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The Multivariable Aleph-function involving the Generalized Mellin-Barnes Contour Integrals

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ABSTRACT

In this paper, we have evaluated three definite integrals involving the product of two hypergeometric functions and multivariable Aleph-function. Certain special cases of the main results are also pointed out.

RESUMEN

En este artículo, hemos evaluado tres integrales definidas que involucran el producto de dos funciones hipergeométricas y la función Aleph multivariada. También se señalan ciertos casos especiales del resultado principal.

Keywords and Phrases: Hypergeometric function, Multivariable Aleph function.

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1 Introduction

The Aleph-function is among very significant special functions and its closely related ones are widely used in physics and engineering. Therefore they are of high interest to physicists and engineers as well as mathematicians. In recent years, many integral formulas involving a diversity of special functions have been presented by many authors (see e.g., [3, 9, 12, 13, 14, 15, 16]). Motivated by these recent papers, three generalized integral formulae involving product of two hypergeometric functions and multivariable Aleph-function are established in the form of three theorems:

For our study, we recall the following three integral formulas (see [5], p. 77, Equations (3.1), (3.2) and (3.3)):

$$\int_0^\infty \left[\left(\alpha x+\frac{\beta}{x}\right)^2+\gamma\right]^{-\rho-1} dx=\frac{\sqrt{\pi}\,\Gamma\left(\rho+\frac{1}{2}\right)}{2\alpha\left(4\alpha\beta+\gamma\right)^{\rho+\frac{1}{2}}\Gamma\left(\rho+1\right)} \tag{1.1}$$

$$\left(\alpha>0;\ \beta\geq 0;\ \gamma+4\alpha\beta>0;\ \Re(\rho)+\frac{1}{2}>0\right).$$

$$\int_0^\infty \frac{1}{x^2}\left[\left(\alpha x+\frac{\beta}{x}\right)^2+\gamma\right]^{-\rho-1} dx=\frac{\sqrt{\pi}\,\Gamma\left(\rho+\frac{1}{2}\right)}{2\beta\left(4\alpha\beta+\gamma\right)^{\rho+\frac{1}{2}}\Gamma\left(\rho+1\right)} \tag{1.2}$$

$$\left(\alpha\geq 0;\ \beta>0;\ \gamma+4\alpha\beta>0;\ \Re(\rho)+\frac{1}{2}>0\right).$$

$$\int_0^\infty \left[\left(\alpha+\frac{\beta}{x^2}\right)\left(\alpha x+\frac{\beta}{x}\right)^2+\gamma\right]^{-\rho-1} dx=\frac{\sqrt{\pi}\,\Gamma\left(\rho+\frac{1}{2}\right)}{\left(4\alpha\beta+\gamma\right)^{\rho+\frac{1}{2}}\Gamma\left(\rho+1\right)} \tag{1.3}$$

$$\left(\alpha>0;\ \beta\geq 0;\ \gamma+4\alpha\beta>0;\ \Re(\rho)+\frac{1}{2}>0\right).$$

We also recall the following identity involving the hypergeometric series ${}_2F_1(.)$ ([8] p. 75, Theorem 1): If

$$(1-y)^{\alpha+\beta-\gamma}\,{}_2F_1\left(2\alpha,2\beta;2\gamma;y\right)=\sum_{k=1}^{\infty}a_k y^k, \tag{1.4}$$

then

$${}_2F_1\left(a,\ b;\ c+\frac{1}{2};\ y\right){}_2F_1\left(c-a,\ c-b;\ c+\frac{1}{2};\ y\right)=\sum_{k=0}^{\infty}\frac{(c)_k}{\left(c+\frac{1}{2}\right)_k}a_k y^k. \tag{1.5}$$

