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# Ideal based graph structures for commutative rings

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**ABSTRACT**

We introduce a graph structure $\Gamma_2^*(R)$ for commutative rings with unity. We study some of the properties of the graph $\Gamma_2^*(R)$. Also we study some parameters of $\Gamma_2^*(R)$ and find rings for which $\Gamma_2^*(R)$ is split.

**RESUMEN**

Introducimos una estructura de grafo $\Gamma_2^*(R)$ para anillos conmutativos con unidad. Estudiamos algunas de las propiedades del grafo $\Gamma_2^*(R)$. También estudiamos algunos parámetros de $\Gamma_2^*(R)$ y encontramos anillos para los cuales $\Gamma_2^*(R)$ se escinde.

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# 1 Introduction

The idea of relating a commutative ring to a graph was introduced by Istvan Beck [3]. He introduced a graph, $\Gamma(R)$, whose vertices are the elements of $R$ and two distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$. In [1], Anderson and Livingston modified the definition of Beck to introduce the *zero-divisor graph*, $\Gamma^*(R)$, and investigated many of its properties. $\Gamma^*(R)$ is the subgraph of $\Gamma(R)$ induced by the set of non-zero zero-divisors of $R$. Cherian Thomas introduced many graph structures for $R$ in [10] and obtained many interesting results.

Throughout the paper, the word 'ring' shall mean a commutative ring with $1 \neq 0$ which is not a field. We denote the Jacobson radical of a ring $R$ by $\mathfrak{J}(R)$ and the set of all maximal ideals by $\max R$.

For the basic concepts from graph theory refer [4, 9]; for commutative ring theory, see [2].

We give two ideal based graphs, $\Gamma_1(R)$ and $\Gamma_2(R)$, introduced in [10].

The graph $\Gamma_1(R)$ has all ideals of $R$ as vertices and two distinct vertices $\mathfrak{a}$ and $\mathfrak{b}$ are adjacent if and only if $\mathfrak{a}\mathfrak{b} = 0$.

The graph $\Gamma_2(R)$ has the same vertex set as that of $\Gamma_1(R)$ and two distinct vertices $\mathfrak{a}$ and $\mathfrak{b}$ are adjacent if and only if $\mathfrak{a} + \mathfrak{b} = R$.

In [5], the authors have studied the subgraph $\Gamma_1^*(R)$ of $\Gamma_1(R)$ induced by all the non-zero proper ideals of $R$.

We state the following result:

**Theorem 1.1** ([5]). *Let $R$ be an Artin ring. $\Gamma_1^*(R)$ is complete if and only if one of the following holds:*

*(i)* $R \cong F_1 \oplus F_2$ *where* $F_1$ *and* $F_2$ *are fields.*

*(ii)* $R$ *is local with maximal ideal* $\mathfrak{m}$ *having index of nilpotency* 2.

*(iii)* $R$ *is local with principal maximal ideal* $\mathfrak{m}$ *having index of nilpotency* 3.

In [8], S. C. Mathew has introduced and studied some basic properties of $\Gamma_2^*(R)$ which is the subgraph of $\Gamma_2(R)$ induced by the set of all non-zero proper ideals of $R$. In this paper we include those results, for the sake of completeness. We compare the graphs $\Gamma_1^*(R)$ and $\Gamma_2^*(R)$ and find the clique number and domination number of $\Gamma_2^*(R)$. Also we investigate the properties of rings for which $\Gamma_2^*(R)$ is split.

## 2 The graph $\Gamma_2^*(R)$ and its properties

In this section we define the graph $\Gamma_2^*(R)$ and investigate some properties of the graph.

**Definition 2.1.** Let $R$ be a ring. We associate a graph $\Gamma_2^*(R)$ to $R$ whose vertex set is the set of all non-zero proper ideals of $R$ and for distinct ideals $\mathfrak{a}$ and $\mathfrak{b}$, the corresponding vertices are adjacent if and only if $\mathfrak{a} + \mathfrak{b} = R$.

**Remark 2.2.** $\Gamma_2^*(R)$ is totally disconnected if and only if $R$ is local.

**Remark 2.3.** $\Gamma_2^*(R) = K_1$ if and only if $(R, \mathfrak{m})$ is local with $\mathfrak{m}$ principal and $\mathfrak{m}^2 = 0$.

