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Odd Harmonious Labeling of Some Classes of Graphs

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ABSTRACT

A graph $G(p,q)$ is said to be odd harmonious if there exists an injection $f: V(G) \rightarrow \{0,1,2,\cdots,2q-1\}$ such that the induced function $f^*: E(G) \rightarrow \{1,3,\cdots,2q-1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. In this paper we prove that T_p- tree, $T\hat{o}P_m$, $T\hat{o}2P_m$, regular bamboo tree, $C_n\hat{o}P_m$, $C_n\hat{o}2P_m$ and subdivided grid graphs are odd harmonious.

RESUMEN

Un grafo $G(p,q)$ se dice impar armonioso si existe una inyección $f: V(G) \rightarrow \{0,1,2,\cdots,2q-1\}$ tal que la función inducida $f^*: E(G) \rightarrow \{1,3,\cdots,2q-1\}$ definida por $f^*(uv) = f(u) + f(v)$ es una biyección. En este artículo probamos que los grafos T_p-árboles, $T\hat{o}P_m$, $T\hat{o}2P_m$, árboles bambú regulares, $C_n\hat{o}P_m$, $C_n\hat{o}2P_m$ y cuadrículas subdivididas son impar armoniosos.

Keywords and Phrases: harmonious labeling, odd harmonious labeling, transformed tree, subdivided grid graph, regular bamboo tree.

2020 AMS Mathematics Subject Classification: 05C78.

1 Introduction

Throughout this paper by a graph is implied as a finite, simple and undirected. For standard terminology and notation we follow Harary [3]. A graph $G(V, E)$ with p vertices and q edges is called a (p, q) – graph. The graph labeling is an assignment of integers to the set of vertices or edges or both, subject to certain conditions. An extensive survey of various graph labeling problems is available in [1]. Graham and Sloane [2] introduced harmonious labeling during their study of modular versions of additive bases problems stemming from error correcting codes. A graph G is said to be harmonious if there exists an injection $f : V(G) \to Z_q$ such that the induced function $f^* : E(G) \to Z_q$ defined by $f^*(uv) = (f(u) + f(v)) \pmod q$ is a bijection and f is called harmonious labeling of G. The concept of an odd harmonious labeling was due to Liang and Bai [14]. A graph G is said to be odd harmonious if there exists an injection $f : V(G) \to \{0, 1, 2, \cdots, 2q-1\}$ such that the induced function $f^* : E(G) \to \{1, 3, \cdots, 2q-1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. If $f : V(G) \to \{0, 1, 2, \cdots, q\}$ then f is called as strongly odd harmonious labeling and G is called a strongly odd harmonious graph. The odd harmoniousness of a graph is useful for the solution of undetermined equations. The following results have been proved in [14]:

1. If G is an odd harmonious graph, then G is a bipartite graph. Hence any graph that contains an odd cycle is not an odd harmonious.
2. If a (p, q) – graph G is odd harmonious, then $2\sqrt{q} \le p \le (2q-1)$.
3. If G is an odd harmonious Eulerian graph with q edges, then $q \equiv 0, 2 (mod\ 4)$.

Followed by this, Vaidya and Shah [18], [19] showed that shadow and splitting graphs are odd harmonious. Selvaraju et al. [17] established that some path related graphs are odd harmonious. Jeyanthi et al. proved that the following graphs are odd harmonious: double quadrilateral snake and banana tree [5], cycle related graphs [6], plus graphs [7], super subdivision graphs [8], subdivided shell graphs [9], spider and necklace graphs [10], m-shadow, m-splitting and m-mirror graphs [11] and [12], grid graphs [13].

We use the following definitions in the subsequent section.

Definition 1.1. *Let $G = (V, E)$ be a graph. G is called a path P_n if $V = \{v_1, v_2, \cdots, v_n\}$ such that $1 \le i \le n$, $(v_i, v_{i+1}) \in E$.*

Definition 1.2. *The Cartesian product of graphs G and H denoted as $G \square H$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) | u \in V(G)$ and $v \in V(H)\}$ and (u, v) is adjacent to $(\acute{u}, \acute{v})$ if and only if either $u = \acute{u}$ and $(v, \acute{v}) \in E(H)$ or $v = \acute{v}$ and $(u, \acute{u}) \in E(G)$. The Cartesian product of two paths P_m and P_n denoted by $P_m \times P_n$ is known as a grid graph on mn vertices and $2mn - (m + n)$ edges.*

Definition 1.3. *Let G be a graph with p vertices and H be any graph and x be a vertex of H. A graph $G\hat{\circ}H$ is obtained from G and p copies of H by identifying vertex x of i^{th} copy of H with i^{th} vertex of G.*

Definition 1.4. [4] *Let T be a tree and u_0 and v_0 be the two adjacent vertices in T. Let u and v be the two pendant vertices of T such that the length of the path $u_0 - u$ is equal to the length of the path $v_0 - v$. If the edge u_0v_0 is deleted from T and u and v are joined by an edge uv, then such a transformation of T is called an elementary parallel transformation (or an ept) and the edge u_0v_0 is called transformable edge. If by some sequence of ept's, T can be reduced to a path, then T is called a T_p- tree (transformed tree) and such sequence regarded as a composition of mappings (ept's) denoted by P is called a parallel transformation of T. The path, the image of T under P is denoted as $P(T)$. A T_p- tree and the sequence of two ept's reducing it to a path are illustrated in Figure 1.*

a) A T_p- tree T b) An *ept* $P_1(T)$ c) Second *ept* $P_2(T)$

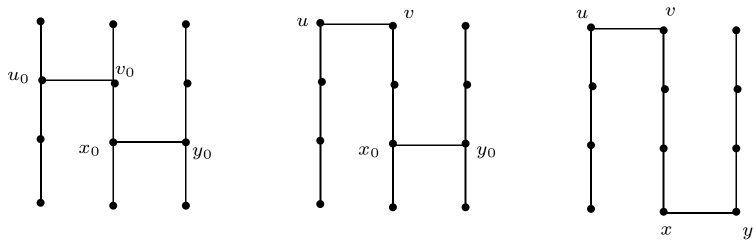

Figure 1: Transformed tree

Definition 1.5. [15] *Let T be a T_p-tree with n vertices $v_1, v_2, \cdots, v_n$. The graph $T\hat{\circ}P_m$ is obtained from T and n copies of P_m by identifying a pendant vertex of i^{th} copy of P_m with vertex v_i of T.*

