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On existence results for hybrid $\psi-$Caputo multi-fractional differential equations with hybrid conditions

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ABSTRACT

In this paper, we study the existence and uniqueness results of a fractional hybrid boundary value problem with multiple fractional derivatives of $\psi-$Caputo with different orders. Using a useful generalization of Krasnoselskii's fixed point theorem, we have established results of at least one solution, while the uniqueness of solution is derived by Banach's fixed point. The last section is devoted to an example that illustrates the applicability of our results.

RESUMEN

En este artículo, estudiamos los resultados de existencia y unicidad de un problema de valor en la frontera fraccional híbrido con múltiples derivadas fraccionarias de $\psi-$Caputo con diferentes órdenes. Usando una generalización útil del teorema del punto fijo de Krasnoselskii, establecemos resultados de al menos una solución, mientras que la unicidad de dicha solución se obtiene a partir del punto fijo de Banach. La última sección está dedicada a un ejemplo que ilustra la aplicabilidad de nuestros resultados.

Keywords and Phrases: $\psi-$fractional derivative, fractional differential equation, hybrid conditions, fixed point, existence, uniqueness.

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1 Introduction

Fractional differential equations have received great attention of many researchers working in different disciplines of science and technology, especially, since they have found that certain thermal [3], electrochemical [4] and viscoelastic [16] systems are governed by fractional differential equations. Recently some publications show the importance of fractional differential equations in the mathematical modeling of many real-world phenomena. For example ecological models [10], economic models [20], physics [12], fluid mechanics [21]. There are many studies on fractional differential equations with distinct kinds of fractional derivatives in the literature, such as Riemann-Liouville fractional derivative, Caputo fractional derivative, and Grunwald Letnikov fractional derivative, etc. For example, see [11, 14, 15]. Very recently, a new kind of fractional derivative the $\psi-$Caputo's derivative, was introduced by Almeida in [1], the main advantage of this derivative is the freedom of choices of the kernels of the derivative by choosing different functions ψ, which gives us some well known fractional derivatives such Caputo, Caputo-Erdelyi-Koper and Caputo Hadamard derivative. For more details on the $\psi-$Caputo and fractional differential equation involving $\psi-$Caputo, we refer the reader to a series of papers [1, 2, 7] and the references cited therein.

Nowadays, many researchers have shown the interest of quadratic perturbations of nonlinear differential equations, these kind of differential equations are known under the name of hybrid differential equations. Some recent works regarding hybrid differential equations can be found in [8, 13, 17, 23] and the references cited therein. Dhage and Lakshmikantham [6] discussed the existence and uniqueness theorems of the solution to the ordinary first-order hybrid differential equation with perturbation of the first type

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{u(t)}{g(t,u(t))}\right) = f(t,u(t)), \quad \text{a.e.} \quad t \in [t_0, t_0+T], \\ \\ u(t_0) = u_0, \quad u_0 \in \mathbb{R}, \end{cases}$$

where $t_0, T \in \mathbb{R}$ with $T > 0$, $g : [t_0, t_0+T] \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $f : [t_0, t_0+T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. By using the fixed point theorem in Banach algebra, the authors obtained the existence results.

In [9], Dong *et al.*, established the existence and the uniqueness of solutions for the following implicit fractional differential equation

$$\begin{cases} {}^cD^p u(t) = f(t, u(t), {}^c D^p u(t)), \quad t \in J := [0,T], \quad 0 < p \leq 1, \\ \\ u(0) = u_0, \end{cases}$$

where ${}^cD^p$ is the Caputo fractional derivative, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

Sitho *et al.* [17] studied existence results for the initial value problems of hybrid fractional sequential integro-differential equations:

$$\begin{cases} D^p\left[\dfrac{D^q u(t)-\displaystyle\sum_{i=1}^{m} I^{\eta_i} g_i(t,u(t))}{h(t,x(t))}\right] = f(t,u(t),I^{\gamma}x(t)), \quad t\in J, \\ \\ u(0)=0, \qquad D^q u(0)=0, \end{cases}$$

where D^p, D^q denotes the Riemann-Liouville fractional derivative of order p, q respectively and $0<p,q\leq 1$, I^{η_i} is the Riemann-Liouville fractional integral of order $\eta_i>0$, $h\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $f\in C(J\times\mathbb{R}^2,\mathbb{R})$ and $g_i\in C(J\times\mathbb{R},\mathbb{R})$ with $g_i(0,0)=0$, $i=1,\ldots,m$.

In 2019, Derbazi *et al.* [8] proved the existence of solutions for the fractional hybrid boundary value problem

$${}^cD^p\left[\frac{u(t)-g(t,u(t))}{h(t,u(t))}\right] = f(t,u(t)), \quad t\in J,$$

with the fractional hybrid boundary value conditions

$$\begin{cases} a_1\left[\dfrac{u(t)-g(t,u(t))}{h(t,u(t))}\right]_{t=0} + b_1\left[\dfrac{u(t)-g(t,u(t))}{h(t,u(t))}\right]_{t=T} = \upsilon_1, \\ \\ a_2\,{}^cD^{\delta}\left[\dfrac{u(t)-g(t,u(t))}{h(t,u(t))}\right]_{t=\xi} + b_2\,{}^cD^{\delta}\left[\dfrac{u(t)-g(t,u(t))}{h(t,u(t))}\right]_{t=T} = \upsilon_2\ ,\ \xi\in J, \end{cases}$$

where $1<p\leq 2$, $0<\delta\leq 1$, $\xi\in J$ and $a_1,a_2,b_1,b_2,\upsilon_1,\upsilon_2$ are real constants. Moreover, two fractional derivatives of Caputo type appeared in the above problem.