**Theorem 2.4.** *Let $R$ be a non-local ring. Then $\Gamma_2^*(R)$ is connected if and only if $\mathfrak{J}(R) = 0$.*

*Proof.* **(⇒):** Assume $\Gamma_2^*(R)$ is connected. If $\mathfrak{J}(R) \neq 0$, then $\mathfrak{J}(R)$ is an isolated vertex in $\Gamma_2^*(R)$.

**(⇐):** Assume that $\mathfrak{J}(R) = 0$. Now, $\max R$ induces a complete subgraph in $\Gamma_2^*(R)$. Let $\mathfrak{a}$ be any proper non-zero non maximal ideal. Since $\mathfrak{J}(R) = 0$, there exists a maximal ideal $\mathfrak{m}$ such that $\mathfrak{a} \nsubseteq \mathfrak{m}$. Thus $\mathfrak{a}$ is adjacent to $\mathfrak{m}$ and hence $\Gamma_2^*(R)$ is connected. □

**Corollary 2.5.** *If $\Gamma_2^*(R)$ is connected,* $\operatorname{diam} \Gamma_2^*(R) \leq 3$.

**Remark 2.6.** $\mathfrak{a}$ is an isolated vertex of $\Gamma_2^*(R)$ if and only if $\mathfrak{a} \subseteq \mathfrak{J}(R)$.

Next result follows from the proof of Theorem 2.4 and Remark 2.6.

**Theorem 2.7.** *$\Gamma_2^*(R)$ is connected except for isolated vertices. That is, $\Gamma_2^*(R)$ has at most one component different from $K_1$.*

**Theorem 2.8.** *$\Gamma_2^*(R) \cong K_2$ if and only if $R$ is a direct sum of two fields.*

*Proof.* **(⇒):** Let $R \cong F_1 \oplus F_2$ where $F_1$ and $F_2$ are fields. Then the ideals of $R$ are $F_1 \oplus 0$, $0 \oplus F_2$, $0 \oplus 0$ and $F_1 \oplus F_2$. Then, $\Gamma_2^*(R) \cong K_2$.

**(⇐):** Suppose $\Gamma_2^*(R) \cong K_2$. Then $R$ is non-local. Also, $R$ cannot have more than two maximal ideals. Therefore $R$ has exactly two maximal ideals, say $\mathfrak{m}_1$ and $\mathfrak{m}_2$ with $\mathfrak{m}_1 \cap \mathfrak{m}_2 = 0$. This implies $R \cong \dfrac{R}{\mathfrak{m}_1} \oplus \dfrac{R}{\mathfrak{m}_2}$, a direct sum of two fields. □

**Theorem 2.9.** *The only triangle free connected graphs that can be realized as $\Gamma_2^*(R)$ are $K_1$ and $K_2$.*

*Proof.* Let $G$ be a triangle free connected graph. Since $G$ is triangle free $R$ can have at most two maximal ideals. Also since $G$ is connected the result follows. □

**Theorem 2.10.** *$\Gamma_2^*(R)$ is complete if and only if either $R$ is a direct sum of two fields or $R$ is local with principal maximal ideal having index of nilpotency* 2.

*Proof.* **(⇒):** If $\Gamma_2^*(R)$ is complete, $R$ can have at most two maximal ideals. For, assume $R$ has 3 maximal ideals say, $\mathfrak{m}_1, \mathfrak{m}_2$ and $\mathfrak{m}_3$. Then $\mathfrak{m}_1\mathfrak{m}_2 = 0$; otherwise $\mathfrak{m}_1\mathfrak{m}_2$ is a vertex of $\Gamma_2^*(R)$ and will not be adjacent to $\mathfrak{m}_1$ and $\mathfrak{m}_2$. For the same reason, $\mathfrak{m}_1\mathfrak{m}_3 = 0$. Then $\mathfrak{m}_1(\mathfrak{m}_2 + \mathfrak{m}_3) = 0$. This implies $\mathfrak{m}_1 = 0$ which is not possible. Now assume that $R$ has exactly 2 maximal ideals say, $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Then $\mathfrak{J}(R) = \mathfrak{m}_1\mathfrak{m}_2 = \{0\}$. Thus $R$ is a direct sum of 2 fields. Now, if $R$ is local with maximal ideal $\mathfrak{m}$, since $\Gamma_2^*(R)$ is complete, $\mathfrak{m}$ must be principal with index of nilpotency 2.