Definition 1.6. [16] *Consider k copies of paths P_n of length $n-1$ and stars S_m with m pendant vertices. Identify one of the two pendant vertices of the j^{th} path with the centre of the j^{th} star. Identify the other pendant vertex of each path with a single vertex u_0 (u_0 is not in any of the star and path). The graph obtained is a regular bamboo tree.*

2 Main Results

In this section, we prove that T_p- tree, $T\hat{o}P_m$, $T\hat{o}2P_m$, regular bamboo tree, $C_n\hat{o}P_m$, $C_n\hat{o}2P_m$ and subdivided grid graphs are odd harmonious.

Theorem 2.1. *Every T_p- tree is strongly odd harmonious.*

Proof. Let T be a T_p-tree with n vertices. By definition, there exists a parallel transformation P of T, we have $V(P(T)) = V(T)$ and $E(P(T)) = (E(T) - E_d) \cup E_a$, where E_d is the set of deleted edges and E_a is the set of newly added edges through the sequence $P = (P_1, P_2, \cdots, P_l)$ of the *ept*'s used to obtain $P(T)$. Hence E_d and E_a have the same number of edges. Let $u_1, u_2, \cdots, u_n$ be the vertices of $P(T)$ successively, from one pendant vertex of $P(T)$ right up to the other. This T_p-tree has n vertices and $n-1$ edges.

We define a labeling $f : V(G) \rightarrow \{0, 1, 2, \cdots, q = n - 1\}$ as follows:
$f(u_i) = i - 1, \ 1 \leq i \leq n.$

Let $(u_i u_j)$ be an edge of T, $1 \leq i<j \leq n$. Let the *ept* P_1 delete the edge $(u_i u_j)$ and adds the edge $(u_{i+t}u_{j-t})$ where t is the distance from u_i to u_{i+t} and also the distance from u_j to u_{j-t}. Let the parallel transformation P contain one of the constituent *ept*'s P_1. Since $(u_{i+t}u_{j-t})$ is an edge of $P(T)$, it follows that $i + t + 1 = j - t$, implies $j = i + 2t + 1$.

The induced edge label of $(u_i u_j)$ is given by
$f^*(u_i u_j) = f^*(u_i u_{i+2t+1}) = f(u_i) + f(u_{i+2t+1}) = 2(i + t) - 1,$
$f^*(u_{i+t}u_{j-t}) = f^*(u_{i+t}u_{i+t+1}) = f(u_{i+t}) + f(u_{i+t+1}) = 2(i + t) - 1,$
$f^*(u_i u_j) = f^*(u_{i+t}u_{j-t}).$

The induced edge label is
$f^*(u_i u_{i+1}) = 2i - 1, \ 1 \leq i \leq n - 1.$

Thus the induced edge labels are $1, 3, \cdots, 2n - 3$. Therefore every T_p-tree is strongly odd harmonious. □

A strongly odd harmonious labeling of a T_p- tree with 12 vertices is shown in Figure 2.

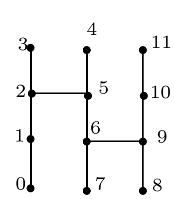

Figure 2: Strongly odd harmonious labeling of T_p- tree with 12 vertices

Theorem 2.2. *If T is a T_p-tree then the graph $T\hat{\circ}P_m$ is strongly odd harmonious.*

Proof. Let T be a T_p-tree with n vertices. By definition there exists parallel transformation $P(T)$, we have $V(P(T)) = V(T)$ and $E(P(T)) = (E(T) - E_d) \cup E_a$, where E_d is the set of deleted edges and E_a is the newly added edges through the sequence $P = (P_1, P_2, \cdots, P_l)$ of the *ept*'s used to obtain $P(T)$. Hence E_d and E_a have the same number of edges. Let $u_1, u_2, \cdots, u_n$ be the vertices of $P(T)$ successively, from one pendant vertex of $P(T)$ right up to the other. Let $x_0^j, x_1^j, \cdots, x_{m-1}^j$, $1 \leq j \leq n$ be the vertices of the j^{th} copy of P_m. Identify x_0^j with u_j, where $1 \leq j \leq n$. Then the graph $T\hat{\circ}P_m$ has nm vertices and $nm - 1$ edges.

We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, q = nm - 1\}$ as follows:

$f(u_j) = mj - 1,$ $\qquad j = 1, 3, \cdots, n-1,$

$f(u_j) = m(j-1),$ $\qquad j = 2, 4, \cdots, n,$

For $1 \leq i \leq m-1$, $f(x_i^j) = mj - i - 1,$ $\qquad j = 1, 3, \cdots, n-1,$

$f(x_i^j) = m(j-1) + i,$ $\qquad j = 2, 4, \cdots, n.$

Let $(u_i u_j)$ be an edge of T, $1 \leq i < j \leq n$. Let the *ept* P_1 delete the edge $(u_i u_j)$ and add the edge $(u_{i+t} u_{j-t})$ where t is the distance from u_i to u_{i+t} and also the distance from u_j to u_{j-t}. Let the parallel transformation P contain one of the constituent *ept*'s P_1. Since $(u_{i+t} u_{j-t})$ is an edge of $P(T)$, it follows that $i + t + 1 = j - t$, implies $j = i + 2t + 1$. Therefore i and j are of opposite equivalence, that is, i is even and j is odd or vice-versa.

