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Graded weakly 1-absorbing prime ideals

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ABSTRACT

In this paper, we introduce and study graded weakly 1-absorbing prime ideals in graded commutative rings. Let G be a group and R be a G-graded commutative ring with a nonzero identity $1 \neq 0$. A proper graded ideal P of R is called a graded weakly 1-absorbing prime ideal if for each nonunits $x, y, z \in h(R)$ with $0 \neq xyz \in P$, then either $xy \in P$ or $z \in P$. We give many properties and characterizations of graded weakly 1-absorbing prime ideals. Moreover, we investigate weakly 1-absorbing prime ideals under homomorphism, in factor ring, in rings of fractions, in idealization.

RESUMEN

En este artículo, introducimos y estudiamos ideales primos débilmente 1-absorbentes en anillos conmutativos gradados. Sea G un grupo y R un anillo conmutativo G-gradado con identidad no cero $1 \neq 0$. Un ideal gradado propio P de R se llama ideal primo gradado débilmente 1-absorbente si para cualquiera $x, y, z \in h(R)$ no-unidades con $0 \neq xyz \in P$, entonces o bien $xy \in P$ o $z \in P$. Entregamos muchas propiedades y caracterizaciones de ideales primos gradados débilmente 1-absorbentes. Más aún, investigamos ideales primos débilmente 1-absorbentes bajo homomorfismo, en anillos cociente, en anillos de fracciones, en idealización.

Keywords and Phrases: graded ideal, 1-absorbing prime ideal, weakly 1-absorbing prime ideal, graded weakly 1-absorbing prime ideal.

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1 Introduction

Throughout the paper, we focus only on graded commutative rings with a nonzero identity. R will always denote such a ring and G denotes a group with identity e. $u(R)$, $N(R)$ and $reg(R)$ denote the set of all unit elements, all nilpotent elements and all regular elements of R, respectively. Over the years, several types of ideals have been developed such as prime, maximal, primary, etc. The concept of prime ideals and its generalizations have a significant place in commutative algebra since they are used in understanding the structure of rings [6, 19, 11, 4]. R is said to be G-graded if $R = \bigoplus_{g\in G} R_g$ with $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$ where R_g is an additive subgroup of R for all $g \in G$. Sometimes we denote the G-graded ring R by $G(R)$. The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Also, we set $h(R) = \bigcup_{g\in G} R_g$. The support of $G(R)$ is defined as $supp(G(R)) = \{g \in G : R_g \neq \{0\}\}$. Moreover, as shown for example in [13] that R_e is a subring of R and $1 \in R_e$. Let P be an ideal of a graded ring R. Then P is said to be graded ideal if $P = \bigoplus_{g\in G}(P \cap R_g)$, *i.e.*, for $x \in P$, $x = \sum_{g\in G} x_g$ where $x_g \in P$ for all $g \in G$. It is known that an ideal of a graded ring need not be graded. Let R be a G-graded ring and P be a graded ideal of R. Then R/P is G-graded by $(R/P)_g = (R_g+P)/P$ for all $g \in G$. If R and S are G-graded rings, then $R\times S$ is a G-graded ring by $(R \times S)_g = R_g \times S_g$ for all $g \in G$.

Lemma 1.1 ([9, Lemma 2.1])**.** *Let R be a G-graded ring.*

(1) *If P and Q are graded ideals of R, then $P + Q$, PQ and $P \bigcap Q$ are graded ideals of R.*

(2) *If $x \in h(R)$, then $Rx = (x)$ is a graded ideal of R.*

Let P be a proper graded ideal of R. Then the graded radical of P is denoted by $\mathrm{Grad}(P)$ and it is defined as follows:

$$\mathrm{Grad}(P) = \left\{ x = \sum_{g\in G} x_g \in R : \text{ for all } g \in G, \text{ there exists } n_g \in \mathbb{N} \text{ such that } x_g^{n_g} \in P \right\}.$$

Note that $\mathrm{Grad}(P)$ is always a graded ideal of R (see [15]).

In [15], Refai *et al.* defined and studied graded prime ideals. A proper graded ideal P of a graded ring R is called graded prime ideal if whenever $xy \in P$ for some $x, y \in h(R)$ then either $x \in P$ or $y \in P$. Clearly, if P is a prime ideal of R and it is a graded ideal of R, then P is a graded prime ideal of R. On the other hand, the importance of graded prime ideals comes from the fact that graded prime ideals are not necessarily prime ideals, as we see in the next example.

Example 1.2. *Consider $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$.*

Consider the graded ideal $I = 17R$ of R. We show that I is a graded prime ideal of R. Let $xy \in I$ for some $x, y \in h(R)$.

Case (1): *Assume that $x, y \in R_0$. In this case, $x, y \in \mathbb{Z}$ such that 17 divides xy, and then either 17 divides x or 17 divides y as 17 is a prime number, which implies that $x \in I$ or $y \in I$.*

Case (2): *Assume that $x, y \in R_1$. In this case, $x = ia$ and $y = ib$ for some $a, b \in \mathbb{Z}$ such that 17 divides $xy = -ab$, and then 17 divides a or 17 divides b in $\mathbb{Z}$, which implies that 17 divides $x = ia$ or 17 divides $y = ib$ in R. Then we have that $x \in I$ or $y \in I$.*

Case (3): *Assume that $x \in R_0$ and $y \in R_1$. In this case, $x \in \mathbb{Z}$ and $y = ib$ for some $b \in \mathbb{Z}$ such that 17 divides $xy = ixb$ in R, that is $ixb = 17(\alpha + i\beta)$ for some $\alpha, \beta \in \mathbb{Z}$. Then we obtain $xb = 17\beta$, that is 17 divides xb in $\mathbb{Z}$, and again 17 divides x or 17 divides b, which implies that 17 divides x or 17 divides $y = ib$ in R. Thus, $x \in I$ or $y \in I$.*

One can similarly show that $x \in I$ or $y \in I$ in other cases. So, I is a graded prime ideal of R. On the other hand, I is not a prime ideal of R since $(4-i)(4+i) \in I$, $(4-i) \notin I$ and $(4+i) \notin I$.

