CUBO, A Mathematical Journal
Vol.22, Nº 01, (137–154). April 2020
http://doi.org/10.4067/S0719-06462020000100137

Level sets regularization with application to optimization problems

Moussa Barro and Sado Traore
Université Nazi BONI,
Burkina Faso.
mousbarro@yahoo.fr, traore.sado@yahoo.fr

ABSTRACT

Given a coupling function c and a non empty subset of $\mathbb{R}$, we define a closure operator. We are interested in extended real-valued functions whose sub-level sets are closed for this operator. Since this class of functions is closed under pointwise suprema, we introduce a regularization for extended real-valued functions. By decomposition of the closure operator using polarity scheme, we recover the regularization by bi-conjugation. We apply our results to derive a strong duality for a minimization problem.

RESUMEN

Dada una función de acoplamiento c y un subconjunto no vacío de $\mathbb{R}$, definimos un operador clausura. Estamos interesados en funciones extendidas a valores reales cuyos conjuntos de sub-nivel son cerrados para este operador. Dado que esta clase de funciones es cerrada bajo supremos puntuales, introducimos una regularización para funciones extendidas a valores reales. Gracias a la descomposición del operador clausura usando el esquema de polaridad, recuperamos la regularización por bi-conjugación. Aplicamos nuestros resultados para derivar una dualidad fuerte para un problema de minimización.

Keywords and Phrases: Duality, regularization, level sets, c-elementary functions, polarity, conjugacy.

2010 AMS Mathematics Subject Classification: 49N15.

1 Introduction

Regularization and conjugation of extended real-valued functions play an important role in optimization theory since it is a base of duality theory. Until the Fenchel's work([4]), many authors have introduced and studied different kinds of regularization and conjugation among which we can cite Moreau ([8]), Crouzeix ([1]), Rockafellar ([13]), Martínez-Legaz ([7]), Singer ([15]), Penot-Volle ([11]), Volle ([16, 17]). In [16], M. Volle used a dual pair of polarities to introduce and study level set regularization and conjugacy.

In this work, we introduce and study level set regularization and conjugacy by means of a coupling function and a nonvoid subset of the real numbers. The outline of the paper is as follows. In Section 2, we recall Moreau conjugation scheme. Section 3 is devoted to the study of the $\Gamma-$regularization of extended real-valued functions and hull of sets. We introduce these notions and give some properties (Proposition 2, 4 and Theorem 3.8). In Section 4, we introduce the level set regularization of extended real-valued functions. By decomposition of a closure operator via a couple of dual polarities, we show that this regularization coincides with the bi-conjugation relative to the polarity couple (Proposition 8 and Theorem 4.6). We derive an analytic expression of level set regularization of extended real-valued functions (Proposition 10). Section 5 is devoted to an application of our theory to a minimization problem. A perturbational dual of this problem is defined and a necessary and sufficient condition is given to ensure a strong duality property for this problem (Theorem 5.1, Corollary 5.2 and Corollary 5.3).

2 Preliminaries

Let us start this section by recalling the Moreau conjugation ([8]). Let U, V two nonvoid sets and $c : U \times V \to \mathbb{R}$, a coupling function. Given an extended real-valued function $h : U \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, we define the $c-$conjugate of h by $h^c(v) := \sup\limits_{u \in U}\{c(u,v) - h(u)\}$, for any $v \in V$. By exchanging the role of U and V, we define the $c-$conjugate of a given function $k : V \to \overline{\mathbb{R}}$ by $k^c(u) := \sup\limits_{v \in V}\{c(u,v) - k(v)\}$ for any $u \in U$. The $c-$bi-conjugate of a given $h : U \to \overline{\mathbb{R}}$ is then defined by $h^{cc}(u) := \sup\limits_{v \in V}\{c(u,v) - h^c(v)\}$ for any $u \in U$.

