


Hyper generalized pseudo Q -symmetric semi-Riemannian manifolds

ADARA M. BLAGA¹ 

MANOJ RAY BAKSHI² 

KANAK KANTI BAISHYA³ 

¹ *Department of Mathematics, West University of Timișoara, Timișoara, România.*
adarablaga@yahoo.com

^{2,3} *Department of Mathematics, Kurseong College, Kurseong, Darjeeling, India.*
raybakshimanoj@gmail.com;
kanakkanti.kc@gmail.com

ABSTRACT

The object of the present paper is to study the properties of a hyper generalized pseudo Q -symmetric semi-Riemannian manifold, proving that under certain assumptions, it is a perfect fluid spacetime.

RESUMEN

El objetivo del presente artículo es estudiar las propiedades de una variedad semi-Riemanniana hiper generalizada pseudo Q -simétrica, probando que bajo ciertas condiciones, es un espacio-tiempo fluido perfecto.

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1 Introduction

Let R , S , L and r denote the curvature tensor, Ricci tensor, Ricci operator and the scalar curvature of a (semi)-Riemannian manifold, respectively. It is Mantica and Suh [5] who have introduced the notion of Q -curvature tensor. In an n -dimensional Riemannian or semi-Riemannian manifold (M^n, g) ($n > 2$), the Q -curvature tensor is defined as

$$R(Y, U, V, W) = Q(Y, U, V, W) + \frac{\psi}{n-1} [g(Y, W)g(U, V) - g(Y, V)g(U, W)], \quad (1.1)$$

where Y, U, V, W are arbitrary vector fields on M^n and ψ is a scalar function. Semi-Riemannian manifolds with Ricci tensor S of the form

$$S(Y, V) = ag(Y, V) + bT(Y)T(V),$$

for any vector fields Y, V , are often termed as perfect fluid spacetimes, where a and b are scalars and the vector field ϱ , metrically equivalent to the 1-form T (that is, $g(Y, \varrho) = T(Y)$), is a unit time like vector field (that is, $g(\varrho, \varrho) = -1$).

An n -dimensional semi-Riemannian manifold is said to be *hyper generalized pseudo Q -symmetric* (which will be abbreviated hereafter as $(HGPQS)_n$) if it satisfies the equation

$$\begin{aligned} & (\nabla_X Q)(Y, U, V, W) \\ = & 2A_1(X)Q(Y, U, V, W) + A_1(Y)Q(X, U, V, W) \\ & + A_1(U)Q(Y, X, V, W) + A_1(V)Q(Y, U, X, W) \\ & + A_1(W)Q(Y, U, V, X) + 2A_2(X)(g \wedge S)(Y, U, V, W) \\ & + A_2(Y)(g \wedge S)(X, U, V, W) + A_2(U)(g \wedge S)(Y, X, V, W) \\ & + A_2(V)(g \wedge S)(Y, U, X, W) + A_2(W)(g \wedge S)(Y, U, V, X), \end{aligned} \quad (1.2)$$

where

$$\begin{aligned} (g \wedge S)(Y, U, V, W) = & g(Y, W)S(U, V) + g(U, V)S(Y, W) \\ & - g(Y, V)S(U, W) - g(U, W)S(Y, V) \end{aligned} \quad (1.3)$$

and A_1, A_2 are non-zero 1-forms whose g -dual vector fields will be denoted by θ_1 and θ_2 , i.e. $A_1(X) = g(X, \theta_1)$ and $A_2(X) = g(X, \theta_2)$.

We organized our paper as follows: section 2 is concerned with preliminaries. After preliminaries, some curvature properties of $(HGPQS)_n$ manifolds are studied in section 3. It is obtained that the Q -curvature tensor in a $(HGPQS)_n$ manifold satisfies 2nd Bianchi's identity. It is further obtained that the scalar function ψ is always constant. In section 4 we investigate properties of divergence-free $(HGPQS)_n$ manifolds and we prove that a divergence-free $(HGPQS)_n$ manifold ($n > 2$) under the assumption $A_1(Q(Y, U)V) = 0$ is a perfect fluid spacetime as well as the integral

curves of the vector field ϱ are geodesics and the vector field ϱ is irrotational, if the associated vector fields ϱ and σ corresponding to the 1-forms T_1 and T_2 are related by $(r - 1)\varrho + n\sigma = 0$.

2 Preliminaries

In this section, some relations useful to the study of $(HGPQS)_n$ manifolds are obtained. Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold, where $1 \leq i \leq n$.