Motivated by these works, we mainly investigate the existence and uniqueness of solutions for a class of hybrid differential equations of arbitrary fractional order of the form

$$\begin{aligned} &{}^cD^{p;\psi}\left[\frac{{}^cD^{q;\psi}u(t)-\displaystyle\sum_{i=1}^{m} I^{\eta_i;\psi} g_i(t,u(t))}{h(t,u(t))}\right] = \\ &\qquad f\left(t,u(t),{}^cD^{p;\psi}\left[\frac{{}^cD^{q;\psi}u(t)-\displaystyle\sum_{i=1}^{m} I^{\eta_i;\psi} g_i(t,u(t))}{h(t,u(t))}\right]\right), \quad t\in J, \quad (1.1) \end{aligned}$$

endowed with the hybrid fractional integral boundary conditions

$$\begin{cases} u(0) = 0, \quad {}^cD^{q;\psi}u(0) = 0, \\ a_1 \left[\dfrac{{}^cD^{q;\psi}u(t) - \sum\limits_{i=1}^{m} I^{\eta_i;\psi} g_i(t,u(t))}{h(t,u(t))} \right]_{t=0} + b_1 \left[\dfrac{{}^cD^{q;\psi}u(t) - \sum\limits_{i=1}^{m} I^{\eta_i;\psi} g_i(t,u(t))}{h(t,u(t))} \right]_{t=T} = \upsilon_1, \\ a_2\, {}^cD^{\delta;\psi} \left[\dfrac{{}^cD^{q;\psi}u(t) - \sum\limits_{i=1}^{m} I^{\eta_i;\psi} g_i(t,u(t))}{h(t,u(t))} \right]_{t=\xi} + \\ \qquad b_2\, {}^cD^{\delta;\psi} \left[\dfrac{{}^cD^{q;\psi}u(t) - \sum\limits_{i=1}^{m} I^{\eta_i;\psi} g_i(t,u(t))}{h(t,u(t))} \right]_{t=T} = \upsilon_2\ , \quad \xi \in J, \end{cases} \tag{1.2}$$

where $J := [0,T]$, $D^{p;\psi}$, $D^{q;\psi}$ and $D^{\delta;\psi}$ denote the $\psi-$Caputo fractional derivative of order $2 < p \leq 3$ and $0 < q, \delta \leq 1$ respectively, $I^{\eta_i;\psi}$ is the $\psi-$Riemann-Liouville fractional integral of order $\eta_i > 0$, $h \in C(J\times\mathbb{R}, \mathbb{R}\setminus\{0\})$, $f \in C(J\times\mathbb{R}^2, \mathbb{R})$ and $g_i \in C(J\times\mathbb{R}, \mathbb{R})$ with $g_i(0,0) = 0$, $i = 1, \ldots, m$, a_1, a_2, b_1, b_2, υ_1, υ_2 are real constants such that $b_1 \neq 0$ and

$$2\big(a_2\Psi_0^{2-\delta}(\xi) + b_2\Psi_0^{2-\delta}(T)\big) - \Psi_0^1(T)(2-\delta)\big(a_2\Psi_0^{1-\delta}(\xi) + b_2\Psi_0^{1-\delta}(T)\big) \neq 0.$$

The rest of the paper is arranged as follows. Section 2 gives some background material needed in this paper, such as fractional differential equations and fixed point theorems. Section 3 treats the main results concerning the existence and uniqueness results of the solution for the given problem (1.1)-(1.2) by employing hybrid fixed point theorem for a sum of two operators in Banach algebra space and Banach's fixed point. In the last section, an example is presented to illustrate our results.

2 Preliminaries

In this section, we introduce some preliminaries and lemmas that will be used throughout this paper. We will prove an auxiliary lemma, which plays a key role in defining a fixed point problem associated with the given problem.

Let $\psi : J \to \mathbb{R}$ an increasing function satisfying $\psi'(t) \neq 0$ for all $t \in J$. For the sake of simplicity, we set $\Psi_0^r(t) = (\psi(t) - \psi(0))^r$.

Definition 2.1 ([2])**.** *The $\psi-$Riemann-Liouville fractional integral of order $(p > 0)$ of an integrable* *function* $g : [0,\infty) \to \mathbb{R}$ *is defined by*

$$I^{p;\psi} g(t) = \frac{1}{\Gamma(p)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} g(s)ds, \qquad 0 < s < t.$$

Definition 2.2 ([2]). *The* $\psi-$*Caputo fractional derivative of order* p $(n-1 < p < n \in \mathbb{N})$ *of a function* $g \in C^n[0,\infty)$ *is defined by*

$$^{c}D^{p;\psi} g(t) = \frac{1}{\Gamma(p-n)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-n-1} D_{\psi}^{n}\ g(s)ds, \qquad 0 < s < t,$$

where $n = [p]+1$ *and* $D_{\psi}^{n} = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}$. *In case, if* $2 < p \leq 3$, *we have*

$$^{c}D^{p;\psi} g(t) = \frac{1}{\Gamma(p-3)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-4} D_{\psi}^{3} g(s)ds, \qquad 0 < s < t.$$

Lemma 2.3 ([2]). *Let* $p > 0$. *The following hold*

- *If* $g \in C(J,\mathbb{R})$, *then*

$$^{c}D^{p;\psi} I^{p;\psi} g(t) = g(t), \quad t \in J.$$

- *If* $g \in C^n(J,\mathbb{R})$, $n-1 < p < n$, *then*

$$I^{p;\psi\, c}D^{p;\psi} g(t) = g(t) - \sum_{k=0}^{n-1} c_k \Psi_0^k(t), \quad t \in J,$$

where $c_k = \dfrac{D_{\psi}^{k} g(0)}{k!}$.