Example 2.1. *The usual Legendre-Fenchel conjugacy is obtained by taking $U := X$, a topological vector space with topological dual $V := X^*$ and c the standard bilinear coupling function.*

Example 2.2. *Other examples of coupling functions have been considered in the literature among which, we can cite:*

(1) $U = V = \{x \in \mathbb{R}^n | x_1 > 0, \ldots, x_n > 0\}$ and $c(u,v) = \min\limits_{1 \le i \le n} v_i u_i$, ([14]),

(2) (U,d) *a metric space,* $\alpha > 0$*,* $V = U$ *and* $c(u,v) = -\alpha d(u,v)$*, ([6]),*

(3) (U,d) *a metric space,* $V = U\times]0,+\infty[$ *and* $c(u,(v,\alpha)) = -\alpha d(u,v)$*, ([6]),*

(4) $U = V = \mathbb{R}^n$*,* $0 < \alpha \leq 1$*,* $\beta > 0$ *and* $c(u,v) = -\beta\|u-v\|^\alpha$*, ([7]),*

(5) U *a topological space,* $V = \mathscr{C}(U,\mathbb{R})$*, space of countinuous real functions on* U *and* $c(u,v) = v(u)$*, ([5],[10]).*

Given a function $h : U \to \overline{\mathbb{R}}$, the following notation and definitions will be needed: dom $h = \{u \in U \mid h(u) < +\infty\}$, the effective domain of h, $[h \leq t] := \{u \in U \mid h(u) \leq t\}$, the t-sub-level set of h(or level set of h in short).

Given a subset A of U, we define its indicator function i_A by $i_A(u) = 0$ if $u \in A$ and $i_A(u) = +\infty$ if $u \in U \setminus A$. Following the terminology introduced in [9] we will also use the valley function v_A of A defined by $v_A(u) = -\infty$ if $u \in A$ and $v_A(u) = +\infty$ if $u \in U \setminus A$.

3 $\Gamma-$regularization of functions and hull of sets

3.1 $\Gamma-$regularization of functions

The notion of continuous affine functions can be generalized by those of $c-$elementary functions. In this work, we call $c-$elementary function on U (resp. on V), the function of the form $c(.,v) - r$ (resp. $c(u,.) - r$) with $v \in V$ (resp. $u \in U$) and $r \in \mathbb{R}$. The upper hull (i.e., the supremum) of a family of $c-$elementary functions is called c-regular. We denote by $\Gamma_c(U)$, the set of $c-$regular functions defined on U. We call c-hull or Γ_c-regularization of $h : U \to \overline{\mathbb{R}}$, the greatest $c-$regular minorant of h. This function is denoted by h^{Γ_c}. It is well known ([8]) that

$$h^{cc} = h^{\Gamma_c}, \text{ for each } h : U \to \overline{\mathbb{R}}. \tag{3.1}$$

Remark 3.1. *The equality* (3.1) *is still valid if the coupling function is an extended real-valued function. In this case, one must interpret the conjugate* h^c *as follows*

$$h^c(v) = -\inf_{u\in U}\{h(u) - c(u,v)\},$$

with the usual conventions $(+\infty) - (+\infty) = (-\infty) - (-\infty) = +\infty$.

There exists an equivalent approach to generalized convex duality in terms of Φ -convexity [2], which consists of working with a set U and a class of functions $\Phi \subset \overline{\mathbb{R}}^U$.

3.2 Hull of sets

Let $\mathbb{P}$ be a nonvoid subset of $\mathbb{R}$. The following definition generalizes the notion of half space.

Definition 1. *We call $(c,\mathbb{P})-$elementary subset of U any subset of U of the form $\{u \in U \mid r - c(u,v) \in \mathbb{P}\}$, where $(v,r) \in V \times \mathbb{R}$. We note it by $E^{\mathbb{P}}_{v,r}$.*

Note that, if $\mathbb{P} = \mathbb{R}$, then $E^{\mathbb{P}}_{v,r} = U$ for any $(v,r) \in V \times \mathbb{R}$. In this case, the only $(c,\mathbb{P})-$elementary subset of U is U itself. The $(c,\mathbb{P})-$elementary subsets of U allow us to define a notion of hull of a subset A of U.

