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# Existence, uniqueness, continuous dependence and Ulam stability of mild solutions for an iterative fractional differential equation

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**ABSTRACT**

In this work, we study the existence, uniqueness, continuous dependence and Ulam stability of mild solutions for an iterative Caputo fractional differential equation by first inverting it as an integral equation. Then we construct an appropriate mapping and employ the Schauder fixed point theorem to prove our new results. At the end we give an example to illustrate our obtained results.

**RESUMEN**

En este trabajo, estudiamos la existencia, unicidad, dependencia continua y estabilidad de Ulam de soluciones mild para una ecuación diferencial fraccionaria de Caputo iterativa, invirtiéndola primero como ecuación integral. Luego construimos una aplicación apropiada y empleamos el teorema del punto fijo de Schauder para demostrar nuestros nuevos resultados. Finalmente damos un ejemplo para ilustrar los resultados obtenidos.

**Keywords and Phrases:** Iterative fractional differential equations, fixed point theorem, existence, uniqueness, continuous dependence, Ulam stability.

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# 1 Introduction

Fractional differential equations have gained considerable importance due to their applications in various sciences, such as physics, mechanics, chemistry, engineering, etc. In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs of Kilbas *et al.* [10], Miller and Ross [12], Podlubny [14]. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1]–[4], [6]–[16], [18] and the references therein.

Recently, iterative functional differential equations of the form

$$x'(t) = H\left(x^{[0]}(t), x^{[1]}(t), x^{[2]}(t), \ldots, x^{[n]}(t)\right),$$

have appeared in several papers, where

$$x^{[0]}(t) = t,\ x^{[1]}(t) = x(t),\ x^{[2]}(t) = x(x(t)), \ldots,\ x^{[n]}(t) = x^{[n-1]}(x(t))$$

are the iterates of the state $x(t)$.

Iterative differential equations often arise in the modeling of a wide range of natural phenomena such as disease transmission models in epidemiology, two-body problem of classical electrodynamics, population models, physical models, mechanical models and other numerous models. This kind of equations which relates an unknown function, its derivatives and its iterates, is a special type of the so-called differential equations with state-dependent delays, see [5, 9, 19] and the references therein.

In this paper, inspired and motivated by the references [1]–[16], [18, 19], we concentrate on the existence, uniqueness, continuous dependence and Ulam stability of mild solutions for the nonlinear iterative fractional differential equation

$$\begin{cases} {}^{C}D_{0^+}^{\alpha} x(t) = f\left(x^{[0]}(t), x^{[1]}(t), x^{[2]}(t), \ldots, x^{[n]}(t)\right),\ t \in J, \\ x(0) = x'(0) = 0, \end{cases} \tag{1.1}$$

where $J = [0, T]$, ${}^{C}D_{0^+}^{\alpha}$ is the standard Caputo fractional derivative of order $\alpha \in (1, 2)$ and $f$ is a positive continuous function with respect to its arguments and satisfies some other conditions that will be specified later. To reach our desired end we have to transform (1.1) into an integral equation and then use the Schauder fixed point theorem to show the existence and uniqueness of mild solutions.

The organization of this paper is as follows. In Section 2, we introduce some definitions and lemmas, and state some preliminary results needed in later sections. Also, we present the inversion of (1.1) and state the Schauder fixed point theorem. For details on the Schauder theorem we refer the reader to [17]. In Section 3, we present our main results on the existence, uniqueness, continuous dependence and Ulam stability of mild solutions for the problem (1.1) and provide an example to illustrate our results.

## 2 Preliminaries

Let $C(J,\mathbb{R})$ be the Banach space of all real-valued continuous functions defined on the compact interval $J$, endowed with the norm

$$\|x\| = \sup_{t\in J} |x(t)|.$$

For $0 < L \leq T$ and $M > 0$, define the sets

$$C(J,L) = \{x \in C(J,\mathbb{R}) : 0 \leq x(t) \leq L,\ \forall t \in J\},$$

and

$$C_M(J,L) = \{x \in C(J,L) : |x(t_2) - x(t_1)| \leq M|t_2 - t_1|,\ \forall t_1, t_2 \in J\}.$$

Then, $C_M(J,L)$ is a closed convex and bounded subset of $C(J,\mathbb{R})$.