The multivariable Aleph-function defined by Sharma and Ahmad [6] as:

$$\aleph\left(z_1,\ z_2,\ ...,\ z_r\right)$$

$$=\aleph^{0,\,n:\,m_1n_1;\ m_2n_2;\,....;m_rn_r}_{p_i,\,q_i,\,\tau_i;\,R;\,p_{i(1)},\,q_{i(1)},\,\tau_{i(1)};\,R^{(1)},\,...,\,p_{i(r)},\,q_{i(r)},\,\tau_{i(r)};\,R^{(r)}}\left\{\begin{array}{c}z_1\\ \vdots\\ z_r\end{array}\middle|\begin{array}{c}B_1:B_2\\ B_3:B_4\end{array}\right\}$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(\varsigma_1, \cdots, \varsigma_r) \prod_{i=1}^{r} (\phi_i(\varsigma_i)(z_i)^{\varsigma_i}) \, d\varsigma_1 \cdots d\varsigma_r \tag{1.6}$$

where, $\omega = \sqrt{-1}$,

$$B_1 = \left(\left(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)}\right)_{1,n} \right), \ \left(\tau_i \left(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)}\right)_{n+1,p_i} \right)$$

$$B_2 = \left(\left(c_j^{(1)}, \gamma_j^{(r)}\right)_{1,n_1} \right), \left(\tau_{i^{(1)}} \left(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_{i^{(1)}}} \right)$$

$$; \cdots ; \left(\left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,n_r} \right), \left(\tau_{i^{(r)}} \left(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_{i^{(r)}}} \right)$$

$$B_3 = \left(\tau_i \left(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)}\right)_{m+1,q_i} \right)$$

$$B_4 = \left(\left(d_j^{(1)}, \delta_j^{(1)}\right)_{1,m_1} \right), \left(\tau_{i^{(1)}} \left(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_{i^{(1)}}} \right)$$

$$; \cdots ; \left(\left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,m_r} \right), \left(\tau_{i^{(r)}} \left(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_{i^{(r)}}} \right)$$

and

$$\psi(\varsigma_1, \cdots, \varsigma_r) = \frac{\prod_{j=1}^{n} \Gamma\left(1 - a_j + \sum_{k=1}^{r} \alpha_j^{(k)} \varsigma_k\right)}{\sum_{i=1}^{R} \left[\tau_i \prod_{j=n+1}^{p_i} \Gamma\left(a_{ji} - \sum_{k=1}^{r} \alpha_{ji}^{(k)} \varsigma_k\right) \prod_{j=1}^{q_i} \Gamma\left(1 - b_{ji} + \sum_{k=1}^{r} \beta_{ji}^{(k)} \varsigma_k\right) \right]}, \tag{1.7}$$

$$\phi_k(\varsigma_k) = \frac{\prod_{j=1}^{m_k} \Gamma\left(d_j^{(k)} - \delta_j^{(k)} \varsigma_k\right) \prod_{j=1}^{n_k} \Gamma\left(1 - c_j^{(k)} + \gamma_j^{(k)} \varsigma_k\right)}{\sum_{i^{(k)}=1}^{R^{(k)}} \left[\tau_{i^{(k)}} \prod_{j=m_k+1}^{q_{i^{(k)}}} \Gamma\left(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} \varsigma_k\right) \prod_{j=n_k+1}^{p_{i^{(k)}}} \Gamma\left(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} \varsigma_k\right) \right]}, \tag{1.8}$$

The parameters $d_{ji^{(k)}}^{(k)}$ $(j = m_k + 1, \cdots, q_{i^{(k)}})$, $(k = 1, \cdots, r;\ i = 1, \cdots, R\ \&\ i^{(k)} = 1, \cdots, R^{(k)}$ are complex numbers. Also positive real numbers α's, β's, γ' sand δ's for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{n} \alpha_j^{(k)} + \tau_i \sum_{j=n+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$- \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leq 0 \tag{1.9}$$

