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Numerical analysis of nonlinear parabolic problems with variable exponent and L^1 data

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ABSTRACT

In this paper, we make the numerical analysis of the mild solution which is also an entropy solution of parabolic problem involving the $p(x)-$Laplacian operator with L^1- data.

RESUMEN

En este artículo, realizamos el análisis numérico de la solución mild que también es una solución de entropía del problema parabólico involucrando el operador $p(x)-$Laplaciano con datos en L^1.

Keywords and Phrases: Elliptic-parabolic, numerical iterative method, variable exponent, mild solution, renormalized solution.

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1 Introduction

We consider a bounded open domain $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ with a Lipschitz boundary denoted by $\partial\Omega$. Let $T > 0$ and $p : \overline{\Omega} \to (1, \infty)$ be a continuous function. In this paper, one of our main goals is the numerical approximation of the mild solution of the following nonlinear parabolic problem involving the $p(x)-$Laplacian operator

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f & \text{in } Q \equiv \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $u_0 \in L^1(\Omega)$, $f \in L^1(Q)$. The assumptions on the variable exponent $p(x)$ will be specified later.

Partial differential equations with nonlinearities involving non-constant exponents have attracted an increasing amount of attention on recent years. Their study is an interesting topic which raises many mathematical difficulties (see [1, 2, 14, 16, 27, 30]). There are many results devoted to questions on existence and uniqueness of solutions to problems like (1.1), we refer for example the reader to the bibliography [3, 4, 5, 9, 24, 29] and references therein. Many of these models have already been analyzed for constant exponents of nonlinearity (see the references therein), but it seems to be more realistic to assume the exponent to be variable. From numerical point of view, in the classical evolution problem case where $p(x) \equiv p$, the numerical analysis was firstly considered in [7, 22]. Afterward, Jäger and Kačur [18] and Kačur [20] studied the numerical approximation. Inspired by these works, Maitre [23] proposed a numerical scheme to approximate the mild solutions. On the other side, for problems with variable exponent, in recent years, there are some papers devoted to their numerical analysis (see for example [8, 10, 12, 13, 17, 19, 26]). Thus, in [13] the authors used a quasi-Newton minimization method to approach the solution of the $p(x)-$Lapacian problems; in [12], they present an inverse power method to compute the first homogeneous eigenpair. In [26], an interior penalty discontinuous Galerkin method has been used by the authors to approximate the minimizer of a variational problem related to the $p(x)-$Laplacian. Other authors use finite elements to approximate the solution (see [10]). Nevertheless, there are scarcely papers about the numerical analysis of nonlinear parabolic problems with variable exponent (see for example [11]).

The importance of investigating the problem (1.1) lies in their occurrence in modeling various physical problems involving strong anisotropic phenomena related to electrorheological fluids (an important class of non-Newtonian fluids, see [27]) which are characterized by their ability to change the mechanical properties under the influence of the exterior electromagnetic field. Other important applications are related to image processing, elasticity [30], the processes of filtration in complex media, stratigraphy problems and also mathematical biology. The study of problem (1.1) involves using of generalized Lebesgue and Sobolev spaces *i.e.*, $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ respectively (see [15]).

Throughout this paper we assume that the exponent $p(.)$ appearing in (1.1) is a continuous function $p : \overline{\Omega} \to (1, \infty)$ such that:

$$\begin{cases} \exists C > 0 \ : |p(x) - p(y)| \leq \dfrac{C}{-\log|x-y|} \text{ for every } x, y \in \Omega \text{ with } |x-y| \leq \dfrac{1}{2} \\ \\ \dfrac{2d}{d+2} < p^- := \min_{x\in\overline{\Omega}} p(x) \leq p^+ := \max_{x\in\overline{\Omega}} p(x) < \infty. \end{cases} \tag{1.2}$$

The first condition says that $p(.)$ belongs to the class of log-Hölder continuous functions. These assumptions are used to obtain several regularity results for Sobolev spaces with variable exponents; in particular, $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$.

Our paper was inspired by the work of Maitre (see [23]) where the author studied the numerical analysis of an elliptic-parabolic problem in the context of constant exponent setting.

The rest of this paper is organized as follows: in Section 2, we give some results for the study of (1.1). In Section 3, we recall the notion of mild solution. In Section 4, we proceed to the numerical study, where we show the existence and uniqueness of solution of numerical scheme for the approximation of mild solution and the study of the convergence of this numerical scheme. We conclude this section by numerical tests.

2 Preliminaries

We first recall in what follows some definitions and basic properties of generalized Lebesgue-Sobolev spaces with variable exponent. We define the Lebesgue space with a variable exponent $p(.)$ by

$$L^{p(\cdot)}(\Omega) = \left\{u : \Omega \to \mathbb{R};\ u \text{ is measurable with } \rho_{p(.)}(u) < \infty\right\},$$

where

$$\rho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)} dx,$$

is called a modular. We define a norm, the so-called *Luxemburg norm,* on this space by the formula

$$|u|_{p(.)} = \inf\left\{\mu > 0 :\ \rho_{p(.)}\left(\frac{u}{\mu}\right) \leq 1\right\}.$$

The space $(L^{p(\cdot)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Moreover, if $1 < p^- \leq p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(.)}|v|_{p'(.)}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

We define also the variable Sobolev space

$$W^{1,p(.)}(\Omega) = \left\{ u \in L^{p(.)}(\Omega) : \ |\nabla u| \in L^{p(.)}(\Omega) \right\}.$$

On $W^{1,p(.)}(\Omega)$ we may consider the following norm

$$\|u\|_{1,p(.)} = |u|_{p(.)} + |\nabla u|_{p(.)}.$$

The space $(W^{1,p(.)}(\Omega), \|u\|_{1,p(.)})$ is a separable and reflexive Banach space. Next, we define $W_0^{1,p(.)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$ under the norm

$$\|u\| := |\nabla u|_{p(.)}.$$

The space $(W_0^{1,p(.)}(\Omega), \|u\|)$ is a separable and reflexive Banach space. For the interested reader, more details about Lebesgue and Sobolev spaces with variable exponent can be found in [15] (see also [21]).

Since Ω is bounded and $p : \Omega \to (1, \infty)$ is log-Hölder continuous, the Poincaré inequality holds (see [28])

$$|u|_{p(.)} \leq C|\nabla u|_{p(.)}, \quad \forall\, u \in W_0^{1,p(.)}(\Omega),$$

where C is a constant which depends on Ω and on the function p.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by modular $\rho_{p(.)}$ of the space $L^{p(.)}$. We have the following result (see [28]).

Lemma 2.1. *If u_n, $u \in L^{p(.)}$ and $p^+ < \infty$, then the following relations hold:*

(1) $|u|_{p(.)} > 1 \Rightarrow |u|_{p(.)}^{p^-} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p^+}$;

(2) $|u|_{p(.)} < 1 \Rightarrow |u|_{p(.)}^{p^+} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p^-}$;

(3) $|u|_{p(.)} < 1$ $(\textit{respectively} = 1; > 1) \Longleftrightarrow \rho_{p(.)}(u) < 1\, (\textit{respectively} = 1; > 1)$;

(4) $|u|_{p(.)} \to 0$ $(\textit{respectively} \to \infty) \Longleftrightarrow \rho_{p(.)}(u) \to 0\, (\textit{respectively} \to \infty)$;

(5) $\rho_{p(.)}\left(u/|u|_{p(.)}\right) = 1$.

Following [4], we extend a variable exponent $p : \overline{\Omega} \to [1, +\infty)$ to $\overline{Q} = [0, T] \times \overline{\Omega}$ by setting $p(t, x) := p(x)$ for all $(t, x) \in \overline{Q}$. We also consider the generalized Lebesgue space

$$L^{p(.)}(Q) = \left\{ u : Q \to \mathbb{R} \text{ measurable such that } \iint_Q |u(x,t)|^{p(x)}\, d(x,t) < \infty \right\}$$

endowed with the norm

$$\|u\|_{L^{p(.)}} := \inf \left\{ \mu > 0 : \ \iint_Q \left| \frac{u(x,t)}{\mu} \right|^{p(x)} d(x,t) < 1 \right\}$$

which shares the same properties as $L^{p(.)}(\Omega)$.