From (1.1) we can easily verify that the tensor Q satisfies the following properties:

$$\begin{aligned} (i) \quad & Q(Y, U)V + Q(U, Y)V = 0, \\ (ii) \quad & Q(Y, U)V + Q(U, V)Y + Q(V, Y)U = 0, \end{aligned} \tag{2.1}$$

where $g(Q(X, Y)U, V) = Q(X, Y, U, V)$.

Also from (1.1) we have

$$\sum_{i=1}^n \epsilon_i Q(X, Y, e_i, e_i) = 0 = \sum_{i=1}^n \epsilon_i Q(e_i, e_i, W, U) \tag{2.2}$$

and

$$\begin{aligned} \sum_{i=1}^n \epsilon_i Q(e_i, Y, V, e_i) &= \sum_{i=1}^n \epsilon_i Q(Y, e_i, e_i, V) = S(Y, V) - \psi g(Y, V) \\ &=: Z(Y, V), \end{aligned} \tag{2.3}$$

where

$$\epsilon_i = g(e_i, e_i) = \pm 1, \quad S(X, Y) = \sum_{i=1}^n \epsilon_i g(R(X, e_i)e_i, Y), \quad r = \sum_{i=1}^n \epsilon_i S(e_i, e_i).$$

From (1.1) and (2.1) it follows that

$$\begin{aligned} (i) \quad & Q(X, Y, U, V) + Q(X, Y, V, U) = 0, \\ (ii) \quad & Q(X, Y, U, V) - Q(U, V, X, Y) = 0. \end{aligned} \tag{2.4}$$

3 Some curvature properties of $(HGPQS)_n$ manifolds

In this section we prove that in a $(HGPQS)_n$ manifold, the Q -curvature tensor satisfies 2nd Bianchi's identity, that is,

$$(\nabla_X Q)(Y, U, V, W) + (\nabla_Y Q)(U, X, V, W) + (\nabla_U Q)(X, Y, V, W) = 0. \tag{3.1}$$

In view of (1.1), (1.2) and (3.1) we get

$$\begin{aligned}
 & (\nabla_X Q)(Y, U, V, W) + (\nabla_Y Q)(U, X, V, W) + (\nabla_U Q)(X, Y, V, W) \\
 = & A_1(V)[Q(Y, U, X, W) + Q(U, X, Y, W) + Q(X, Y, U, W)] \\
 & + A_1(W)[Q(Y, U, V, X) + Q(U, X, V, Y) + Q(X, Y, V, U)] \\
 & + A_2(V)[(g \wedge S)(Y, U, X, W) + (g \wedge S)(U, X, Y, W) \\
 & + (g \wedge S)(X, Y, U, W)] + A_2(W)[(g \wedge S)(Y, U, V, X) \\
 & + (g \wedge S)(U, X, V, Y) + (g \wedge S)(X, Y, V, U)].
 \end{aligned} \tag{3.2}$$

Using (1.3) and 1st Bianchi's identity for the Q -curvature tensor in (3.2) and then simplifying, we obtain (3.1).

Thus we can state the following:

Theorem 3.1. *The Q -curvature tensor in a $(HGPQS)_n$ manifold satisfies 2nd Bianchi's identity.*

Using (1.1) in (3.1), we have

$$\begin{aligned}
 (\nabla_X R)(Y, U, V, W) & + (\nabla_Y R)(U, X, V, W) + (\nabla_U R)(X, Y, V, W) \\
 & - \frac{d\psi(X)}{(n-1)}[g(Y, W)g(U, V) - g(Y, V)g(U, W)] \\
 & - \frac{d\psi(Y)}{(n-1)}[g(U, W)g(X, V) - g(U, V)g(X, W)] \\
 & - \frac{d\psi(U)}{(n-1)}[g(X, W)g(Y, V) - g(X, V)g(Y, W)] = 0.
 \end{aligned} \tag{3.3}$$

By virtue of 2nd Bianchi's identity for the Riemannian curvature tensor, (3.3) yields

$$\begin{aligned}
 & \frac{d\psi(X)}{(n-1)}[g(Y, W)g(U, V) - g(Y, V)g(U, W)] \\
 & + \frac{d\psi(Y)}{(n-1)}[g(U, W)g(X, V) - g(U, V)g(X, W)] \\
 & + \frac{d\psi(U)}{(n-1)}[g(X, W)g(Y, V) - g(X, V)g(Y, W)] = 0.
 \end{aligned} \tag{3.4}$$