The induced edge label of $(u_i u_j)$ is given by

$f^*(u_i u_j) = f^*(u_i u_{i+2t+1}) = f(u_i) + f(u_{i+2t+1}) = 2m(i+t) - 1,$

$f^*(u_{i+t} u_{j-t}) = f^*(u_{i+t} u_{i+t+1}) = f(u_{i+t}) + f(u_{i+t+1}) = 2m(i+t) - 1,$

$f^*(u_i u_j) = f^*(u_{i+t} u_{j-t}).$

The induced edge labels are

$f^*(u_j u_{j+1}) = 2mj - 1,$ $\qquad 1 \leq j \leq n-1,$

For $1 \leq i \leq m-2$, $f^*(x_i^j x_{i+1}^j) = 2mj - 2i - 3,$ $\qquad j = 1, 3, \cdots, n-1,$

$f^*(x_i^j x_{i+1}^j) = 2m(j-1) + 2i + 1,$ $\qquad j = 2, 4, \cdots, n,$

$f^*(u_j x_1^j) = 2mj - 3,$ $\qquad j = 1, 3, \cdots, n-1,$

$f^*(u_j x_1^j) = 2m(j-1) + i,$ $\qquad j = 2, 4, \cdots, n.$

Thus the induced edge labels are $1, 3, \cdots, 2mn - 3$. Hence every $T\hat{\circ}P_m$ is strongly odd harmonious. $\square$

A strongly odd harmonious labeling of $T\hat{\circ}P_4$ where T is a T_p-tree with 10 vertices is shown in Figure 3.

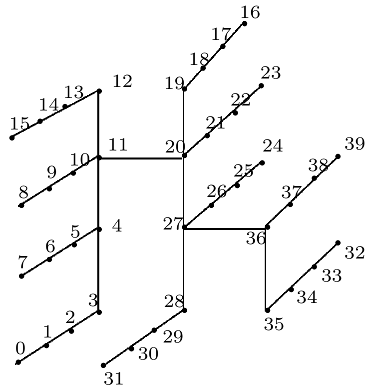

Figure 3: Strongly odd harmonious labeling of $T\hat{o}P_4$

Theorem 2.3. *If T is a T_p-tree then the graph $T\hat{o}2P_m$ is strongly odd harmonious.*

Proof. Let T be a T_p-tree with n vertices. By definition there exists a parallel transformation $P(T)$, we have $V(P(T)) = V(T)$ and $E(P(T)) = (E(T) - E_d) \cup E_a$, where E_d is the set of deleted edges and E_a is the set of newly added edges through the sequence $P = (P_1, P_2, \cdots, P_l)$ of the *ept*'s used to obtain $P(T)$. Hence E_d and E_a have the same number of edges. Let $x^j_{1,0}, x^j_{1,1}, x^j_{1,2}, \cdots, x^j_{1,m-1}$ and $x^j_{2,0}, x^j_{2,1}, x^j_{2,2}, \cdots, x^j_{2,m-1}$, $1 \le j \le n$ be the vertices of two disjoint paths P_m. Identify $x^j_{1,0}$ and $x^j_{2,0}$ with u_j, $1 \le j \le n$ to obtain $T\hat{o}2P_m$. Then the graph $T\hat{o}2P_m$ has $n(2m-1)$ vertices and $n(2m-1)-1$ edges.

We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, q = n(2m-1)-1\}$ as follows:

$f(u_j) = m - 1 + (2m-1)(j-1)$, if j is odd,
$f(u_j) = 3m - 2 + (2m-1)(j-2)$, if j is even,
$f(x^j_{1,i}) = m - 1 + (2m-1)(j-1) - i$, if j is odd,
$f(x^j_{1,i}) = 3m - 2 + (2m-1)(j-2) - i$, if j is even,
$f(x^j_{2,i}) = m - 1 + (2m-1)(j-1) + i$, if j is odd,
$f(x^j_{2,i}) = 3m - 2 + (2m-1)(j-2) + i$, if j is even.

Let (u_iu_j) be an edge of T, $1 \le i<j \le n$. Let the *ept* P_1 delete the edge (u_iu_j) and add the edge $(u_{i+t}u_{j-t})$ where t is the distance from u_i to u_{i+t} and also the distance from u_j to u_{j-t}. Let the parallel transformation P contain one of the constituent *ept$'$s* P_1. Since $(u_{i+t}u_{j-t})$ is an edge of $P(T)$, it follows that $i + t + 1 = j - t$, implies $j = i + 2t + 1$. Therefore i and j are of opposite equivalence, that is, i is even and j is odd or vice-versa.

The induced edge label of (u_iu_j) is given by
$f^*(u_iu_j) = f^*(u_iu_{i+2t+1}) = f(u_i) + f(u_{i+2t+1}) = 4mi + 4mt - 2i - 2t - 1$,
$f^*(u_{i+t}u_{j-t}) = f^*(u_{i+t}u_{i+t+1}) = f(u_{i+t}) + f(u_{i+t+1}) = 4mi + 4mt - 2i - 2t - 1$,
$f^*(u_iu_j) = f^*(u_{i+t}u_{j-t})$.