Now, we recall the main results for the study of (1.1).

In order to approximate the mild solution of (1.1), let us recall that Ouaro and Traoré have studied in [25] the existence and uniqueness of weak energy and entropy solutions of the following stationary problem associated to the problem (1.1)

$$\begin{cases} u - \operatorname{div} a(x, \nabla u) = f & \text{in} \quad \Omega, \\ \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary and $f \in L^1(\Omega)$. For the vector field $a(x,\xi) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, in addition to be Carathéodory, is the continuous derivative with respect to ξ of the mapping $A : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, *i.e.* $a(x,\xi) = \nabla_\xi A(x,\xi)$ such that:

$$A(x,0) = 0 \quad \text{for almost every } x \in \Omega. \tag{2.2}$$

There exists a positive constant C_1 such that

$$|a(x,\xi)| \leq C_1(j(x) + |\xi|^{p(x)-1}), \tag{2.3}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^d$ where j is a non-negative function in $L^{p'(.)}(\Omega)$, with $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$.

The following inequalities hold

$$(a(x,\xi) - a(x,\eta)).(\xi - \eta) > 0, \tag{2.4}$$

for almost every $x \in \Omega$ and for every ξ, $\eta \in \mathbb{R}^d$, with $\xi \neq \eta$ and

$$\frac{1}{C}|\xi|^{p(x)} \leq a(x,\xi).\xi \leq Cp(x)A(x,\xi), \tag{2.5}$$

for almost every $x \in \Omega$, $C > 0$ and for every $\xi \in \mathbb{R}^d$.

The exponent appearing in (2.3) and (2.5) is defined as follows.

$$\begin{cases} p(.) : \Omega \to \mathbb{R} \ \text{ is a measurable function such that} \\ \\ 1 < p^- := ess\inf_{x\in\Omega} p(x) \leq p^+ := ess\sup_{x\in\Omega} p(x) < \infty. \end{cases} \tag{2.6}$$

For more details, see [24, 25].

As example of models with respect to above assumptions, we can give the following.

Set $A(x,\xi) = \dfrac{1}{p(x)}|\xi|^{p(x)}$, $a(x,\xi) = |\xi|^{p(x)-2}\xi$. Then, we get the $p(x)-$Laplace operator

$$\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u).$$

Note that the weak solution of (2.1) is defined as follows.

Definition 2.2. *A weak solution of* (2.1) *is a function* $u \in W_0^{1,1}(\Omega)$ *such that* $a(.,\nabla u) \in \left(L^1_{loc}(\Omega)\right)^d$ *and*

$$\int_\Omega a(.,\nabla u).\nabla\varphi\,dx + \int_\Omega u\varphi\,dx = \int_\Omega f\varphi\,dx, \tag{2.7}$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$.

A weak energy solution is a weak solution such that $u \in W_0^{1,p(.)}(\Omega)$.

Now, we recall one of main results.

Theorem 2.3. *Assume that* (2.2)–(2.6) *hold and* $f \in L^\infty(\Omega)$. *Then there exists a unique weak energy solution of* (2.1).

We also recall a useful result needed in this paper (see [23]).

Lemma 2.4 ([23])**.** *Let* X *be a Banach space and* C *a convex subset of* X, *containing* 0. *Let* $\bar{T}$ *be a non-expansive map on* C *such that* $\bar{T}(C) \subset C$, *admitting a unique fixed point* x^* *in* C. *Let* λ_k *be a sequence of* $(0,1)$ *verifying*

$$\lim_{k\to\infty}\lambda_k = 1, \quad \prod_{k\geq 0}\lambda_k = 0, \quad \sum_{k\geq 0}|\lambda_{k+1}-\lambda_k| < \infty.$$

Then the sequence (x^k) *generated by the iterative scheme*

$$x^0 \in C, \quad x^{k+1} = \lambda_{k+1}\bar{T}(x^k) \tag{2.8}$$

verifies $\lim_{k\to\infty} x^k - \bar{T}(x^k) = 0$. *Consequently, if all subsequences of* (x^k) *have in turn a subsequence converging to a point of* C, *then the whole sequence* (x^k) *converges toward* x^*.

Recall that a self-mapping $\bar{T}$ *of* C *is non-expansive if*

$$\|\bar{T}(x) - \bar{T}(y)\| \leq \|x-y\| \ \text{ for all } \ x,y \in C.$$

In the next section, we give the definition of mild solution.

3 Notion of mild solution

Let $f \in L^1(0,T;L^1(\Omega))$, $u_0 \in L^1(\Omega)$ and $\varepsilon > 0$ be given. We consider the time discretization of problem (1.1) by an implicit Euler scheme

$$\begin{cases} \dfrac{u^\varepsilon_{n+1} - u^\varepsilon_n}{t_{n+1}-t_n} - \operatorname{div}(|\nabla u^\varepsilon_{n+1}|^{p(x)-2}\nabla u^\varepsilon_{n+1}) = f^\varepsilon_{n+1} \text{ in } \mathcal{D}'(\Omega) \text{ for } n = 0,\ldots,N-1, \\ \\ u^\varepsilon_{n+1} \in W_0^{1,p(.)}(\Omega)\cap L^\infty(\Omega); \end{cases} \tag{3.1}$$

where

$$\begin{cases} N \in \mathbb{N}^*, 0 = t_0 < t_1 < \cdots < t_N \leq T \text{ is a partition of } [0,T]. \\ \\ f_n^\varepsilon \in L^\infty(\Omega) \text{ for } n = 1, \ldots, N \text{ such that } \displaystyle\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|f(t) - f_n^\varepsilon\|_{L^1(\Omega)} dt \to 0 \text{ as } \varepsilon \to 0, \\ \\ \max_{n=1,\ldots,N}(t_n - t_{n-1}) \to 0, T - t_N \to 0 \text{ as } \varepsilon \to 0,\ u_0^\varepsilon \in L^\infty(\Omega) \text{ such that} \\ \\ \|u_0 - u_0^\varepsilon\|_{L^1(\Omega)} \to 0 \text{ as } \varepsilon \to 0, \\ \\ \text{with } u^\varepsilon \text{ the piecewise constant function defined by} \\ \\ u^\varepsilon(t) = u_n^\varepsilon \text{ on } (t_{n-1}, t_n] \text{ with } n = 1, \ldots, N; \quad u^\varepsilon(0) = u_0^\varepsilon. \end{cases} \tag{3.2}$$

Definition 3.1. *A mild solution of* (1.1) *is a function* $u \in C([0,T]; L^1(\Omega))$ *with* $u(0) = u_0 \in L^1(\Omega)$ *such that, for all* $\varepsilon > 0$*, there exists* $(t_0, t_1, \ldots, t_N\,;\, f_1^\varepsilon, f_2^\varepsilon, \ldots, f_N^\varepsilon)$ *and* u_0^ε *verifying* (3.2)*; and for which there exists* $(u_1^\varepsilon, \ldots, u_N^\varepsilon)$ *verifying* (3.1) *such that* $\|u(t) - u_n^\varepsilon\|_{L^1(\Omega)} \leq \varepsilon$ *for all* $t \in (t_{n-1}, t_n]$, $n = 1, \ldots, N$.

Remark 3.2. *In this paper, for the sake of simplicity and readability, we chose to present the constant step subdivision algorithm, i.e. that we set* $t_{n+1} - t_n = h = \frac{T}{N}$ *for all* $n = 0, \ldots, N-1$. *However, the techniques developed thereafter can be adapted to a varying step subdivision without difficulty.*

Note that using the nonlinear semigroups theory [6], Ouaro and Ouédraogo have proved in [24] the existence and uniqueness of mild solutions of the following parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div} a(x, \nabla u) = f & \text{in } Q \equiv \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases}$$

where $u_0 \in L^1(\Omega)$ and $f \in L^1(Q)$. The assumptions on the vector field are the same than those given in (2.2)–(2.5) and those on the variable exponent $p(x)$ are the same as (2.6). Thanks to their paper, one has the existence and uniqueness of the mild solution of problem (1.1).