Lemma 2.4. *Let* $2 < p < 3$, $0 < q < 1$. *For any functions* $F \in C(J,\mathbb{R})$, $H \in C(J, \mathbb{R}\setminus\{0\})$ *and* $G_i \in C(J,\mathbb{R})$ *with* $G_i(0) = 0$, $i = 1,\ldots,m$, *the following linear fractional boundary value problem*

$$D^{p;\psi}\left[\frac{^{c}D^{q;\psi} u(t) - \sum_{i=1}^{m} I^{\eta_i;\psi} G_i(t)}{H(t)}\right] = F(t), \quad 2 < p \leq 3, \quad 0 < q \leq 1, \quad t \in J, \tag{2.1}$$

supplemented with the following conditions

$$
\begin{cases}
u(0)=0, \quad {}^cD^{q;\psi}u(0)=0,\\[2ex]
a_1\left[\dfrac{{}^cD^{q;\psi}u(t)-\displaystyle\sum_{i=1}^{m}I^{\eta_i;\psi}G_i(t)}{H(t)}\right]_{t=0}+b_1\left[\dfrac{{}^cD^{q;\psi}u(t)-\displaystyle\sum_{i=1}^{m}I^{\eta_i;\psi}G_i(t)}{H(t)}\right]_{t=T}=\upsilon_1,\\[2ex]
a_2\,{}^cD^{\delta;\psi}\left[\dfrac{{}^cD^{q;\psi}u(t)-\displaystyle\sum_{i=1}^{m}I^{\eta_i;\psi}G_i(t)}{H(t)}\right]_{t=\xi}+\\
\qquad\qquad b_2\,{}^cD^{\delta;\psi}\left[\dfrac{{}^cD^{q;\psi}u(t)-\displaystyle\sum_{i=1}^{m}I^{\eta_i;\psi}G_i(t)}{H(t)}\right]_{t=T}=\upsilon_2, \quad \xi\in J,
\end{cases}
\tag{2.2}
$$

has a unique solution, which is given by

$$
\begin{aligned}
u(t)&=I^{q;\psi}\Big(H(s)I^{p;\psi}F(s)\Big)(t)+\sum_{i=1}^{m}I^{\eta_i+q;\psi}G_i(s)(t)\\
&+I^{q;\psi}\left(H(s)\Big(\Psi_0^1(s)\Omega_3-\Psi_0^2(s)\Omega_2\Big)\Big(\frac{\upsilon_1}{b_1}-I^{p;\psi}F(s)\Big)\right)(t)\\
&+\Omega_1\Big(\upsilon_2-a_2I^{p-\delta;\psi}F(\xi)-b_2I^{p-\delta;\psi}F(T)\Big)I^{q;\psi}\left(H(s)\Big(\Psi_0^2(s)-\Psi_0^1(T)\Psi_0^1(s)\Big)\right)(t),
\end{aligned}
\tag{2.3}
$$

where

$$
\Omega_1=\frac{\Gamma(3-\delta)}{2\big(a_2\Psi_0^{2-\delta}(\xi)+b_2\Psi_0^{2-\delta}(T)\big)-\Psi_0^1(T)(2-\delta)\big(a_2\Psi_0^{1-\delta}(\xi)+b_2\Psi_0^{1-\delta}(T)\big)},
$$

$$
\Omega_2=\frac{a_2\Psi_0^{1-\delta}(\xi)+b_2\Psi_0^{1-\delta}(T)}{\Gamma(2-\delta)\Omega_1}, \quad \Omega_3=1+\Omega_2\Psi_0^1(T).
$$

Proof. Applying the $\psi-$Caputo fractional integral of order p to both sides of equation in (2.1) and using Lemma 2.3, we get

$$
\frac{{}^cD^{q;\psi}u(t)-\displaystyle\sum_{i=1}^{m}I^{\eta_i;\psi}G_i(t)}{H(t)}=I^{p;\psi}F(t)+c_0+c_1\Psi_0^1(t)+c_2\Psi_0^2(t),
\tag{2.4}
$$

where $c_0,c_1,c_2\in\mathbb{R}$.

Next, applying the $\psi-$Caputo fractional integral of order q to both sides (2.4), we get

$$
\begin{aligned}
u(t) = {} & I^{q;\psi}\Big(H(s)I^{p;\psi}F(s)\Big)(t)+\sum_{i=1}^{m} I^{\eta_i+q;\psi}G_i(s)(t) \\
& + I^{q;\psi}\Big(H(s)\Big(c_0+c_1\Psi_0^1(s)+c_2\Psi_0^2(s)\Big)\Big)(t)+c_3, \qquad c_3\in\mathbb{R}.
\end{aligned} \tag{2.5}
$$

With the help of conditions $u(0)=0$ and ${}^cD^{q;\psi}u(0)=0$, we find, $c_3=0$ and $c_0=0$ respectively.

