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Toric, $U(2)$, and LeBrun metrics

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ABSTRACT

The LeBrun ansatz was designed for scalar-flat Kähler metrics with a continuous symmetry; here we show it is generalizable to much broader classes of metrics with a symmetry. We state the conditions for a metric to be (locally) expressible in LeBrun ansatz form, the conditions under which its natural complex structure is integrable, and the conditions that produce a metric that is Kähler, scalar-flat, or extremal Kähler. Second, toric Kähler metrics (such as the generalized Taub-NUTs) and $U(2)$-invariant metrics (such as the Fubini-Study or Page metrics) are certainly expressible in the LeBrun ansatz. We give general formulas for such transitions. We close the paper with examples, and find expressions for two examples—the exceptional half-plane metric and the Page metric—in terms of the LeBrun ansatz.

RESUMEN

El ansatz de LeBrun fue diseñado para métricas Kähler escalares-planas con una simetría continua; acá mostramos que es generalizable a clases mucho más amplias de métricas con una simetría. Establecemos las condiciones para que una métrica sea (localmente) expresable con la forma de ansatz de LeBrun, las condiciones bajo las cuales su estructura compleja natural es integrable, y las condiciones que producen una métrica que es Kähler, escalar-plana, o Kähler extremal. En segundo lugar, métricas tóricas Kähler (tales como las Taub-NUT generalizadas) y métricas $U(2)$-invariantes (tales como la métrica de Fubini-Study o la de Page) son ciertamente expresables en el ansatz de LeBrun. Damos fórmulas generales para tales transiciones. Concluimos el artículo con ejemplos, y encontramos expresiones para dos ejemplos—la métrica excepcional del semiplano y la métrica de Page—en términos del ansatz de LeBrun.

Keywords and Phrases: Differential geometry, Kähler geometry, canonical metrics, ansatz.

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1 Introduction

LeBrun [19] created an ansatz for scalar-flat Kähler metrics with a continuous symmetry. This was an expansion of the Gibbons-Hawking ansatz for Ricci-flat Kähler metrics with a symmetry, itself a version of the Kaluza ansatz [18] [6]. In the original construction Kaluza showed that if a Lorentzian 5-metric is endowed with a spacelike continuous symmetry, the Einstein equations will partially linearize, with the linear part being the Maxwell equations. The Gibbons-Hawking construction utilized this idea except in Euclidean signature and a dimension lower, where the Maxwell equations reduce to just the Laplace equation on a potential, and the "gravity" equations (the Ricci-flat equations) fully linearize.

LeBrun's ansatz, which also works for 4-dimensional Riemannian metrics with a circle symmetry, partially linearizes the scalar-flat Kähler (SFK) equations. These SFK equations, normally exceedingly complicated and nonlinear, were shown to reduce to a pair of second order equations, one linear and the other quasilinear.

We show that LeBrun's ansatz is much more general than this original use, and is suitable for expressing interesting 4-metrics that are not scalar-flat, Kähler, or even have an integrable complex structure. We show the conditions under which a metric is expressible in terms of the LeBrun ansatz, and give the explicit transformations into the LeBrun ansatz from two toric Kähler ansätze, and from the $U(2)$-invariant ansatz. In the last section we use these translations to express several common metrics in the LeBrun ansatz. Finally we indicate how the LeBrun ansatz can be used, at least in principle, to create new metrics of special kinds, a subject we shall take up elsewhere.

2 The LeBrun ansatz

We lay out the basic definitions in the LeBrun ansatz and determine when the ansatz possesses an integrable complex structure and when it possesses a closed Kähler 2-form. We end with some expressions for curvature quantities of such metrics, and state when such a metric is extremal Kähler. The reference for this section is [19].

2.1 The ansatz

The LeBrun ansatz is an $\mathbb{S}^1$-fibration $\pi : M^4 \to N^3$ along with the metric

$$g \;=\; we^u\left(dx^2\,+\,dy^2\right)\;+\;w\,dz^2\;+\;w^{-1}\left(d\tau\,+\,\pi^*A\right)^2 \tag{2.1}$$

where (x, y, z) are local coordinates on N^3, $w = w(x, y, z)$ and $u = u(x, y, z)$ are functions, and A is a 1-form $A = A_x(x, y, z)dx + A_y(x, y, z)dy + A_z(x, y, z)dz$ on N^3.[1] The coordinate τ is defined after a choice of a transversal: after setting $\tau = 0$ on this transversal, τ is pushed forward via the $\mathbb{S}^1$-action. The field $\frac{d}{d\tau}$ is invariant under rechoosing the transversal so it is globally defined, and it is Killing.

The exterior derivative of A will be important. Because $d\pi^*A = \pi^*dA$, it is immaterial whether we compute on M^4 or N^3. Letting $B = dA$ we have

$$\begin{aligned} B \;&=\; B_x\, dy \wedge dz \,-\, B_y\, dx \wedge dz \,+\, B_z\, dx \wedge dy, \quad \text{where} \\ B_x \;&=\; A_{y,x} - A_{x,y}, \quad B_y \;=\; A_{x,z} - A_{z,x}, \quad B_z \;=\; A_{z,y} - A_{y,z}. \end{aligned} \tag{2.2}$$

In the spirit of Kaluza's work, we may interpret A as a vector potential over 3-space and $B = dA$ as the corresponding Maxwell field strength. It so closely resembles a magnetostatic field that we will sometimes call it the metric's magnetic field. In all curvature computations A never appears; only its field B appears.