**(⇐):** If $R$ is a direct sum of two fields, $\Gamma_2^*(R) \cong K_2$ and if $R$ is local with principal maximal ideal having index of nilpotency 2, $\Gamma_2^*(R) \cong K_1$. □

The following corollary is immediate.

**Corollary 2.11.** *The only complete graphs that can be realized as* $\Gamma_2^*(R)$ *are* $K_1$ *and* $K_2$*.*

## 3 Comparison between $\Gamma_1^*(R)$ and $\Gamma_2^*(R)$

**Theorem 3.1.** *Assume* $\operatorname{diam}\Gamma_2^*(R) = 2$*. Then any two vertices in* $\Gamma_2^*(R)$ *which are not adjacent are also not adjacent in* $\Gamma_1^*(R)$*. That is,* $\Gamma_1^*(R)$ *is a subgraph of* $\Gamma_2^*(R)$*.*

*Proof.* Let $\operatorname{diam}\Gamma_2^*(R) = 2$. Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are not adjacent in $\Gamma_2^*(R)$. Then, there exists a maximal ideal $\mathfrak{m}$ such that $\mathfrak{a} + \mathfrak{m} = R = \mathfrak{b} + \mathfrak{m}$. Therefore, $(\mathfrak{a} + \mathfrak{m})(\mathfrak{b} + \mathfrak{m}) = R$. That is, $\mathfrak{ab} + \mathfrak{am} + \mathfrak{bm} + \mathfrak{m}^2 = R$.

But, $\mathfrak{ab} + \mathfrak{am} + \mathfrak{bm} + \mathfrak{m}^2 \subseteq \mathfrak{ab} + \mathfrak{m}$. Therefore, $\mathfrak{ab} + \mathfrak{m} = R$. This implies, in particular, $\mathfrak{ab} \neq 0$. Thus, $\mathfrak{a}$ and $\mathfrak{b}$ are not adjacent in $\Gamma_1^*(R)$. □

**Remark 3.2.** Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are adjacent in $\Gamma_2^*(R)$. Then, $\mathfrak{a} + \mathfrak{b} = R$. This implies $\mathfrak{ab} = \mathfrak{a} \cap \mathfrak{b}$. Hence $\mathfrak{a}$ is adjacent to $\mathfrak{b}$ in $\Gamma_1^*(R)$ if and only if $\mathfrak{a} \cap \mathfrak{b} = 0$. This must hold for every pair of comaximal ideals $\mathfrak{a}$ and $\mathfrak{b}$.

**Theorem 3.3.** *Let $R$ be a non-local ring. Then,* $\Gamma_2^*(R)$ *is a subgraph of* $\Gamma_1^*(R)$ *if and only if $R$ is a direct sum of two fields; and hence* $\Gamma_1^*(R) = \Gamma_2^*(R)$ *only when $R$ is a direct sum of two fields.*

*Proof.* **(⇒):** $\Gamma_2^*(R)$ is a subgraph of $\Gamma_1^*(R)$ if and only if for any pair of comaximal ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $R$, $\mathfrak{ab} = 0$. So, if $\Gamma_2^*(R)$ is a subgraph of $\Gamma_1^*(R)$, in particular, $\mathfrak{m}_1\mathfrak{m}_2 = 0$ where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are two maximal ideals of $R$. Hence, $R \cong \frac{R}{\mathfrak{m}_1} \oplus \frac{R}{\mathfrak{m}_2}$.

**(⇐):** If $R$ is a direct sum of two fields, $\Gamma_1^*(R) = \Gamma_2^*(R) = K_2$. □

**Theorem 3.4.** *If $R$ is a finite direct sum of fields,* $\Gamma_1^*(R) \cong \Gamma_2^*(R)$*.*

*Proof.* Let $R = F_1 \oplus F_2 \oplus \cdots \oplus F_n$ where $F_i$'s are fields. Thus, an ideal $\mathfrak{a}$ of $R$ is of the form, $\mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_n$ where, $\mathfrak{a}_i = 0$ or $F_i$.