Applying the boundary conditions (2.2), and from (2.4), we obtain

$$
c_1\Psi_0^1(T)+c_2\Psi_0^2(T)=\frac{\upsilon_1}{b_1}-I^{p;\psi}F(T),
$$

and

$$
\begin{aligned}
\frac{c_1}{\Gamma(2-\delta)}\Big(a_2\Psi_0^{1-\delta}(\xi)+b_2\Psi_0^{1-\delta}(T)\Big)&+\frac{2c_2}{\Gamma(3-\delta)}\Big(a_2\Psi_0^{2-\delta}(\xi)+b_2\Psi_0^{2-\delta}(T)\Big) \\
&=\upsilon_2-a_2I^{p-\delta;\psi}F(\xi)-b_2I^{p-\delta;\psi}F(T).
\end{aligned}
$$

Solving the resulting equations for c_1 and c_2, we find that

$$
c_1=\Big(\frac{\upsilon_1}{b_1}-I^{p;\psi}F(T)\Big)\Omega_3-\Big(\upsilon_2-a_2I^{p-\delta;\psi}F(\xi)-b_2I^{p-\delta;\psi}F(T)\Big)\Omega_1\Psi_0^1(T),
$$

$$
c_2=\Big(\upsilon_2-a_2I^{p-\delta;\psi}F(\xi)-b_2I^{p-\delta;\psi}F(T)\Big)\Omega_1-\Big(\frac{\upsilon_1}{b_1}-I^{p;\psi}F(T)\Big)\Omega_2.
$$

Inserting c_1 and c_2 in (2.5), which leads to the solution system (2.3). □

Let $E=C(J,\mathbb{R})$ be the Banach space of continuous real-valued functions defined on J. We define in E a norm $\|\cdot\|$ by

$$
\|u\|=\sup_{t\in J}|u(t)|,
$$

and a multiplication by

$$
(uv)(t)=u(t)v(t), \quad \forall t\in J.
$$

Clearly E is a Banach algebra with above defined supremum norm and multiplication.

Lemma 2.5 ([5]). *Let S be a nonempty, convex, closed, and bounded set such that $S\subseteq E$, and let $A:E\to E$ and $B:S\to E$ be two operators which satisfy the following:*

(1) A is contraction,

(2) B is completely continuous, and

(3) $u=Au+Bv$, for all $v\in S\Rightarrow u\in S$.

Then the operator equation $u=Au+Bu$ has at least one solution in S.

Theorem 2.6 ([18]). *Let S be a non-empty closed convex subset of a Banach space E, then any contraction mapping A of S into itself has a unique fixed point.*

3 Main result

In this section, we derive conditions for the existence and uniqueness of a solution for the problem (1.1)-(1.2).

The following assumptions are necessary in obtaining the main results.

(H1) The functions $h \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, and $f \in C(J \times \mathbb{R}^2, \mathbb{R})$ are continuous, and there exist bounded functions $L, M : J \to [0, \infty)$, such that

$$|h(t, u(t)) - h(t, v(t))| \leq L(t)|u(t) - v(t)|,$$

and

$$|f(t, u(t), v(t)) - f(t, \overline{u}(t), \overline{v}(t))| \leq M(t)\big(|u(t) - \overline{u}(t)| + |v(t) - \overline{v}(t)|\big),$$

for $t \in J$ and $u, v, \overline{u}, \overline{v} \in \mathbb{R}$.

(H2) There exist functions $\vartheta, \chi, \varphi_i \in C(J, [0, \infty))$ such that

$$|f(t, u(t), v(t))| \leq \vartheta(t) \quad \text{for each } t, u \in J \times \mathbb{R},$$

$$|h(t, u(t))| \leq \chi(t) \quad \text{for each } t, u \in J \times \mathbb{R},$$

$$|g_i(t, u(t))| \leq \varphi_i(t) \quad \text{for each } t, u \in J \times \mathbb{R}, i = 1, \dots, m,$$

for $t \in J$ and $u \in \mathbb{R}$.

(H3) The functions $g_i \in C(J \times \mathbb{R}, \mathbb{R})$ are continuous, and there exist bounded functions $K_i : J \to (0, \infty)$, such that

$$|g_i(t, u(t)) - g_i(t, v(t))| \leq K_i(t)|u(t) - v(t)|.$$

We set $L^* = \sup_{t \in J} |L(t)|$, $M^* = \sup_{t \in J} |M(t)|$, $\chi^* = \sup_{t \in J} |\chi(t)|$, $\vartheta^* = \sup_{t \in J} |\vartheta(t)|$ and $\varphi_i^* = \sup_{t \in J} |\varphi_i(t)|$, $K_i^* = \sup_{t \in J} |K_i(t)|$, $i = 1, 2, \dots, m$.

3.1 Existence of solutions

In this subsection, we prove the existence of a solution for the problem (1.1)–(1.2) by applying a generalization of Krasnoselskii's fixed point theorem.

Theorem 3.1. *Assume that hypotheses (H1)–(H2) hold and if*

$$\begin{aligned} \Lambda =& \frac{\Psi_0^p(T)}{\Gamma(p+1)}\left(\frac{\chi^* M^*}{1 - M^*} + \vartheta^* L^*\right)\left(\frac{\Psi_0^q(T)}{\Gamma(q+1)} + \frac{|\Omega_3|\Psi_0^{q+1}(T)}{\Gamma(q+2)} + \frac{2|\Omega_2|\Psi_0^{q+2}(T)}{\Gamma(q+3)}\right) \\ &+ |\Omega_1|(q+4)\frac{\Psi_0^{q+2}(T)}{\Gamma(q+3)}\left(|\upsilon_2|L^* + \frac{|a_2|\Psi_0^{p-\delta}(\xi) + |b_2|\Psi_0^{p-\delta}(T)}{\Gamma(p-\delta+1)}\right. \\ &\times \left.\left(\frac{\chi^* M^*}{1 - M^*} + \vartheta^* L^*\right)\right) + \frac{|\upsilon_1|L^*}{|b_1|}\left(\frac{|\Omega_3|\Psi_0^{q+1}(T)}{\Gamma(q+2)} + \frac{2|\Omega_2|\Psi_0^{q+2}(T)}{\Gamma(q+3)}\right) < 1. \end{aligned} \tag{3.1}$$

Then the problem (1.1)–(1.2) *has at least one solution on* J.