Definition 2. *The $(c,\mathbb{P})-$hull of $A \subset U$ is the intersection of all $(c,\mathbb{P})-$elementary subsets of U containing A. The $(c,\mathbb{P})-$hull of A is denoted by $\langle A\rangle_{c,\mathbb{P}}$.*

Remark 3.2. *If there is not $(c,\mathbb{P})-$elementary subset of U containing A, then $\langle A\rangle_{c,\mathbb{P}} = U$ by convention.*

Proposition 1. *If $\mathbb{P} \neq \mathbb{R}$, then $\langle \emptyset\rangle_{c,\mathbb{P}} = \emptyset$.*

Proof Let $s \in \mathbb{R} \setminus \mathbb{P}$. Assume $\langle \emptyset\rangle_{c,\mathbb{P}} \neq \emptyset$. Let $a \in \langle \emptyset\rangle_{c,\mathbb{P}}$. Then $r - c(a,v) \in \mathbb{P}$ for any $(v,r) \in V \times \mathbb{R}$. In particular $s = (s + c(a,v)) - c(a,v) \in \mathbb{P}$, absurd. $\square$

It follows from the definition of $\langle . \rangle_{c,\mathbb{P}}$ that, for each $A \subset U$, and for each $u \in U$, one has

$$u \notin \langle A\rangle_{c,\mathbb{P}} \Longleftrightarrow \exists (v,r) \in V \times \mathbb{R} \ : \ A \subset E^{\mathbb{P}}_{v,r} \ \text{ and } r - c(u,v) \notin \mathbb{P}. \tag{3.2}$$

By definition 1, one has $A \subset \langle A\rangle_{c,\mathbb{P}}$, for any $A \subset U$. Moreover, if $A \subset B$ then $\langle A\rangle_{c,\mathbb{P}} \subset \langle B\rangle_{c,\mathbb{P}}$. Therefore, $\langle\langle A\rangle_{c,\mathbb{P}}\rangle_{c,\mathbb{P}} = \langle A\rangle_{c,\mathbb{P}}$, $\forall A \subset U$. We deduce that $\langle . \rangle_{c,\mathbb{P}}$ is an algebraic closure operator.

Definition 3. *A subset A of U is said to be $(c,\mathbb{P})-$regular if $A = \langle A\rangle_{c,\mathbb{P}}$. We denote $\mathcal{R}_{c,\mathbb{P}}(U)$, the set of all $(c,\mathbb{P})$-regular subsets of U.*

Observe that $(c,\mathbb{P})-$elementary sets are $(c,\mathbb{P})-$regular. More generally, any intersection of $(c,\mathbb{P})-$regular subsets is $(c,\mathbb{P})-$regular and the $(c,\mathbb{P})-$regular hull of $A \subset U$ coincides with the intersection of all $(c,\mathbb{P})-$regular subsets of U containing A.

In what follows, we will use the following values for $\mathbb{P}$: $\mathbb{P}_1 = \mathbb{R}_+ := [0,+\infty[$, $\mathbb{P}_2 = \mathbb{R}^*_+ :=]0,+\infty[$, $\mathbb{P}_3 = \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $\mathbb{P}_4 = \{0\}$. For $i = 1,2,3,4$, $\langle . \rangle_{c,i} := \langle . \rangle_{c,\mathbb{P}_i}$ for short. For $i = 1,2,3$, the set $\langle A\rangle_{c,i}$ can be explained as follows:

Proposition 2. *For any $A \subset U$, one has:*

$$\langle A\rangle_{c,1} = \{u \in U \mid c(u,v) \leq \sup_{a\in A} c(a,v), \ \forall v \in V\}, \tag{3.3}$$

$$\langle A\rangle_{c,2} = \{u \in U \mid \forall v \in V, \exists a \in A \mid c(u,v) \leq c(a,v)\}, \tag{3.4}$$

$$\langle A\rangle_{c,3} = \{u \in U \mid \forall v \in V, \exists a \in A \mid c(u,v) = c(a,v)\}. \tag{3.5}$$

Proof By (3.2), one has:

$$\begin{aligned} a \notin \langle A\rangle_{c,1} &\iff \exists (v,r) \in V \times \mathbb{R} : A \subset [c(.,v) \le r] \ \text{ and } \ r < c(a,v) \\ &\iff \exists (v,r) \in V \times \mathbb{R} : \sup_{u\in A} c(u,v) \le r < c(a,v) \\ &\iff \exists v \in V : \ \sup_{u\in A} c(u,v) < c(a,v). \end{aligned}$$