A g-compatible almost-complex structure on (M^4, g) is

$$J(dx) \;=\; -dy, \quad J(dz) \;=\; -w^{-1}(d\tau + \pi^*A), \tag{2.3}$$

which dualizes to

$$J(\nabla x) \;=\; \nabla y, \quad J(\nabla z) \;=\; \frac{\partial}{\partial \tau} \tag{2.4}$$

where the duality convention is $J(\eta) \triangleq \eta \circ J$ for $\eta \in \bigwedge^1$. The corresponding antisymmetric form is

$$\omega \;=\; g(J\cdot, \cdot) \;=\; we^u dx \wedge dy \,+\, dz \wedge (d\tau + \pi^*A)\,. \tag{2.5}$$

2.2 The complex and symplectic structures

As usual, the almost complex structure splits $\bigwedge^1_{\mathbb{C}} = \bigwedge^1(M^4) \otimes \mathbb{C}$ into holomorphic and antiholomorphic bundles, where $\bigwedge^1_{\mathbb{C}} = \bigwedge^{1,0} \oplus \bigwedge^{0,1}$ are the respective $\pm\sqrt{-1}$ eigenspaces of J. In bases,

$$\begin{aligned} \bigwedge{}^{1,0} \;&=\; span_{\mathbb{C}} \left\{ dx + \sqrt{-1}dy, \;\; dz + \sqrt{-1}w^{-1}(d\tau + \pi^*A) \right\}, \\ \bigwedge{}^{0,1} \;&=\; span_{\mathbb{C}} \left\{ dx - \sqrt{-1}dy, \;\; dz - \sqrt{-1}w^{-1}(d\tau + \pi^*A) \right\}. \end{aligned} \tag{2.6}$$

Of the many ways to check the integrability of an almost-complex structure, the most convenient will be verifying that $d : \bigwedge^{0,1} \to \bigwedge^1_{\mathbb{C}} \wedge \bigwedge^{0,1}$.

Lemma 2.1. *The complex structure (2.3) is integrable if and only if*

$$w_x \;=\; B_x \quad \textit{and} \quad w_y \;=\; B_y. \tag{2.7}$$

[1] LeBrun denotes $\omega = d\tau + \pi^*A$, and interprets this as a connection. Following a different but very standard convention, we shall prefer using the symbol ω for the 2-form $\omega = g(J\cdot, \cdot)$.

Proof. This comes from out of the proof of Proposition 1 of [19]. We compute on bases. Certainly $d(dx - \sqrt{-1}dy) = 0$. Then

$$\begin{aligned} & d\left(dz - \sqrt{-1}w^{-1}(d\tau + \pi^* A)\right) \\ & \quad = \; w^{-1}\Big(dw \wedge \left(dz - \sqrt{-1}w^{-1}(d\tau + \pi^* A)\right) \; - \; dw \wedge dz \; - \; \sqrt{-1}B\Big). \end{aligned} \tag{2.8}$$

From (2.6), the first term is in $\bigwedge^1_{\mathbb{C}} \wedge \bigwedge^{0,1}$. The second and third terms become

$$\begin{aligned} & - dw \wedge dz - \sqrt{-1}B \\ & \quad = \; -(w_x - \sqrt{-1}B_y)dx \wedge dz - (w_y + \sqrt{-1}B_x)dy \wedge dz - \sqrt{-1}B_z dx \wedge dy \\ & \quad = \; \frac{1}{2}\Big((w_x - B_x) - \sqrt{-1}(w_y - B_y)\Big)dz \wedge (dx + \sqrt{-1}dy) \\ & \qquad + \frac{1}{2}\Big((w_x + B_x)dz - \sqrt{-1}(B_y + w_y)dz \\ & \qquad\qquad\qquad - \; \sqrt{-1}B_z(dx + \sqrt{-1}dy)\Big) \wedge (dx - \sqrt{-1}dy). \end{aligned} \tag{2.9}$$

Because $dx - \sqrt{-1}dy \in \bigwedge^{0,1}$ the second term on the right is in $\bigwedge^1_{\mathbb{C}} \wedge \bigwedge^{0,1}$. But the first term is in $\bigwedge^1_{\mathbb{C}} \wedge \bigwedge^{1,0}$. We conclude J is integrable if and only if this term is zero, which is the same as $(w_x - B_x) - \sqrt{-1}(w_y - B_y) = 0$. ☐

Lemma 2.2. *We have $d\omega = (-B_z + (we^u)_z)\, dz \wedge dx \wedge dy$. In particular, the antisymmetric form ω of (2.5) is closed if and only if $B_z = (we^u)_z$.*

Proof. Using $\omega = dz \wedge (d\tau + \pi^* A) + we^u dx \wedge dy$ and $d\pi^* A = \pi^* dA = \pi^* B$,

$$\begin{aligned} d\omega \; &= \; -dz \wedge d\pi^* A \; + \; (we^u)_z dz \wedge dx \wedge dy \\ &= \; (-B_z \; + \; (we^u)_z)\, dz \wedge dx \wedge dy, \end{aligned} \tag{2.10}$$

from which the assertion follows. ☐

Theorem 2.1. *The triple (g, J, ω) always has $g(J\cdot, J\cdot) = g(\cdot, \cdot)$. It is*

i) Hermitian if and only if $B_x = w_x$ and $B_y = w_y$,

ii) symplectic if and only if $B_z = (we^u)_z$, and

iii) Kähler if and only if $B_x = w_x$, $B_y = w_y$, and $B_z = (we^u)_z$.