Several generalizations of graded prime ideals attracted attention by many authors. In [14], Refai and Al-Zoubi introduced graded primary ideals which is a generalization of graded prime ideals. A proper graded ideal P of a graded ring R is called graded primary ideal if $xy \in P$ for some $x, y \in h(R)$ implies that either $x \in P$ or $y \in \mathrm{Grad}(P)$. They also studied graded primary G-decomposition related to graded primary ideals. Atani defined a generalization of graded prime ideals as graded weakly prime ideals in [5]. A proper graded ideal P of a graded ring R is said to be graded weakly prime ideal if whenever $x, y \in h(R)$ such that $0 \neq xy \in P$ then either $x \in P$ or $y \in P$. They gave some characterizations of graded weakly prime ideals and their homogeneous components. In [12], Naghani and Moghimi introduced 2-absorbing version of graded prime ideals and graded weakly prime ideals. A proper graded ideal P of a graded ring R is called graded 2-absorbing (graded weakly 2-absorbing) if whenever $x, y, z \in h(R)$ such that $xyz \in P$ ($0 \neq xyz \in P$) then $xy \in P$ or $yz \in P$ or $xz \in P$. They investigated some properties of this new class of graded ideals. Yassine *et al.* studied 1-absorbing prime ideals which is a generalization of prime ideals in [19]. A proper ideal P of R is said to be 1-absorbing prime ideal if for some nonunit elements $x, y, z \in R$ such that $xyz \in P$ implies that either $xy \in P$ or $z \in P$. Authors determined 1-absorbing prime ideals in some special rings such as principal ideal domains, valuation domains and Dedekind domains. Currently, Koç *et al.* defined weakly 1-absorbing prime ideals which is a generalization of 1-absorbing prime ideals in [11]. A proper ideal P of R is called weakly 1-absorbing prime ideal if $0 \neq xyz \in P$ for some nonunits $x, y, z \in R$ implies $xy \in P$ or $z \in P$. They gave certain properties of this new concept and characterized rings that every proper ideal is weakly 1-absorbing ideal. More recently, in [1], Dawwas *et al.* defined graded version of 1-absorbing prime ideals which is a generalization of both graded prime ideals and 1-absorbing prime ideals. A proper graded ideal P of a graded ring R is called graded 1-absorbing prime ideal if whenever for some nonunits x, y, z in $h(R)$ such that $xyz \in P$ then either $xy \in P$ or $z \in P$. Moreover, many studies have been made by researchers related to graded versions of known structures [3, 7, 8, 10, 16].

In this paper, we define graded weakly 1-absorbing prime ideal which is a generalization of graded 1-absorbing prime ideals. A proper graded ideal P of a graded ring R is said to be graded weakly 1-absorbing prime ideal if whenever for some nonunits x, y, z in $h(R)$ such that $0 \neq xyz \in P$ then either $xy \in P$ or $z \in P$. Every graded 1-absorbing prime ideal is a graded weakly 1- absorbing prime ideals but the converse is not true in general (see, Example 3.2). In addition to many properties of this new class of graded ideals, we also investigate behavior of graded weakly 1-absorbing ideals under homomorphism, in factor ring, in rings of fractions, in idealization (see, Theorem 3.15, Proposition 3.14, Theorem 3.16, Theorem 3.18 and Theorem 3.23).

2 Motivation

Graded prime ideals play an essential role in graded commutative ring theory. Indeed, graded prime ideals are interesting because they correspond to irreducible varieties and schemes in the graded case and because of their connection to factorization. Also, graded prime ideals are important because they have applications to combinatorics and they have structural significance in graded ring theory. Thus, this concept has been generalized and studied in several directions. The significance of some of these generalizations is same as the graded prime ideals. In a feeling of animate being, they determine how far an ideal is from being graded prime. Several generalizations of graded prime ideals attracted attention by many authors. For instance, graded weakly prime ideals, graded primary ideals, graded almost prime ideals, graded 2-absorbing ideals, graded 2-absorbing primary ideals and graded 1-absorbing prime ideals. In continuation of these generalizations, we present the concept of graded weakly 1-absorbing prime ideals, as a new generalization to graded prime ideals, in order to benefit from this new concept in more applications, and to make the study of graded prime ideals more flexible.

3 Graded weakly 1-absorbing prime ideals

Definition 3.1. *Let R be a G-graded ring and P be a proper graded ideal of R. Then, P is called graded weakly 1-absorbing prime ideal of R if whenever $0 \neq xyz \in P$ for some nonunit elements x, y, z in $h(R)$ then $xy \in P$ or $z \in P$.*

Example 3.2. *Every graded 1-absorbing prime ideal is a graded weakly 1-absorbing prime ideal. The converse may not be true. Let $R = \mathbb{Z}_{21}$ and consider the trivial grading on R. $P = (\bar{0})$ is graded* *weakly 1-absorbing prime ideal. But it is not graded 1-absorbing prime ideal since* $\bar{3}.\bar{3}.\bar{7} = \bar{0} \in P$, $\bar{3}.\bar{3} \notin P$ *and* $\bar{7} \notin P$.

Example 3.3. *Let* $R = \mathbb{Z}_8[i] = \mathbb{Z}_8 \oplus i\mathbb{Z}_8$. *Then note that* R *is a* $\mathbb{Z}_2$*-graded ring and* $h(R) = \mathbb{Z}_8 \cup i\mathbb{Z}_8$. *Now, put* $P = (\overline{4})$. *Since* $\overline{2}(\overline{1}+i)(\overline{1}-i) = \overline{4} \in P$ *but* $\overline{2}(\overline{1}+i) \notin P$ *and* $(\overline{1}-i) \notin P$, *it follows that* P *is not a weakly 1-absorbing prime ideal of* R. *However, the set of nonunit homogeneous elements of* R *is* $\{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{2}i, \overline{4}i, \overline{6}i\}$. *Let* $x, y, z \in h(R)$ *be nonunit elements. Then note that* $xyz = \overline{0} \in P$, *which implies that* P *is a graded weakly 1-absorbing prime ideal of* R.

$N(R)$ denotes the set of all nilpotent elements of R. Recall that a ring R is said to be reduced if $N(R) = 0$.

Theorem 3.4. *Let* R *be a* G*-graded reduced ring and* P *be a graded weakly 1-absorbing prime ideal of* R. *Then,* $\mathrm{Grad}(P)$ *is a graded weakly prime ideal of* R.