The real numbers $\tau_i > 0$ $(i = 1, ..., R)$ and $\tau_{i^{(k)}} > 0$ $(i = 1, \cdots, R)$. The contour is in the s_k-plane and run from $\sigma - \omega\,\infty$to$\sigma + \omega\,\infty$, where σ is real number with loop, if necessary, ensure that the poles of $\Gamma\left(d_j^{(k)} - \delta_j^{(k)}\varsigma_k\right)$ with $j = 1, \ldots, m_k$ are separated from those of $\Gamma\left(1 - a_j + \sum_{k=1}^{r}\alpha_j^{(k)}\varsigma_k\right)$ with $j = 1, \ldots, n$ and $\Gamma\left(1 - c_j^{(k)} + \gamma_j^{(k)}\varsigma_k\right)$ with $j = 1, \ldots, n_k$ to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contours (1.6) can be obtained by extension of corresponding conditions for multi variable H-function as: $|\arg\, z_k| < \frac{1}{2}A_i^{(k)}\pi$ where.

$$A_i^{(k)} = \sum_{j=1}^{n}\alpha_j^{(k)} - \tau_i\sum_{j=n+1}^{p_i}\alpha_{ji}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} - \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0 \tag{1.10}$$

with $k = 1, \cdots, r$; $i = 1, \cdots, R$ and $i^{(k)} = 1, \cdots, R^{(k)}$.

Remark 1: By setting $\tau_i = \tau_{i^{(k)}} = 1$, the multivariable Aleph function reduces to multivariable I-function (see [4, 7]).

Remark 2: By setting $\tau_i = \tau_{i^{(k)}} = 1\,(k = 1, ..., r)$ and $R = R^{(1)} =, ..., R^{(r)} = 1$, the multivariable Aleph-function reduces to multivariable H-function defined by Srivastava and Panda [10].

Remark 3: When we set $r = 1$, the multivariable Aleph function reduces to Aleph-function of one variable defined by Sudland [11].

2 Main Results

Theorem 2.1. *Let* $\alpha > 0$, $\beta \geq 0$, $\gamma + 4\alpha\beta > 0$, $\mu_i > 0$, $\eta \geq 0$, $\Re(\rho) + \frac{1}{2} > 0$; $-\frac{1}{2} < \alpha - \beta - \gamma < \frac{1}{2}$; $\Re\left(\lambda + \mu_i \min_{1\leq j\leq m_i}\left\{\frac{Re\left(d_j^{(i)}\right)}{\delta_j^{(i)}}\right\}\right) > 0$ $(i = 1, \cdots, r)$, *and* $\sigma = \left[\left(\alpha\, x + \frac{\beta}{x}\right)^2 + \gamma\right]$ *then the following formula holds:*

$$\int_0^\infty \sigma^{-\rho-1}{}_2F_1\left(\alpha,\ \beta;\ \gamma + \frac{1}{2};\ \sigma\right)\ {}_2F_1\left(\gamma - \alpha,\ \gamma - \beta;\ \gamma + \frac{1}{2};\ \sigma\right)$$

$$\times\aleph\left(z_1\sigma^{-\eta_1},\ ...,\ z_r\sigma^{-\eta_r}\right)dx$$

$$= \frac{\sqrt{\pi}}{2\alpha\left(4\alpha\beta + \gamma\right)^{\rho+\frac{1}{2}}}\sum_{h=0}^{\infty}\frac{1}{\left(4\alpha\beta + \gamma\right)^{-h}}\frac{(\gamma)_h}{\left(\gamma + \frac{1}{2}\right)_h}a_h$$

$$\times\,\aleph^{0,\,n+1:\,m_1n_1;\,m_2n_2;\,\ldots;m_rn_r}_{p_i+1,\,q_i+1,\,\tau_i;\,R;\,p_{i^{(1)}},\,q_{i^{(1)}},\,\tau_{i^{(1)}};\,R^{(1)},\,\ldots,\,p_{i^{(r)}},\,q_{i^{(r)}},\,\tau_{i^{(r)}};\,R^{(r)}}$$

$$\left\{ \begin{array}{c} \frac{z_1}{(4\alpha\beta+\gamma)^{\eta_1}} \\ \vdots \\ \frac{z_r}{(4\alpha\beta+\gamma)^{\eta_r}} \end{array} \middle| \begin{array}{c} \left(-\frac{1}{2}-\rho+h;\eta_1,\cdots,\eta_r\right), B_1 : B_2 \\ \left(-\rho+h;\eta_1,\cdots,\eta_r\right), B_3 : B_4 \end{array} \right\}. \tag{2.1}$$