Define $\varphi : V(\Gamma_1^*(R)) \to V(\Gamma_2^*(R))$ by $\varphi(\mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_n) = \mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \cdots \oplus \mathfrak{b}_n$ where

$$\mathfrak{b}_i = \begin{cases} F_i, & \text{if } \mathfrak{a}_i = (0) \\ 0, & \text{if } \mathfrak{a}_i = F_i. \end{cases}$$

Clearly, $\varphi$ is a bijection.

Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are adjacent in $\Gamma_1^*(R)$. Thus $\mathfrak{b}$ must contain 0 at the positions in which $\mathfrak{a}$ contains $F_i$'s. Therefore, $\varphi(\mathfrak{b})$ contains $F_i$'s at the positions where $\varphi(\mathfrak{a})$ contains 0. Then, $\varphi(\mathfrak{a})$ is adjacent to $\varphi(\mathfrak{b})$.

Similarly, if $\varphi(\mathfrak{a})$ and $\varphi(\mathfrak{b})$ are adjacent in $\Gamma_2^*(R)$ then, $\mathfrak{a}$ and $\mathfrak{b}$ are adjacent in $\Gamma_1^*(R)$. Thus, $\varphi$ is a graph isomorphism. That is, $\Gamma_1^*(R) \cong \Gamma_2^*(R)$. □

**Remark 3.5.** *In the context of Theorem 3.4, we can explicitly determine $\Gamma_1^*(R)$ and $\Gamma_2^*(R)$ by identifying the vertex set with the power set $P(X) \setminus \{X, \varnothing\}$ where $X = \{1, 2, \ldots, n\}$ and $A \subset X$ with $\bigoplus_{i \in A} F_i$. Then $A$ and $B$ are adjacent in $\Gamma_1^*(R)$ if and only if $A \cap B = \varnothing$ and $A$ and $B$ are adjacent in $\Gamma_2^*(R)$ if and only if $A \cup B = X$.*

**Theorem 3.6.** *$\Gamma_1^*(R)$ and $\Gamma_2^*(R)$ are edge disjoint if and only if $R$ has no non-trivial idempotents.*

*Proof.* **(⇒):** Suppose that $R$ contains a non-trivial idempotent $e$. Then, $R = Re \oplus R(1-e)$. This implies, $Re + R(1-e) = R$ and $Re \cap R(1-e) = ReR(1-e) = 0$. That is, $\Gamma_1^*(R)$ and $\Gamma_2^*(R)$ are not edge disjoint.

**(⇐):** Assume that $\Gamma_1^*(R)$ and $\Gamma_2^*(R)$ are not edge disjoint and then there exist two ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a} + \mathfrak{b} = R$ and $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b} = 0$. Then, $R = \mathfrak{a} \oplus \mathfrak{b}$ and hence, $\mathfrak{a} = Re$ and $\mathfrak{b} = R(1-e)$ for some idempotent $e$. Since $\mathfrak{a}$ and $\mathfrak{b}$ are non-zero proper ideals, $e$ must be non-trivial. □

**Theorem 3.7.** *Let $R$ be a non-local ring. If $\Gamma_1^*(R) = \overline{\Gamma_2^*(R)}$, $R$ is not semi-local.*

*Proof.* Assume that $R$ is semi-local with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$. Then, there are the following possibilities.

**Case (I):** $\Gamma_2^*(R)$ is connected.

This assumption implies $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n = 0$, by Theorem 2.4. Therefore, $(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1}), \mathfrak{m}_n$ are adjacent in $\Gamma_1^*(R)$ as well as in $\Gamma_2^*(R)$, which means $\Gamma_1^*(R) \neq \overline{\Gamma_2^*(R)}$.

**Case (II):** $\Gamma_2^*(R)$ is disconnected.

This implies $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n = \mathfrak{J}(R) \neq 0$. We subdivide this case into two.

**Case (II)(a):** $\mathfrak{J}(R)$ is nilpotent.

Then there exist least positive integers $k_1, k_2, \ldots, k_n$ such that $\mathfrak{m}_1^{k_1}\mathfrak{m}_2^{k_2}\cdots\mathfrak{m}_n^{k_n} = 0$ with at least one $k_j > 1$ for $1 \leq j \leq n$, say $k_n > 1$.