Condition (iii) implies

$$w_{xx} + w_{yy} + (we^u)_{zz} = 0. \tag{2.11}$$

Proof. After Lemmas 2.2 and 2.1, we must only verify equation (2.11). But with $B = dA$, after assuming the relations in (*iii*) then equation (2.11) is just $dB = 0$. □

Remark. The metric is *almost Kähler* if (*ii*) holds but (*i*) does not.

Remark. The original approach of LeBrun [19] was essentially the reverse of this. LeBrun solves (2.11) for w first, and then finds a 1-form A (which will have Dirac string singularities) whose field B satisfies (*iii*). This contrasts with our method which starts with a metric of the form (2.1), finds conditions on A and w that give it special traits, and from such traits *derives* equation (2.11).

We have the following characterization of the LeBrun ansatz.

Theorem 2.2. *Let g be a metric on M^4. Then g can be expressed locally via the LeBrun ansatz if and only if the following three conditions hold:*

i) M^4 has a vector field v and an almost-complex structure J compatible with g so that, letting $\omega = g(J\cdot, \cdot)$ be the associated antisymmetric form, then ω, g, and J are all v-invariant,

ii) Given any simply connected domain $\Omega \subset M^4$, there is a function $z : \Omega \to \mathbb{R}$ with $i_v\omega = dz$, and

iii) The action of ∇z on J, when restricted to the rank-2 distribution $P \subset \bigwedge^1 M^4$ that is null on span$\{v, Jv\}$, is zero.

Remark. Regarding condition (*iii*), P is specifically the distribution $P = \{\eta \in \bigwedge^1_{M^4}$ such that $\eta(v) = 0$ and $\eta(Jv) = 0\}$.

Remark. Condition (*iii*) is certainly the most technical; it exists so that the first two terms in the ansatz can be written in the form $f(x,y,z)(dx^2 + dy^2)$, instead of $f_1dx^2 + f_2(dxdy + dydx) + f_3dy^2$. Condition (*iii*) could also be written $\mathcal{L}_{\nabla z}(J\big|_P) = 0$ where $\mathcal{L}$ is the Lie derivative.

Proof. Supposing g can be expressed via the LeBrun ansatz, we simply set $v = \frac{\partial}{\partial t}$ and let J be as in (2.3) or equivalently (2.4). The work above shows J and ω are v-invariant and $i_v\omega = dz$. We compute $\mathcal{L}_{\nabla z}J\big|_P$ by

$$(\mathcal{L}_{\nabla z}J)(dx) = \mathcal{L}_{\nabla z}(Jdx) - J\mathcal{L}_{\nabla z}dx = \mathcal{L}_{\nabla z}(dy) - J\mathcal{L}_{\nabla z}dx. \tag{2.12}$$

The Cartan formula gives $\mathcal{L}_{\nabla z}dx = di_{\nabla z}dx = d\langle dz, dx\rangle$. But this inner product is zero, as is easily verified after computing the inverse matrix g^{ij}. Similarly $\mathcal{L}_{\nabla z}dy = 0$, so we have shown $\mathcal{L}_{\nabla z}J(dx) = 0$. The same argument works for $\mathcal{L}_{\nabla z}J(dy)$, so we have shown that $\mathcal{L}_{\nabla z}(J\big|_P) = 0$.

For the converse we assume g, J, ω are v-invariant, and that $i_v\omega = dz$ for some function z. This allows us to perform a version of the Kähler reduction. Because z is itself v-invariant (due to the fact that $\mathcal{L}_vz = i_vi_v\omega = 0$), the function z passes to the quotient manifold $N^3 = M^4/v$

where the quotient is by the action of the Killing field v—this works if the orbits of v are closed; if not then a second Killing field must exist, and we can take an appropriate linear combination to find a Killing field with closed orbits. Pick a level-set $\Sigma^2_z = \{z = const\}$ on which to place isothermal coordinates (x, y), and then extend (x, y) along trajectories of ∇z so the functions x, y are now defined on some region of N^3. We show that (x, y) remains isothermal on all other nearby level-sets of z; this is a consequence of $J|_P$ being invariant under trajectories of ∇z. To see this, note that $J|_P$ restricts to the Hodge-star $*_2$ on any level-set of z, and x, y are isothermal if and only if $d *_2 dx = d *_2 dy = 0$ and $dx \wedge *dy = 0$. By construction, $d *_2 dx = d *_2 dy = 0$ and $dx \wedge *dy = 0$ holds on *one* level-set of z; to see it is true on all nearby level-sets we compute

$$\mathcal{L}_{\nabla z} d *_2 dx \;=\; d\mathcal{L}_{\nabla z} J|_P dx \;=\; dJ|_P \mathcal{L}_{\nabla z} dx \;=\; dJ|_P d\mathcal{L}_{\nabla z} x \;=\; 0. \tag{2.13}$$

where we used the facts that d always commutes with $\mathcal{L}_{\nabla z}$, that by hypothesis $\mathcal{L}_{\nabla z} J|_P = 0$, and that by construction $\mathcal{L}_{\nabla z} x = 0$. Therefore $d *_2 dx$ remains zero on all level-sets. Similarly we compute

$$\begin{aligned} \mathcal{L}_{\nabla z}\left(dx \wedge *_2 dy\right) \;&=\; \left(\mathcal{L}_{\nabla z} dx\right) \wedge *_2 dy + dx \wedge \left(\mathcal{L}_{\nabla z} *_2 dy\right) \\ &=\; dx \wedge *_2 \left(\mathcal{L}_{\nabla z} dy\right) \;=\; 0 \end{aligned} \tag{2.14}$$

where again we used $\mathcal{L}_{\nabla z} dx = \mathcal{L}_{\nabla z} dy = 0$ and $\mathcal{L}_{\nabla z} *_2 = \mathcal{L}_{\nabla z} J|_P = 0$.