4 Numerical study

4.1 Numerical scheme

We are now interested in the numerical resolution of (3.1). Let $f_1, f_2, \dots, f_N$, u_0 be some functions satisfying (3.2), we use the following iterative scheme (proposed by Maitre in [23]) to get u^ε_{n+1} from u^ε_n.

$$\begin{cases} \text{Let } u^{\varepsilon,0}_{n+1} = u^\varepsilon_n \in L^\infty(\Omega), \text{ solve for } k = 0, 1, \dots, \\ \\ u^{\varepsilon,k+1}_{n+1} - \rho\,\mathrm{div}(|\nabla u^{\varepsilon,k+1}_{n+1}|^{p(x)-2}\nabla u^{\varepsilon,k+1}_{n+1}) = \lambda_k u^{\varepsilon,k}_{n+1} - \dfrac{\rho}{h}(\lambda_k u^{\varepsilon,k}_{n+1} - u^\varepsilon_n) + \rho f^\varepsilon_{n+1}, \end{cases} \tag{4.1}$$

where $\rho > 0$ is a given parameter and (λ_k) is a sequence of $(0,1)$ such that

$$\lim_{k\to\infty} \lambda_k = 1, \quad \prod_{k\geq 0} \lambda_k = 0, \quad \sum_{k\geq 0} |\lambda_{k+1} - \lambda_k| < \infty. \tag{4.2}$$

For example, we can take $\lambda_k = 1 - \dfrac{1}{k+1}$.

Remark 4.1. *For the sake of simplicity, we could take $\rho = h$, but in this paper our idea is to build a non-expansive map and use the Halpern algorithm to approach the solution of* (3.1)*. In the numerical simulation one will give examples where $\rho = h$.*

4.2 Existence and uniqueness of solution of (4.1)

In this section, we state and prove the well-posedness of our scheme.

Definition 4.2. *For any $n = 0, \dots, N-1$, $\varepsilon > 0$ and $u^\varepsilon_n \in L^\infty(\Omega)$, a weak solution of* (4.1) *is a sequence $\left(u^{\varepsilon,k+1}_{n+1}\right)_{k\geq 0}$ such that $u^{\varepsilon,k+1}_{n+1} \in W^{1,p(.)}_0(\Omega) \cap L^\infty(\Omega)$ for all $k = 0, 1, \dots,$ and*

$$\int_\Omega u^{\varepsilon,k+1}_{n+1}\varphi\, dx + \rho \int_\Omega |\nabla u^{\varepsilon,k+1}_{n+1}|^{p(x)-2}\nabla u^{\varepsilon,k+1}_{n+1}.\nabla\varphi\, dx = \int_\Omega g^\varepsilon_{n,k}\varphi\, dx, \tag{4.3}$$

for all $\varphi \in W^{1,p(.)}_0(\Omega)$, where

$$g^{\varepsilon,k}_n := \lambda_k u^{\varepsilon,k}_{n+1} - \frac{\rho}{h}(\lambda_k u^{\varepsilon,k}_{n+1} - u^\varepsilon_n) + \rho f^\varepsilon_{n+1}.$$

Theorem 4.3. *Let $\varepsilon > 0$. For any $n = 0, \dots, N-1$ let $u^{\varepsilon,0}_{n+1} = u^\varepsilon_n \in L^\infty(\Omega)$ and $f^\varepsilon_{n+1} \in L^\infty(\Omega)$. Then, problem* (4.1) *admits a unique weak solution $u^{\varepsilon,k+1}_{n+1} \in W^{1,p(.)}_0(\Omega)$ for all $k = 0, 1, \dots$*

Furthermore, for $k = 0, 1, \dots,$ $u^{\varepsilon,k+1}_{n+1} \in L^\infty(\Omega)$.

Proof. Let $\varepsilon > 0$ and fix n. For $k = 0$ we rewrite problem (4.1) as

$$\begin{cases} u^{\varepsilon,1}_{n+1} - \rho\,\mathrm{div}(|\nabla u^{\varepsilon,1}_{n+1}|^{p(x)-2}\nabla u^{\varepsilon,1}_{n+1}) = g^{\varepsilon,0}_n & \text{in } \Omega \\ \\ u^{\varepsilon,1}_{n+1} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

where

$$g_n^{\varepsilon,0} = \left[\lambda_0 \left(1 - \frac{\rho}{h}\right) + 1\right] u_n^\varepsilon + \rho f_{n+1}^\varepsilon.$$

Consider the energy functional J_ρ on $W_0^{1,p(.)}(\Omega)$ associated to (4.4) given by

$$J_\rho(U) = \frac{1}{2}\int_\Omega U^2 dx + \rho \int_\Omega \frac{|\nabla U|^{p(x)}}{p(x)} dx - \int_\Omega g_n^{\varepsilon,0} U\, dx.$$

We will establish that $J_\rho(U)$ has a minimizer $u_{n+1}^{\varepsilon,1}$ in $W_0^{1,p(.)}(\Omega)$.

Note that J_ρ is well-defined and Gateaux differentiable on $W_0^{1,p(.)}(\Omega)$, since $W_0^{1,p(.)}(\Omega) \hookrightarrow L^2(\Omega)$ thanks to (1.2).

For $\|U\|_{W_0^{1,p(.)}(\Omega)} \geq 1$ we have from the continuous embedding of $W_0^{1,p(.)}(\Omega)$ in $L^{p^-}(\Omega)$ and $g_n^{\varepsilon,0} \in L^\infty(\Omega)$,

$$J_\rho(U) = \frac{1}{2}\int_\Omega U^2 dx + \rho \int_\Omega \frac{|\nabla U|^{p(x)}}{p(x)} dx - \int_\Omega g_n^{\varepsilon,0} U\, dx \geq \frac{\rho}{p^+}\|U\|^{p^-}_{W_0^{1,p(x)}(\Omega)} - C\|U\|_{W_0^{1,p(x)}(\Omega)}.$$

As $p^- > 1$, then J_ρ is coercive. $J_\rho(U)$ is lower bounded and furthermore weakly lower semi-continuous; therefore, admits a global minimizer $u_{n+1}^{\varepsilon,1} \in W_0^{1,p(.)}(\Omega)$ which is a weak solution to (4.4). The global minimizer $u_{n+1}^{\varepsilon,1}$ is also unique.

It remains to show that $u_{n+1}^{\varepsilon,1} \in L^\infty(\Omega)$. To do this, let us show that $\|u_{n+1}^{\varepsilon,1}\|_\infty \leq \|g_n^{\varepsilon,0}\|_\infty$.

As $u_{n+1}^{\varepsilon,1}$ is a weak solution of (4.4), we have

$$\int_\Omega u_{n+1}^{\varepsilon,1} \varphi\, dx + \rho \int_\Omega |\nabla u_{n+1}^{\varepsilon,1}|^{p(x)-2} \nabla u_{n+1}^{\varepsilon,1}.\nabla \varphi\, dx = \int_\Omega g_n^{\varepsilon,0} \varphi\, dx, \tag{4.5}$$

for all $\varphi \in W_0^{1,p(.)}(\Omega)$.

Let $\tau \in \mathbb{R}^+$. Then, $u_{n+1}^{\varepsilon,1} - \tau \in W_0^{1,p(.)}(\Omega)$ and $\left(u_{n+1}^{\varepsilon,1} - \tau\right)^+ \in W_0^{1,p(.)}(\Omega)$.
Note that for $r \in \mathbb{R}$, $r^+ := \max(r, 0)$ and $r^- := \min(r, 0)$.