If $k_n > 2$, we have $(\mathfrak{m}_1 \cdots \mathfrak{m}_n) + \mathfrak{m}_n \neq R$ and $(\mathfrak{m}_1 \cdots \mathfrak{m}_n)\mathfrak{m}_n \neq 0$. That is, $\Gamma_1^*(R) \neq \overline{\Gamma_2^*(R)}$.

Now consider the case when $k_n = 2$. If $k_i > 1$ for some $i \neq n$, $(\mathfrak{m}_1 \cdots \mathfrak{m}_n) + \mathfrak{m}_n \neq R$ and $(\mathfrak{m}_1 \cdots \mathfrak{m}_n)\mathfrak{m}_n \neq 0$. If $k_i = 1$ $\forall i \neq n$, $(\mathfrak{m}_1 \cdots \mathfrak{m}_{l-1}\mathfrak{m}_{l+1} \cdots \mathfrak{m}_n^2) + \mathfrak{m}_l = R$ where $l \neq n$. But, $(\mathfrak{m}_1 \cdots \mathfrak{m}_{l-1}\mathfrak{m}_{l+1} \cdots \mathfrak{m}_n^2)\mathfrak{m}_l = 0$. So, $\Gamma_1^*(R) \neq \overline{\Gamma_2^*(R)}$.

**Case (II)(b):** $\mathfrak{J}(R)$ is not nilpotent.

In this case we have $(\mathfrak{m}_1 \cdots \mathfrak{m}_n) + \mathfrak{m}_1 \neq R$ and $(\mathfrak{m}_1 \cdots \mathfrak{m}_n)\mathfrak{m}_1 \neq 0$.

Thus, if $\Gamma_1^*(R) = \overline{\Gamma_2^*(R)}$, $R$ cannot be semi-local. □

**Theorem 3.8.** *Let $(R, \mathfrak{m})$ be an Artin local ring. Then, $\Gamma_1^*(R) = \overline{\Gamma_2^*(R)}$ if and only if either $\mathfrak{m}$ has index of nilpotency 2 or $\mathfrak{m}$ is principal with index of nilpotency 3.*

*Proof.* Follows from Remark 2.2 and Theorem 1.1. □

# 4 Some parameters of $\Gamma_2^*(R)$

In this section we find the clique number and the domination number of $\Gamma_2^*(R)$.

**Theorem 4.1.** $cl(\Gamma_2^*(R)) = |\max R|$.

*Proof.* Clearly $\max R$ induces a complete subgraph. Let $\mathfrak{a}$ be any non-zero non-maximal proper ideal of $R$. Then $\mathfrak{a}$ is contained in a maximal ideal. That is, there exists a maximal ideal $\mathfrak{m}$ such that $\mathfrak{a}$ is not adjacent to $\mathfrak{m}$. Thus, $\max R$ induces a maximal complete subgraph.

Now suppose $S = \{\mathfrak{a}_i : i \in \Lambda\}$, where $\Lambda$ is an index set, induces a complete subgraph in $\Gamma_2^*(R)$. Then one maximal ideal can contain at most one $\mathfrak{a}_i \in S$. That is, there exists an injective map from $S$ to $\max R$. This implies, $|S| \leq |\max R|$. Thus, $cl(\Gamma_2^*(R)) = |\max R|$. □

**Theorem 4.2.** *Let $R$ be a semi local ring with $|\max R| = n > 2$. Then, $\gamma(\Gamma_2^*(R)) = |\max R| +$ Number of isolated vertices in $\Gamma_2^*(R)$.*

*Proof.* Let $\Gamma_2^{**}(R)$ be the connected component of $\Gamma_2^*(R)$ induced by the non-isolated vertices of $\Gamma_2^*(R)$. Now, by Theorem 2.7, it is enough to show that $\gamma(\Gamma_2^{**}(R)) = |\max R|$.