Now, because the functions x, y remain an isothermal system on any level-set of z, we may express the metric g_3 on the quotient manifold N^3 in the form $g_3 = f_1(x, y, z) dz^2 + f_2(x, y, z)\left(dx^2 + dy^2\right)$. We define the functions w, e^u by

$$\begin{aligned} w \;&\triangleq\; |dz|^{-2}_{g_3} \;=\; f_1 \\ we^u \;&\triangleq\; |dx|^{-2}_{g_3} \;=\; |dy|^{-2}_{g_3} = f_2. \end{aligned} \tag{2.15}$$

The functions x and y pull back from N^3 to M^4, where we now have three coordinate functions x, y, and z. For the fourth coordinate τ, after choosing a transversal to v, we may set $\tau = 0$ along this transversal, and push τ along trajectories of v—incidentally, this establishes $\frac{\partial}{\partial \tau} = v$ and $J\nabla z = \frac{\partial}{\partial \tau}$. We now have coordinates (x, y, z, τ) on M^4.

From (2.15) we have $w^{-1} = |dz|^2 = |\nabla z|^2 = |J\nabla z|^2 = |\partial/\partial\tau|^2$. We define functions C, A_x, A_y, and A_z in terms of the complex structure J by

$$-C\left(d\tau + A_x dx + A_y dy + A_z dz\right) \;=\; Jdz. \tag{2.16}$$

We can compute the value of C. Transvecting both sides of (2.16) with $\frac{\partial}{\partial \tau}$ gives

$$-C \;=\; Jdz\left(\tfrac{\partial}{\partial \tau}\right) \;=\; \left\langle \nabla z,\, J\tfrac{\partial}{\partial \tau}\right\rangle \;=\; -|\nabla z|^2 \;=\; -|dz|^2 = -w^{-1}. \tag{2.17}$$

Therefore $C = w^{-1}$. Finally because the distribution $\{\nabla x, \nabla y\}$ is perpendicular to the distribution $\{\nabla z,\, \partial/\partial\tau\}$, we arrive at the expression

$$g \;=\; we^u\left(dx^2+dy^2\right) \;+\; wdz^2 \;+\; w^{-1}\left(d\tau+A_xdz+A_ydy+A_zdz\right)^2. \tag{2.18}$$

□

2.3 Curvature quantities

Proposition 2.1. *Assume the metric (2.1) is Kähler, meaning (iii) of Theorem 2.1 holds. Then the Ricci curvature of g is*

$$\mathrm{Ric} \;=\; -\frac{1}{2}\Big(\mathrm{Hess}\;u\;(\cdot,\,\cdot)+\mathrm{Hess}\;u\;(J\cdot,\,J\cdot)\Big) \tag{2.19}$$

Proof. The proof of Proposition 1 of [19] gives Ricci form and Ricci curvature

$$\begin{aligned}\rho \;&=\; -\sqrt{-1}\partial\bar{\partial}u, \quad \text{and}\\ \mathrm{Ric} \;&=\; \rho(\cdot,\,J\cdot) \;=\; -\frac{1}{2}\big(\mathrm{Hess}\,u\;(\cdot,\,\cdot)+\mathrm{Hess}\,u\;(J\cdot,\,J\cdot)\big).\end{aligned} \tag{2.20}$$

□

Proposition 2.2. *Assume the metric (2.1) is Kähler, meaning (iii) of Theorem 2.1 holds. Then the scalar curvature s of g is*

$$s \;=\; -\frac{1}{we^u}\left(u_{xx}+u_{yy}+(e^u)_{zz}\right). \tag{2.21}$$

Proof. This is computed in the proof of Proposition 1 of [19]. □

Proposition 2.3 (The extremal condition)**.** *Assume the metric (2.1) is Kähler. Then it is an extremal Kähler metric if constants $m, b\in\mathbb{R}$ exist so*

$$-\frac{1}{we^u}\left(u_{xx}+u_{yy}+(e^u)_{zz}\right) \;=\; mz\;+\;b. \tag{2.22}$$

Proof. If (2.22) holds then $s=mz+b$ and so $\nabla s=m\nabla z$ and $J\nabla s=m\frac{\partial}{\partial\tau}$; thus $J\nabla s$ is a Killing field. The proposition is established after recalling that a Kähler metric is extremal if and only if $J\nabla s$ is Killing [7] [8]. □

Remark. Whether g is Kähler or not, its scalar curvature is

$$\begin{aligned}s \;=\; -\frac{1}{we^u}\Bigg(&\Big(u_{xx}+u_{yy}+(e^u)_{zz}\Big)+\frac{1}{w}\Big(w_{xx}+w_{yy}+(we^u)_{zz}\Big)\\ &+\frac{1}{2w^2}(B_x^2-(w_x)^2)+\frac{1}{2w^2}(B_y^2-(w_y)^2)+\frac{e^{-u}}{2w^2}(B_z^2-\big((we^u)_z\big)^2\Bigg).\end{aligned} \tag{2.23}$$

3 Expressing Toric Kähler metrics using the LeBrun ansatz

The LeBrun ansatz operates on 4-manifolds with one symmetry. On Kähler 4-manifolds with *two* holomorphic symmetries, there are more specialized ansätze. Letting $\mathcal{X}^1$, $\mathcal{X}^2$ be commuting holomorphic Killing fields (recall that "holomorphic" means $\mathcal{L}_{\mathcal{X}^i}J = 0$, just as Killing means $\mathcal{L}_{\mathcal{X}^i}g = 0$), then $(M^4, g, J, \mathcal{X}^1, \mathcal{X}^2)$ can be considered a *toric Kähler* 4-manifold. This situation has been studied in [17] [1] [13] [14] [2] [9] and many other works. Certainly a toric Kähler metric can be translated into the LeBrun ansatz once a distinguished Killing field is chosen. We do this here.