The induced edge labels are

$f^*(u_j x_{1,1}^j) = 2(m-1) + 2(2m-1)(j-1) - 1,$ if j is odd,
$f^*(u_j x_{1,1}^j) = 2(3m-2) + 2(2m-1)(j-2) - 1,$ if j is even,
$f^*(u_j x_{2,1}^j) = 2(m-1) + 2(2m-1)(j-1) + 1,$ if j is odd,
$f^*(u_j x_{2,1}^j) = 2(3m-2) + 2(2m-1)(j-2) + 1,$ if j is even,
$f^*(u_j u_{j+1}) = 2(2m-1)(j-1) + 4m + 3,$ $1 \leq j \leq n-1,$
For $1 \leq i \leq m-2,$
$f^*(x_{1,i}^j x_{1,i+1}^j) = 2(2m-1)(j-1) + 2(m-1) - 2i - 1,$ if j is odd,
$f^*(x_{1,i}^j x_{1,i+1}^j) = 2(2m-1)(j-2) + 2(3m-2) - 2i - 1,$ if j is even,
$f^*(x_{2,i}^j x_{2,i+1}^j) = 2(2m-1)(j-1) + 2(m-1) + 2i + 1,$ if j is odd,
$f^*(x_{2,i}^j x_{2,i+1}^j) = 2(2m-1)(j-2) + 2(3m-2) + 2i + 1,$ if j is even.

Hence $T\hat{o}2P_m$ is strongly odd harmonious. □

The strongly odd harmonious labeling of $T\hat{o}2P_4$ where T is a T_p-tree with 13 vertices is shown in Figure 4.

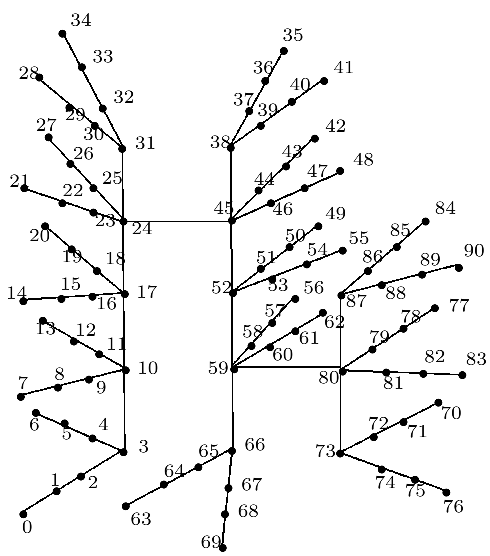

Figure 4: strongly odd harmonious labeling of $T\hat{o}2P_4$

Theorem 2.4. *Every regular bamboo tree is odd harmonious.*

Proof. Let $v_0^j, v_1^j, v_2^j, \cdots, v_{n-1}^j$ be the vertices of the j^{th} path P_n, $1 \leq j \leq m$ where v_0^j is identified with the apex vertex v_0 and v_{n-1}^j is identified with u_0^j which is the centre of the j^{th} star. Let $u_1^j, u_2^j, \cdots, u_t^j$ be the pendant vertices of the j^{th} star. The regular bamboo tree has $m(t+n-1)+1$ vertices and $m(t+n-1)$ edges.

We define the labeling $f : V(G) \to \{0, 1, 2, \cdots, 2m(n+t-1)-1\}$ as follows:

Case(i): m is odd

$f(v_0) = 0$,
For $1 \le j \le m$,
$f(v_i^j) = 2j - 1 + m(i-1)$, if i is odd,
$f(v_i^j) = 2 + 4(m-j) + m(i-2)$, if i is even,
$f(u_i^j) = m(n-1) + 2m - 1 + 2m(t-i) - 2(m-j)$, $1 \le i \le t$.

The induced edge labels are
For $1 \le j \le m$,
$f^*(v_0 v_1^j) = 2j - 1$,
$f^*(v_i^j v_{i+1}^j) = 2j + 2m(i-1) + 4(m-j) + 1$, if i is odd,
$f^*(v_i^j v_{i+1}^j) = 4(m-j) + 2m(i-1) + 2j + 1$, if i is even,
$f^*(v_{n-1}^j u_i^j) = 2m(n+t-i) - 2j + 1$, $1 \le i \le t$.

Case (ii): m is even

$f(v_0) = n - 1$,
$f(v_i^1) = n - 1 - i$, $1 \le i \le n-1$,
For $2 \le j \le m$ and $1 \le i \le t$,
$f(v_i^j) = n + 2(j-2) + (m-1)(i-1)$, if i is odd,
$f(v_i^j) = n + 1 + 4(m-j) + (m-1)(i-2)$, if i is even,
If n is odd, $f(u_i^1) = 2mn + 2(m-1)(t-1) - 2m + 2(t-i) + 7$,
If n is even, $f(u_i^1) = 2mn + 2(m-1)(t-1) + 2(t-i) - 1$,
If n is odd, $f(u_i^j) = m(n-1) + 5 + 2(m-1)(t-i) - 2(m-j)$,
If n is even, $f(u_i^j) = m(n-2) + 3 + 2(m-1)(t-i) + 4(m-j)$.

The induced edge labels are
$f^*(v_0 v_1^1) = 2n - 3$,
$f^*(v_0 v_1^j) = 2n - 1 + 2(j-2)$, $2 \le j \le m$,
$f^*(v_i^j v_{i+1}^j) = 2n + 2(j-2) + 2(m-1)(i-1) + 4(m-j) + 1$, $2 \le j \le m$.