Proof. Suppose that $0 \neq xy \in \mathrm{Grad}(P)$ where $x, y \in h(R)$. Then there exists $n \in \mathbb{N}$ such that $(xy)^n \in P$. We have $0 \neq (xy)^n = x^k x^{n-k} y^n \in P$ for some positive integer $k < n$. If x or y is unit in $h(R)$, we are done. So, assume that x and y are nonunit elements in $h(R)$. As P is graded weakly 1-absorbing prime ideal, $x^n \in P$ or $y^n \in P$ showing that $x \in \mathrm{Grad}(P)$ or $y \in \mathrm{Grad}(P)$. □

Theorem 3.5. *Let* R *be a* G*-graded ring and* P *be a graded weakly 1-absorbing prime ideal of* R. *Then,* $(P : a)$ *is a graded weakly prime ideal of* R *where* a *is a regular nonunit element in* $h(R) - P$.

Proof. From [1, Lemma 2.4], $(P : a)$ is a graded ideal of R. Suppose $0 \neq xy \in (P : a)$ for some $x, y \in h(R)$. Then $0 \neq (xa)y \in P$ where $xa, y \in h(R)$. If x or y is unit, there is nothing to prove. So, we can assume that x and y are nonunit elements in $h(R)$. Since P is graded weakly 1-absorbing prime ideal of R, we get either $xa \in P$ or $y \in P$. It gives $x \in (P : a)$ or $y \in (P : a)$, as needed. □

Definition 3.6. *Let* R *be a* G*-graded ring and* P *be a graded ideal of* R. *Then,* P *is called* g*-weakly 1-absorbing prime ideal of* R *for* $g \in G$ *with* $P_g \neq R_g$ *if* $0 \neq xyz \in P$ *for some nonunit elements* x, y, z *in* R_g *implies that* $xy \in P$ *or* $z \in P$.

We say that a proper graded ideal P of a G-graded ring R is said to be a g-weakly prime for $g \in G$ if $P_g \neq R_g$ and whenever $0 \neq xy \in P$ for some $x, y \in R_g$ implies $x \in P$ or $y \in P$.

Proposition 3.7. *Let* R *be a* G*-graded reduced ring and* P *be a* g^n*-weakly 1-absorbing prime ideal of* R *for each* $n \in \mathbb{N}$. *Then, Grad*(P) *is a* g*-weakly prime ideal of* R.

Proof. It immediately follows from Theorem 3.4. □

Recall from [1] that a proper graded ideal P of a G-graded ring R is said to be a g-1-absorbing prime for $g \in G$ if $P_g \neq R_g$ and whenever $xyz \in P$ for some nonunits $x, y, z \in R_g$ implies $xy \in P$ or $z \in P$.

Proposition 3.8. *Let R be a G-graded ring. If R has a g-weakly-1-absorbing prime ideal that is not a g-weakly prime ideal of R and (0) is a g-1-absorbing prime ideal of R, then, for each unit element u in R_g and for each nonunit element v in R_g the sum $u+v$ is a unit element in R_g.*

Proof. Assume that P is a g-weakly 1-absorbing prime ideal of R that is not a g-weakly prime ideal of R. Then, there exist nonunit elements $x, y \in R_g$ such that $xy \in P$ but $x \notin P$ and $y \notin P$. Then we have $vxy \in P$ where v is a nonunit element in R_g. If $vxy = 0 \in (0)$, then $vx \in P$ since (0) is a g-1 absorbing prime ideal and $y \notin P$. If $0 \neq vxy \in P$, we have $vx \in P$ since P is a g-weakly 1-absorbing prime ideal of R. Now we will show that $u+v$ is a unit element in R_g where u is a unit element in R_g. Suppose to the contrary. If $(u+v)xy = 0 \in (0)$, we get $(u+v)x \in P$. This implies $ux \in P$ giving that $x \in P$ which is a contadiction. If we assume $0 \neq (u+v)xy \in P$, then again we get a contradiction by using the fact that P is a g-weakly 1-absorbing prime ideal and so it completes the proof. □

Theorem 3.9. *Let R be a G-graded ring and P be a proper graded ideal of R. Consider the following statements.*

(i) *P is a graded weakly 1-absorbing prime ideal of R.*

(ii) *If $xy \notin P$ for some nonunits $x, y \in h(R)$, then $(P : xy) = P \cup (0 : xy)$.*

(iii) *If $xy \notin P$ for some nonunits $x, y \in h(R)$, then either $(P : xy) = P$ or $(P : xy) = (0 : xy)$.*

(iv) *If $0 \neq xyK \subseteq P$ for some nonunits $x, y \in h(R)$ and proper graded ideal K of R, then either $xy \in P$ or $K \subseteq P$.*

(v) *If $0 \neq xJK \subseteq P$ for some nonunit $x \in h(R)$ and proper graded ideals J, K of R, then either $xJ \subseteq P$ or $K \subseteq P$.*

(vi) *If $0 \neq IJK \subseteq P$ for proper graded ideals I, J, K of R, then either $IJ \subseteq P$ or $K \subseteq P$.*

Then, $(vi) \Rightarrow (v) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$.

Proof. $(vi) \Rightarrow (v):$ Suppose that $0 \neq xJK \subseteq P$ for some nonunit $x \in h(R)$ and proper graded ideals J, K of R. Now, put $I = (x)$. Then I is a proper graded ideal of R and $0 \neq IJK \subseteq P$. By (vi), we have $xJ \subseteq IJ \subseteq P$ or $K \subseteq P$, which completes the proof.

$(v) \Rightarrow (iv):$ Suppose that $0 \neq xyK \subseteq P$ for some nonunits $x, y \in h(R)$ and proper graded ideal K of R. Now, consider the proper graded ideal $J = (y)$ of R and note that $0 \neq xJK \subseteq P$. So by (v), we get $xy \in xJ \subseteq P$ or $K \subseteq P$.

$(iv) \Rightarrow (iii):$ Let $x, y \in h(R)$ be nonunit elements such that $xy \notin P$. It is easy to see that $xy(P : xy) \subseteq P$.