Taking $\left(u_{n+1}^{\varepsilon,1} - \tau\right)^+$ as a test function, it follows from (4.5) that

$$\int_\Omega u_{n+1}^{\varepsilon,1}(u_{n+1}^{\varepsilon,1} - \tau)^+ dx + \rho \int_\Omega |\nabla u_{n+1}^{\varepsilon,1}|^{p(x)-2} \nabla u_{n+1}^{\varepsilon,1}.\nabla (u_{n+1}^{\varepsilon,1} - \tau)^+ dx = \int_\Omega g_n^{\varepsilon,0}(u_{n+1}^{\varepsilon,1} - \tau)^+ dx.$$

Setting $A_\tau = \left\{x \in \Omega : u_{n+1}^{\varepsilon,1} \geq \tau\right\}$, we have

$$\begin{aligned} \rho \int_\Omega |\nabla u_{n+1}^{\varepsilon,1}|^{p(x)-2} \nabla u_{n+1}^{\varepsilon,1}.\nabla (u_{n+1}^{\varepsilon,1} - \tau)^+ dx &= \rho \int_{A_\tau} |\nabla u_{n+1}^{\varepsilon,1}|^{p(x)-2} \nabla u_{n+1}^{\varepsilon,1}.\nabla (u_{n+1}^{\varepsilon,1} - \tau)\, dx \\ &= \rho \int_{A_\tau} |\nabla u_{n+1}^{\varepsilon,1}|^{p(x)} dx \geq 0. \end{aligned}$$

Therefore,

$$\int_\Omega u_{n+1}^{\varepsilon,1}(u_{n+1}^{\varepsilon,1} - \tau)^+ dx \leq \int_\Omega g_n^{\varepsilon,0}(u_{n+1}^{\varepsilon,1} - \tau)^+ dx.$$

As Ω is a bounded open domain, we have

$$\int_\Omega [(u_{n+1}^{\varepsilon,1}-\tau)^+]^2\,dx \le \int_\Omega (g_n^{\varepsilon,0}-\tau)(u_{n+1}^{\varepsilon,1}-\tau)^+\,dx.$$

Taking $\tau = \|g_n^{\varepsilon,0}\|_\infty$, then $g_n^{\varepsilon,0}-\tau \le 0$ a.e. in Ω.

Therefore, we have $(u_{n+1}^{\varepsilon,1}-\tau)^+ = 0$ a.e. in Ω for all $\tau = \|g_n^{\varepsilon,0}\|_\infty$ which is equivalent to saying

$$u_{n+1}^{\varepsilon,1} \le \|g_n^{\varepsilon,0}\|_\infty \quad \text{a.e. in } \Omega.$$

It remains to prove that $u_{n+1}^{\varepsilon,1} \ge -\|g_n^{\varepsilon,0}\|_\infty$ a.e. in Ω. To do this we take $(u_{n+1}^{\varepsilon,1}+\tau)^-$ as test function in (4.5) and use the same argument as previously. Thus, setting $C = \|g_n^{\varepsilon,0}\|_\infty$ implies that $u_{n+1}^{\varepsilon,1} \in L^\infty(\Omega)$.

In short $u_{n+1}^{\varepsilon,1} \in W_0^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$.

By induction, we deduce in the same manner that the problem (4.1) has a unique weak solution $\left(u_{n+1}^{\varepsilon,k+1}\right)_{k\ge 0}$ such that $u_{n+1}^{\varepsilon,k+1} \in W_0^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$ for all $k\in\mathbb{N}$. □

4.3 Study of the convergence

We begin with the following lemma which provides a crucial L^∞ uniform bound for the sequence $\left(u_{n+1}^{\varepsilon,k}\right)_{k\ge 0}$.

Lemma 4.4. *Let $\varepsilon > 0$ and fix n. If $\rho \le h$, there exists $M > 0$ independent of k such that $\|u_{n+1}^{\varepsilon,k}\|_\infty \le M$.*

Proof. Let $M = \max\left(\|u_{n+1}^{\varepsilon,0}\|_\infty, \|hf_{n+1}^\varepsilon + u_n^\varepsilon\|_\infty\right)$.

Now let us show by induction that $\|u_{n+1}^{\varepsilon,k}\|_\infty \le M$. We first note that $\|u_{n+1}^{\varepsilon,0}\|_\infty \le M$.
One assumes that $\|u_{n+1}^{\varepsilon,k}\|_\infty \le M$, and one shows that $\|u_{n+1}^{\varepsilon,k+1}\|_\infty \le M$.

As $u_{n+1}^{\varepsilon,k+1} \in L^\infty(\Omega)$ and verifies

$$u_{n+1}^{\varepsilon,k+1} - \operatorname{div}\left(\rho|\nabla u_{n+1}^{\varepsilon,k+1}|^{p(x)-2}\nabla u_{n+1}^{\varepsilon,k+1}\right) = \lambda_k u_{n+1}^{\varepsilon,k} - \frac{\rho}{h}(\lambda_k u_{n+1}^{\varepsilon,k} - u_n^\varepsilon) + \rho f_{n+1}^\varepsilon,$$

then, from the previous proof, it is established that for all $k = 1, 2, \ldots,$

$$\|u_{n+1}^{\varepsilon,k+1}\|_\infty \le \left\|\lambda_k u_{n+1}^{\varepsilon,k} - \frac{\rho}{h}(\lambda_k u_{n+1}^{\varepsilon,k} - u_n^\varepsilon) + \rho f_{n+1}^\varepsilon\right\|_\infty.$$

Since $\rho \le h$, we then obtain using the induction assumption

$$\|u_{n+1}^{\varepsilon,k+1}\|_\infty \le \left(1-\frac{\rho}{h}\right)M + \frac{\rho}{h}\|hf_{n+1}^\varepsilon + u_n^\varepsilon\|_\infty \le M. \quad \square$$

Thanks to M defined in the above proof we have the following convergence result.

Theorem 4.5. *Assume that conditions in Theorem 4.3 are satisfied. Then, for $\rho \le h$, the iterative scheme* (4.1) *converges, i.e.*

$$u^{\varepsilon,k}_{n+1} \longrightarrow u^{\varepsilon}_{n+1} \quad \textit{strongly in } L^1(\Omega) \textit{ as } k \to +\infty,$$

where u^{ε}_{n+1} verifies (3.1).

Proof. Thanks to Lemma 4.4, we can write (4.1) as

$$\frac{1}{\lambda_{k+1}}\bar{u}^{\varepsilon,k+1}_{n+1} - \rho\,\mathrm{div}\left(|\nabla \frac{1}{\lambda_{k+1}}\bar{u}^{\varepsilon,k+1}_{n+1}|^{p(x)-2}\nabla \frac{1}{\lambda_{k+1}}\bar{u}^{\varepsilon,k+1}_{n+1}\right) = \bar{u}^{\varepsilon,k}_{n+1} - \frac{\rho}{h}(\bar{u}^{\varepsilon,k}_{n+1} - u^{\varepsilon}_{n}) + \rho f^{\varepsilon}_{n+1}, \quad (4.6)$$

where we put $\bar{u}^{\varepsilon,k}_{n+1} = \lambda_k u^{\varepsilon,k}_{n+1}$ and $\bar{u}^{\varepsilon,k+1}_{n+1} = \lambda_{k+1} u^{\varepsilon,k+1}_{n+1}$.

Let $A(u) = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$. We identify the operator $A : L^1(\Omega) \to L^1(\Omega)$ associated with the $p(x)-$Laplacian problem (1.1) with its graph *i.e.*

$$G(A) = \left\{(u,v) \in L^1(\Omega) \times L^1(\Omega);\ v \in A(u)\right\}.$$

Therefore, A is $T-$accretive as soon as u is an entropy solution of problem (2.1) where $a(x,\nabla u) = (|\nabla u|^{p(x)-2}\nabla u)$. For more details, see [6] and [24, Proposition 4.3]. A is called $T-$accretive if $\|(u-\hat{u})^+\|_1 \le \|(u-\hat{u}+\rho(v-\hat{v}))^+)\|_1$, for any (u,v), $(\hat{u},\hat{v}) \in A$, $\rho > 0$; equivalently, if $\displaystyle\int_{\{u>\hat{u}\}}(v-\hat{v}) + \int_{\{u=\hat{u}\}}(v-\hat{v})^+ \ge 0$ for any (u,v), $(\hat{u},\hat{v}) \in A$.

Hence, (4.6) yields

$$(I+\rho A)\left(\frac{1}{\lambda_{k+1}}\bar{u}^{\varepsilon,k+1}_{n+1}\right) = \bar{u}^{\varepsilon,k}_{n+1} - \frac{\rho}{h}(\bar{u}^{\varepsilon,k}_{n+1} - u^{\varepsilon}_{n}) + \rho f^{\varepsilon}_{n+1}. \quad (4.7)$$

To complete the proof of Theorem 4.5, we use the following technical lemma.