For $2 \le j \le m$ and $1 \le i \le t$,
If n is even, $f^*(v_{n-1}^j u_i^j) = 2mn - 2j + 1 + 2(m-1)(t-i)$.
If n is odd, $f^*(v_{n-1}^j u_i^j) = 2m(n-1) - 2j + 2(m-1)(t-i) + 9$.

$$f^*(v_{n-1}^1 u_i^1) = \begin{cases} 2mn + 2(m-1)(t-1) - 2m + 2(t-i) + 7 & \text{if } n \text{ is odd} \\ 2mn + 2(m-1)(t-1) + 2(t-i) - 1 & \text{if } n \text{ is even.} \end{cases}$$

Thus every regular bamboo tree is odd harmonious. $\square$

An odd harmonious labeling of a regular bamboo tree with $m = 5, n = 6, t = 2$ is shown in Figure 5.

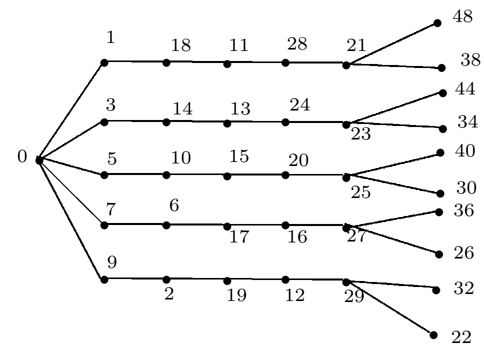

Figure 5: A regular bamboo tree with $m = 5, n = 6, t = 2$

An odd harmonious labeling of a regular bamboo tree with $m = 4, n = 5, t = 2$ is shown in Figure 6.

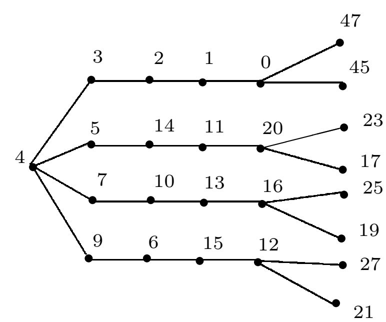

Figure 6: A regular bamboo tree with $m = 4, n = 5, t = 2$

Theorem 2.5. *The graph $C_n \hat{o} P_m$, $n \equiv 0(mod\ 4)$ is odd harmonious.*

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of cycle C_n. Let $u_i^0, u_i^1, \cdots, u_i^{m-1}$ be the vertices of path P_m. We identify u_i^0 with u_i, $1 \leq i \leq n$ to obtain $C_n \hat{o} P_m$. Then the graph $C_n \hat{o} P_m$ has mn edges and vertices.

We define the labeling $f : V(G) \to \{0, 1, 2, \cdots, 2nm - 1\}$ as follows:

Case (i): m is odd

$f(u_i) = mi,$ $\quad i = 1, 3, \cdots, n-1,$

$f(u_i) = mi - m - 1,$ $\quad i = 2, 4, \cdots, \frac{n}{2},$

$f(u_i) = mi - m + 1,$ $\quad i = \frac{n}{2} + 2, \cdots, n;$

$f(u_i^j) = mi - j,$ $\quad$ if i is odd and j is even,

$$f(u_i^j) = \begin{cases} mi - j - 2 & \text{if } 1 \le i \le \frac{n}{2} - 1 \\ mi - j & \text{if } \frac{n}{2} + 1 \le i \le n - 1 \end{cases} \qquad \text{if both } i \text{ and } j \text{ are odd,}$$

$$f(u_i^j) = \begin{cases} mi - m + j - 1 & \text{if } 2 \le i \le \frac{n}{2} \\ mi - m + j + 1 & \text{if } \frac{n}{2} + 2 \le i \le n \end{cases} \qquad \text{if both } i \text{ and } j \text{ are even,}$$

$$f(u_i^j) = mi - m + j + 1, \qquad \text{if } i \text{ is even and } j \text{ is odd.}$$

The induced edge labels are

$$f^*(u_i u_{i+1}) = 2im - 1, \qquad 1 \le i \le \frac{n}{2},$$

$$f^*(u_i u_{i+1}) = 2im + 1, \qquad \frac{n}{2} + 1 \le i \le n - 1,$$

$$f^*(u_1 u_n) = mn + 1,$$

$$f(u_i u_i^1) = \begin{cases} 2mi - 3 & \text{if } 1 \le i \le \frac{n}{2} - 1 \\ 2mi - 1 & \text{if } \frac{n}{2} + 1 \le i \le n - 1 \end{cases}$$

$$f(u_i u_i^1) = \begin{cases} 2mi - 2m + 1 & \text{if } 2 \le i \le \frac{n}{2} \\ 2mi - 2m + 3 & \text{if } \frac{n}{2} + 2 \le i \le n \end{cases} \qquad \text{if } i \text{ is even and } j \text{ is odd,}$$

$$f(u_i^j u_i^{j+1}) = \begin{cases} 2mi - 2j - 3 & \text{if } 1 \le i \le \frac{n}{2} - 1 \\ 2mi - 2j - 1 & \text{if } \frac{n}{2} + 1 \le i \le n - 1 \end{cases} \qquad \text{if } i \text{ is odd,}$$

$$f(u_i^j u_i^{j+1}) = \begin{cases} 2mi - 2m + 2j - 1 & \text{if } 2 \le i \le \frac{n}{2} \\ 2mi - 2m + 2j + 1 & \text{if } \frac{n}{2} + 2 \le i \le n \end{cases} \qquad \text{if } i \text{ is even.}$$