Case 1: Assume that $xy(P : xy) = 0$. This gives $(P : xy) \subseteq (0 : xy) \subseteq (P : xy)$, that is, $(P : xy) = (0 : xy)$.

Case 2: Assume that $xy(P : xy) \neq 0$. Then by (iv), we have $(P : xy) \subseteq P$ which implies that $(P : xy) = P$.

$(iii) \Rightarrow (ii):$ It is straightforward.

$(ii) \Rightarrow (i):$ Let $x, y, z \in h(R)$ be nonunits such that $0 \neq xyz \in P$. If $xy \in P$, then we are done. So assume that $xy \notin P$. Since $z \in (P : xy) - (0 : xy)$ and $(P : xy) = (0 : xy) \cup P$, we have $z \in P$ which completes the proof. □

In the following example, we show that the condition "P is a graded weakly 1-absorbing prime ideal" does not ensure that the conditions (ii)-(vi) in Theorem 3.9 hold. In fact, we will show that $(i) \not\Rightarrow (ii)$.

Example 3.10. *Let $R = \mathbb{Z}_{12}[X]$, where X is an indeterminate over $\mathbb{Z}_{12}$. Then $R = \bigoplus\limits_{n\in\mathbb{Z}} R_n$ is a $\mathbb{Z}$-graded ring, where $R_0 = \mathbb{Z}_{12}$ and $R_n = \mathbb{Z}_{12}X^n$ if $n > 0$, otherwise $R_n = 0$. Then note that $h(R) = \bigcup\limits_{n\geq 0} \mathbb{Z}_{12}X^n$ and the set of nonunits homogeneous elements of R is $nh(R) = \{\overline{2k}, \overline{3k}, \overline{a}X^n : k, a \in \mathbb{Z}$ and $n \in \mathbb{N}\}$. Consider the graded ideal $P = (X, \overline{4})$ of R. Let $f, g, h \in nh(R)$ such that $0 \neq fgh \in P$. If at least one of the f, g, h is of the form $\overline{a}X^n$, then we are done. So assume that $f, g, h \in \{\overline{2k}, \overline{3k} : k \in \mathbb{Z}\}$. Since $0 \neq fgh \in P = (X, \overline{4})$, we have $0 \neq fgh = \overline{4k}$ for some $k \in \mathbb{Z}$ with $\gcd(k, 3) = 1$. Since $4|fgh$ and $3 \nmid fgh$, we conclude that $f, g, h \in \{\overline{2}, \overline{4}, \overline{8}, \overline{10}\}$. This implies that $fg \in P$, that is, P is a graded weakly 1-absorbing prime ideal of R. Now, we will show that P does not satisfy the condition (ii) in Theorem 3.9. Take nonunits homogeneous elements $c = \overline{2}, d = \overline{3}$ of R. Then note that $cd \notin P$. On the other hand, it is clear that $\overline{2} \in (0 : cd) - P$ and $X \in P - (0 : cd)$. This gives $z = \overline{2} + X \in (P : cd) - ((0 : cd) \cup P)$. Thus, we have $(P : cd) \supsetneq (0 : cd) \cup P$, i.e., P does not satisfy the condition (ii) in Theorem 3.9.*

Definition 3.11. *Let P be a graded weakly 1-absorbing prime ideal of R and $x_{g_1}, y_{g_2}, z_{g_3}$ be nonunits in $h(R)$. Then, $(x_{g_1}, y_{g_2}, z_{g_3})$ is called graded 1-triple zero if $x_{g_1}y_{g_2}z_{g_3} = 0$, $x_{g_1}y_{g_2} \notin P$ and $z_{g_3} \notin P$, where $g_1, g_2, g_3 \in G$.*

Theorem 3.12. *Let $P = \bigoplus_{g\in G} P_g$ be a graded weakly 1-absorbing prime ideal that is not graded 1-absorbing prime ideal and $(x_{g_1}, y_{g_2}, z_{g_3})$ be a graded 1-triple zero of P, where $g_1, g_2, g_3 \in G$. Then,*

(i) $x_{g_1}y_{g_2}P_{g_3} = 0$.

(ii) $x_{g_1}z_{g_3} \notin P_{g_1g_3}$ *and* $y_{g_2}z_{g_3} \notin P_{g_2g_3}$ *imply that* $x_{g_1}z_{g_3}P_{g_2} = y_{g_2}z_{g_3}P_{g_1} = x_{g_1}P_{g_2}P_{g_3} = y_{g_2}P_{g_1}P_{g_3} = z_{g_3}P_{g_1}P_{g_2} = 0$. *In particular,* $P_{g_1}P_{g_2}P_{g_3} = 0$.

Proof. $(i):$ Let $P = \bigoplus_{g\in G} P_g$ be a graded weakly 1-absorbing prime ideal that is not graded 1-absorbing prime ideal and $(x_{g_1}, y_{g_2}, z_{g_3})$ be a graded 1-triple zero of P. Assume that $x_{g_1}y_{g_2}P_{g_3} \neq 0$. Then, there exists $a \in P_{g_3} = P \cap R_{g_3}$ such that $0 \neq x_{g_1}y_{g_2}a$. So, we have $0 \neq x_{g_1}y_{g_2}a = x_{g_1}y_{g_2}(z_{g_3} + a) \in P$. If $z_{g_3} + a$ is unit, then $x_{g_1}y_{g_2} \in P$ which gives a contradiction. Since P is graded weakly 1-absorbing prime ideal and $x_{g_1}y_{g_2} \notin P$, $z_{g_3} + a \in P$. This shows $z_{g_3} \in P$, a contradiction.

$(ii):$ Let $x_{g_1}z_{g_3} \notin P_{g_1g_3}$ and $y_{g_2}z_{g_3} \notin P_{g_2g_3}$. Then, $x_{g_1}z_{g_3}, y_{g_2}z_{g_3} \notin P$. Now choose $a \in P_{g_2}$. So, we have $x_{g_1}(y_{g_2} + a)z_{g_3} = x_{g_1}az_{g_3} \in P$ since $x_{g_1}y_{g_2}z_{g_3} = 0$. If $y_{g_2} + a$ is unit, then we obtain $x_{g_1}z_{g_3} \in P$, which is a contradiction. Thus, $y_{g_2} + a$ is not unit. If $x_{g_1}az_{g_3} \neq 0$, then $0 \neq x_{g_1}(y_{g_2} + a)z_{g_3} \in P$. Thus, $x_{g_1}(y_{g_2} + a) \in P$ or $z_{g_3} \in P$ implying that $x_{g_1}y_{g_2} \in P$ or $z_{g_3} \in P$, a contradiction. This shows $x_{g_1}az_{g_3} = 0$ and so $x_{g_1}z_{g_3}P_{g_2} = 0$. Similarly, $y_{g_2}z_{g_3}P_{g_1} = 0$.