3.1 The two toric ansätze

There are two standard presentations for toric Kähler 4-manifolds. These were originally explored by Guillemin [17], who also discovered that they are equivalent via a Legendre transform. The LeBrun ansatz is a mixture of the two.

The first of the two presentations is the *symplectic ansatz*. If $\{\mathcal{X}^1, \mathcal{X}^2\}$ are independent commuting holomorphic Killing fields, we can use the Arnold-Liouville construction [3] to produce the so-called action-angle coordinates on M^4. To execute this construction, one defines *action variables* (up to a constant) by $\nabla\varphi^i = -J\mathcal{X}^i$ or equivalently by $d\varphi^i = i_{\mathcal{X}^i}\omega$, and defines *angle variables*, denoted θ_1, θ_2, by choosing a transversal and then pushing forward the action of the fields $\mathcal{X}^1$, $\mathcal{X}^2$. In these coordinates, the ansatz demands the metric be expressed

$$g \;=\; U_{ij}d\varphi^i \otimes d\varphi^j \;+\; U^{ij}d\theta_i \otimes d\theta_j \tag{3.1}$$

where $U = U(\varphi^1, \varphi^2)$ is a convex function of the action variables. The matrix (U_{ij}) is defined by $U_{ij} \triangleq \frac{\partial^2 U}{\partial\varphi^i\varphi^j}$, and we define $(U^{ij}) \triangleq (U_{ij})^{-1}$.

The map $M^4 \to \mathbb{R}^2$ given by $p \mapsto (\varphi^1(p), \varphi^2(p))$ sends M^4 to a region $\Sigma^2 \subset \mathbb{R}^2$; this is sometimes called the *Arnold-Liouville reduction* or, by abuse of terminology, the *moment map*. If M^4 is compact then its image Σ^2 is a compact polygon in $\mathbb{R}^2$. This polygon encodes the topology of M^4, via the Delzant gluing rules [11]. If M^4 is non-compact, then Σ^2 need not be a polygon nor even be topologically closed.

The second ansatz, the *holomorphic ansatz*, also begins with the fields $\{\mathcal{X}^1, \mathcal{X}^2\}$. Again we may produce corresponding coordinates θ_1, θ_2 after choosing a transversal. Because $\mathcal{X}^1$, $\mathcal{X}^2$ are not only symplectomorphic but holomorphic, the variables θ^i are actually pluriharmonic, meaning $d(Jd\theta_i) = 0$. The Poincaré lemma then guarantees functions ξ_1, ξ_2 exist (at least locally) so that $d\xi^i = Jd\theta_i$, and we have two holomorphic functions $f_i = \xi_i + \sqrt{-1}\theta_i$ which constitute a holomorphic chart $(f_1, f_2) : \Omega \to \mathbb{C}^2$ on some subdomain $\Omega \subseteq M^4$. The Kähler form on this chart, as usual, can be expressed $\omega = \sqrt{-1}\partial\bar{\partial}V$ for some pseudoconvex function V. Because V is θ_1-θ_2 invariant, it is convex instead of just pseudoconvex. The metric is then

$$g \ = \ V^{ij} d\xi_i \otimes d\xi_i \ + \ V^{ij} d\theta_i \otimes d\theta_j \tag{3.2}$$

where (V^{ij}) is the matrix with components $V^{ij} \triangleq \frac{\partial^2 V}{\partial \xi_i \partial \xi_j}$.

We might consider the map $p \mapsto (\xi_1(p),\, \xi_2(p))$ for $p \in M^4$, just as we considered the moment map $p \mapsto (\varphi^1(p), \varphi^2(p))$. But it is much less interesting than the moment map. If M^4 is compact then its image is all of $\mathbb{R}^2$. In particular there is no way to read off the topology of M^4 from its image.

A duality relationship exists between the symplectic system $(\varphi^1, \theta_1, \varphi^2, \theta_2)$ with its symplectic potential U and the holomorphic system $(\xi_1, \theta_1, \xi_2, \theta_2)$ with its Kähler potential V. As shown in [17], they are Legendre transforms of each other:

$$\begin{aligned} \xi_i \ &= \ \frac{\partial U}{\partial \varphi^i}, \quad \varphi^i \ = \ \frac{\partial V}{\partial \xi_i}, \quad \text{and} \\ U(\varphi^i) \ + \ V(\xi_i) \ &= \ \sum_i \varphi^i \xi_i. \end{aligned} \tag{3.3}$$