Let $\max R = \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n\}$. Clearly $\max R$ is a dominating set for $\Gamma_2^{**}(R)$. Now consider, $S = \{\mathfrak{m}_2 \cdots \mathfrak{m}_n, \mathfrak{m}_1\mathfrak{m}_3 \cdots \mathfrak{m}_n, \ldots, \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{n-1}\}$, which is an independent set in $\Gamma_2^{**}(R)$. Note that any ideal $\mathfrak{a} \notin S$ can be adjacent only to at most one element of $S$. So every dominating set in $\Gamma_2^{**}(R)$ must contain at least $n$ elements. Thus, $\gamma(\Gamma_2^{**}(R)) = n = |\max R|$. Hence the result follows. ☐

**Remark 4.3.** If $R$ is a semi-local ring with $|\max R| = 2$ then, the above result is not true. For example, if $R$ is a direct sum of two fields, $\gamma(\Gamma_2^*(R)) = \gamma(K_2) = 1$ but $|\max R| = 2$.

# 5 Splitness

A graph $(V, E)$ is said to be a *split graph* if $V$ is the disjoint union of two sets $K$ and $S$ where $K$ induces a complete subgraph and $S$ is an independent set. Then, we can assume either $K$ is a clique or $S$ is a maximal independent set. In [6] & [7], the authors have carried out a detailed study on splitness of some graphs associated with a ring. In this section we continue the study in the case of $\Gamma_2^*(R)$.

**Lemma 5.1.** *Let $R = R_1 \times R_2 \times R_3$ be a ring. If $\Gamma_2^*(R)$ is split, each $R_i$ must be a field.*

*Proof.* Suppose $R_1$ is not a field. Then there exists a proper non-zero ideal $I$ of $R_1$. Then, $\{I \times R_2 \times R_3, R_1 \times R_2 \times 0, 0 \times R_2 \times R_3, R_1 \times 0 \times 0\}$ induces a $C_4$ in $\Gamma_2^*(R)$, a contradiction. ☐

**Lemma 5.2.** *If $F_i$ $(1 \leq i \leq 3)$ are fields and $R = F_1 \times F_2 \times F_3$ then $\Gamma_2^*(R)$ is split.*

*Proof.* $V(\Gamma_2^*(R))$ can be partitioned into $K = \{F_1 \times F_2 \times 0, F_1 \times 0 \times F_3, 0 \times F_2 \times F_3\}$ and $S = \{F_1 \times 0 \times 0, 0 \times F_2 \times 0, 0 \times 0 \times F_3\}$ where $K$ induces a complete subgraph and $S$ is an independent set. ☐

**Lemma 5.3.** *Let $F$ be a field and $R_1$ a local ring. Let $R = R_1 \times F$. Then $\Gamma_2^*(R)$ is split.*

*Proof.* Let $\{I_j : j \in J\}$ be the collection of non-zero proper ideals of $R_1$. Then $\{I_j \times F : j \in J\} \cup \{I_j \times 0 : j \in J\}$ is an independent set and $\{0 \times F, R_1 \times 0\}$ is a $K_2$. This forms a partition of $V(\Gamma_2^*(R))$. Thus, $\Gamma_2^*(R)$ is split. ☐

**Lemma 5.4.** *Suppose $R$ has exactly $n$ maximal ideals $\mathfrak{m}_i$ $(1 \leq i \leq n)$ with each $\mathfrak{m}_i$ being generated by an idempotent $e_i$. Then $R \cong \prod_{i=1}^{n} F_i$ where each $F_i \cong R/\mathfrak{m}_i$, a field.*

*Proof.* Let $e = \prod_{i=1}^{n} e_i$. Then $e \in \mathfrak{J}(R)$. Therefore, $1 - e$ is a unit (and an idempotent). So, $1 - e = 1 \Rightarrow e = 0$. Then by the Chinese Remainder Theorem,

$$R \cong \frac{R}{\prod_{i=1}^{n} Re_i} \cong \frac{R}{\bigcap_{i=1}^{n} Re_i} \cong \prod_{i=1}^{n} \frac{R}{Re_i}.$$ ☐

**Theorem 5.5.** *Let $R$ be a ring. $\Gamma_2^*(R)$ is a split graph if and only if one of the following conditions holds:*

*(i)* *$R$ is local.*

*(ii)* *$R \cong R_1 \times F$ where $R_1$ is a local ring and $F$ is a field.*

*(iii)* *$R \cong F_1 \times F_2 \times F_3$ where $F_i$'s are fields.*

*Proof.* First we note that $\Gamma_2^*(R)$ is split if and only if $\Gamma_2(R)$ is split. Also, if $R$ is local, $\Gamma_2^*(R)$ is split. Sufficiency of other conditions follows from the lemmas. To prove the necessity of the conditions, we assume that $R$ is not local and $V(\Gamma_2(R))$ is the disjoint union of two sets $K$ and $S$ where $K$ induces a complete subgraph and $S$ is an independent set. We assume that $K$ and $S$ are non-empty. Also, $S$ can contain at most one maximal ideal.