Now assume that $x_{g_1}P_{g_2}P_{g_3} \neq 0$. Then there exist $a_{g_2} \in P_{g_2}, b_{g_3} \in P_{g_3}$ such that $x_{g_1}a_{g_2}b_{g_3} \neq 0$. This gives $0 \neq x_{g_1}(y_{g_2} + a_{g_2})(z_{g_3} + b_{g_3}) = x_{g_1}y_{g_2}z_{g_3} + x_{g_1}y_{g_2}b_{g_3} + x_{g_1}a_{g_2}z_{g_3} + x_{g_1}a_{g_2}b_{g_3} = x_{g_1}a_{g_2}b_{g_3} \in P$. If $(y_{g_2} + a_{g_2})$ is unit, $x_{g_1}(z_{g_3} + b_{g_3}) \in P$. It means that $x_{g_1}z_{g_3} \in P$, which is a contradiction. Hence, $(y_{g_2} + a_{g_2})$ is nonunit. Similar argument shows that $(z_{g_3} + b_{g_3})$ is nonunit. Since P is graded weakly 1-absorbing prime ideal, $x_{g_1}(y_{g_2} + a_{g_2}) \in P$ or $z_{g_3} + b_{g_3} \in P$. This proves $x_{g_1}y_{g_2} \in P$ or $z_{g_3} \in P$ which is a contradiction. So, $x_{g_1}P_{g_2}P_{g_3} = 0$. Similarly we have $y_{g_2}P_{g_1}P_{g_3} = z_{g_3}P_{g_1}P_{g_2} = 0$.

Suppose $P_{g_1}P_{g_2}P_{g_3} \neq 0$. Then there exist $a_{g_1} \in P_{g_1}, b_{g_2} \in P_{g_2}, c_{g_3} \in P_{g_3}$ such that $a_{g_1}b_{g_2}c_{g_3} \neq 0$. So, we have $0 \neq (a_{g_1} + x_{g_1})(b_{g_2} + y_{g_2})(c_{g_3} + z_{g_3}) = a_{g_1}b_{g_2}c_{g_3} \in P$ since $x_{g_1}z_{g_3}P_{g_2} = y_{g_2}z_{g_3}P_{g_1} = x_{g_1}P_{g_2}P_{g_3} = y_{g_2}P_{g_1}P_{g_3} = z_{g_3}P_{g_1}P_{g_2} = 0$ and $x_{g_1}y_{g_2}z_{g_3} = 0$. If $a_{g_1} + x_{g_1}$ is unit, $(b_{g_2} + y_{g_2})(c_{g_3} + z_{g_3}) \in P$ and it implies $y_{g_2}z_{g_3} \in P$, a contradiction. So, $a_{g_1} + x_{g_1}$ is not unit. Similar argument shows that $b_{g_2} + y_{g_2}, c_{g_3} + z_{g_3}$ are nonunits. Since P is graded weakly 1-absorbing prime ideal, we have either $(a_{g_1} + x_{g_1})(b_{g_2} + y_{g_2}) \in P$ or $c_{g_3} + z_{g_3} \in P$. Thus, we conclude that $x_{g_1}y_{g_2} \in P$ or $z_{g_3} \in P$ giving a contradiction. Therefore, $P_{g_1}P_{g_2}P_{g_3} = 0$. □

Let R be a G-graded ring. It is clear that for each $g \in G$, R_g is an R_e-module and P_g is an R_e-submodule of R_g.

Theorem 3.13. *Let $P = \bigoplus_{g\in G} P_g$ be a graded 1-absorbing prime ideal of $G(R)$ and $g \in G$. If $x, y \in R_g$ are nonunits such that $xy \notin P$, then $(P_{g^2} :_{R_e} xy) = P_e$.*

Proof. Let $z \in (P_{g^2} :_{R_e} xy)$, where $x, y \in R_g$ are nonunits. Then, $xyz \in P_{g^2} \subseteq P$. If z is a unit, $xy \in P$ which gives a contradiction. So, z is not unit. As P is graded 1-absorbing prime ideal and $xy \notin P$ we get $z \in P$. Thus, $z \in P \cap R_e = P_e$. This shows $(P_{g^2} :_{R_e} xy) \subseteq P_e$.

On the other hand, suppose $z \in P_e \subseteq P$. Then, $xyz \in P \cap R_{g^2} = P_{g^2}$ proving $z \in (P_{g^2} :_{R_e} xy)$, as desired. $\square$

Proposition 3.14. *Let R be a G-graded ring and $J \subseteq I$ be proper graded ideals of R. Then the followings statements are satisfied.*

(i) If I is graded weakly 1-absorbing prime ideal, then I/J is graded weakly 1-absorbing prime ideal of R/J.

(ii) Suppose that J consists of all nilpotent elements of R. If J is a graded weakly 1-absorbing prime ideal of R and I/J is a graded weakly 1-absorbing prime ideal of R/J, then I is a graded weakly 1-absorbing prime ideal of R.

(iii) If (0) is graded 1-absorbing prime ideal of R and I is graded weakly 1-absorbing prime ideal of R, then I is graded 1-absorbing prime ideal of R.

Proof. (i) : Let $0+J \neq (x+J)(y+J)(z+J) \in I/J$ for some nonunits $x+J, y+J, z+J \in h(R/J)$. Then, $0 \neq xyz + J \in I/J$ and so $0 \neq xyz \in I$ where x, y, z are nonunits in $h(R)$. As I is a graded weakly 1-absorbing prime ideal, either $xy \in I$ or $z \in I$. Hence, $xy + J \in I/J$ or $z + J \in I/J$, as desired.