3.2 Translation to the LeBrun Ansatz

It is now possible to relate these two systems to the LeBrun ansatz, which is a mixed symplectic-holomorphic system. We define the LeBrun variable τ to be the angle variable θ_1 corresponding to $\mathcal{X}^1$, and y the angle variable θ_2 corresponding to $\mathcal{X}^2$. Let z be the *symplectic* variable corresponding to the angle τ, meaning $z = \varphi^1$, and x the *holomorphic* variable corresponding the angle variable y, meaning $x = \xi_2$. Then we create the LeBrun functions w and u, and determine the 1-form A. We record the change of frame from the symplectic frame $\left\{\frac{\partial}{\partial \varphi^1}, \frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \varphi^2}, \frac{\partial}{\partial \theta_2}\right\}$ to the LeBrun frame $\left\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$. One easily computes

$$\begin{array}{rclcrcl} \frac{\partial}{\partial \varphi^1} &=& \frac{\partial}{\partial z} + U_{21}\frac{\partial}{\partial x} & \qquad & d\varphi^1 &=& dz \\ \frac{\partial}{\partial \theta_1} &=& \frac{\partial}{\partial \tau} & & d\theta^1 &=& d\tau \\ \frac{\partial}{\partial \varphi^2} &=& U_{22}\frac{\partial}{\partial x} & & d\varphi^2 &=& -\frac{U_{21}}{U_{22}}dz \ + \ \frac{1}{U_{22}}dx \\ \frac{\partial}{\partial \theta_1} &=& \frac{\partial}{\partial y} & & d\theta_2 &=& dy. \end{array} \tag{3.4}$$

Upon substituting the symplectic frame components into the LeBrun metric (2.1), we find the functions w, u and the components A_x, A_y, and A_z to be

$$\begin{aligned} & w \ = \ 1/U^{11}, \quad u \ = \ \log\left(U^{11}U^{22} - (U^{12})^2\right) \\ & A_x \ = \ 0, \quad A_y \ = \ \frac{U^{12}}{U^{11}}, \quad A_z \ = \ 0. \end{aligned} \tag{3.5}$$

We express this in the form of a proposition.

Proposition 3.1. *Assume $(M^4, J, g, \mathcal{X}^1, \mathcal{X}^2)$ is a toric Kähler manifold. Let $(\varphi^1, \theta_1, \varphi^2, \theta_2)$ be symplectic coordinates and $(\xi_1, \theta_1, \xi_2, \theta_2)$ holomorphic coordinates on M^4. There exists a convex function $U(\varphi^1, \varphi^2)$ on Σ^2, where Σ^2 is the image of the moment map $(\varphi^1, \varphi^2) : M^4 \to \mathbb{R}^2$, so that*

$$g \;=\; U_{ij}d\varphi^i \otimes d\varphi^j + U^{ij}d\theta_i \otimes d\theta_j \tag{3.6}$$

where $U_{ij} = \frac{\partial^2 U}{\partial\varphi^i\varphi^j}$ and $(U^{ij}) = (U_{ij})^{-1}$. There also exists a convex function $V = V(\xi_1, \xi_2)$ on $\mathbb{R}^2$ so that

$$g \;=\; V^{ij}d\xi_i \otimes d\xi_j + V^{ij}d\theta_i \otimes d\theta_j \tag{3.7}$$

where $V^{ij} = \frac{\partial^2 V}{\partial\xi_i\xi_j}$. These systems are related via the Legendre transform:

$$\begin{aligned} \varphi^i \;&=\; \frac{\partial V}{\partial \xi_i}, \quad \xi_i \;=\; \frac{\partial U}{\partial \varphi^i}, \\ U(\varphi^1, \varphi^2) + V(\xi_1, \xi_2) \;&=\; \varphi^1\xi_1 + \varphi^2\xi_2. \end{aligned} \tag{3.8}$$

The metric $(M^4, g, J, \mathcal{X}^1, \mathcal{X}^2)$ can be expressed in the LeBrun ansatz after setting

$$\big(z,\,\tau,\,x,\,y\big) \;=\; \big(\varphi^1,\,\theta_1,\,\xi_2,\,\theta_2\big). \tag{3.9}$$

A LeBrun ansatz expression of g is obtained by setting

$$\begin{aligned} u \;&=\; \log\det U^{ij} \;=\; \log\left(U^{11}U^{22} - (U^{12})^2\right), \\ w \;&=\; \frac{1}{U^{11}}, \quad \text{and} \quad A \;=\; A_y dy \;=\; \frac{U^{12}}{U^{11}}dy \end{aligned} \tag{3.10}$$

(the components A_x and A_z are zero). The components of the magnetic 2-form are $B_x = -A_{y,z}$, $B_y = 0$, and $B_z = A_{y,x}$.

3.3 Variation of LeBrun structures

In our construction of Section 3.2 we began by setting $\tau = \theta_1$, but we could have chosen $\tau = \theta_2$ or indeed any linear combination of the cyclic variables. Up to scale a toric metric automatically has a 1-parameter family of distinct LeBrun structures. If $\alpha \in [0, \pi/2]$ is a constant and $\mathcal{X}^1$, $\mathcal{X}^2$ are symplectomorphic Killing fields, then for each α we may select the field

$$\mathcal{X} \;=\; \cos(\alpha)\mathcal{X}^1 \,+\, \sin(\alpha)\mathcal{X}^2. \tag{3.11}$$

Then, referring to the construction of Section 3.2, the corresponding angle variable is $\tau = \cos(\alpha)\theta_1 + \sin(\alpha)\theta_2$ with conjugate momentum variable $z = \cos(\alpha)\varphi^1 + \sin(\alpha)\varphi^2$. The holomorphic variables are then $x = -\sin(\alpha)\xi_1 + \cos(\alpha)\xi_2$ and $y = -\sin(\alpha)\theta_1 + \cos(\alpha)\theta_2$.