Furthermore, we suppose that the positive function $f$ is globally Lipschitz in $x_i$, that is, there exist positive constants $c_1$, $c_2$, $\ldots$, $c_n$ such that

$$|f(t,x_1,x_2,\ldots,x_n) - f(t,y_1,y_2,\ldots,y_n)| \leq \sum_{i=1}^{n} c_i |x_i - y_i|. \tag{2.1}$$

We introduce the constants

$$\begin{aligned} \rho &= \sup_{t\in J}\{f(t,0,0,\ldots,0)\}, \\ \zeta &= \rho + L\sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j, \end{aligned}$$

where $M^j = M \times M^{j-1}$.

**Definition 2.1** ([10])**.** *The fractional integral of order $\alpha > 0$ of a function $x : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is given by*

$$I_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\,ds,$$

*provided the right side is pointwise defined on $\mathbb{R}^+$, where $\Gamma$ is the gamma function.*

For instance, $I_{0^+}^{\alpha} x$ exists for all $\alpha > 0$, when $x \in C(\mathbb{R}^+)$ then $I_{0^+}^{\alpha} x \in C(\mathbb{R}^+)$ and moreover $I_{0^+}^{\alpha} x(0) = 0$.

**Definition 2.2** ([10])**.** *The Caputo fractional derivative of order $\alpha > 0$ of a function $x : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is given by*

$${}^{C}D_{0^+}^{\alpha} x(t) = I_{0^+}^{n-\alpha} x^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s)\,ds,$$

*where $n = [\alpha] + 1$, provided the right side is pointwise defined on $\mathbb{R}^+$.*

**Lemma 2.3** ([10])**.** *Suppose that $x \in C^{n-1}([0,+\infty))$ and $x^{(n)}$ exists almost everywhere on any bounded interval of $\mathbb{R}^+$. Then*

$$\left(I_{0^+}^{\alpha}\, {}^{C}D_{0^+}^{\alpha} x\right)(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k.$$

*In particular, when $\alpha \in (1,2)$, $\left(I_{0^+}^{\alpha}\, {}^{C}D_{0^+}^{\alpha} x\right)(t) = x(t) - x(0) - x'(0)\, t$.*

**Definition 2.4.** *A function $x \in C_M(J,L)$ is a mild solution of the problem (1.1) if $x$ satisfies the corresponding integral equation of (1.1).*

From Lemma 2.3, we deduce the following lemma.

**Lemma 2.5.** *Let $x \in C_M(J,L)$ is a mild solution of (1.1) if $x$ satisfies*

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), \dots, x^{[n]}(s)\right) ds, \ t \in J. \tag{2.2}$$

**Lemma 2.6** ([19])**.** *If $\varphi, \psi \in C_M(J,L)$, then*

$$\left\|\varphi^{[m]} - \psi^{[m]}\right\| \le \sum_{j=0}^{m-1} M^j \|\varphi - \psi\|, \ m = 1, 2, \dots$$

**Theorem 2.7** (Schauder fixed point theorem [17])**.** *Let $\mathbb{M}$ be a nonempty compact convex subset of a Banach space $(\mathbb{B}, \|\cdot\|)$ and $\mathcal{A} : \mathbb{M} \to \mathbb{M}$ is a continuous mapping. Then $\mathcal{A}$ has a fixed point.*

## 3 Main results

In this section, we use Theorem 2.7 to prove the existence of mild solutions for (1.1). Moreover, we will introduce the sufficient conditions of the uniqueness of mild solutions of (1.1).

To transform (2.2) to be applicable to the Schauder fixed point, we define an operator $\mathcal{A} : C_M(J,L) \to C(J,\mathbb{R})$ by

$$(\mathcal{A}\varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) ds, \ t \in J. \tag{3.1}$$

Since $C_M(J,L)$ is a compact set as a uniformly bounded, equicontinuous and closed subset of the space $C(J,\mathbb{R})$. To prove that operator $\mathcal{A}$ has at least one fixed point, we will prove that $\mathcal{A}$ is well defined, continuous and $\mathcal{A}(C_M(J,L)) \subset C_M(J,L)$, *i. e.*

$$\mathcal{A}\varphi \in C_M(J,L) \text{ for all } \varphi \in C_M(J,L).$$

**Lemma 3.1.** *Suppose that (2.1) holds. Then the operator $\mathcal{A} : C_M(J,L) \to C(J,\mathbb{R})$ given by (3.1) is well defined and continuous.*

*Proof.* Let $\mathcal{A}$ be defined by (3.1). Clearly, $\mathcal{A}$ is well defined. To show the continuity of $\mathcal{A}$. Let $\varphi, \psi \in C_M(J, L)$, we have

$$\begin{aligned}|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big| & f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right) \\ & - f\left(\psi^{[0]}(s), \psi^{[1]}(s), \psi^{[2]}(s), \ldots, \psi^{[n]}(s)\right)\Big| \, ds.\end{aligned}$$

