metric spaces”. Atti Sem. Mat. Fis. Univ. Modena, vol. 46, no. 2, pp. 263–276, 1998.

[18] L. J. Ćirić, N. Cakić, M. Rajović and J. S. Ume, “Monotone generalized nonlinear contractions in partially ordered metric spaces”, Fixed Point Theory Appl., Art. ID 131294, 11 pages, 2008.

[19] L. J. Ćirić, “Some recent results in metrical fixed point theory; Fixed point theorems for mixed monotone operators and applications to integral equations”, Belgrade: University of Belgrade, 2003.

[20] P. Debnath, Z. D. Mitrović and S. Radenović, “Interpolative Hardy-Rogers and Reich-Rus-Ćirić type contractions in b-metric and rectangular b-metric spaces”, Mat. Vesnik, vol. 72, no. 4, pp. 368–374, 2020.

[21] D. Đorić, “Common fixed point for generalized (ψ, ϕ)-weak contractions”, Appl. Math. Lett., vol. 22, no. 12, pp. 1896–1900, 2009.

[22] H. Faraji, D. Savić, S. Dragana and S. N. Radenović, “Fixed point theorems for Geraghty contraction type mappings in b-metric spaces and applications”, Axioms, vol. 8, no. 1, Article ID 34, 12 pages, 2019.

[23] T. Gnana Bhaskar and V. Lakshmikantham, “Fixed point theorems in partially ordered metric spaces and applications”, Nonlinear Anal., vol. 65, no. 7, pp. 1379–1393, 2006.

[24] E. Graily, S. M. Vaezpour, R. Saadati and Y. J. Cho, “Generalization of fixed point theorems in ordered metric spaces concerning generalized distance”, Fixed Point Theory Appl., 30, 8 pages, 2011.

[25] J. Harjani, B. López and K. Sadarangani, “Fixed point theorems for mixed monotone operators and applications to integral equations”, Nonlinear Anal., vol. 74, no. 5, pp. 1749–1760, 2011.

[26] N. T. Hieu and N. V. Dung, "Some fixed point results for generalized rational type contraction mappings in partially ordered b-metric spaces", Facta Univ. Ser. Math. Inform., vol. 30, no. 1, pp. 49–66, 2015.

[27] H. Huang, S. Radenović and J. Vujaković, "On some recent coincidence and immediate consequences in partially ordered b-metric spaces", Fixed Point Theory and Appl., 2015:63, 18 pages, 2015

[28] H. Huang and S. Xu, "Fixed point theorems of contractive mappings in cone b-metric spaces and applications", Fixed Point Theory Appl., 2013:112, 10 pages, 2013.

[29] N. Hussain, D. Đorić, Z. Kadelburg and S. Radenović, "Suzuki-type fixed point results in metric type spaces", Fixed Point Theory Appl., 2012:126, 12 pages, 2012.

[30] N. Hussain, Z. D. Mitrović and S. Radenović, "A common fixed point theorem of Fisher in b-metric spaces", Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM., vol. 113, no. 2, pp. 949–956, 2019.

[31] J. Jachymski, "Equivalent conditions for generalized contractions on (ordered) metric spaces", Nonlinear Anal., vol. 74, no. 3, pp. 768–774, 2011.

[32] M. Jovanović, Z. Kadelburg and S. Radenović, "Common fixed point results in metric-type spaces", Fixed Point Theory Appl., Article ID 978121, 15 pages, 2010.

[33] K. Kalyani, N. Seshagiri Rao and L. N. Mishra, "Coupled fixed points theorems for generalized weak contractions in ordered b-metric spaces", Asian-Eur. J. Math., vol. 15, no. 3, Paper No. 2250050, 22 pages, 2022.

[34] E. Karapınar, Z. D. Mitrović, A. Öztürk and S. Radenović, "On a theorem of Ćirić in b-metric spaces", Rend. Circ. Mat. Palermo (2), vol. 70, no. 1, pp. 217–225, 2021.

[35] M. A. Khamsi and N. Hussain, "KKM mappings in metric type spaces", Nonlinear Anal., vol. 73, no. 9, pp. 3123–3129, 2010.

[36] W. A. Kirk and N. Shahzad, *Fixed point theory in distance spaces*, Cham: Springer, 2014.