This allows for a "tuning" or selection of a distinguished 1-parameter symmetry field form which the LeBrun ansatz metric can be constructed. The variable y remains cyclic (that is, its field remains a symmetry direction), and u, w will remain functions of x and z. These functions will change with α, so we may write $u = u_\alpha(x,z)$ and $w = w_\alpha(x,z)$. We remark that a third auxiliary function $\dot{u}_\alpha \triangleq \frac{d}{d\alpha}u_\alpha$ exists. If the u_α solve the LeBrun equation $(u_\alpha)_{xx} + (e^{u_\alpha})_{zz} = 0$ then $\dot{u}_\alpha$ will solve the linearized equation $(\dot{u}_\alpha)_{xx} + (\dot{u}_\alpha e^{u_\alpha})_{zz} = 0$. Under some conditions u_α will be positive, and setting $w = \dot{u}_\alpha$ we have an entirely new LeBrun metric.

4 Expressing $U(2)$-invariant metrics in the LeBrun ansatz

The usual ansatz for $U(2)$-invariant metrics is

$$g \;=\; A\,dr^2 \;+\; B\,(\eta_1)^2 \;+\; C\left((\eta_2)^2 + (\eta_3)^2\right) \tag{4.1}$$

where $\{\eta_1, \eta_2, \eta_3\}$ is a standard left-invariant coframe on $\mathbb{S}^3$, and A, B, C are functions of the radial variable r. If (ψ, φ, θ) are Euler coordinates on on $\mathbb{S}^3$, the usual frame transitions are

$$\begin{aligned} \eta_1 \;&=\; \frac{1}{2}\,(d\psi \;+\; \cos(\theta)d\varphi) \\ \eta_2 \;&=\; \frac{1}{2}\,(\sin(\theta)\cos(\psi)d\varphi - \sin(\psi)d\theta) \\ \eta_3 \;&=\; \frac{1}{2}\,(\sin(\theta)\sin(\psi)d\varphi + \cos(\psi)d\theta)\,. \end{aligned} \tag{4.2}$$

From this we deduce $(\eta_2)^2 + (\eta_3)^2 = \frac{1}{4}\left(d\theta^2 + \sin^2(\theta)d\varphi^2\right)$, so in Euler coordinates

$$g \;=\; A\,dr^2 \;+\; \frac{B}{4}\,(d\psi + \cos(\theta)d\varphi)^2 \;+\; \frac{C}{4}\left(d\theta^2 \;+\; \sin^2(\theta)d\varphi^2\right) \tag{4.3}$$

This is already close to LeBrun ansatz form. To place it precisely in LeBrun ansatz form we make the change of variables

$$x \;=\; \log\cot\frac{\theta}{2}, \qquad y \;=\; \varphi, \qquad z \;=\; \frac{1}{2}\int\sqrt{AB}\,dr, \qquad \tau \;=\; \psi. \tag{4.4}$$

This gives $d\theta^2 + \sin^2(\theta)d\varphi^2 = \operatorname{sech}^2(x)(dx^2 + dy^2)$, and the metric now reads

$$g \;=\; \frac{4}{B}dz^2 \;+\; \frac{B}{4}\,(d\tau + \tanh(x)\,dy)^2 \;+\; \frac{C}{4}\operatorname{sech}^2(x)\left(dx^2 + dy^2\right). \tag{4.5}$$

Reading off the LeBrun ansatz quantities from (2.1), we have

$$\begin{aligned} &w \;=\; \frac{4}{B}, \quad u \;=\; \log\left(\frac{BC}{16}\operatorname{sech}^2(x)\right) \\ &A_x \;=\; 0, \quad A_y \;=\; \tanh(x), \quad A_z \;=\; 0 \end{aligned} \tag{4.6}$$

where B and C are now functions of the new variable z, via the transition from r to z given in (4.4). Because $U(2)$ has a rank 2 toral subgroup, any $U(2)$-invariant metric is also $\mathbb{T}^2$-invariant—if the metric is Kähler then it is toric. One can see directly that the metric (4.5) has no τ- or y-dependency so has $\mathbb{T}^2$ symmetry.

5 Examples

We give two examples of our method. The exceptional half-plane metric from [21] was originally written in a toric ansatz, and the Page metric on $\mathbb{C}P^2\sharp\overline{\mathbb{C}P}^2$ was originally written in the $U(2)$ ansatz. We use our methods to express both in the LeBrun ansatz. In the last section we outline methods for creating new metrics that are Einstein, half-conformally flat, or Bach-flat.

5.1 The exceptional half-plane metric on $\mathbb{C}^2$.

This toric SFK metric on $\mathbb{C}^2$ appears in [21]. It has one translational and one rotational field. In rectangular coordinates (x_1, y_1, x_2, y_2) on $\mathbb{C}^2$, these fields are $\mathcal{X}^1 = \frac{\partial}{\partial y_1}$ and $\mathcal{X}^2 = -y_2\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial y_2}$, which are clearly translational and rotational, respectively. Let $U = U(\varphi^1, \varphi^2)$ be the symplectic potential

$$U \;=\; \frac{1}{2}\left(\frac{(\varphi^2)^2}{1+2M\varphi^1} + \varphi^1\log(\varphi^1) + M(\varphi^1)^2\right) \tag{5.1}$$

where $M \geq 0$ is a constant. The case $M = 0$ produces the flat metric. When $M > 0$, the resulting metric is the exceptional half-plane metric; the fact that (5.1) is the correct symplectic potential for the exceptional half-plane metric can be verified directly from equations (6-1) and (6-3) of [21]. The Kähler potential V is difficult to write explicitly, as it involves inverting a function with transcendental and algebraic parts. However it is possible to find LeBrun coordinates, which in terms of the symplectic coordinates are

$$x \;=\; \frac{\varphi^2}{1+2M\varphi^1}, \quad y \;=\; \theta_2, \quad z \;=\; \varphi^1, \quad \tau = \theta_1. \tag{5.2}$$