Proof. Assume that Ω in L.H.S. of (2.1), then by virtue of equation (1.5) and (1.6), we have the following

$$\Omega = \int_0^\infty \sigma^{-\rho-1} \sum_{h=0}^{\infty} \frac{(\gamma)_h}{\left(\gamma+\frac{1}{2}\right)_h} a_h \sigma^h \aleph^{0,n:m_1n_1,\ m_2n_2,\ \dots,m_rn_r}_{p_i,q_i,\tau_i;R;p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)},\dots,p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)}}$$

$$\left\{ \begin{array}{c} z_1\sigma^{-\eta_1} \\ \vdots \\ z_r\sigma^{-\eta_r} \end{array} \middle| \begin{array}{c} B_1 : B_2 \\ B_3 : B_4 \end{array} \right\} dx$$

$$= \int_0^\infty \sigma^{-\rho-1} \sum_{h=0}^{\infty} \frac{(\gamma)_h}{\left(\gamma+\frac{1}{2}\right)_h} a_h \sigma^h$$

$$\times \left\{ \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi\left(\varsigma_1,\cdots,\varsigma_r\right) \prod_{i=1}^{r} \left(\phi_i\left(\varsigma_i\right)\left(z_i\sigma^{-\eta_i}\right)^{\varsigma_i}\right) d\varsigma_1 \cdots d\varsigma_r \right\} dx$$

$$= \sum_{h=0}^{\infty} \frac{(\gamma)_h}{\left(\gamma+\frac{1}{2}\right)_h} a_h \left\{ \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi\left(\varsigma_1,\cdots,\varsigma_r\right) \prod_{i=1}^{r} \left(\phi_i\left(\varsigma_i\right)\left(z_i\right)^{\varsigma_i}\right) d\varsigma_1 \cdots d\varsigma_r \right\}$$

$$\times \int_0^\infty \sigma^{-\rho-1+h-\sum_{k=1}^{s}\eta_k\varsigma_k} dx$$

By using equation (1.1), we can obtain the following equation

$$\Omega = \sum_{h=0}^{\infty} \frac{(\gamma)_h}{\left(\gamma+\frac{1}{2}\right)_h} a_h \left\{ \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi\left(\varsigma_1,\cdots,\varsigma_r\right) \prod_{i=1}^{r} \left(\varphi_i\left(\varsigma_i\right)\left(z_i\right)^{\varsigma_i}\right) d\varsigma_1 \cdots d\varsigma_r \right\}$$

$$\times \frac{\sqrt{\pi}\,\Gamma\left(\rho-h+\sum_{k=1}^{s}\eta_k\varsigma_k+\frac{1}{2}\right)}{2\alpha\left(4\alpha\beta+\gamma\right)^{\rho+\frac{1}{2}-h+\sum_{k=1}^{s}\eta_k\varsigma_k}\,\Gamma\left(\rho-h+\sum_{k=1}^{s}\eta_k\varsigma_k+1\right)}$$

$$= \sum_{h=0}^{\infty} \frac{(\gamma)_h\ a_h}{\left(\gamma+\frac{1}{2}\right)_h} \frac{\sqrt{\pi}\,\Gamma\left(\rho-h+\sum_{k=1}^{s}\eta_k\varsigma_k+\frac{1}{2}\right)}{2\alpha\left(4\alpha\beta+\gamma\right)^{\rho+\frac{1}{2}-h+\sum_{k=1}^{s}\eta_k\varsigma_k}\,\Gamma\left(\rho-h+\sum_{k=1}^{s}\eta_k\varsigma_k+1\right)}$$

$$\times \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi\left(\varsigma_1,\cdots,\varsigma_r\right) \prod_{i=1}^{r} \left(\varphi_i\left(\varsigma_i\right)\left(z_i\right)^{\varsigma_i}\right) d\varsigma_1 \cdots d\varsigma_r$$