Lemma 4.6. *Let $\rho \le 2h$ and M defined in the above proof such that $C_M = \left\{u \in L^1(\Omega), \|u\|_\infty \le M\right\}$. The iteration operator*

$$\tilde{T}(\bar{u}) = (I+\rho A)^{-1}\left(\bar{u} - \frac{\rho}{h}(\bar{u} - u^{\varepsilon}_{n}) + \rho f^{\varepsilon}_{n+1}\right)$$

is an L^1-non-expanding operator from C_M to C_M.

Proof. The fact that $\tilde{T}$ maps C_M to C_M is easily seen thanks to the proof of the Lemma 4.4 and (4.7). Now let $(\bar{u},\bar{v}) \in C^2_M$. One has from the $T-$accretiveness of A on $L^1(\Omega)$ that $(I+\rho A)^{-1}$ is a $T-$contraction in $L^1(\Omega)$ (see [6]), thus, a contraction. Therefore,

$$\begin{aligned}
\|\tilde{T}(\bar{u}) - \tilde{T}(\bar{v})\|_1 &= \Big\|(I+\rho A)^{-1}\left(\bar{u} - \frac{\rho}{h}(\bar{u} - u_n) + \rho f_{n+1}\right) \\
&- (I+\rho A)^{-1}\left(\bar{v} - \frac{\rho}{h}(\bar{v} - u_n) + \rho f_{n+1}\right)\Big\|_1 \\
&\le \left\|\left(1-\frac{\rho}{h}\right)\bar{u} - \left(1-\frac{\rho}{h}\right)\bar{v}\right\|_1.
\end{aligned}$$

Since $\rho \leq 2h$, we obtain

$$\|\tilde{T}(\bar{u}) - \tilde{T}(\bar{v})\|_1 \leq \|\bar{u} - \bar{v}\|_1. \quad \square$$

Consequently, from (4.7) one has the iteration $\bar{u}_{n+1}^{\varepsilon,k+1} = \lambda_{k+1}\tilde{T}(\bar{u}_{n+1}^{\varepsilon,k})$ where $\tilde{T}$ is a non-expansive operator in $L^1(\Omega)$ defined as in Lemma 4.6. Now, we are going to apply the Lemma 2.4 with $X = L^1(\Omega)$ and $C = C_M$ which is clearly a convex subset of $L^1(\Omega)$ containing 0. The uniqueness of a fixed point is verified thanks to Theorem 2.3. Indeed a fixed point u^* of $\tilde{T}$ verifies

$$u^* - \rho \operatorname{div}(|\nabla u^*|^{p(x)-2}\nabla u^*) = u^* - \frac{\rho}{h}(u^* - u_n^\varepsilon) + \rho f_{n+1}^\varepsilon.$$

Thus, $u^* - h \operatorname{div}(|\nabla u^*|^{p(x)-2}\nabla u^*) = u_n^\varepsilon + h f_{n+1}^\varepsilon$. From Theorem 2.3 this equation has a unique solution and from the definition of mild solution it is u_{n+1}^ε.

To conclude the proof of convergence of (4.1), we point out that each subsequence of $\bar{u}_{n+1}^{\varepsilon,k}$ has a convergent subsequence to an element of C_M, using the L^∞ bound of $\bar{u}_{n+1}^{\varepsilon,k}$ and the monotonicity of $(|\nabla \bar{u}_{n+1}^{\varepsilon,k}|^{p(x)-2}\nabla \bar{u}_{n+1}^{\varepsilon,k})$, to the equation (4.6). Applying Lemma 2.4, we conclude that the sequence $\bar{u}_{n+1}^{\varepsilon,k}$ converges strongly in $L^1(\Omega)$ toward u_{n+1}^ε. The same occurs for $u_{n+1}^{\varepsilon,k} = \dfrac{1}{\lambda_k}\bar{u}_{n+1}^{\varepsilon,k}$. $\square$

4.4 Convergence when $\varepsilon \to 0$ toward a solution of (1.1)

Note that for a mild solution we do not need to show the convergence in time since it is included in its definition: once convergence in k is achieved for u_{n+1}^ε, then, by the definition of mild solution, u_{n+1}^ε approaches $u^\varepsilon(t)$ on $(t_n, t_{n+1}]$ up to ε. Thus, our scheme converges to the mild solution when ε goes to zero.

We can state also the following result.

Proposition 4.7. *Let $u_0 \in L^\infty(\Omega)$, $f \in L^\infty(Q)$ and u the unique mild solution of* (1.1). *Then u is a weak solution of* (1.1). *By a weak solution we understand a solution in the sense of distributions that belongs to the energy space, i.e.,*

$$\begin{aligned} &u \in V := \left\{ v \in L^{p^-}(0,T; W_0^{1,p(.)}(\Omega)); |\nabla v| \in L^{p(.)}(Q) \right\}, \\ &\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f \text{ in } \mathcal{D}'(Q), \quad u(.,0) = u_0. \end{aligned} \tag{4.8}$$

Remark 4.8. *Note that a proof of the above proposition exists in [24]. Here, we use L^∞ uniform boundedness and the strong convergence in $L^1(\Omega)$ of the solution of our numerical scheme to prove Proposition 4.7.*

Moreover, these two results lead to the L^∞ uniform boundedness of the weak solution.

Proof of Proposition 4.7. Let u be the mild solution of (1.1). For $n = 0, \ldots, N-1$, u_{n+1}^ε is the unique weak solution of (3.1). We have

$$\int_\Omega \frac{u_{n+1}^\varepsilon - u_n^\varepsilon}{h}\varphi\, dx + \int_\Omega |\nabla u_{n+1}^\varepsilon|^{p(x)-2}\nabla u_{n+1}^\varepsilon.\nabla\varphi\, dx = \int_\Omega f_{n+1}^\varepsilon \varphi\, dx, \tag{4.9}$$

$\forall \varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ and

$$\begin{cases} \bullet \quad 0 = t_0 < \cdots < t_N = T \text{ such that } t_n - t_{n-1} = h \leq \varepsilon \text{ for } n = 1, \ldots, N, \\ \bullet \quad \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|f(t) - f_n^\varepsilon\|_{L^1(\Omega)}\, dt \leq \varepsilon \Rightarrow \|f_n^\varepsilon\|_{L^\infty(\Omega)} \leq \|f(t)\|_{L^\infty(\Omega)}, \\ \bullet \quad \sum_{n=1}^{N} h \|f_n^\varepsilon\|_{L^\infty(\Omega)} \leq \int_0^T \|f(.,t)\|_{L^\infty(\Omega)}\, dt, \\ \bullet \quad \|u_0 - u_0^\varepsilon\|_{L^1(\Omega)} \leq \varepsilon \Rightarrow \|u_0^\varepsilon\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}. \end{cases} \tag{4.10}$$

Note that relations in (4.10) are equivalent to relations in (3.2).

Let us set $u_\varepsilon(t) = u_{n+1}^\varepsilon \ \forall t \in (t_n, t_{n+1}]$, $u_\varepsilon(0) = u_0^\varepsilon$ and $f_\varepsilon(t) = f_{n+1}^\varepsilon$, $\forall t \in (t_n, t_{n+1}]$.

Lemma 4.4, Theorem 4.5 and the above relations in (4.10) imply that

$$\|u_\varepsilon\|_{L^\infty(Q)} \leq C(\|u_0\|_{L^\infty(\Omega)}; \|f\|_{L^\infty(Q)}). \tag{4.11}$$

Let ζ be the function defined by $\zeta(r) = \dfrac{r^2}{2}$ that satisfies $\zeta(r) - \zeta(\tilde{r}) \leq (r - \tilde{r})r$.