Proof. First, we choose $r>0$ such that

$$\begin{aligned}
r \geq & \chi^*\vartheta^*\frac{\Psi_0^{p+q}(T)}{\Gamma(p+1)\Gamma(q+1)}+\chi^*\Big(\frac{|\Omega_3|\Psi_0^{q+1}(T)}{\Gamma(q+2)}+\frac{2|\Omega_2|\Psi_0^{q+2}(T)}{\Gamma(q+3)}\Big)\Big(\frac{|\upsilon_1|}{|b_1|}+\frac{\Psi_0^{p}(T)}{\Gamma(p+1)}\vartheta^*\Big)\\
&+\chi^*|\Omega_1|\frac{(q+4)\Psi_0^{q+2}(T)}{\Gamma(q+3)}\Big(|\upsilon_2|+\vartheta^*\frac{|a_2|\Psi_0^{p-\delta}(\xi)+|b_2|\Psi_0^{p-\delta}(T)}{\Gamma(p-\delta+1)}\Big)\\
&+\sum_{i=1}^{n}\varphi_i^*\frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q+1)}.
\end{aligned}$$

Set

$$B_r=\{u\in E:\|u\|\leq r\}.$$

Clearly B_r is a closed, convex and bounded subset of the Banach space E.

Let $u(t)$ be a solution of the problem (1.1)–(1.2). Define

$$F_u(t):=f\left(t,u(t),{}^cD^{p;\psi}\left[\frac{{}^cD^{q;\psi}u(t)-\sum_{i=1}^{m}I^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]\right).$$

Then

$${}^cD^{p;\psi}\left[\frac{{}^cD^{q;\psi}u(t)-\sum_{i=1}^{m}I^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]=F_u(t),$$

supplemented with the conditions (1.2), then by Lemma 2.4, we get

$$\begin{aligned}
u(t)=&I^{q;\psi}\Big(h(s,u(s))I^{p;\psi}F_u(s)\Big)(t)+\sum_{i=1}^{m}I^{\eta_i+q;\psi}g_i(s,u(s))(t)+\\
&+I^{q;\psi}\Big(h(s,u(s))\Big(\Psi_0^1(s)\Omega_3-\Psi_0^2(s)\Omega_2\Big)\Big(\frac{\upsilon_1}{b_1}-I^{p;\psi}F_u(s)\Big)\Big)(t)\\
&+\Omega_1\Big(\upsilon_2-a_2I^{p-\delta;\psi}F_u(\xi)-b_2I^{p-\delta;\psi}F_u(T)\Big)I^{q;\psi}\Big(h(s,u(s))\Big(\Psi_0^2(s)-\Psi_0^1(T)\Psi_0^1(s)\Big)\Big)(t),
\end{aligned}$$

Let us define three operators $C_p, C_{p-\delta}:E\to E$ and $D:E\to E$ such that

$$\begin{aligned}
C_pu(t)&=\frac{1}{\Gamma(p)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{p-1}F_u(s)ds,\quad t\in J,\\
C_{p-\delta}u(t)&=\frac{1}{\Gamma(p-\delta)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{p-\delta-1}F_u(s)ds,\quad t\in J,
\end{aligned}$$

and

$$Du(t)=h(t,u(t)),\quad t\in J.$$

Then, using assumptions (H1)–(H2) , we have

$$|C_pu(t)-C_pv(t)|\leq\frac{1}{\Gamma(p)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{p-1}|F_u(s)-F_v(s)|ds, \tag{3.2}$$

and

$$\begin{aligned}|F_u(t)-F_v(t)| &\le |f(t,u(t),F_u(t))-f(t,v(t),F_v(t))| \\ &\le M(t)\big(|u(t)-v(t)|+|F_u(t)-F_v(t)|\big) \\ &\le \frac{M(t)}{1-M(t)}\|u(\cdot)-v(\cdot)\|.\end{aligned} \tag{3.3}$$

By replacing (3.3) in (3.2), we obtain

$$|C_pu(t)-C_pv(t)| \le \frac{M^*\Psi_0^p(T)}{(1-M^*)\Gamma(p+1)}\|u(\cdot)-v(\cdot)\|,$$

and

$$|Du(t)-Dv(t)| \le L^*\|u(\cdot)-v(\cdot)\|,$$

$$|C_pu(t)| \le \frac{\Psi_0^p(T)}{\Gamma(p+1)}\vartheta^*,$$

and

$$|Du(t)| \le \chi^*.$$

Now we define two more operators $A: E \to E$ and $B: B_r \to E$ such that

$$\begin{aligned}Au(t) &= I^{q;\psi}\Big(Du(s)C_pu(s)\Big)(t) + I^{q;\psi}\bigg(Du(s)\Big(\Psi_0^1(s)\Omega_3-\Psi_0^2(s)\Omega_2\Big)\Big(\frac{\upsilon_1}{b_1}-C_pu(s)\Big)\bigg)(t) \\ &\quad + \Omega_1\Big(\upsilon_2-a_2C_{p-\delta}u(\xi)-b_2C_{p-\delta}u(T)\Big)I^{q;\psi}\bigg(Du(s)\Big(\Psi_0^2(s)-\Psi_0^1(T)\Psi_0^1(s)\Big)\bigg)(t),\end{aligned}$$

and

$$Bu(t) = \sum_{i=1}^{m} I^{\eta_i+q;\psi}g_i(s,u(s))(t).$$

We need to show that the two operators A and B satisfy all conditions of Lemma 2.5. This can be achieved in the following steps.