By (2.1), we obtain

$$|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^{n} c_i \left\|\varphi^{[i]} - \psi^{[i]}\right\| ds.$$

It follows from Lemma 2.6 that

$$\begin{aligned}|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \|\varphi - \psi\| \, ds \\ &\leq \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \|\varphi - \psi\|,\end{aligned}$$

which proves that the operator $\mathcal{A}$ is continuous. □

**Lemma 3.2.** *Suppose that (2.1) holds. If*

$$\frac{\zeta T^\alpha}{\Gamma(\alpha+1)} \leq L, \tag{3.2}$$

*and*

$$\frac{\zeta T^{\alpha-1}}{\Gamma(\alpha)} \leq M, \tag{3.3}$$

*then* $\mathcal{A}(C_M(J, L)) \subset C_M(J, L)$.

*Proof.* For $\varphi \in C_M(J, L)$, we get

$$|(\mathcal{A}\varphi)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right)\right| ds.$$

But

$$\begin{aligned}&\left| f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right)\right| \\ &= \left| f\left(s, \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right) - f(s, 0, 0, \ldots, 0) + f(s, 0, 0, \ldots, 0)\right| \\ &\leq \left| f\left(s, \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right) - f(s, 0, 0, \ldots, 0)\right| + |f(s, 0, 0, \ldots, 0)| \\ &\leq \rho + \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \|\varphi\| \\ &\leq \rho + L \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j = \zeta,\end{aligned}$$

then

$$|(\mathcal{A}\varphi)(t)| \leq \frac{\zeta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \leq \frac{\zeta T^{\alpha}}{\Gamma(\alpha+1)} \leq L.$$

From (3.2), we have

$$0 \leq (\mathcal{A}\varphi)(t) \leq |(\mathcal{A}\varphi)(t)| \leq L.$$

Let $t_1, t_2 \in J$ with $t_1 < t_2$, we have

$$\begin{aligned}
&\left|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)\right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right| \left|f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right)\right| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \left|f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right)\right| ds \\
&\leq \frac{\zeta}{\Gamma(\alpha)} \left(\int_0^{t_1} \left((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right) ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds\right) \\
&\leq \frac{\zeta}{\Gamma(\alpha+1)} \left(t_2^{\alpha} - t_1^{\alpha}\right) \\
&\leq \frac{\zeta T^{\alpha-1}}{\Gamma(\alpha)} \left|t_2 - t_1\right|.
\end{aligned}$$

Using (3.3), we obtain

$$\left|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)\right| \leq M \left|t_2 - t_1\right|.$$

Therefore, $\mathcal{A}\varphi \in C_M(J, L)$ for all $\varphi \in C_M(J, L)$. So, we conclude that $\mathcal{A}(C_M(J, L)) \subset C_M(J, L)$. □

**Theorem 3.3.** *Suppose that conditions (2.1), (3.2) and (3.3) hold. Then (1.1) has at least one mild solution $x$ in $C_M(J, L)$.*

*Proof.* From Lemma 2.5, the problem (1.1) has a mild solution $x$ on $C_M(J, L)$ if and only if the operator $\mathcal{A}$ defined by (3.1) has a fixed point. From Lemmas 3.1 and 3.2, all conditions of the Schauder fixed point theorem are satisfied. Consequently, $\mathcal{A}$ has at least one fixed point on $C_M(J, L)$ which is a mild solution of (1.1). □

**Theorem 3.4.** *In addition to the assumptions of Theorem 3.3, if we suppose that*

$$\frac{T^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j < 1, \tag{3.4}$$

*then (1.1) has a unique mild solution in $C_M(J, L)$.*

*Proof.* Let $\varphi$ and $\psi$ be two distinct fixed points of the operator $\mathcal{A}$. Similarly as in the proof of Lemma 3.1 we have

$$|\varphi(t) - \psi(t)| = |(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \|\varphi - \psi\|.$$

It follows from (3.4) that

$$\|\varphi-\psi\|<\|\varphi-\psi\|.$$

Therefore, we arrive at a contradiction. We conclude that $\mathcal{A}$ has a unique fixed point which is the unique mild solution of (1.1). □

**Theorem 3.5.** *Suppose that the conditions of Theorem 3.4 hold. The unique mild solution of (1.1) depends continuously on the function $f$.*