(ii) : Suppose $0 \neq xyz \in I$ for some nonunits $x, y, z \in h(R)$. Then, $xyz+J = (x+J)(y+J)(z+J) \in I/J$. If $xyz \in J$, then $xy \in J \subseteq I$ or $z \in J$ since $J \subseteq I$ is graded weakly 1-absorbing prime ideal. So we can assume $xyz \notin J$. Then we have $0 + J \neq (x + J)(y + J)(z + J) \in I/J$. As I/J is graded weakly 1-absorbing prime ideal of R/J, $(x + J)(y + J) \in I/J$ or $z + J \in I/J$. It implies either $xy \in I$ or $z \in I$.

(iii) : Suppose that $xyz \in I$ for some nonunits $x, y, z \in h(R)$. If $xyz \neq 0$, then we are done. So, we can assume $xyz = 0 \in (0)$. Then, we get either $xy = 0 \in I$ or $z = 0 \in I$ since (0) is graded 1-absorbing prime ideal. Therefore, we conclude that $xy \in I$ or $z \in I$. $\square$

Let R and S be two G-graded rings. A ring homomorphism $f : R \to S$ is said to be graded homomorphism if $f(R_g) \subseteq S_g$ for all $g \in G$.

Theorem 3.15. *Let R_1 and R_2 be two G-graded rings and $f : R_1 \longrightarrow R_2$ be a graded homomorphism such that $f(1_{R_1}) = 1_{R_2}$. The following statements are satisfied.*

(i) If f is injective, J is a graded weakly 1-absorbing prime ideal of R_2 and $f(x)$ is a nonunit element of R_2 for all nonunit elements $x \in h(R_1)$, then $f^{-1}(J)$ is a graded weakly 1-absorbing prime ideal of R_1.

(ii) If f is surjective and I is a graded weakly 1-absorbing prime ideal of R_1 with $\ker(f) \subseteq I$, then $f(I)$ is a graded weakly 1-absorbing prime ideal of R_2.

Proof. (i) : It is clear that $f^{-1}(J)$ is a graded ideal of R_1. Let $0 \neq xyz \in f^{-1}(J)$ for some nonunits x, y, z in $h(R_1)$. So, $f(x), f(y)$ and $f(z)$ are nonunits in $h(R_2)$ by the assumption. Since f is injective and $xyz \neq 0$, we have $f(xyz) \neq 0$. Then we get $0 \neq f(x)f(y)f(z) = f(xyz) \in J$. As J is a graded weakly 1-absorbing prime ideal of R_2, $f(x)f(y) \in J$ or $f(z) \in J$. It implies that we have either $xy \in f^{-1}(J)$ or $z \in f^{-1}(J)$.

(ii) : Suppose that $0 \neq abc \in f(I)$ for some nonunits $a, b, c \in h(R_2)$. Then, there exist nonunits $x, y, z \in h(R_1)$ such that $f(x) = a, f(y) = b$ and $f(z) = c$. It gives that $0 \neq f(x)f(y)f(z) = abc \in f(I)$. So, there exists $i \in I$ such that $f(xyz) = f(i)$. This means $xyz - i \in \ker(f) \subseteq I$ giving $xyz \in I$. Since I is a graded weakly 1-absorbing prime ideal and $0 \neq xyz \in I$, we conclude that $xy \in I$ or $z \in I$. It shows $f(x)f(y) = ab \in f(I)$ or $f(z) = c \in f(I)$, as needed. □

Let $S \subseteq h(R)$ be a multiplicative set and R be a G-graded ring. Then $S^{-1}R$ is a G-graded ring with $(S^{-1}R)_g = \left\{\frac{a}{s} : a \in R_h, s \in S \cap R_{hg^{-1}}\right\}$. Let I be a graded ideal of R. Then we denote the set $\{a \in R : ab \in I \text{ for some } b \in R - I\}$ by $Z_I(R)$.

Theorem 3.16. *Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset. The following statements are satisfied.*

(i) *If I is a graded weakly 1-absorbing prime ideal of R with $I \cap S = \emptyset$, then $S^{-1}I$ is a graded weakly 1-absorbing prime ideal of $S^{-1}R$.*

(ii) *If $S^{-1}I$ is a graded weakly 1-absorbing prime ideal of $S^{-1}R$, $u(S^{-1}R) = \{\frac{x}{s} : x \in u(R), s \in S\}$, $S \subseteq reg(R)$ and $S \cap Z_I(R) = \emptyset$, then I is a graded weakly 1-absorbing prime ideal of R.*

Proof. (i) : Suppose that $0 \neq \frac{x}{s}\frac{y}{t}\frac{z}{u} \in S^{-1}I$ for some nonunits $\frac{x}{s}, \frac{y}{t}, \frac{z}{u} \in h(S^{-1}R)$. Then $0 \neq a(xyz) = (ax)yz \in I$ for some $a \in S$. Here, ax, y, z are nonunits in $h(R)$. Otherwise, we would have $\frac{x}{s}, \frac{y}{t}, \frac{z}{u}$ are units in $S^{-1}R$, a contradiction. As I is a graded weakly 1-absorbing prime ideal of R, we have either $axy \in I$ or $z \in I$. This implies that $\frac{x}{s}\frac{y}{t} = \frac{axy}{ast} \in S^{-1}I$ or $\frac{z}{u} \in S^{-1}I$. Thus, $S^{-1}I$ is a graded weakly 1-absorbing prime ideal of $S^{-1}R$.