$$= \frac{\sqrt{\pi}}{2\alpha\left(4\alpha\beta+\gamma\right)^{\rho+\frac{1}{2}}} \sum_{h=0}^{\infty} \frac{(\gamma)_h}{\left(\gamma+\frac{1}{2}\right)_h} \frac{a_h}{\left(4\alpha\beta+\gamma\right)^{-h}}$$

$$\times \frac{\Gamma\left(\rho - h + \sum_{k=1}^{s} \eta_k \varsigma_k + \frac{1}{2}\right)}{(4\alpha\beta+\gamma)^{\sum_{k=1}^{s} \eta_k \varsigma_k} \Gamma\left(\rho - h + \sum_{k=1}^{s} \eta_k \varsigma_k + 1\right)}$$

$$\times \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi\left(\varsigma_1, \cdots, \varsigma_r\right) \prod_{i=1}^{r} \left(\varphi_i\left(\varsigma_i\right)\left(z_i\right)^{\varsigma_i}\right) d\varsigma_1 \cdots d\varsigma_r$$

$$= \frac{\sqrt{\pi}}{2\alpha\left(4\alpha\beta+\gamma\right)^{\rho+\frac{1}{2}}} \sum_{h=0}^{\infty} \frac{1}{(4\alpha\beta+\gamma)^{-h}} \frac{(\gamma)_h}{\left(\gamma+\frac{1}{2}\right)_h} a_h x \left\{ \frac{\Gamma\left(\rho - h + \sum_{k=1}^{s} \eta_k \varsigma_k + \frac{1}{2}\right)}{\Gamma\left(\rho - h + \sum_{k=1}^{s} \eta_k \varsigma_k + 1\right)} \right\}$$

$$\times \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi\left(\varsigma_1, \cdots, \varsigma_r\right) \prod_{i=1}^{r} \left(\phi_i\left(\varsigma_i\right)\left[\frac{z_i}{(4\alpha\beta+\gamma)^{\eta_i}}\right]^{\varsigma_i}\right) d\varsigma_1 \cdots d\varsigma_r$$

we readily arrive at the right hand side of (2.1) in view of the presentation of Aleph function in Mellin Barnes contour integral. □

Theorem 2.2. *Let* $\alpha \geq 0,\ \beta > 0,\ \gamma + 4\alpha\beta > 0,\ \mu_i > 0,\ \eta \geq 0,\ \Re(\rho) + \frac{1}{2} > 0;\quad -\frac{1}{2} < \alpha - \beta - \gamma < \frac{1}{2}$ $\Re\left(\lambda + \mu_i \min_{1 \leq j \leq m_i} \left\{\frac{Re\left(d_j^{(i)}\right)}{\delta_j^{(i)}}\right\}\right) > 0\ (i = 1, \cdots, r)$, *and* $\sigma = \left[\left(\alpha x + \frac{\beta}{x}\right)^2 + \gamma\right]$ *then the following formula holds:*

$$\int_0^{\infty} \frac{1}{x^2} \sigma^{-\rho-1} {}_2F_1\left(\alpha,\, \beta;\ \gamma + \frac{1}{2}; \sigma\right) {}_2F_1\left(\gamma - \alpha,\ \gamma - \beta;\ \gamma + \frac{1}{2}; \sigma\right)$$
$$\times \aleph\left(z_1 \sigma^{-\eta_1},\, ...,\, z_r \sigma^{-\eta_r}\right) dx$$