Taking $\varphi = u_{n+1}^\varepsilon$ as test function in (4.9) and integrating over $(t_n, t_{n+1}]$ and summing over $n = 0, \ldots, N-1$, we get

$$\int_\Omega \zeta(u_\varepsilon(t))\, dx + \int_Q |\nabla u_\varepsilon|^{p(x)}\, dx\, dt \leq \int_Q f_\varepsilon u_\varepsilon\, dx\, dt + \int_\Omega \zeta(u_0^\varepsilon)\, dx.$$

Thanks to the uniform boundedness of u_ε in ε and as $u_0^\varepsilon \in L^\infty(\Omega)$, we have

$$\int_Q |\nabla u_\varepsilon|^{p(x)}\, dx\, dt \leq C.$$

Moreover,

$$\int_0^T \|\nabla u_\varepsilon\|_{L^{p(.)}(\Omega)}^{p^-}\, dt \leq \int_0^T \max\left(\int_\Omega |\nabla u_\varepsilon|^{p(x)}; \left(\int_\Omega |\nabla u_\varepsilon|^{p(x)} \right)^{\frac{p^-}{p^+}} \right) dt.$$

Hence,

$$\int_0^T \|u_\varepsilon\|_{W_0^{1,p(.)}(\Omega)}^{p^-}\, dt \leq C.$$

As a consequence, there exists a subsequence still denoted $(u_\varepsilon)_{\varepsilon>0}$, such that

$$\begin{aligned} u_\varepsilon &\rightharpoonup u, \quad \text{weakly-* in } L^\infty(Q), \\ u_\varepsilon &\rightharpoonup u, \quad \text{weakly in } L^{p^-}(0,T; W_0^{1,p(.)}(\Omega)), \\ |\nabla u_\varepsilon|^{p(.)-2}\nabla u_\varepsilon &\rightharpoonup \Phi, \quad \text{weakly in } \left(L^{p'(.)}(Q)\right)^d. \end{aligned}$$

Using the monotonicity method we show that $\Phi = |\nabla u|^{p(.)-2}\nabla u$ a.e. in Q.

Now, let $\tilde{u}_\varepsilon$ be the piecewise linear function defined by

$$\tilde{u}_\varepsilon(t) = u_n^\varepsilon + \frac{t - t_n}{h}(u_{n+1}^\varepsilon - u_n^\varepsilon) \text{ for } t \in [t_n, t_{n+1}],\ n = 0, \ldots, N-1.$$

The function $\tilde{u}_\varepsilon$ verifies $(\tilde{u}_\varepsilon)_t(t) = \dfrac{u^\varepsilon_{n+1} - u^\varepsilon_n}{h}$ and $\tilde{u}_\varepsilon \to u$ in $L^\infty(0,T;L^1(\Omega))$. Hence,

$$u \in C([0,T];L^1(\Omega)).$$

Integrating (4.9) over (t_n, t_{n+1}) and summing over $n = 0,\dots,N-1$, we find

$$\begin{aligned} - & \int_0^T \int_\Omega \varphi_t \tilde{u}_\varepsilon \, dx\, dt - \int_\Omega \varphi(0) u_0^\varepsilon \, dx + \int_0^T \int_\Omega \left(|\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \right) . \nabla \varphi \, dx\, dt \\ = & \int_0^T \int_\Omega f_\varepsilon \varphi \, dx\, dt. \end{aligned} \tag{4.12}$$

Using the convergence results and passing to the limit in (4.12) as $\varepsilon \to 0$, we get the result. □

Remark 4.9. *For $u_0 \in L^1(\Omega)$, $f \in L^1(Q)$ the unique mild solution u of (1.1) is also an entropy solution. Indeed, since L^∞ is dense in L^1, we consider two sequences of functions $(f_m)_{m\geq 1} \subset L^\infty(Q)$ and $(u_{0m})_{m\geq 1} \subset L^\infty(\Omega)$ satisfying*

$$\begin{cases} f_m \to f \text{ in } L^1(Q), \qquad u_{0m} \to u_0 \text{ in } L^1(\Omega), \quad \text{as } m \to \infty, \\ \\ \|f_m\|_{L^1(Q)} \leq \|f\|_{L^1(Q)}, \quad \|u_{0m}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}. \end{cases} \tag{4.13}$$

Then, we get the following approximate problem of (1.1).

$$\begin{cases} \dfrac{\partial u_m}{\partial t} - \operatorname{div}(|\nabla u_m|^{p(x)-2} \nabla u_m) = f_m & \text{in } Q, \\ u_m = 0 & \text{on } \partial\Omega \times (0,T), \\ u_m(x,0) = u_{0m} & \text{in } \Omega. \end{cases} \tag{4.14}$$

Thanks to [24], for each $m = 1,2,\dots,$ we can find a unique mild solution $u_m \in C([0,T];L^1(\Omega))$ for problem (4.14) which verifies the L^1-contraction principle, i.e. the following estimate holds for almost all $t \in (0,T)$,

$$\begin{aligned} \|u_m(.,t)\|_{L^1(\Omega)} & \leq \|u_{0m}\|_{L^1(\Omega)} + \int_0^t \|f_m(.,s)\|_{L^1(\Omega)} \, ds \\ & \leq \|u_0\|_{L^1(\Omega)} + \int_0^t \|f(.,s)\|_{L^1(\Omega)} \, ds. \end{aligned}$$

By Proposition 4.7, and following the proof of [24, Theorem 5.1] we get the result.

Note that this entropy solution is equivalent to the renormalized solution of (1.1). Indeed, in [29], Zhang and Zhou have proved thanks to the assumptions (1.2) the existence and uniqueness of renormalized and entropy solutions of (1.1). In their paper, they have showed the equivalence between entropy and renormalized solutions.

4.5 Numerical tests

4.5.1 Implementation

We know that solving the equation (4.1) is equivalent to solve the following minimization problem for $n = 0, 1, \dots, N-1$ and $k = 0, 1, \dots$

$$u_{n+1}^{\varepsilon,k+1} = \operatorname{argmin}_{v\in\mathbb{W}} J(v), \tag{4.15}$$

where,

$$\mathbb{W} := \left\{ v \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega) \right\}$$

and the functional J is

$$\begin{aligned} J(v) \quad = \quad & \frac{1}{2}\int_\Omega v^2\,dx + \rho\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx - \left(1-\frac{\rho}{h}\right)\lambda_k\int_\Omega u_{n+1}^{\varepsilon,k} v - \frac{\rho}{h}\int_\Omega u_n^\varepsilon v\,dx \\ & -\rho\int_\Omega f_{n+1}^\varepsilon v\,dx. \end{aligned} \tag{4.16}$$

We formulate a basic procedure for solving problem (4.15) following the split Bregman technique (see [17]). We solve the minimization problem by introducing an auxiliary variable b. We have

$$\begin{aligned} \min_v \Bigg\{ & \frac{1}{2}\int_\Omega v^2\,dx + \rho\int_\Omega \frac{1}{p(x)}|b|^{p(x)}\,dx - \left(1-\frac{\rho}{h}\right)\lambda_k\int_\Omega u_{n+1}^{\varepsilon,k} v\,dx - \frac{\rho}{h}\int_\Omega u_n^\varepsilon v\,dx \\ & - \rho\int_\Omega f_{n+1}^\varepsilon v\,dx \quad \text{subject to } b = \nabla v \Bigg\}. \end{aligned} \tag{4.17}$$

By adding one quadratic penalty function term, we convert equation (4.17) to an unconstrained splitting formulation as follow.

$$\begin{aligned} \min_{v,b} \Bigg\{ & \frac{1}{2}\int_\Omega v^2\,dx + \rho\int_\Omega \frac{1}{p(x)}|b|^{p(x)}\,dx + \frac{\gamma}{2}\int_\Omega |b-\nabla v|^2\,dx - \left(1-\frac{\rho}{h}\right)\lambda_k\int_\Omega u_{n+1}^{\varepsilon,k} v\,dx \\ & -\frac{\rho}{h}\int_\Omega u_n^\varepsilon v\,dx - \rho\int_\Omega f_{n+1}^\varepsilon v\,dx \Bigg\}, \end{aligned} \tag{4.18}$$

where γ is a positive parameter which controls the weight of the penalty term. Similar to the split Bregman iteration, we propose the following scheme.

$$\left\{ \begin{aligned} (v^{l+1}, b^{l+1}) \quad = \quad & \operatorname{argmin}_{v,b}\Bigg\{ \frac{1}{2}\int_\Omega v^2\,dx + \rho\int_\Omega \frac{1}{p(x)}|b|^{p(x)}\,dx + \frac{\gamma}{2}\int_\Omega |b-\nabla v-\delta^l|^2\,dx \\ & - \left(1-\frac{\rho}{h}\right)\lambda_k\int_\Omega u_{n+1}^{\varepsilon,k} v\,dx - \frac{\rho}{h}\int_\Omega u_n^\varepsilon v\,dx - \rho\int_\Omega f_{n+1}^\varepsilon v\,dx \Bigg\}, \\ \delta^{l+1} \quad = \quad & \delta^l + \nabla v^{l+1} - b^{l+1}. \end{aligned} \right. \tag{4.19}$$

Alternatively, this joint minimization problem can be solved by decomposing into several subproblems.