Contracting U and V in (3.4), we have

$$(n-2)[d\psi(X)g(Y, W) - d\psi(Y)g(X, W)] = 0 \tag{3.5}$$

which yields after further contraction

$$(n-1)(n-2)d\psi(X) = 0.$$

This implies that $d\psi(X) = 0$, that is, ψ is constant since $n > 2$ and leads to the following:

Theorem 3.2. *In a $(HGPQS)_n$ manifold, the scalar function ψ is always constant.*

Consequently, one can easily bring out the following:

Theorem 3.3. *In a $(HG PQS)_n$ manifold, $(divQ)(X, Y)Z$ and $(divR)(X, Y)Z$ are equivalent.*

In view of (1.1), (1.2) and Theorem 3.2 we have

$$\begin{aligned}
 & (\nabla_X R)(Y, U, V, W) \tag{3.6} \\
 = & 2A_1(X)Q(Y, U, V, W) + A_1(Y)Q(X, U, V, W) \\
 & + A_1(U)Q(Y, X, V, W) + A_1(V)Q(Y, U, X, W) \\
 & + A_1(W)Q(Y, U, V, X) + 2A_2(X)(g \wedge S)(Y, U, V, W) \\
 & + A_2(Y)(g \wedge S)(X, U, V, W) + A_2(U)(g \wedge S)(Y, X, V, W) \\
 & + A_2(V)(g \wedge S)(Y, U, X, W) + A_2(W)(g \wedge S)(Y, U, V, X)
 \end{aligned}$$

which yields

$$\begin{aligned}
 & (\nabla_X S)(U, V) \tag{3.7} \\
 = & [F_1(X) + F_2(X)]S(U, V) + F_2(U)S(X, V) + F_2(V)S(U, X) \\
 & + [F_3(X) + F_4(X)]g(U, V) + F_4(U)g(X, V) + F_4(V)g(U, X) \\
 & + A_1(Q(X, U)V) - A_1(Q(V, X)U)
 \end{aligned}$$

after contraction over Y and W , where

$$\begin{aligned}
 F_1(X) &= A_1(X) + (n + 1)A_2(X), \\
 F_2(X) &= A_1(X) + (n - 3)A_2(X), \\
 F_3(X) &= rA_2(X) - \psi A_1(X) + 3A_2(LX), \\
 F_4(X) &= rA_2(X) - \psi A_1(X) - A_2(LX),
 \end{aligned}$$

where L is the Ricci operator defined by $g(LX, Y) = S(X, Y)$.

Definition 3.4. *An n -dimensional semi-Riemannian manifold is called almost generalized pseudo Ricci symmetric if the non-flat Ricci curvature tensor satisfies the equation*

$$\begin{aligned}
 & (\nabla_X S)(U, V) \\
 = & [A(X) + B(X)]S(U, V) + A(U)S(X, V) + A(V)S(U, X) \\
 & + [C(X) + D(X)]g(U, V) + C(U)g(X, V) + C(V)g(U, X),
 \end{aligned}$$

where A, B, C and D are non-zero 1-forms whose g -dual vector fields will be denoted by $\gamma_1, \gamma_2, \delta_1$ and δ_2 , i.e. $A(X) = g(X, \gamma_1)$, $B(X) = g(X, \gamma_2)$, $C(X) = g(X, \delta_1)$ and $D(X) = g(X, \delta_2)$.

Thus we can state the following:

Theorem 3.5. A $(HGPQS)_n$ manifold ($n > 2$) under the assumption $A_1(Q(X, U)V) = A_1(Q(V, X)U)$ is necessarily almost generalized pseudo Ricci symmetric.

Making use of (2.3) in (3.7), we get

$$\begin{aligned}
 & (\nabla_X Z)(U, V) \\
 = & [F_1(X) + F_2(X)]Z(U, V) + F_2(U)Z(X, V) + F_2(V)Z(U, X) \\
 & + [F_3(X) + \psi F_1(X) + F_4(X) + \psi F_2(X)]g(U, V) \\
 & + [F_4(U) + \psi F_2(U)]g(X, V) + [F_4(V) + \psi F_2(V)]g(U, X),
 \end{aligned} \tag{3.8}$$

where $Z = S - \psi g$ is the tensor considered in ([4], [6], [7]). This leads to the following:

Theorem 3.6. A $(HGPQS)_n$ manifold ($n > 2$) under the assumption $A_1(Q(X, U)V) = A_1(Q(V, X)U)$ is necessarily almost generalized pseudo Z -symmetric.