Thus, $a \in \langle A\rangle_{c,1} \iff \forall v \in V, \ c(a,v) \le \sup_{u\in A} c(u,v)$, and (3.3) holds.

$$\begin{aligned} a \notin \langle A\rangle_{c,2} &\iff \exists (v,r) \in V \times \mathbb{R} : A \subset [c(.,v) < r] \ \text{ and } \ r \le c(a,v) \\ &\iff \exists (v,r) \in V \times \mathbb{R} : c(u,v) < r \le c(a,v), \ \forall u \in A \\ &\iff \exists v \in V : \ c(u,v) < c(a,v), \ \forall u \in A. \end{aligned}$$

Thus, $a \in \langle A\rangle_{c,2} \iff \forall v \in V, \exists u \in A : \ c(a,v) \le c(u,v)$, and (3.4) holds.

$$\begin{aligned} a \notin \langle A\rangle_{c,3} &\iff \exists (v,r) \in V \times \mathbb{R} : A \subset [c(.,v) \neq r] \ \text{ and } \ r = c(a,v) \\ &\iff \exists v \in V : \ c(u,v) \neq c(a,v), \ \forall u \in A. \end{aligned}$$

Thus $a \in \langle A\rangle_{c,3} \iff \forall v \in V, \ \exists u \in A : \ c(a,v) = c(u,v)$, and (3.5) holds. □

Remark 3.3. *Observe that one cannot remove the real parameter r in the definition of $\langle A\rangle_{c,4}$.*

Example 3.4. *We observe the situation in topological vector case. Assume U is a topological vector space with topological dual V and c the standard coupling function. The $c-$elementary functions are affine continuous functions, and we have:*

1. *$(c, \mathbb{P}_1)-$elementary sets are $\emptyset$, U and closed half spaces. Moreover, if U is locally convex, then by Hahn-Banach separation theorem and* (3.3)*, $\langle A\rangle_{c,1} = \overline{\mathrm{conv}} A$, the closed convex hull of A.*

2. *$(c, \mathbb{P}_2)-$elementary sets are $\emptyset$, U and half open spaces. The $(c, \mathbb{P}_2)-$hull of a subset of U is its evenly convex hull ([4],[7],. . .).*

3. *$(c, \mathbb{P}_3)-$elementary sets are $\emptyset$, U and complementary set of closed hyperplane. The $(c, \mathbb{P}_3)-$hull of a subset of U is its evenly co-affine hull ([15]). Observe that ([15], corollary 2.2) A is evenly convex if and only if A is evenly co-affine and convex.*

4. *$(c, \mathbb{P}_4)-$elementary sets are $\emptyset$, U and closed hyperplane. Moreover, if U is locally convex, then by the Hahn-Banach separation theorem and* (3.2)*, the $(c, \mathbb{P}_4)-$hull of a non empty subset of U is its closed affine hull.*

Proposition 3. *Let P and Q be two nonvoid subsets of $\mathbb{R}$. Assume that any (c, P)-elementary set is (c, Q)-regular. Then, $\langle A\rangle_{c,Q} \subset \langle A\rangle_{c,P}, \forall A \subset U$.*

Proof Let $a \notin \langle A\rangle_{c,P}$. By definition, there exists an (c,P)-elementary set E such that $A \subset E$ and $a \notin E$. Since E is also (c,Q)-regular, it follows from (3.2) that $a \notin \langle A\rangle_{c,Q}$, and we are done. $\square$

Corollary 3.5. *For any $A \subset U$, one has:*

$$\langle A\rangle_{c,1} \supset \langle A\rangle_{c,2} \supset \langle A\rangle_{c,3} \quad \textit{and } \langle A\rangle_{c,4} \supset \langle A\rangle_{c,3}.$$

Proof Let $v \in V$ and $r \in \mathbb{R}$, it is obvious that

$$\{u \in U \mid c(u,v) \leq r\} = \bigcap_{s>r}\{u \in U \mid c(u,v) < s\}.$$

Consequently, any $(c,\mathbb{P}_1)-$elementary subset is $(c,\mathbb{P}_2)-$regular, and by Proposition 3, one has $\langle A\rangle_{c,1} \supset \langle A\rangle_{c,2}$ for any $A \subset U$.