Step 1. First we show that A is a contraction mapping. Let $u(t), v(t) \in B_r$, then we have

$$\begin{aligned}&|Au(t)-Av(t)| \\ &\le I^{q;\psi}\bigg(\big|Du(s)C_pu(s)-Dv(s)C_pv(s)\big|\Big(1+\big|\Psi_0^1(s)\Omega_3-\Psi_0^2(s)\Omega_2\big|\Big)\bigg)(t) \\ &+ I^{q;\psi}\bigg(\frac{|\upsilon_1|}{|b_1|}\big|\Psi_0^1(s)\Omega_3-\Psi_0^2(s)\Omega_2\big|\big|Du(s)-Dv(s)\big|\bigg)(t) \\ &+ |\Omega_1|I^{q;\psi}\bigg(\big|\Psi_0^2(s)-\Psi_0^1(T)\Psi_0^1(s)\big|\Big(|\upsilon_2|\big|Du(s)-Dv(s)\big|+|a_2|\big|Du(s)C_{p-\delta}u(\xi) \\ &- Dv(s)C_{p-\delta}v(\xi)\big|+|b_2|\big|Du(s)C_{p-\delta}u(T)-Dv(s)C_{p-\delta}v(T)\big|\Big)\bigg)(t)\end{aligned}$$

$$\leq I^{q;\psi}\Big(\Big(\big|Du(s)\big|\big|C_pu(s)-C_pv(s)\big|+\big|C_pv(s)\big|\big|Du(s)-Dv(s)\big|\Big)$$
$$\times\Big(1+\big|\Psi_0^1(s)\Omega_3-\Psi_0^2(s)\Omega_2\big|\Big)\Big)(t)+I^{q;\psi}\Big(\frac{|\upsilon_1|}{|b_1|}\big|\Psi_0^1(s)\Omega_3-\Psi_0^2(s)\Omega_2\big|\big|Du(s)-Dv(s)\big|\Big)(t)$$
$$+|\Omega_1|I^{q;\psi}\Big(\big|\Psi_0^2(s)-\Psi_0^1(T)\Psi_0^1(s)\big|\Big(\big|Du(s)-Dv(s)\big|\Big(|\upsilon_2|+|a_2|\big|C_{p-\delta}v(\xi)\big|+|b_2|\big|C_{p-\delta}v(T)\big|\Big)$$
$$+\big|Du(s)\big|\Big(|a_2|\big|C_{p-\delta}u(\xi)-C_{p-\delta}v(\xi)\big|+|b_2|\big|C_{p-\delta}u(T)-C_{p-\delta}v(T)\big|\Big)\Big)\Big)(t)$$

Using the hypotheses (H1)–(H2) and taking the supremum over t, we get

$$\|Au(\cdot)-Av(\cdot)\|\leq\Lambda\|u(\cdot)-v(\cdot)\|. \tag{3.4}$$

Therefore from (3.1), we conclude that the operator A is a contraction mapping.

Step 2. Next, we prove that the operator B satisfies condition (2) of Lemma 2.5, that is, the operator B is compact and continuous on B_r. Therefore first, we show that the operator B is continuous on B_r.

Let $u_n(t)$ be a sequence of functions in B_r converging to a function $u(t)\in B_r$. Then, by the Lebesgue dominant convergence theorem, for all $t\in J$, we have

$$\begin{aligned}\lim_{n\to\infty}Bu_n(t)&=\lim_{n\to\infty}\sum_{i=1}^m\frac{1}{\Gamma(\eta_i+q)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\eta_i+q-1}g_i(s,u_n(s))ds\\&=\sum_{i=1}^m\frac{1}{\Gamma(\eta_i+q)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\eta_i+q-1}\lim_{n\to\infty}g_i(s,u_n(s))ds\\&=\sum_{i=1}^m\frac{1}{\Gamma(\eta_i+q)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\eta_i+q-1}g_i(s,u(s))ds.\end{aligned}$$

Hence $\lim_{n\to\infty}Bu_n(t)=Bu(t)$. Thus B is a continuous operator on B_r.

Further, we show that the operator B is uniformly bounded on B_r. For any $u\in B_r$, we have

$$\begin{aligned}\|Bu(\cdot)\|&\leq\sup_{t\in J}\Big\{\sum_{i=1}^m\frac{1}{\Gamma(\eta_i+q)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\eta_i+q-1}|g_i(s,u(s))|ds\Big\}\\&\leq\sum_{i=1}^m\frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q+1)}\varphi_i^*\leq r.\end{aligned}$$

Therefore $Bu(t)\leq r$, for all $t\in J$, which shows that B is uniformly bounded on B_r.

Now, we show that the operator B is equi-continuous. Let $t_1,t_2\in J$ with $t_1>t_2$. Then for any

$u(t) \in B_r$, we have

$$\begin{aligned}
&|Bu(t_1) - Bu(t_2)| \\
&\leq \sum_{i=1}^{m} \frac{1}{\Gamma(\eta_i + q)} \left| \int_0^{t_2} \psi'(s)\Big((\psi(t_1) - \psi(s))^{\eta_i+q-1} - (\psi(t_2) - \psi(s))^{\eta_i+q-1} \Big) g_i(s, u(s)) ds \right| \\
&+ \sum_{i=1}^{m} \frac{1}{\Gamma(\eta_i + q)} \left| \int_{t_2}^{t_1} \psi'(s)(\psi(t_1) - \psi(s))^{\eta_i+q-1} g_i(s, u(s)) ds \right| \\
&\leq \sum_{i=1}^{m} \frac{\varphi_i^*}{\Gamma(\eta_i + q + 1)} \Big(2|\psi(t_1) - \psi(t_2)|^{\eta_i+q} + \big| \Psi_0^{\eta_i+q}(t_2) - \Psi_0^{\eta_i+q}(t_1) \big| \Big).
\end{aligned}$$

As $t_2 \to t_1$, so the right-hand side tends to zero. Thus B is equi-continuous. Therefore, it follows from the Arzelá–Ascoli theorem that B is a compact operator on B_r. We conclude that B is completely continuous.