[37] V. Lakshmikantham and L. Ćirić, "Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces", Nonlinear Anal., vol. 70, no. 12, pp. 4341–4349, 2009.

[38] N. V. Luong and N. X. Thuan, "Coupled fixed point theorems in partially ordered metric spaces", Bull. Math. Anal. Appl., vol. 2, no. 4, pp. 16–24, 2010.

[39] B. Mitiku, K. Kalyani and N. Seshagiri Rao, "Some fixed point results of generalized (ϕ, ψ)-contractive mappings in ordered b-metric spaces", BMC Research Notes, vol. 13, Article No. 537, 2020.

[40] Z. D. Mitrović, S. Radenović, F. Vetro and J. Vujaković, "Some remark on TAC-contractive mappings in b-metric spaces", Mat. Vesnik, vol. 70, no. 2, pp. 167–175, 2018.

[41] N. Mlaiki, N. Dedovic, H. Aydi, M. Gardašević -Filipović, B. Bin-Mohsin and S. Radenović, "Some new observations on Geraghty and Ćirić type results in b-metric spaces", Mathematics, vol. 7, Article ID 643, 11 pages, 2019.

[42] J. J. Nieto and R. Rodríguez-López, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations", Order, vol. 22, no. 3, pp. 223–239, 2005.

[43] M. V. Pavlović and S. N. Radenović, "A note on the Meir-Keeler theorem in the context of b-metric spaces", Vojnotehnički Glasnik/Military Technical Courier, vol. 67, no. 1, pp. 1–12, 2019.

[44] O. Popescu, "Fixed points for (ψ, ϕ)-weak contractions", Appl. Math. Lett., vol. 24, no. 1, pp. 1–4, 2011.

[45] A. C. M. Ran and M. C. B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations", Proc. Amer. Math. Soc., vol. 132, no. 5, pp. 1435–1443, 2004.

[46] J. R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei and W. Shatanawi, "Common fixed points of almost generalized $(\psi, \varphi)_s$-contractive mappings in ordered b-metric spaces", Fixed Point Theory Appl., 2013:159, 23 pages, 2013.

[47] J. R. Roshan, V. Parvaneh and I. Altun, "Some coincidence point results in ordered b-metric spaces and applications in a system of integral equations", Appl. Math. Comput., vol. 226, pp. 725–737, 2014.

[48] B. Samet and H. Yazidi, "Coupled fixed point theorems in partially ordered ϵ-chainable metric spaces", J. Math. Comput. Sci., vol. 1, no. 3, pp. 142–151, 2010.

[49] N. Seshagiri Rao and K. Kalyani, "Unique fixed point theorems in partially ordered metric spaces", Heliyon, vol. 6, e05563, 7 pages, 2020.

[50] N. Seshagiri Rao and K. Kalyani, "Generalized contractions to coupled fixed point theorems in partially ordered metric spaces", Zh. Sib. Fed. Univ. Mat. Fiz., vol. 13, no. 4, pp. 492–502, 2020.

[51] N. Seshagiri Rao and K. Kalyani, "Coupled fixed point theorems with rational expressions in partially ordered metric spaces", J. Anal., vol. 28, no. 4, pp. 1085–1095, 2020.

[52] N. Seshagiri Rao, K. Kalyani and K. Khatri, "Contractive mapping theorems in partially ordered metric spaces", CUBO, vol. 22, no. 2, pp. 203–214, 2020.

[53] N. Seshagiri Rao, K. Kalyani and B. Mitiku, “Fixed point theorems for nonlinear contractive mappings in ordered b-metric space with auxiliary function”, BMC Research Notes, vol. 13, Paper No. 451, 2020.

[54] N. Seshagiri Rao, K. Kalyani and B. Mitiku, “Fixed point results of almost generalized $(\phi, \psi, \theta)_s$-contractive mappings in ordered b-metric spaces”, Afr. Mat., vol. 33, no. 2, Paper No. 64, 19 pages, 2022.

[55] I. M. Vul’pe, D. Ostraĭh and F. Hoĭman, “The topological structure of a quasimetric space”, (Russian) in *Investigations in functional analysis and differential equations*, Math. Sci., Interuniv. Work Collect., Shtiintsa, pp. 14–19, 1981.