4 $(HGPQS)_n$ manifolds ($n > 2$) with $div Q = 0$

Let (M^n, g) be a semi-Riemannian manifold of dimension n and let $\{e_i\}$ be an orthonormal basis of the tangent space $T_p M$ at any point $p \in M$ and $\epsilon_i = \pm 1$. Then the divergence of a vector field U is defined as

$$div U = \sum_{i=1}^n \epsilon_i g(\nabla_{e_i} U, e_i),$$

and the divergence of a tensor field of type $(1, 3)$, which is a tensor field of type $(0, 3)$, is defined as

$$(div K)(X, Y)Z = \sum_{i=1}^n \epsilon_i g((\nabla_{e_i} K)(X, Y)Z, e_i).$$

Now

$$\begin{aligned}
 (div Q)(Y, U)V &= \sum_{i=1}^n \epsilon_i g((\nabla_{e_i} Q)(Y, U)V, e_i) \\
 &= \sum_{i=1}^n \epsilon_i [2A_1(e_i)Q(Y, U, V, e_i) + A_1(Y)Q(e_i, U, V, e_i) \\
 &\quad + A_1(U)Q(Y, e_i, V, e_i) + A_1(V)Q(Y, U, e_i, e_i) \\
 &\quad + A_1(e_i)Q(Y, U, V, e_i) + 2A_2(e_i)(g \wedge S)(Y, U, V, e_i) \\
 &\quad + A_2(Y)(g \wedge S)(e_i, U, V, e_i) + A_2(U)(g \wedge S)(Y, e_i, V, e_i) \\
 &\quad + A_2(V)(g \wedge S)(Y, U, e_i, e_i) + A_2(e_i)(g \wedge S)(Y, U, V, e_i)]
 \end{aligned}$$

$$\begin{aligned}
 &= 3A_1(Q(Y,U)V) + A_1(Y)[S(U,V) - \psi g(U,V)] \\
 &\quad - A_1(U)[S(Y,V) - \psi g(Y,V)] + 3A_2(Y)S(U,V) \\
 &\quad + 3A_2(LY)g(U,V) - 3A_2(LU)g(Y,V) - 3A_2(U)S(Y,V) \\
 &\quad + A_2(Y)[(n-2)S(U,V) + rg(U,V)] \\
 &\quad - A_2(U)[(n-2)S(Y,V) + rg(Y,V)] \\
 &= 3A_1(Q(Y,U)V) + S(U,V)[A_1(Y) + (n+1)A_2(Y)] \\
 &\quad - S(Y,V)[A_1(U) + (n+1)A_2(U)] \\
 &\quad + g(U,V)[3A_2(LY) + rA_2(Y) - \psi A_1(Y)] \\
 &\quad - g(Y,V)[3A_2(LU) + rA_2(U) - \psi A_1(U)] \\
 &= 3A_1(Q(Y,U)V) + T_1(Y)S(U,V) - T_1(U)S(Y,V) \\
 &\quad + T_2(Y)g(U,V) - T_2(U)g(Y,V),
 \end{aligned}$$

hence

$$\begin{aligned}
 (divQ)(Y,U)V &= 3A_1(Q(Y,U)V) + T_1(Y)S(U,V) - T_1(U)S(Y,V) \\
 &\quad + T_2(Y)g(U,V) - T_2(U)g(Y,V),
 \end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
 T_1(Y) &= A_1(Y) + (n+1)A_2(Y) =: g(Y, \varrho), \text{ for } \varrho = \theta_1 + (n+1)\theta_2, \\
 T_2(Y) &= 3A_2(LY) + rA_2(Y) - \psi A_1(Y) =: g(Y, \sigma), \text{ for } \sigma = 3L\theta_2 + r\theta_2 - \psi\theta_1.
 \end{aligned}$$

Assuming $(divQ)(Y,U)V = 0$ and $A_1(Q(Y,U)V) = 0$, we get from the above equation

$$T_1(Y)S(U,V) + T_2(Y)g(U,V) = T_1(U)S(Y,V) + T_2(U)g(Y,V). \tag{4.2}$$

Now contracting (4.2) over U and V we get

$$S(Y, \varrho) = rT_1(Y) + (n-1)T_2(Y). \tag{4.3}$$

Again putting $V = \varrho$ in (4.2) we get

$$(n-2)[T_1(Y)T_2(U) - T_1(U)T_2(Y)] = 0, \tag{4.4}$$

which under the assumption $n > 2$ implies $T_1(Y)T_2(U) = T_1(U)T_2(Y)$.