Step 3. It remains to verify the condition (3) of Lemma 2.5. For any $v \in B_r$, we have

$$\begin{aligned}
\|u(\cdot)\| &= \|Au(\cdot) + Bv(\cdot)\| \\
&\leq \|Au(\cdot)\| + \|Bv(\cdot)\| \\
&\leq \sup_{t \in J} \Bigg\{ \Bigg| I^{q;\psi}\Big(Du(s) C_p u(s) \Big)(t) + I^{q;\psi}\Bigg(Du(s) \Big(\Psi_0^1(s)\Omega_3 - \Psi_0^2(s)\Omega_2 \Big) \Big(\frac{\upsilon_1}{b_1} - C_p u(s) \Big) \Bigg)(t) \\
&+ \Omega_1 \Big(\upsilon_2 - a_2 C_{p-\delta} u(\xi) - b_2 C_{p-\delta} u(T) \Big) I^{q;\psi}\Bigg(Du(s) \Big(\Psi_0^2(s) - \Psi_0^1(T)\Psi_0^1(s) \Big) \Bigg)(t) \Bigg| \Bigg\} \\
&+ \sup_{t \in J} \Bigg\{ \sum_{i=1}^{m} I^{\eta_i+q;\psi} \big| g_i(s, v(s)) \big|(t) \Bigg\} \\
&\leq \chi^* \vartheta^* \frac{\Psi_0^{p+q}(T)}{\Gamma(p+1)\Gamma(q+1)} + \chi^* \left(\frac{|\Omega_3| \Psi_0^{q+1}(T)}{\Gamma(q+2)} + \frac{2|\Omega_2| \Psi_0^{q+2}(T)}{\Gamma(q+3)} \right) \left(\frac{|\upsilon_1|}{|b_1|} + \frac{\Psi_0^p(T)}{\Gamma(p+1)} \vartheta^* \right) \\
&+ \chi^* |\Omega_1| \frac{(q+4)\Psi_0^{q+2}(T)}{\Gamma(q+3)} \left(|\upsilon_2| + \vartheta^* \frac{|a_2| \Psi_0^{p-\delta}(\xi) + |b_2| \Psi_0^{p-\delta}(T)}{\Gamma(p-\delta+1)} \right) \\
&+ \sum_{i=1}^{n} \varphi_i^* \frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i + q + 1)}.
\end{aligned}$$

Which implies, from the choice of r that $\|u\| \leq r$, and so $u \in B_r$. Hence all conditions of Lemma 2.5 are satisfied. Therefore, the operator equation $u(t) = Au(t) + Bu(t)$ has at least one solution in B_r. Consequently, the problem (1.1)–(1.2) has at least on solution on J. Thus the proof is completed. □

3.2 Uniqueness of solutions

In the next result, we apply the Banach fixed theorem to prove the uniqueness of solutions for the problem (1.1)–(1.2).

Theorem 3.2. *Assume that the hypotheses(H1)–(H3) together with the inequality*

$$\Lambda + \sum_{i=1}^{m} K_i^* \frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q)} < 1.$$

are satisfied, then the problem (1.1)–(1.2) *has an unique solution.*

Proof. According to Lemma 2.4, we define the operator $Q : E \to E$ by

$$Qu(t) = Au(t) + Bu(t).$$

First, we show that $Q(B_r) \subset B_r$. As in the previous proof (**step 3**) of Theorem 3.1, we can obtain for $u \in B_r$ and $t \in J$

$$\begin{aligned}\|Qu(\cdot)\| \leq{}& \chi^*\vartheta^* \frac{\Psi_0^{p+q}(T)}{\Gamma(p+1)\Gamma(q+1)} + \chi^*\left(\frac{|\Omega_3|\Psi_0^{q+1}(T)}{\Gamma(q+2)} + \frac{2|\Omega_2|\Psi_0^{q+2}(T)}{\Gamma(q+3)}\right)\left(\frac{|\upsilon_1|}{|b_1|} + \frac{\Psi_0^{p}(T)}{\Gamma(p+1)}\vartheta^*\right)\\ &+ \chi^*|\Omega_1|\frac{(q+4)\Psi_0^{q+2}(T)}{\Gamma(q+3)}\left(|\upsilon_2| + \vartheta^*\frac{|a_2|\Psi_0^{p-\delta}(\xi) + |b_2|\Psi_0^{p-\delta}(T)}{\Gamma(p-\delta+1)}\right)\\ &+ \sum_{i=1}^{n}\varphi_i^*\frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q+1)} \leq r.\end{aligned}$$

This shows that $Q(B_r) \subset B_r$.
Next, we prove that the operator Q is a contractive operator. For $u, v \in B_r$

$$\|Qu(\cdot) - Qv(\cdot)\| \leq \|Au(\cdot) - Av(\cdot)\| + \|Bu(\cdot) - Bv(\cdot)\|,$$

and

$$\begin{aligned}&\|Bu(\cdot) - Bv(\cdot)\|\\ &\leq \sup_{t\in J}\left\{\sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i+q)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\eta_i+q-1}\big|g_i(s,u(s)) - g_i(s,v(s))\big|ds\right\}\\ &\leq \sum_{i=1}^{m} K_i^*\frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q+1)}\|u(\cdot)-v(\cdot)\|.\end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we get

$$\|Qu(\cdot) - Qv(\cdot)\| \leq \left(\Lambda + \sum_{i=1}^{m} K_i^*\frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q+1)}\right)\|u(\cdot)-v(\cdot)\|.$$

This implies that the operator Q is a contractive operator. Consequently, by Theorem 3.2, we conclude that Q has an unique fixed point, which is a solution of the problem (1.1)–(1.2). This completes the proof. $\square$