Case (ii): m is even

$$f(u_i) = mi - 2, \qquad i = 1, 3, \cdots, n - 1,$$

$$f(u_i) = \begin{cases} mi - m + 1 & \text{if } 2 \le i \le \frac{n}{2} \\ mi - m + 3 & \text{if } \frac{n}{2} + 2 \le i \le n \end{cases} \qquad \text{if } i \text{ is even,}$$

$$f(u_i^j) = \begin{cases} mi - j & \text{if } 1 \le i \le \frac{n}{2} - 1 \\ mi - j + 2 & \text{if } \frac{n}{2} + 1 \le i \le n - 1 \end{cases} \qquad \text{if both } i \text{ and } j \text{ are odd,}$$

$$f(u_i^j) = mi - j - 2, \ 1 \le i \le n - 1, \qquad \text{if } i \text{ is odd and } j \text{ is even,}$$

$$f(u_i^j) = mi - m + j - 1, \ 2 \le i \le n - 2, \qquad \text{if } i \text{ is even and } j \text{ is odd,}$$

$$f(u_i^j) = \begin{cases} mi - m + j + 1 & \text{if } 2 \le i \le \frac{n}{2} \\ mi - m + j - 1 & \text{if } \frac{n}{2} + 2 \le i \le n \end{cases} \qquad \text{if both } i \text{ and } j \text{ are even.}$$

The induced edge labels are

$$f^*(u_i u_{i+1}) = \begin{cases} 2mi - 1 & \text{if } 1 \le i \le \frac{n}{2} \\ 2mi + 1 & \text{if } \frac{n}{2} + 1 \le i \le n \end{cases} \qquad \text{if } i \text{ is even and } j \text{ is odd,}$$

$$f^*(u_1 u_n) = mn + 1;$$

$$f(u_i u_i^1) = \begin{cases} 2mi - 3 & \text{if } 1 \le i \le \frac{n}{2} - 1 \\ 2mi - 1 & \text{if } \frac{n}{2} + 1 \le i \le n - 1 \end{cases} \qquad \text{if } i \text{ is odd,}$$

$$f^*(u_iu_i^1) = \begin{cases} 2mi-2m+1 & \text{if } 2 \le i \le \dfrac{n}{2} \\ 2mi-2m+3 & \text{if } \dfrac{n}{2}+2 \le i \le n \end{cases} \qquad \text{if } i \text{ is even,}$$

$$f^*(u_i^ju_i^{j+1}) = \begin{cases} 2mi-2j-3 & \text{if } 1 \le i \le \dfrac{n}{2}-1 \\ 2mi-2j-1 & \text{if } \dfrac{n}{2}+1 \le i \le n-1 \end{cases} \qquad \text{if } i \text{ is odd,}$$

$$f^*(u_i^ju_i^{j+1}) = \begin{cases} 2mi-2m+2j+1 & \text{if } 2 \le i \le \dfrac{n}{2} \\ 2mi-2m+2j-1 & \text{if } \dfrac{n}{2}+2 \le i \le n \end{cases} \qquad \text{if } i \text{ is even.}$$

Therefore $C_n\hat{\circ}P_m$ is odd harmonious. □

An odd harmonious labeling of $C_4\hat{\circ}P_5$ and $C_{12}\hat{\circ}P_4$ are shown in Figure 7.

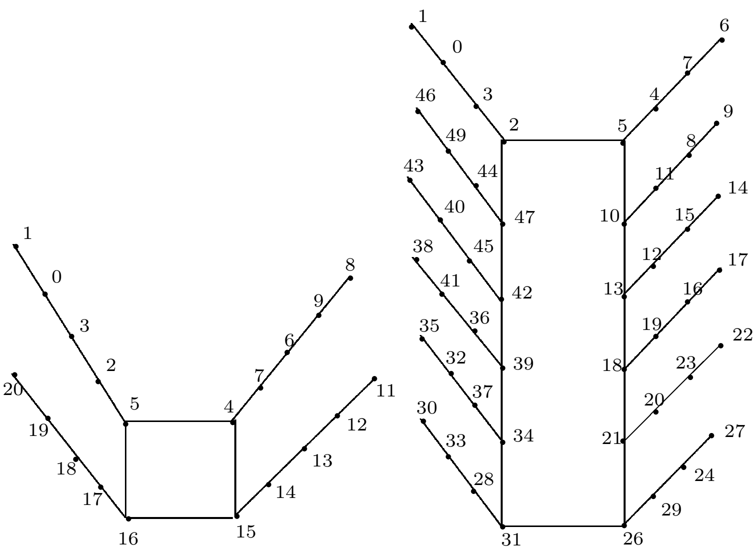

Figure 7: An odd harmonious labeling of $C_4\hat{\circ}P_5$ and $C_{12}\hat{\circ}P_4$

Theorem 2.6. *The graph $C_n\hat{\circ}2P_m$, $n \equiv 0(mod\ 4)$ is odd harmonious.*

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of C_n. Let $x_{1,0}^j, x_{1,1}^j, x_{1,2}^j, \cdots, x_{1,m-1}^j$ and $x_{2,0}^j, x_{2,1}^j, x_{2,2}^j, \cdots, x_{2,m-1}^j$, $1 \le j \le n$ be the vertices of two disjoint paths P_m. Identify $x_{1,0}^j$ and $x_{2,0}^j$ with u_j, $1 \le j \le n$ to obtain $C_n\hat{\circ}2P_m$. Then the graph $C_n\hat{\circ}2P_m$ has $n(2m-1)$ edges and vertices.