4.5.2 Subproblem v with fixed b and δ

Given the fixed variable b^l and δ^l, our aim is to find the solution of the following problem

$$\begin{aligned} v^{l+1} \quad = \quad & \operatorname{argmin}_v \left\{ \frac{1}{2}\int_\Omega v^2\,dx + \frac{\gamma}{2}\int_\Omega |b^l - \nabla v - \delta^l|^2\,dx - \left(1-\frac{\rho}{h}\right)\lambda_k \int_\Omega u^{\varepsilon,k}_{n+1} v\,dx \right. \\ & \left. -\frac{\rho}{h}\int_\Omega u^\varepsilon_n v\,dx - \rho\int_\Omega f^\varepsilon_{n+1} v\,dx \right\}. \end{aligned} \tag{4.20}$$

We know that solve (4.20) is equivalent to solve the following optimality condition.

$$v - \gamma\Delta v = \gamma\nabla.(\delta^l - b^l) + \left(1-\frac{\rho}{h}\right)\lambda_k u^{\varepsilon,k}_{n+1} + \frac{\rho}{h}u^\varepsilon_n + \rho f^\varepsilon_{n+1}. \tag{4.21}$$

Since the discrete system is strictly diagonally dominant with Neumann boundary condition, the most natural choice is the Gauss-Seidel method.

4.5.3 Subproblem b with fixed v and δ

Similarly, we solve

$$b^{l+1} = \operatorname{argmin}_b \left\{ \rho\int_\Omega \frac{1}{p(x)}|b|^{p(x)}\,dx + \frac{\gamma}{2}\int_\Omega |b - \nabla v^{l+1} - \delta^l|^2\,dx \right\} \tag{4.22}$$

In two dimensional space.

Here, setting $b = (b_x, b_y)$ and $\delta = (\delta_x, \delta_y)$.

Then, the resolution of (4.22) is equivalent to solve the following optimality condition.

$$\begin{cases} \rho|b|^{p(x,y)-2}b_x + \gamma(b_x - \nabla_x v^{l+1} - \delta^l_x) & = 0 \\ \\ \rho|b|^{p(x,y)-2}b_y + \gamma(b_y - \nabla_y v^{l+1} - \delta^l_y) & = 0, \end{cases} \tag{4.23}$$

where $\nabla v = (\nabla_x v, \nabla_y v)$.

If b_x and b_y are not zero, then,

$$b_x = \frac{\nabla_x v^{l+1} + \delta^l_x}{\nabla_y v^{l+1} + \delta^l_y} b_y. \tag{4.24}$$

Substituting (4.24) into (4.23), we obtain

$$\operatorname{sign}(b_y) T |b_y|^{p(x,y)-1} + \gamma(b_y - \nabla_y v^{l+1} - \delta^l_y) = 0, \tag{4.25}$$

where $T = \rho\left(\left(\frac{\nabla_x v^{l+1} + \delta^l_x}{\nabla_y v^{l+1} + \delta^l_y}\right)^2 + 1\right)^{\frac{p(x,y)-2}{2}}$. Here, sign is defined as follows.

$$\operatorname{sign}(\omega) := \begin{cases} 1 & \text{if} \quad \omega > 0, \\ 0 & \text{if} \quad \omega = 0, \\ -1 & \text{if} \quad \omega < 0. \end{cases}$$

Note that

$$\operatorname{sign}(b_x) = \operatorname{sign}(\nabla_x v^{l+1} + \delta_x^l) \tag{4.26}$$

and

$$\operatorname{sign}(b_y) = \operatorname{sign}(\nabla_y v^{l+1} + \delta_y^l). \tag{4.27}$$

So, (4.25) can be expressed as

$$\operatorname{sign}(\nabla_y v^{l+1} + \delta_y^l) T |b_y|^{p(x,y)-1} + \gamma(b_y - \nabla_y v^{l+1} - \delta_y^l) = 0. \tag{4.28}$$

Unfortunately, we cannot obtain the explicit solution of the equation (4.28). We can use Newton method to get an approximate solution. If b_y is solved, b_x can be easily determined using (4.24) and (4.26).

4.5.4 Applications

In the following numerical simulation the iteration process stops when the following condition is satisfied

$$\frac{\|u_{n+1}^{k+1} - u_{n+1}^k\|_2}{\|u_{n+1}^{k+1}\|_2} \leq \text{stop} := 10^{-5}, \tag{4.29}$$

where $\|.\|_2$ is the Euclidean norm and u_{n+1}^k the vector approaching, at iteration k, the space-discretization of u_{n+1}. After stopping the iterations at $k = k_{last}$, we denote $u_{n+1} = u_{n+1}^{k_{last}}$ and switch to the next time step.

Note that for implementation, finite difference method is used to approximate the partial derivatives. Moreover, for sake of simplicity, the domain Ω will be a square. The domain Ω will be subdivided into N_x^2 uniform squares.

For numerical simulation, we will use the following parameters

$$N_x = 80 \quad \text{and} \quad h = 0.02.$$

Let us recall that h is the time step. The space step is easily computed thanks to N_x and Ω.

Example 4.10. *In this example, we take* $\Omega = (0,1) \times (0,1)$, $T = 1$, $p(x,y) = 2$, *and* $f = xy(1-x)(1-y) + 2t((1-y)y + (1-x)x)$. *As initial condition, we set*

$$u_0(x,y) = 0.$$

Let us note that with these data p, u_0 *and* f, *the exact solution is*

$$u(x,y,t) = txy(1-x)(1-y).$$

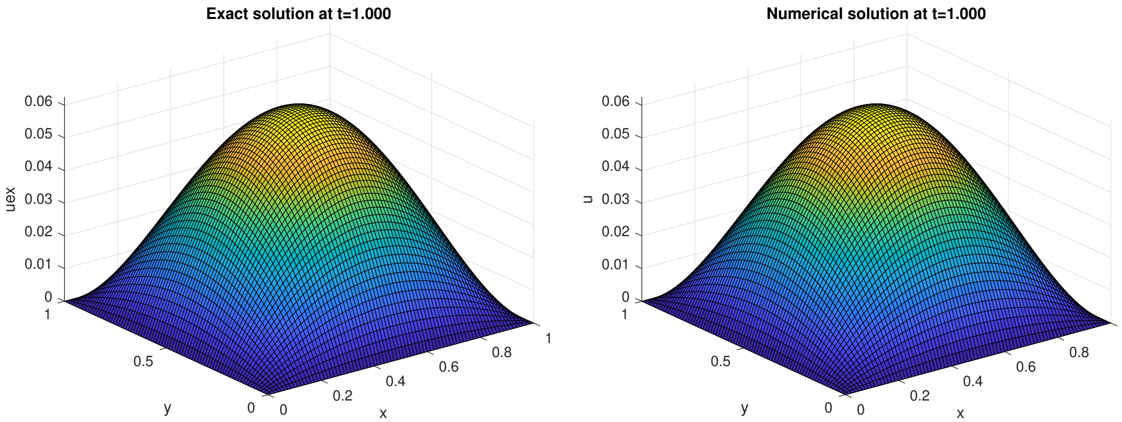

Figure 1: left: $u(x, y, t) = txy(1 - x)(1 - y)$ right: For $\rho = h$ and $\gamma = 0.02$