$$= \frac{\sqrt{\pi}}{2\beta\left(4\alpha\beta+\gamma\right)^{\rho+\frac{1}{2}}} \sum_{h=0}^{\infty} \frac{1}{(4\alpha\beta+\gamma)^{-h}} \frac{(\gamma)_h}{\left(\gamma+\frac{1}{2}\right)_h} a_h$$
$$\times \aleph^{0,\, n+1:\, m_1 n_1;\ m_2 n_2;\ \ldots;m_r n_r}_{p_i+1,\, q_i+1,\, \tau_i;\, R;\, p_{i^{(1)}},\, q_{i^{(1)}},\, \tau_{i^{(1)}};\, R^{(1)},\, \ldots,\, p_{i^{(r)}},\, q_{i^{(r)}},\, \tau_{i^{(r)}};\, R^{(r)}}$$
$$\times \left\{ \begin{array}{c|c} \frac{z_1}{(4\alpha\beta+\gamma)^{\eta_1}} & \left(-\frac{1}{2} - \rho + h; \eta_1, \cdots, \eta_r\right), B_1 : B_2 \\ \vdots & \left(-\rho + h; \eta_1, \cdots, \eta_r\right), B_3 : B_4 \\ \frac{z_r}{(4\alpha\beta+\gamma)^{\eta_r}} & \end{array} \right\}. \tag{2.2}$$

Proof. In the similar manner of Theorem 2.1 and using (1.2) we easily arrive at the result (2.2). □

Theorem 2.3. *Let* $\alpha > 0,\ \beta > 0,\ \gamma + 4\alpha\beta > 0,\ \mu_i > 0,\ \eta \geq 0,\ \Re(\rho) + \frac{1}{2} > 0;\quad -\frac{1}{2} < \alpha - \beta - \gamma < \frac{1}{2};$ $\Re\left(\lambda + \mu_i \min_{1 \leq j \leq m_i} \left\{\frac{Re\left(d_j^{(i)}\right)}{\delta_j^{(i)}}\right\}\right) > 0\ (i = 1, \cdots, r)$, *and* $\sigma = \left[\left(\alpha x + \frac{\beta}{x}\right)^2 + \gamma\right]$ *then the following formula holds:*

$$\int_0^{\infty} \left(\alpha + \frac{\beta}{x^2}\right) \sigma^{-\rho-1} {}_2F_1\left(\alpha,\ \beta;\ \gamma + \frac{1}{2}; \sigma\right) {}_2F_1\left(\gamma - \alpha,\ \gamma - \beta;\ \gamma + \frac{1}{2}; \sigma\right)$$
$$\times \aleph\left(z_1 \sigma^{-\eta_1},\, ...,\, z_r \sigma^{-\eta_r}\right) dx$$

$$= \frac{\sqrt{\pi}}{(4\alpha\beta+\gamma)^{\rho+\frac{1}{2}}} \sum_{h=0}^{\infty} \frac{1}{(4\alpha\beta+\gamma)^{-h}} \frac{(\gamma)_h}{\left(\gamma+\frac{1}{2}\right)_h} a_h$$

$$\times \aleph^{0,n+1:m_1 n_1;\, m_2 n_2;\, \ldots; m_r n_r}_{p_i+1,q_i+1,\tau_i;R;p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)},\ldots,p_{i(r)},q_{i(r)},\tau_{i(r)};R^{(r)}}$$

$$\left\{ \begin{array}{c} \frac{z_1}{(4\alpha\beta+\gamma)^{\eta_1}} \\ \vdots \\ \frac{z_r}{(4\alpha\beta+\gamma)^{\eta_r}} \end{array} \middle| \begin{array}{c} \left(-\frac{1}{2}-\rho+h;\eta_1,\cdots,\eta_r\right), B_1 : B_2 \\ \left(-\rho+h;\eta_1,\cdots,\eta_r\right), B_3 : B_4 \end{array} \right\}. \quad (2.3)$$

Proof. In the similar way of Theorem 2.1 and using (1.3) we easily arrive at the result (2.3). □

3 Special Cases

(1) If we put $\tau_i = 1$, in (2.1), (2.2) and (2.3), we get the results in terms of multivariable I-function [4, 7].

(2) Some suitable parametric changes in (1.1), we obtain single variable I-function, then we arrive at the results due to Chand [1].

(3) Also, multivariable Aleph function reduces to multivariable H-function with some suitable parameters;we get the known result due to Daiya et al. [2].

4 Conclusion

In this article, we analyze the generalized fractional calculus involving definite integrals of Gradshteyn-Ryzhik of the Multivariable Aleph-function. As the special cases of our main results, which are related to I-function, H-function and G-function, we can also get the number of special functions.

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