(ii) : Let $0 \neq xyz \in I$ for some nonunits $x, y, z \in h(R)$. Since $S \subseteq reg(R)$, we conclude that $0 \neq \frac{x}{1}\frac{y}{1}\frac{z}{1} \in S^{-1}I$. Here, $\frac{x}{1}, \frac{y}{1}, \frac{z}{1}$ are nonunits in $h(S^{-1}R)$. Since $S^{-1}I$ is a graded weakly 1-absorbing prime ideal of $S^{-1}R$, we conclude either $\frac{x}{1}\frac{y}{1} = \frac{xy}{1} \in S^{-1}I$ or $\frac{z}{1} \in S^{-1}I$. Then there exists $s \in S$ such that $sxy \in I$ or $sz \in I$. We can assume that $sxy \in I$. If $xy \notin I$, then we have $s \in Z_I(R) \cap S$ which is a contradiction. Thus we have $xy \in I$. In other case, similarly, we get $z \in I$. Therefore, I is a graded weakly 1-absorbing prime ideal of R. □

Theorem 3.17. *Let $P = \bigoplus_{g\in G} P_g$ be a graded weakly 1-absorbing prime ideal of R and $g \in G$. Then, $(P_{g^2} :_{R_e} xy) = P_e \cup (0 :_{R_e} xy)$ where $x, y \in R_g$ are nonunits such that $xy \notin P$.*

Proof. Clearly $(0:_{R_e} xy) \subseteq (P_{g^2} :_{R_e} xy)$. Let $z \in P_e \subseteq P$. This implies that $xyz \in P \cap R_{g^2} = P_{g^2}$ and so $z \in (P_{g^2} :_{R_e} xy)$. Hence, $P_e \cup (0 :_{R_e} xy) \subseteq (P_{g^2} :_{R_e} xy)$. Now, we will show that $(P_{g^2} :_{R_e} xy) \subseteq (0 :_{R_e} xy) \cup P_e$. Let $z \in (P_{g^2} :_{R_e} xy)$. Then, we have $xyz \in P_{g^2} \subseteq P$. If z is a unit, then we have $xy \in P$, a contradiction. Suppose that z is a nonunit of R. If $xyz \neq 0$, then $z \in P \cap R_e = P_e$. So assume that $xyz = 0$. It gives $z \in (0 :_{R_e} xy)$. Thus we have $z \in P_e \cup (0 :_{R_e} xy)$. Therefore, $(P_{g^2} :_{R_e} xy) = P_e \cup (0 :_{R_e} xy)$. □

Let $R = \bigoplus_{g\in G} R_g$ be a graded ring. Recall from [18] that R is said to be a graded field if every nonzero homogenous element is a unit in R.

Theorem 3.18. *Suppose that R_1, R_2 be two G-graded commutative rings that are not graded fields and $R = R_1 \times R_2$. Let P be a nonzero proper graded ideal of R. The following statements are equivalent.*

- (i) *P is a graded weakly 1-absorbing prime ideal of R.*
- (ii) *$P = P_1 \times R_2$ for some graded prime ideal P_1 of R_1 or $P = R_1 \times P_2$ for some graded prime ideal P_2 of R_2.*
- (iii) *P is a graded prime ideal of R.*
- (iv) *P is a graded weakly prime ideal of R.*
- (v) *P is a graded 1-absorbing prime ideal of R.*

Proof. $(i) \Rightarrow (ii):$ Let P be a nonzero proper graded ideal of R. Then we can write $P = P_1 \times P_2$ for some graded ideals P_1 of R_1 and P_2 of R_2. Since P is nonzero, $P_1 \neq 0$ or $P_2 \neq 0$. Without loss of generality, we may assume that $P_1 \neq 0$. Then there exists a homogeneous element $0 \neq x \in P_1$. Since P is a graded weakly 1-absorbing prime ideal and $(0, 0) \neq (1, 0)(1, 0)(x, 1) \in P$, we conclude either $(1, 0) \in P$ or $(x, 1) \in P$. Then we have either $P_1 = R_1$ or $P_2 = R_2$. Assume that $P_1 = R_1$. Now we will show that P_2 is a graded prime ideal of R_2. Let $yz \in P_2$ for some $y, z \in h(R_2)$. If y or z is a unit, then we have either $y \in P_2$ or $z \in P_2$. So assume that y, z are nonunits in $h(R_2)$. Since R_1 is not a graded field, there exists a nonzero nonunit $t \in h(R_1)$. This implies that $(0, 0) \neq (t, 1)(1, y)(1, z) = (t, yz) \in P$. As P is a graded weakly 1-absorbing prime ideal of R, we conclude either $(t, 1)(1, y) = (t, y) \in P$ or $(1, z) \in P$. Thus we get $y \in P_2$ or $z \in P_2$ and so P_2 is a graded prime ideal of R_2. In other case, one can similarly show that $P = P_1 \times R_2$ and P_1 is a graded prime ideal of R_1.

$(ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i):$ It is obvious.

$(iii) \Rightarrow (v):$ It is clear.

$(v) \Rightarrow (i):$ It is straightforward. □

Definition 3.19. *Let R be a ring and M be an R-module. A proper submodule N of M is called a 1-absorbing R-submodule if whenever $xym \in N$ where $x, y \in R$ are nonunits, $m \in M$, then either $xy \in (N :_R M)$ or $m \in N$.*

Theorem 3.20. *Let $P = \bigoplus_{g\in G} P_g$ be a graded 1-absorbing prime ideal of $G(R)$. If $P_g \neq R_g$, then P_g is a 1-absorbing R_e-submodule of R_g.*

Proof. Let $xyr \in P_g \subseteq P$ for some nonunits $x, y \in R_e$ and $r \in R_g$. As P is graded 1-absorbing prime ideal, $xy \in P$ or $r \in P$. This implies that $xy \in (P_g :_{R_e} R_g)$ since $xyR_g \subseteq PR_g \subseteq P \cap R_g = P_g$ or $r \in P \cap R_g = P_g$. □

Definition 3.21. *Let $P = \bigoplus_{g\in G} P_g$ be a graded ideal of $G(R)$. A graded component P_g of P is called 1-absorbing prime subgroup of R_g if $xyz \in P_g$ for some nonunits $x, y, x \in h(R)$ implies either $xy \in P_g$ or $z \in P_g$.*

Proposition 3.22. *Let $P = \bigoplus_{g\in G} P_g$ be a graded ideal of $G(R)$. If P_g is a 1-absorbing prime subgroup of R_g for all $g \in G$, then P is a graded 1-absorbing prime ideal of R.*

Proof. Suppose $xyz \in P$ for some nonunits $x, y, z \in h(R)$. Then, $xyz \in P_g$ for some $g \in G$. Since P_g is 1-absorbing prime subgroup of R_g, $xy \in P_g$ or $z \in P_g$. This gives $xy \in P$ or $z \in P$, as needed. □

Let M be an R-module. The idealization $R \ltimes M = \{(r, m) : r \in R \text{ and } m \in M\}$ of M is a commutative ring with componentwise addition and multiplication: $(x, m_1)+(y, m_2) = (x+y, m_1+m_2)$ and $(x, m_1)(y, m_2) = (xy, xm_2 + ym_1)$ for each $x, y \in R$ and $m_1, m_2 \in M$. Let G be an Abelian group and M be a G-graded R-module. Then $X = R \ltimes M$ is a G-graded ring by $X_g = R_g \ltimes M_g = R_g \oplus M_g$ for all $g \in G$. Note that, X_g is an additive subgroup of X for all $g \in G$. Also, for $g, h \in G$, $X_gX_h = (R_g \ltimes M_g)(R_h \ltimes M_h) = R_gR_h \ltimes (R_gM_h+R_hM_g) \subseteq R_{gh} \ltimes (M_{gh}+M_{hg}) \subseteq R_{gh} \ltimes M_{gh} = X_{gh}$ as G is Abelian (see [2, 17]).