The LeBrun functions w and u are

$$w \;=\; M + \frac{1}{2z}, \quad u \;=\; \log(2z) \tag{5.3}$$

and the vector potential and field strength are

$$\begin{aligned} A \;&=\; 2Mx\,dy, \qquad \text{which is} \quad A_x = 0,\, A_y = 2Mx,\, A_z = 0, \\ B \;&=\; 2M dx\wedge dy, \;\text{which is} \quad B_x = 0,\, B_y = 0,\, B_z = 2M. \end{aligned} \tag{5.4}$$

We notice that $u = \log(2z)$ gives what LeBrun called the *hyperbolic ansatz* in section 4 of [19]. If $M = 0$ this is the flat metric, which LeBrun wrote down on p. 233 of [19] (unfortunately LeBrun's equations are mostly unnumbered). The exceptional half-plane metric in LeBrun ansatz form is

$$g = (1+2Mz)(dx^2+dy^2) + \frac{1+2Mz}{2z}dz^2 + \frac{2z}{1+2Mz}(d\tau + 2Mx\,dy)^2. \tag{5.5}$$

5.2 The Page metric

The Page metric was originally developed in [20], and can be found explicitly in (3.25) of [16] (unfortunately its expression in the appendix of [15] has a typo). Methods for building Ricci-flat metrics, including the Page metric, can be found [4]; see also 9.125 of [5]. This metric exists on $\mathbb{C}P^2\sharp\overline{\mathbb{C}P}^2$; it is Einstein, Hermitian, and Bach-flat, but not half-conformally flat. It is conformal to an extremal Kähler metric, which Calabi [7] [8] independently wrote down; see [10] for the specific conformal transformation, or [12] for a more general theory of conformal transformations between extremal Kähler and Einstein metrics on 4-manifolds. From [16] the Page metric is

$$\begin{aligned} g \;=\; & \frac{3(1+\nu^2)}{\Lambda}\left[\frac{1-\nu\cos^2(r)}{3-\nu^2-\nu^2(1+\nu^2)\cos^2(r)}\,dr^2+\right. \\ & \left. +\frac{3-\nu^2-\nu^2(1+\nu^2)\cos^2(r)}{(3+\nu^2)^2(1-\nu\cos^2(r))}\sin^2(r)\eta_1^2+4\frac{1-\nu^2\cos^2(r)}{3+6\nu^2-\nu^4}\left(\eta_2^2+\eta_3^2\right)\right]. \end{aligned} \tag{5.6}$$

The method of Section 4 gives its expression in the LeBrun ansatz:

$$\begin{aligned} & g = we^u\left(dx^2+dy^2\right)\;+\;w\,dz^2\;+\;\frac{1}{w}\left(d\tau+\tanh(x)dy\right)^2, \quad \text{where} \\ & w=\frac{F(z)}{G(z)}\;\text{ and }\; we^u=\frac{1}{3\Lambda(1+\nu^2)(3+6\nu^2-\nu^4)}H(z)\,\mathrm{sech}^2(x) \end{aligned} \tag{5.7}$$

and F, G, H are the polynomials

$$\begin{aligned} F(z) \;&=\; 27(1+\nu^2-\nu^4-\nu^6)+36(4\nu^2+4\nu^4+\nu^6)\Lambda z \\ &\qquad -12(9\nu^2+6\nu^4+\nu^6)\Lambda^2z^2 \\ G(z) \;&=\; 27(1+\nu^2-\nu^4-\nu^6)+3(-9+9\nu^2+11\nu^4+15\nu^6)\Lambda z \\ &\qquad -24(3\nu^2+3\nu^4-\nu^6)\Lambda^2z^2+4(9\nu^2+6\nu^3+\nu^6)\Lambda^3z^3 \\ H(z) \;&=\; 9(1+\nu^2-\nu^4-\nu^6)+12(3\nu^2+16\nu^4+\nu^6)\Lambda z \\ &\qquad -4(9\nu^2+6\nu^4+\nu^6)\Lambda^2z^2. \end{aligned} \tag{5.8}$$

The domain for (x,z) is $x\in\mathbb{R}$ and $z\in\left[0,\,\frac{3(1+\nu^2)}{\Lambda(3+\nu^2)}\right]$.

5.3 New metrics

Creation of special metrics, namely Einstein, half-conformally flat, or Bach-flat metrics are of considerable importance in differential geometry. One may regard the metric g, if expressed in the LeBrun ansatz, as a dynamic variable with five unknowns w, u, B_x, B_y, B_z which are each functions of the coordinates (x,y,z). These values can be specified independently, subject to the single requirement that $B_{x,x}+B_{y,y}+B_{z,z}=0$ which is equivalent to the definition of B from (2.2), which is that $B = dA$ for a 1-form A. In a sense, there are four completely independent variables that may be chosen, with the choice of a fifth being partially constrained.

Letting W^+ be the self-dual part of the Weyl tensor, one might consider the condition $W^+ = 0$. Because the operator $W^+ : \bigwedge^+ \to \bigwedge^+$ has three eigenvalues which are subject to the condition that they sum to zero, the condition $W^+ = 0$ imposes two differential identities on our five variables. With the fifth constraint discussed above, we arrive at an underdetermined system, which surely has a large solutions space. There remain many obstacles, both technical and theoretical, to fully understanding this system. Similar comments hold for systems like $\mathrm{R\mathring{i}c} = 0$ and $\mathrm{B} = 0$ where $\mathrm{R\mathring{i}c}$ is the trace-free Ricci tensor and B is the Bach tensor. This subject will be taken up elsewhere.

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