4 Example

Consider the following fractional hybrid differential equation

$$\begin{cases} {}^cD^{\frac{5}{2};t}\left[\dfrac{{}^cD^{\frac{3}{4};t}u(t)-\sum\limits_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]=f\left(t,u(t),{}^cD^{\frac{5}{2};t}\left[\dfrac{{}^cD^{\frac{3}{4};t}u(t)-\sum\limits_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]\right), \\ u(0)=0, \quad {}^cD^{\frac{3}{4};t}u(0)=0, \\ 2\left[\dfrac{{}^cD^{\frac{3}{4};t}u(t)-\sum\limits_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]_{t=0}+\dfrac{2}{7}\left[\dfrac{{}^cD^{\frac{3}{4};t}u(t)-\sum\limits_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]_{t=1}=\dfrac{7}{2}, \\ \dfrac{7}{13}\,{}^cD^{\frac{4}{5};t}\left[\dfrac{{}^cD^{\frac{3}{4};t}u(t)-\sum\limits_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]_{t=\frac{4}{5}}+\dfrac{1}{2}\,{}^cD^{\frac{4}{5};t}\left[\dfrac{{}^cD^{\frac{3}{4};t}u(t)-\sum\limits_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]_{t=1}=2\ , \end{cases} \tag{4.1}$$

where

$$\begin{aligned} \sum_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))(s) &= I^{\frac{1}{3};t}\left(\frac{\sin^2 x(s)}{8(s+1)^2}\right)(t)+I^{\frac{3}{2};t}\left(\frac{1}{2\pi\sqrt{81+s^2}}\frac{|x(s)|}{2+|x(s)|}\right)(t) \\ &\quad + I^{\frac{7}{3};t}\left(\frac{\sin x(s)}{3\pi\sqrt{49+s^2}}\right)(t), \end{aligned}$$

$$h(t,u(t))=\frac{e^{-3t}\cos u(t)}{2t+40}+\frac{1}{80}(t^3+1),$$

and

$$\begin{aligned} & f\left(t,u(t),{}^cD^{\frac{5}{2};t}\left[\frac{{}^cD^{\frac{3}{4};t}u(t)-\sum\limits_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]\right) \\ &= \frac{1}{60\sqrt{t+81}}\left(\frac{|x(t)|}{3+|x(t)|}-\arctan\left({}^cD^{\frac{5}{2};t}\left[\frac{{}^cD^{\frac{3}{4};t}u(t)-\sum\limits_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]\right)\right). \end{aligned}$$

Here $T = 1, p = \frac{5}{2}, q = \frac{3}{4}, m = 3, \eta_1 = \frac{1}{3}, \eta_2 = \frac{3}{2}, \eta_3 = \frac{7}{3}, \delta = \frac{4}{5}, a_1 = 2, a_2 = \frac{7}{13}, b_1 = \frac{2}{7}$, $b_2 = \frac{1}{2}, \upsilon_1 = \frac{7}{2}, \upsilon_2 = 2, \xi = \frac{4}{5}$, $g_1 = \dfrac{\sin^2 x(t)}{8(t+1)^2}$, $g_2 = \dfrac{1}{2\pi\sqrt{81+t^2}} \dfrac{|x(t)|}{2+|x(t)|}$, $g_3 = \dfrac{\sin x(t)}{3\pi\sqrt{49+t^2}}$.

The hypotheses (H1), (H2) and (H4) are satisfied with the following positives functions: $L(t) = \dfrac{e^{-3}}{2t+40}$, $M(t) = \vartheta(t) = \dfrac{1}{60\sqrt{t+81}}$, $\varphi_1(t) = K_1(t) = \dfrac{1}{8(t+1)^2}$, $\varphi_2(t) = K_2(t) = \dfrac{1}{2\pi\sqrt{81+t^2}}$, $\varphi_3(t) = K_3(t) = \dfrac{1}{3\pi\sqrt{49+t^2}}$ and $\chi(t) = \dfrac{e^{-3}}{2t+40} + \dfrac{1}{80}(t^3+1)$, which gives us $L^* = \frac{1}{40}$, $M^* = \vartheta^* = \frac{1}{540}, \chi^* = \frac{3}{80}$, $\varphi_1^* = K_1^* = \frac{1}{8}, \varphi_2^* = K_2^* = \frac{1}{18\pi}$, $\varphi_3^* = K_3^* = \frac{1}{21\pi}$.

With the given data, we find that

$$\Omega_1 \simeq 1.81820508, \quad \Omega_2 \simeq 0.60797139, \quad \Omega_3 \simeq 1.60797139,$$

and

$$\Lambda \simeq 0.48820986 < 1.$$

By Theorem 3.1, the problem (4.1) has a solution on $[0, 1]$.

Also, we have

$$\Lambda + \sum_{i=1}^{3} K_i^* \frac{\Psi_0^{\eta_i+q}(1)}{\Gamma(\eta_i + \frac{7}{4})} \simeq 0.61782704 < 1.$$

In view of Theorem 3.2 the problem (4.1) has an unique solution.

5 Conclusion

In this manuscript, we have successfully investigated the existence, uniqueness of the solutions for a new class of $\psi-$Caputo type hybrid fractional differential equations with hybrid conditions. The existence of solutions is provided by using a generalization of Krasnoselskii's fixed point theorem due to Dhage [5], whereas the uniqueness result is achieved by Banach's contraction mapping principle. Also, we have presented an illustrative example to support our main results. In future works, many results can be established when one takes a more generalized operator. Precisely, it will be of interest to study the current problem in this work for the fractional operator with variable order [22], and ψ-Hilfer fractional operator [19].

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