*Proof.* Let $f_1, f_2 : J\times\mathbb{R}^n \to [0,+\infty)$ two continuous functions with respect to their arguments. From Theorem 3.4, it follows that there exist two unique corresponding functions $x_1$ and $x_2$ in $C_M(J,L)$ such that

$$x_1(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s),\ldots,x_1^{[n]}(s)\right)ds,$$

and

$$x_2(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s),\ldots,x_2^{[n]}(s)\right)ds.$$

We get

$$\begin{aligned}|x_2(t)-x_1(t)|\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big| & f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s),\ldots,x_2^{[n]}(s)\right)\\ &-f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s),\ldots,x_1^{[n]}(s)\right)\Big|\,ds.\end{aligned}$$

But

$$\begin{aligned}&\left|f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s),\ldots,x_2^{[n]}(s)\right)-f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s),\ldots,x_1^{[n]}(s)\right)\right|\\ &=\Big|f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s),\ldots,x_2^{[n]}(s)\right)-f_2\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s),\ldots,x_1^{[n]}(s)\right)\\ &+f_2\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s),\ldots,x_1^{[n]}(s)\right)-f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s),\ldots,x_1^{[n]}(s)\right)\Big|.\end{aligned}$$

Using (2.1) and Lemma 2.6, we arrive at

$$\begin{aligned}&\left|f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s),\ldots,x_2^{[n]}(s)\right)-f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s),\ldots,x_1^{[n]}(s)\right)\right|\\ &\le \|f_2-f_1\|+\sum_{i=1}^n c_i\sum_{j=0}^{i-1}M^j\|x_2-x_1\|.\end{aligned}$$

Hence

$$\|x_2-x_1\|\le\frac{T^\alpha}{\Gamma(\alpha+1)}\|f_2-f_1\|+\frac{T^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^n c_i\sum_{j=0}^{i-1}M^j\|x_2-x_1\|.$$

Therefore

$$\|x_2-x_1\|\le\frac{\frac{T^\alpha}{\Gamma(\alpha+1)}}{1-\frac{T^\alpha}{\Gamma(\alpha+1)}\sum\limits_{i=1}^n c_i\sum\limits_{j=0}^{i-1}M^j}\|f_2-f_1\|.$$

This completes the proof. □

Now, we investigate the Ulam-Hyers stability and generalized Ulam-Hyers stability for the problem (1.1).

**Definition 3.6** ([18]). *The problem (1.1) is said to be Ulam-Hyers stable if there exists a real number $K_f > 0$ such that for each $\epsilon > 0$ and for each mild solution $y \in C_M(J,L)$ of the inequality*

$$\left| {}^C D^{\alpha}_{0+} y(t) - f\left(y^{[0]}(t), y^{[1]}(t), y^{[2]}(t), \ldots, y^{[n]}(t)\right)\right| \leq \epsilon,\ t \in J, \tag{3.5}$$

*with $y(0) = y'(0) = 0$, there exists a mild solution $x \in C_M(J,L)$ of the problem (1.1) with*

$$|y(t) - x(t)| \leq K_f \epsilon,\ t \in J.$$

**Definition 3.7** ([18]). *The problem (1.1) is generalized Ulam-Hyers stable if there exists $\psi \in C(J, \mathbb{R}^+)$ with $\psi(0) = 0$ such that for each $\epsilon > 0$ and for each mild solution $y \in C_M(J,L)$ of the inequality (3.5) with $y(0) = y'(0) = 0$, there exists a mild solution $x \in C_M(J,L)$ of the problem (1.1) with*

$$|y(t) - x(t)| \leq \psi(\epsilon),\ t \in J.$$

**Theorem 3.8.** *Assume that the assumptions of Theorem 3.4 hold. Then the problem (1.1) is Ulam-Hyers stable.*

*Proof.* Let $y \in C_M(J,L)$ be a mild solution of the inequality (3.5) with $y(0) = y'(0) = 0$, i.e.

$$\begin{cases} \left|{}^C D^{\alpha}_{0+} y(t) - f\left(y^{[0]}(t), y^{[1]}(t), y^{[2]}(t), \ldots, y^{[n]}(t)\right)\right| \leq \epsilon,\ t \in J, \\ y(0) = y'(0) = 0. \end{cases} \tag{3.6}$$

Let us denote by $x \in C_M(J,L)$ the unique mild solution of the problem (1.1). By using Lemma 2.5, we get

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), \ldots, x^{[n]}(s)\right) ds,\ t \in J.$$

By integration of (3.6), we have

$$\left| y(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(y^{[0]}(s), y^{[1]}(s), y^{[2]}(s), \ldots, y^{[n]}(s)\right) ds \right| \leq \frac{t^{\alpha}}{\Gamma(\alpha+1)} \epsilon \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)} \epsilon.$$