Now putting $U = \varrho$ in (4.2) and using (4.3) and (4.4) we get

$$T_1(\varrho)S(Y,V) + T_2(\varrho)g(Y,V) = T_1(Y)[rT_1(V) + nT_2(V)] \tag{4.5}$$

and we can state:

Theorem 4.1. *A divergence-free (HGPQS)_n manifold ($n > 2$) under the assumption $A_1(Q(Y, U)V) = 0$ is a perfect fluid spacetime with unit timelike vector field ϱ , provided the associated vector fields ϱ and σ corresponding to the 1-forms T_1 and T_2 are related by $(r - 1)\varrho + n\sigma = 0$.*

In this case, (4.5) becomes

$$S(Y, V) = ag(Y, V) - T_1(Y)T_1(V), \quad (4.6)$$

where $a =: T_2(\varrho)$.

Again, $(\text{div}Q)(Y, U)V = 0$ gives

$$(\nabla_Y S)(U, V) - (\nabla_U S)(Y, V) = 0. \quad (4.7)$$

Now using (4.6) in (4.7) we find

$$\begin{aligned} & da(Y)g(U, V) - da(U)g(Y, V) \\ & - [T_1(V)(\nabla_Y T_1)(U) + T_1(U)(\nabla_Y T_1)(V)] \\ & + [T_1(V)(\nabla_U T_1)(Y) + T_1(Y)(\nabla_U T_1)(V)] = 0. \end{aligned} \quad (4.8)$$

Taking a frame field and contracting Y and V we get

$$(n - 1)da(U) + [T_1(U)(\delta T_1) + (\nabla_\varrho T_1)(U)] = 0, \quad (4.9)$$

where

$$\delta T_1 = \sum_{i=1}^n \epsilon_i (\nabla_{e_i} T_1)(e_i).$$

Setting $V = Y = \varrho$ in (4.8) we find

$$(\nabla_\varrho T_1)(U) = -da(U) - da(\varrho)T_1(U). \quad (4.10)$$

Substituting (4.10) in (4.9) we get

$$(n - 2)da(U) + T_1(U)(\delta T_1) - da(\varrho)T_1(U) = 0 \quad (4.11)$$

which yields

$$\delta T_1 = (n - 1)da(\varrho) \quad (4.12)$$

for $U = \varrho$.

Using (4.12) in (4.11) we obtain

$$da(U) = -T_1(U)da(\varrho), \quad (4.13)$$

provided $n > 2$.

Putting $V = \varrho$ in (4.8) and using (4.13) we get

$$(\nabla_Y T_1)(U) - (\nabla_U T_1)(Y) = 0.$$

This means that the 1-form T_1 is closed, that is,

$$dT_1(Y, U) = 0.$$

Hence

$$g(\nabla_U \varrho, Y) = g(\nabla_Y \varrho, U) \text{ for all } U, Y, \tag{4.14}$$

which yields

$$g(\nabla_{\varrho} \varrho, Y) = g(\nabla_Y \varrho, \varrho), \tag{4.15}$$

for $U = \varrho$. Since $g(\nabla_Y \varrho, \varrho) = 0$, from (4.15) it follows that $g(\nabla_{\varrho} \varrho, Y) = 0$ for all Y . Hence $\nabla_{\varrho} \varrho = 0$. This implies that the integral curves of the vector field ϱ are geodesics. Therefore we can state the following:

Theorem 4.2. *In a divergence-free $(HG PQS)_n$ manifold ($n > 2$) under the assumption $A_1(Q(Y, U)V) = 0$, the integral curves of the unit timelike vector field ϱ are geodesics, provided the associated vector fields ϱ and σ corresponding to the 1-forms T_1 and T_2 are related by $(r - 1)\varrho + n\sigma = 0$.*

Taking into account that the divergence of the conformal curvature tensor of a Riemannian manifold (M^n, g) is ([3], [6]):

$$\begin{aligned} (div C)(X, Y)Z &= \frac{n-3}{n-2} [(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)] \\ &= \frac{n-3}{n-2} (div Q)(X, Y)Z, \end{aligned} \tag{4.16}$$

for any vector fields X, Y, Z on M^n , from the Lemma 2.1 of [2] we infer

Theorem 4.3. *Let (M, g) be a $(HG PQS)_n$ perfect fluid spacetime ($n > 2$). If $(div Q)(X, Y)Z = 0$, for any vector fields X, Y, Z on M , then the unit timelike vector field ϱ is irrotational.*

Also, in [2] was proved the following result:

Theorem 4.4. [2] *Let (M, g) be a $(HG PQS)_n$ perfect fluid spacetime ($n > 2$). If $(div Q)(X, Y)Z = 0$, for any vector fields X, Y, Z on M , then (M, g) is a GRW spacetime whose fiber is Einstein.*

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