Theorem 3.23. *Let G be an Abelian group, M be a G-graded R-module and P be an ideal of R. Then, the following statements are equivalent.*

(*i*) *$P \ltimes M$ is a graded weakly 1-absorbing prime ideal of $R \ltimes M$.*

(*ii*) *P is a graded weakly 1-absorbing prime ideal of R and if $x_{g_1}y_{g_2}z_{g_3} = 0$ such that $x_{g_1}y_{g_2} \notin P$ and $z_{g_3} \notin P$ for some nonunit elements $x_{g_1}, y_{g_2}, z_{g_3}$ in $h(R)$, where $g_1, g_2, g_3 \in G$, then $x_{g_1}y_{g_2}M_{g_3} = x_{g_1}z_{g_3}M_{g_2} = y_{g_2}z_{g_3}M_{g_1} = 0$.*

Proof. $(i) \Rightarrow (ii)$: By [17, Theorem 3.3], P is a graded ideal of R. Suppose that $0 \neq abc \in P$ where a, b, c are nonunits in $h(R)$. Since $(0, 0) \neq (a, 0)(b, 0)(c, 0) \in P \ltimes M$ and $(a, 0), (b, 0), (c, 0)$

are nonunits in $h(R \ltimes M)$, we get $(a,0)(b,0) \in P \ltimes M$ or $(c,0) \in P \ltimes M$. Thus, we conclude that $ab \in P$ or $c \in P$, as needed. Now suppose $x_{g_1}y_{g_2}z_{g_3} = 0$ such that $x_{g_1}y_{g_2} \notin P$ and $z_{g_3} \notin P$ for some nonunit elements $x_{g_1}, y_{g_2}, z_{g_3}$ in $h(R)$, where $g_1, g_2, g_3 \in G$. Let $x_{g_1}y_{g_2}M_{g_3} \neq 0$. Then there exists $m_{g_3} \in M_{g_3}$ such that $x_{g_1}y_{g_2}m_{g_3} \neq 0$. This gives $(0,0) \neq (x_{g_1},0)(y_{g_2},0)(z_{g_3},m_{g_3}) = (0, x_{g_1}y_{g_2}m_{g_3}) \in P \ltimes M$ for some nonunits $(x_{g_1},0)$, $(y_{g_2},0)$, $(z_{g_3},m_{g_3}) \in h(R \ltimes M)$ and $P \ltimes M$ is a graded weakly 1-absorbing prime ideal, we have $(x_{g_1},0)(y_{g_2},0) = (x_{g_1}y_{g_2},0) \in P \ltimes M$ or $(z_{g_3},m_{g_3}) \in P \ltimes M$. This gives $x_{g_1}y_{g_2} \in P$ or $z_{g_3} \in P$, a contradiction. Hence, $x_{g_1}y_{g_2}M_{g_3} = 0$. Similar argument shows that $x_{g_1}z_{g_3}M_{g_2} = y_{g_2}z_{g_3}M_{g_1} = 0$.

$(ii) \Rightarrow (i)$: By [17, Theorem 3.3], $P \ltimes M$ is a graded ideal of $R \ltimes M$. Assume that $(0,0) \neq (x_{g_1},m_{g_1})(y_{g_2},m_{g_2})(z_{g_3},m_{g_3}) = (x_{g_1}y_{g_2}z_{g_3}, x_{g_1}y_{g_2}m_{g_3} + x_{g_1}z_{g_3}m_{g_2} + y_{g_2}z_{g_3}m_{g_1}) \in P \ltimes M$ for some nonunits $(x_{g_1},m_{g_1}), (y_{g_2},m_{g_2}), (z_{g_3},m_{g_3})$ in $h(R \ltimes M)$. Then we get $x_{g_1}y_{g_2}z_{g_3} \in P$ for some nonunits $x_{g_1}, y_{g_2}, z_{g_3} \in h(R)$.

Case 1: Assume that $x_{g_1}y_{g_2}z_{g_3} = 0$. If $x_{g_1}y_{g_2} \notin P$ and $z_{g_3} \notin P$, we have $x_{g_1}y_{g_2}M_{g_3} = x_{g_1}z_{g_3}M_{g_2} = y_{g_2}z_{g_3}M_{g_1} = 0$. This implies that $x_{g_1}y_{g_2}m_{g_3} + x_{g_1}z_{g_3}m_{g_2} + y_{g_2}z_{g_3}m_{g_1} = 0$ and so $(x_{g_1},m_{g_1})(y_{g_2},m_{g_2})(z_{g_3},m_{g_3}) = (0,0)$ giving a contradiction. Hence, we must have $x_{g_1}y_{g_2} \in P$ or $z_{g_3} \in P$. This gives $(x_{g_1},m_{g_1})(y_{g_2},m_{g_2}) \in P \ltimes M$ or $(z_{g_3},m_{g_3}) \in P \ltimes M$.

Case 2: Now, assume that $x_{g_1}y_{g_2}z_{g_3} \neq 0$. This gives $x_{g_1}y_{g_2} \in P$ or $z_{g_3} \in P$ since P is graded weakly 1-absorbing prime ideal. Then we conclude that $(x_{g_1},m_{g_1})(y_{g_2},m_{g_2}) \in P \ltimes M$ or $(z_{g_3},m_{g_3}) \in P \ltimes M$ which completes the proof. □

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