We define the labeling $f : V(G) \to \{0, 1, 2, \cdots, 2n(2m-1)-1\}$ as follows:

$$f(u_j) = m-1+(2m-1)(j-1), \qquad j = 1, 3, \cdots, n-1,$$

$$f(u_j) = \begin{cases} 3m-2+(2m-1)(j-2) & \text{if } 2 \le j \le \dfrac{n}{2} \\ 7m-2+(2m-1)(j-4) & \text{if } \dfrac{n}{2}+2 \le j \le n \end{cases} \qquad \text{if } j \text{ is even,}$$

$$f(x_{1,i}^j) = (m-1)+(2m-1)(j-1)-i, \qquad \text{if } i \text{ is even and } j \text{ is odd,}$$

$f(x_{1,i}^j) = \begin{cases} m+(2m-1)(j-1)-i-1 & \text{if } 1 \le j \le \frac{n}{2}-1 \\ m+(2m-1)(j-1)-i+1 & \text{if } \frac{n}{2}+1 \le j \le n-1 \end{cases}$ if both i and j are odd,

$f(x_{2,i}^j) = m+(2m-1)(j-1)+i-1$, $1 \le j \le n-1$, if i is even and j is odd,

$f(x_{2,i}^j) = \begin{cases} m+(2m-1)(j-1)+i-1 & \text{if } 1 \le j \le \frac{n}{2}-1 \\ m+(2m-1)(j-1)+i+1 & \text{if } \frac{n}{2}+1 \le j \le n-1 \end{cases}$ if both i and j are odd,

$f(x_{1,i}^j) = \begin{cases} 3m+(2m-1)(j-2)-i-2 & \text{if } 2 \le j \le \frac{n}{2} \\ 7m+(2m-1)(j-4)-i-2 & \text{if } \frac{n}{2}+2 \le j \le n \end{cases}$ if both i and j are even,

$f(x_{1,i}^j) = \begin{cases} 3m+(2m-1)(j-2)-i-2 & \text{if } 2 \le j \le \frac{n}{2} \\ 7m+(2m-1)(j-4)-i-4 & \text{if } \frac{n}{2}+2 \le j \le n \end{cases}$ if i is odd and j is even,

$f(x_{2,i}^j) = \begin{cases} 3m+(2m-1)(j-2)+i-2 & \text{if } 2 \le j \le \frac{n}{2} \\ 7m+(2m-1)(j-4)+i-2 & \text{if } \frac{n}{2}+2 \le j \le n \end{cases}$ if both i and j are even,

$f(x_{2,i}^j) = \begin{cases} 3m+(2m-1)(j-2)+i-2 & \text{if } 2 \le j \le \frac{n}{2} \\ 7m+(2m-1)(j-4)+i-4 & \text{if } \frac{n}{2}+2 \le j \le n \end{cases}$ if i is odd and j is even.

The induced edge labels are

$f^*(u_ju_{j+1}) = 4m+2(2m-1)(j-1)-3,$ $\quad 1 \le j \le \frac{n}{2}-1,$

$f^*(u_ju_{j+1}) = 8m+2(2m-1)(j-2)-3,$ $\quad \frac{n}{2}+1 \le j \le n-1,$

$f^*(u_jx_{1,1}^j) = \begin{cases} 2m+2(2m-1)(j-1)-3 & \text{if } 1 \le j \le \frac{n}{2}-1 \\ 2m+2(2m-1)(j-1)-1 & \text{if } \frac{n}{2}+1 \le j \le n-1 \end{cases}$ if j is odd,

$f^*(u_jx_{1,1}^j) = \begin{cases} 6m+2(2m-1)(j-2)-5 & \text{if } 2 \le j \le \frac{n}{2} \\ 14m+2(2m-1)(j-4)-7 & \text{if } \frac{n}{2}+2 \le j \le n \end{cases}$ if j is even,

$f^*(u_jx_{2,1}^j) = \begin{cases} 2m+2(2m-1)(j-1)-1 & \text{if } 1 \le j \le \frac{n}{2}-1 \\ 2m+2(2m-1)(j-1)+1 & \text{if } \frac{n}{2}+1 \le j \le n-1 \end{cases}$ if j is odd,

$f^*(u_jx_{2,1}^j) = \begin{cases} 6m+2(2m-1)(j-2)-3 & \text{if } 2 \le j \le \frac{n}{2} \\ 14m+2(2m-1)(j-4)-5 & \text{if } \frac{n}{2}+2 \le j \le n \end{cases}$ if j is even,

$f^*(x_{1,i}^jx_{1,i+1}^j) = \begin{cases} 4mj-2m-2j-2i-1 & \text{if } 1 \le j \le \frac{n}{2}-1 \\ 4mj-2m-2j-2i+1 & \text{if } \frac{n}{2}+1 \le j \le n-1 \end{cases}$ if j is odd,

$f^*(x_{1,i}^jx_{1,i+1}^j) = \begin{cases} 6m+2(2m-1)(j-2)-2i-5 & \text{if } 2 \le j \le \frac{n}{2} \\ 14m+2(2m-1)(j-4)-2i-7 & \text{if } \frac{n}{2}+2 \le j \le n \end{cases}$ if j is even,

$f^*(x_{2,i}^jx_{2,i+1}^j) = \begin{cases} 2m+2(2m-1)(j-1)+2i-1 & \text{if } 1 \le j \le \frac{n}{2}-1 \\ 2m+2(2m-1)(j-1)+2i+1 & \text{if } \frac{n}{2}+1 \le j \le n-1 \end{cases}$ if j is odd,

$$f^*(x^j_{2,i}x^j_{2,i+1}) = \begin{cases} 6m + 2(2m-1)(j-2) + 2i - 3 & \text{if } 2 \leq j \leq \frac{n}{2} \\ 14m + 2(2m-1)(j-4) + 2i - 5 & \text{if } \frac{n}{2} + 2 \leq j \leq n \end{cases} \quad \text{if } j \text{ is even,}$$

$f^*(u_1u_n) = 8m + (2m-1)(n-4) - 3.$

Thus $C_n\hat{o}2P_m$ is odd harmonious. □

An odd harmonious labeling of $C_8\hat{o}2P_3$ is shown in Figure 8.