On the other hand, we obtain, for each $t \in J$

$$\begin{aligned} |y(t) - x(t)| &= \left| y(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), \ldots, x^{[n]}(s)\right) ds \right| \\ &\leq \left| y(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(y^{[0]}(s), y^{[1]}(s), y^{[2]}(s), \ldots, y^{[n]}(s)\right) ds \right| \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f\left(y^{[0]}(s), y^{[1]}(s), y^{[2]}(s), \ldots, y^{[n]}(s)\right) \right. \\ &\left. - f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), \ldots, x^{[n]}(s)\right)\right| ds \\ &\leq \frac{T^{\alpha}}{\Gamma(\alpha+1)} \epsilon + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \|y - x\|. \end{aligned}$$

Thus, in view of (3.4)

$$\|y-x\| \leq \frac{\frac{T^\alpha}{\Gamma(\alpha+1)}}{1-\frac{T^\alpha}{\Gamma(\alpha+1)}\sum\limits_{i=1}^{n} c_i \sum\limits_{j=0}^{i-1} M^j}\epsilon.$$

Then, there exists a real number $K_f = T^\alpha / \left(\Gamma(\alpha+1) - T^\alpha \sum\limits_{i=1}^{n} c_i \sum\limits_{j=0}^{i-1} M^j\right) > 0$ such that

$$|y(t) - x(t)| \leq K_f \epsilon, \ t \in J. \tag{3.7}$$

Thus, the problem (1.1) is Ulam-Hyers stable, which completes the proof. □

**Corollary 3.9.** *Suppose that all the assumptions of Theorem 3.8 are satisfied. Then the problem (1.1) is generalized Ulam-Hyers stable.*

*Proof.* Let $\psi(\epsilon) = K_f \epsilon$ in (3.7) then $\psi(0) = 0$ and the problem (1.1) is generalized Ulam-Hyers stable. □

**Example 3.10.** *Let us consider the following nonlinear fractional initial value problem*

$$\begin{cases} {}^C D_{0^+}^{\frac{3}{2}} x(t) = \frac{1}{4} + \frac{1}{4}\cos t + \frac{1}{18}\cos^2(t)\, x^{[1]}(t) + \frac{1}{19}\sin^2(t)\, x^{[2]}(t), \ t \in [0,1], \\ x(0) = x'(0) = 0, \end{cases} \tag{3.8}$$

*where* $T = 1$, $J = [0,1]$ *and*

$$f(t,x,y) = \frac{1}{4} + \frac{1}{4}\cos t + \frac{1}{18} x \cos^2(t) + \frac{1}{19} y \sin^2(t).$$

*We have*

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \leq \frac{1}{18}|x_1 - y_1| + \frac{1}{19}|x_2 - y_2|,$$

*then*

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \leq \sum_{i=1}^{2} c_i \|x_i - y_i\|.$$

*with* $c_1 = \frac{1}{18}$, $c_2 = \frac{1}{19}$. *Furthermore, if* $L = 1$ *and* $M = 4$ *in the definition of* $C_M(J,L)$, *then* $f$ *is positive,* $\rho = \sup\limits_{t \in J}\{f(t,0,0)\} = \frac{1}{2}$ *and* $\zeta = 0.5 + \left(\frac{1}{18} + \frac{4}{19}\right) \simeq 0.766$. *For* $\alpha = \frac{3}{2}$, *we get*

$$\frac{\zeta T^\alpha}{\Gamma(\alpha+1)} = \frac{0.766}{\Gamma\left(\frac{5}{2}\right)} \simeq 0.576 \leq L = 1,$$

*and*

$$\frac{\zeta T^{\alpha-1}}{\Gamma(\alpha)} = \frac{0.766}{\Gamma\left(\frac{3}{2}\right)} \simeq 0.864 \leq M = 4.$$

*So,*

$$\frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j = \frac{1}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{1}{18} + \frac{4}{19}\right) \simeq 0.2 < 1.$$

*Then, by Theorems 3.4 and 3.5, (3.8) has a unique mild solution which depends continuously on the function* $f$. *Also, from Theorem 3.8, (3.8) is Ulam-Hyers stable, and from Corollary 3.9, (3.8) is generalized Ulam-Hyers stable.*

# 4 Conclusion

In the current paper, under some sufficient conditions on the nonlinearity, we established the existence, uniqueness, continuous dependence and Ulam stability of a mild solution for an iterative Caputo fractional differential equation. The main tool of this work is the Schauder fixed point theorem. The obtained results have a contribution to the related literature.

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