**Case (I):** $S$ contains a maximal ideal, say $\mathfrak{m}_1$.

In this case, $R$ can have only one maximal ideal other than $\mathfrak{m}_1$. For, if $\mathfrak{m}_2$ and $\mathfrak{m}_3$ are distinct maximal ideals other than $\mathfrak{m}_1$, then $\mathfrak{m}_2$ and $\mathfrak{m}_3$ are in $K$. Then, $\mathfrak{m}_2\mathfrak{m}_3 \in S$, $\mathfrak{m}_1 \in S$. Clearly, $\mathfrak{m}_1 + \mathfrak{m}_2\mathfrak{m}_3 = R$, a contradiction. Thus, $R$ contains only one maximal ideal other than $\mathfrak{m}_1$, say $\mathfrak{m}_2$ which belongs to $K$. Let $x_i \in \mathfrak{m}_i$ $(i = 1, 2)$ with $x_1 + x_2 = 1$. As $\mathfrak{m}_2^2 + \mathfrak{m}_1 = R$, $\mathfrak{m}_2^2 \in K$ which implies $\mathfrak{m}_2^2 = \mathfrak{m}_2$. Similarly, as $Rx_2 + \mathfrak{m}_1 = R$, $Rx_2 \in K$ which implies $\mathfrak{m}_2 = Rx_2$. Then, $\mathfrak{m}_2$ is a finitely generated maximal ideal which is idempotent. Hence, $\mathfrak{m}_2$ is generated by an idempotent. So, $R \cong R_1 \times F$ where $F$ is a field and $\mathfrak{m}_2$ is isomorphic to the ideal $R_1 \times \{0\}$. Further, $R_1$ must be local.

**Case (II):** $S$ contains no maximal ideal.

In this case, $R$ can have at most three maximal ideals, for, if $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ and $\mathfrak{m}_4$ are distinct maximal ideals, $\mathfrak{m}_1\mathfrak{m}_2$ and $\mathfrak{m}_3\mathfrak{m}_4$ are in $S$ which leads to a contradiction. If $R$ has only two maximal ideals, say, $\mathfrak{m}_1$ and $\mathfrak{m}_2$, then $\mathfrak{m}_1, \mathfrak{m}_2 \in K$. Since, $\mathfrak{m}_i^2 + \mathfrak{m}_i \neq R$ $(i = 1, 2)$, we have $\mathfrak{m}_1^2, \mathfrak{m}_2^2 \in S$. But $\mathfrak{m}_1^2 + \mathfrak{m}_2^2 = R$. So, to avoid a contradiction we have to assume $\mathfrak{m}_1^2 = \mathfrak{m}_1$ or $\mathfrak{m}_2^2 = \mathfrak{m}_2$. That is, $R \cong R_1 \times F$ where $F$ is a field and $R_1$ is a local ring. So, let us assume $R$ has exactly 3 maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2$ and $\mathfrak{m}_3$. Note that $\mathfrak{m}_i \in K$ $(i = 1, 2, 3)$. Then, as $\mathfrak{m}_1 + \mathfrak{m}_2\mathfrak{m}_3 = R$, there exists $x_1 \in \mathfrak{m}_1$ such that $Rx_1 + \mathfrak{m}_2\mathfrak{m}_3 = R$ which implies $Rx_1 \in K$ and hence, $Rx_1 = \mathfrak{m}_1$. Similarly arguing with $\mathfrak{m}_1^2 + \mathfrak{m}_2\mathfrak{m}_3 = R$, we get $\mathfrak{m}_1 = \mathfrak{m}_1^2$. Then $\mathfrak{m}_1$ is generated by an idempotent. Similarly each $\mathfrak{m}_j$ $(j = 2, 3)$ is generated by an idempotent. Then by the Lemma 5.4, $R \cong F_1 \times F_2 \times F_3$ where $F_i$ $(1 \leq i \leq 3)$ are fields. ☐

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