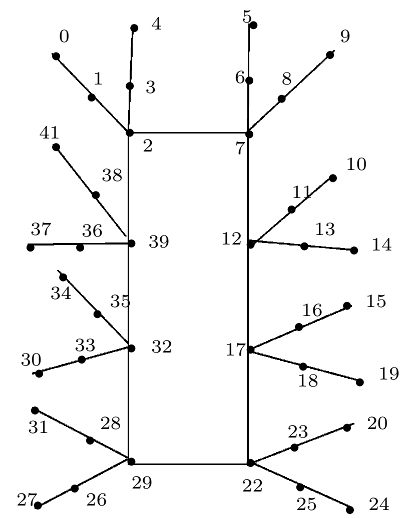

Figure 8: An odd harmonious labeling of $C_8\hat{o}2P_3$

Theorem 2.7. *Every subdivided grid $P_m \times P_m$, $m \geq 2$ is strongly odd harmonious.*

Proof. Let $v_{i,1}, v_{i,2}, \cdots, v_{i,m}$, $1 \leq i \leq m$ be the vertices of the i^{th} row of $P_m \times P_m$. Let $u_{1,i}$, $u_{2,i}, \cdots, u_{m-1,i}$, $1 \leq i \leq m$ be the vertices of the subdivided of i^{th} column and $w_{1,i}, w_{2,i}, \cdots, w_{m-1,i}$, $1 \leq i \leq m$ be the vertices of the subdivided of i^{th} row. Then the subdivided grid graph has $m(3m-2)$ and $4m(m-1)$ vertices and edges respectively.

We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, q = 4m(m-1)\}$ as follows:

$f(v_{i,j}) = 2(j-1) + 2(2m-1)(i-1),$ $\quad 1 \leq j \leq m$ and i is odd,

$f(v_{i,j}) = 2(3m-j-1) + 2(2m-1)(i-2),$ $\quad 1 \leq j \leq m$ and i is even,

$f(u_{i,j}) = 2m - 1 + 4(m-j) + 2(2m-1)(i-1),$ $\quad 1 \leq j \leq m$ and i is odd,

$f(u_{i,j}) = 6m - 3 + 4(j-1) + 2(2m-1)(i-2),$ $\quad 1 \leq j \leq m$ and i is even,

$f(w_{i,1}) = 2i - 1,$ $\quad 1 \leq i \leq m-1,$

$f(w_{i,j}) = (m-1)(4j-2) - 4i + 2j - 1,$ $\quad 1 \leq i \leq m-1$ and j is even,

$f(w_{i,j}) = (m-1)(4j-6) + 4i + 2j - 5,$ $\quad 1 \leq i \leq m-1$ and j is odd.

The induced edge labels are

$f^*(v_{i,j}u_{i,j}) = 2(j-1) + 4(2m-1)(i-1) + 2m - 1 + 4(m-j)$, $1 \le j \le m$ and i is odd,
$f^*(u_{i,j}v_{i+1,j}) = 2m - 1 + 2(5m - 3j - 1) + 4(2m-1)(i-1)$, $1 \le j \le m$ and i is odd,
$f^*(u_{i,j}v_{i+1,j}) = 6m - 3 + 6(j-1) + 4(2m-1)(i-1)$, $1 \le j \le m$ and i is even,
$f^*(v_{i,j}w_{t,k}) = 2(j-1) + 2(2m-1)(i-1) + (m-1)(4k-2) - 4t - 2k - 1$, i is odd and k is even,
$f^*(v_{i,j}w_{t,k}) = 2(j-1) + 2(2m-1)(i-1) + (m-1)(4k-6) + 4t + 2k - 5$, i is odd and k is odd,
$f^*(v_{i,j}w_{t,k}) = 2(3m-j-1) + 2(2m-1)(i-2) + 2(m-1)(2k-1) - 4t - 2k - 1$, i is even and k is even,
$f^*(v_{i,j}w_{t,k}) = 2(3m-j-1) + 2(2m-1)(i-2) + 2(m-1)(2k-3) + 4t + 2k - 5$, i is even and k is odd,
$f^*(v_{1,j}w_{i,1}) = 2(j-1) + 2i - 1$, $1 \le i$ and $j \le m-1$,
$f^*(w_{i,1}v_{1,j}) = 2(j-1) + 2i - 1$, $1 \le i \le m-1$ and $2 \le j \le m$.

Therefore every subdivided grid graph is strongly odd harmonious. □

A strongly odd harmonious labeling of subdivided grid $P_4 \times P_4$ is shown in Figure 9.

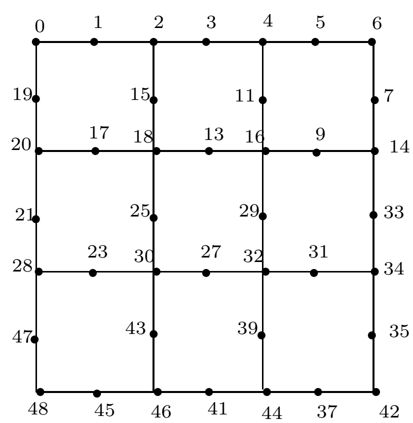

Figure 9: Strongly odd harmonious labeling of subdivided grid $P_4 \times P_4$

3 Acknowledgements

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