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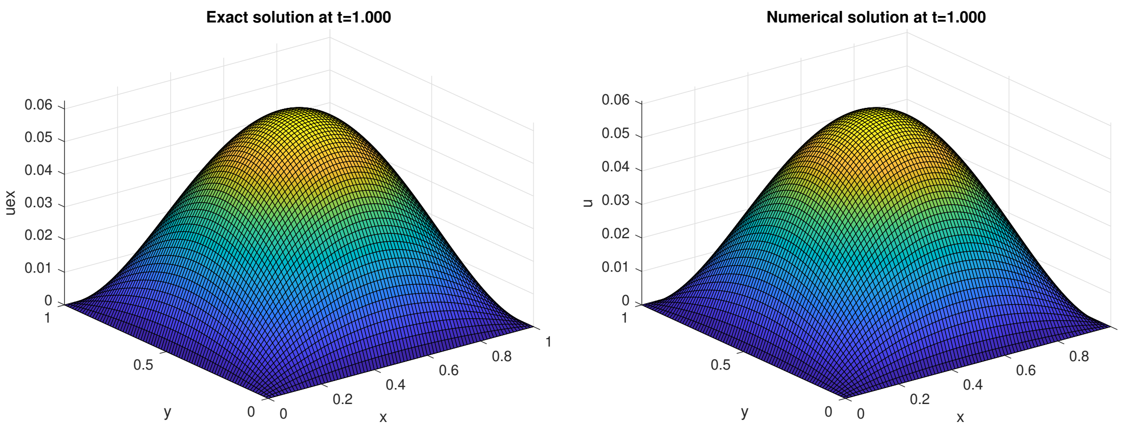

Figure 2: left: $u(x, y, t) = txy(1 - x)(1 - y)$ right: For $\rho = h/2$ and $\gamma = 0.02$

Figure 1 shows the exact solution and the numerical solution for $\gamma = 0.02$ and $\rho = h$. While, Figure 2 shows the exact solution and the numerical solution for $\gamma = 0.02$ and $\rho = h/2$.

As we can see, we always get a good numerical approximation of the solution even if ρ varies.

Denoting u_h the numerical solution and u the exact solution of Example 4.10, with $\rho = h$ and $\gamma = 0.02$, we get the following table of the error approximation.

t	0.1	0.2	0.3	0.4	0.5
$\|u_h - u\|_1$	$2.5099.10^{-5}$	$5.6941.10^{-5}$	$7.9789.10^{-5}$	$1.0717.10^{-4}$	$1.345.10^{-4}$

t	0.6	0.7	0.8	0.9	1
$\|u_h - u\|_1$	$1.6192.10^{-4}$	$1.8930.10^{-4}$	$2.1668.10^{-4}$	$2.4406.10^{-4}$	$2.7144.10^{-4}$

Example 4.11. *In this example, we set* $\Omega = (0,1) \times (0,1)$*,* $T = 5$*,* $p(x,y) = 2 + \dfrac{|x|}{2}$*, and* $f = 1$*. As initial condition we set*

$$u_0(x,y) = 0.$$

As parameters we set $\rho = h$ *and* $\gamma = 0.02$*.*

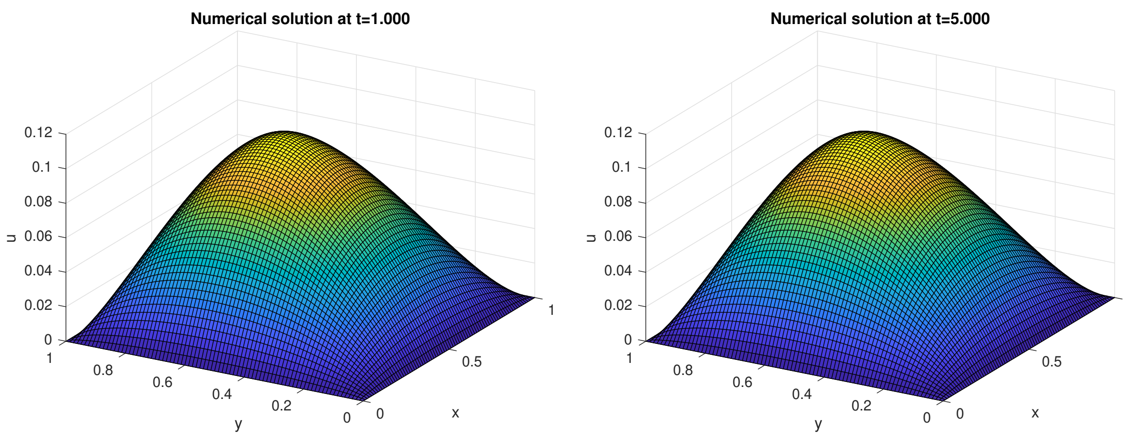

Figure 3: Numerical solution for $p(x,y) = 2 + \dfrac{|x|}{2}$, $\rho = h$ and $\gamma = 0.02$.

Figure 3 shows the numerical solution at $t = 1$ and at $t = 5$. One can see that both figures are the same.

Example 4.12. *In this example, we take* $\Omega = (-1,1) \times (-1,1)$*,* $T = 5$*,* $p(x,y) = \dfrac{9}{5} - \dfrac{x^2}{2}$ *and*

$$f = \begin{cases} 1 & \textit{if} \quad x \geq 0 \\ 0 & \textit{if} \quad x < 0. \end{cases}$$

As the initial condition, we set

$$u_0(x,y) = e^{(1-x^2)(1-y^2)} - 1.$$

We use the same parameters ρ *and* γ *as previously.*

Figure 4 shows the numerical solution at $t = 1$ *and* $t = 5$*.*

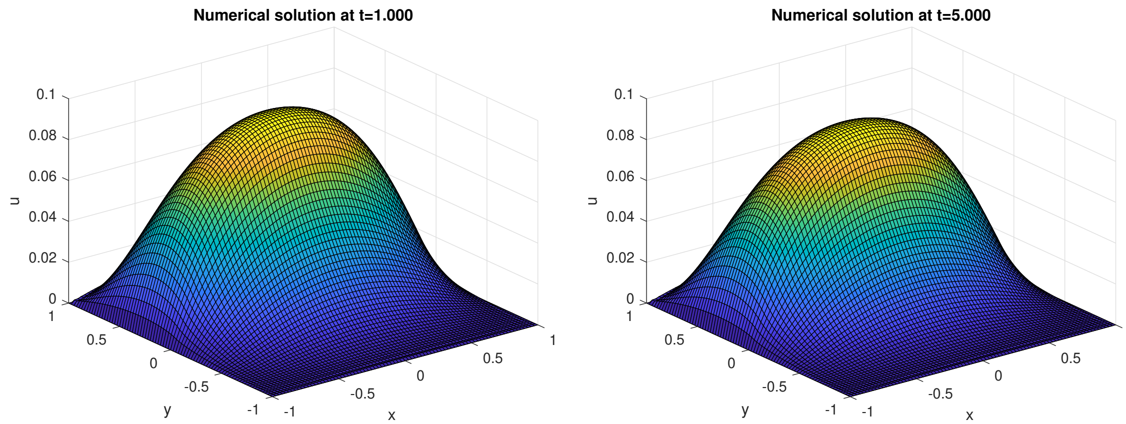

Figure 4: Numerical solution for $p(x,y) = \dfrac{9}{5} - \dfrac{x^2}{2}$, $\rho = h$ and $\gamma = 0.02$.

We remark that the exponents $p(x)$ considered in the three examples satisfy the condition 1.2. Also, note that the choice of γ results from the knowledge of the explicit solution of the Example 4.10. Indeed, knowing the explicit solution, we choose γ so as to obtain a better approximation of this explicit solution. This leads to the choice of $\gamma = 0.02$.

Conclusion and discussion

Inspired by the work of Maitre (see [23]), we have in this paper made a numerical analysis of the mild solution of parabolic problem involving the $p(x)-$Laplacian operator. Using the works of Zhang and Zhou (see [29]), and Ouaro and Ouédraogo (see [24]), we have shown that the mild solution is also an entropy solution which is equivalent to the renormalized solution. For the numerical tests, we have used the split Bregman iteration.

In a forthcoming paper, we will make a comparison of the solutions of our numerical scheme (4.1) to those of the classical backward Euler scheme.

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