It is easy to verify that:

$$\{u \in U \mid c(u,v) < r\} = \bigcap_{s \geq r}\{u \in U \mid c(u,v) \neq s\},$$
$$\{u \in U \mid c(u,v) = r\} = \bigcap_{s \neq r}\{u \in U \mid c(u,v) \neq s\},$$

therefore, $\langle A\rangle_{c,2} \supset \langle A\rangle_{c,3} \subset \langle A\rangle_{c,4}$. $\square$

We derive from Corollary 3.5, the following comparison between the sets $\mathcal{R}_{c,\mathbb{P}_i}(U)$:

$$\mathcal{R}_{c,\mathbb{P}_1}(U) \subset \mathcal{R}_{c,\mathbb{P}_2}(U) \subset \mathcal{R}_{c,\mathbb{P}_3}(U) \text{ and } \mathcal{R}_{c,P_4}(U) \subset \mathcal{R}_{c,\mathbb{P}_3}(U). \tag{3.6}$$

Remark 3.6. *Observe that*

1. $\mathbb{P}_1 \supset \mathbb{P}_2$ *and* $\mathcal{R}_{c,\mathbb{P}_1}(U) \subset \mathcal{R}_{c,\mathbb{P}_2}(U)$.
2. $\mathbb{P}_3 \supset \mathbb{P}_2$ *and* $\mathcal{R}_{c,\mathbb{P}_3}(U) \supset \mathcal{R}_{c,\mathbb{P}_2}(U)$.
3. $\mathcal{R}_{c,\mathbb{P}_1}(U) \subset \mathcal{R}_{c,\mathbb{P}_3}(U)$ *whereas* $\mathbb{P}_1$ *and* $\mathbb{P}_3$ *are not comparable in the sense of inclusion.*
4. *In particular, in the case of locally convex vector space, we recover the fact that every closed convex subset is evenly convex.*

Proposition 4. *Assume that the coupling function c satisfies the property:*

$$\forall v \in V,\ \exists w \in V \quad | \quad -c(.,v) = c(.,w). \tag{3.7}$$

Then, one has:

$$\mathcal{R}_{c,\mathbb{P}_4}(U) \subset \mathcal{R}_{c,\mathbb{P}_1}(U) \subset \mathcal{R}_{c,\mathbb{P}_2}(U) \subset \mathcal{R}_{c,\mathbb{P}_3}(U). \tag{3.8}$$

Proof By (3.6), we only need to show that $\mathcal{R}_{c,\mathbb{P}_4}(U) \subset \mathcal{R}_{c,\mathbb{P}_1}(U)$. Let $(v,r) \in V \times \mathbb{R}$. We have

$$[c(.,v) = r] = [c(.,v) \leq r] \bigcap [-c(.,v) \leq -r].$$

By assumption on the coupling function, there exists $w \in V$ such that $-c(.,v) = c(.,w)$. Consequently,

$$[c(.,v) = r] = [c(.,v) \leq r] \bigcap [c(.,w) \leq -r].$$

We conclude with Proposition 3. □

Example 3.7. *Assume that $U = V = \mathbb{R}^n$, and coupling function c is defined by $c(u,v) = \|u - v\|$, where $\|.\|$ is the euclidean norm. The non trivial $(c, \mathbb{P}_4)-$elementary sets are spheres (not convex) whereas the non trivial $(c, \mathbb{P}_1)-$elementary sets are closed balls (closed convex). In this case, $\mathcal{R}_{c,\mathbb{P}_1}(U)$ and $\mathcal{R}_{c,\mathbb{P}_4}(U)$ are not comparable. Observe that in this case assumption* (3.7) *does not hold.*

Proposition 5. *For any $A \subset U$, one has $\langle A \rangle_{c,1} = [i_A^{\Gamma_c} \leq 0]$.*

Proof By (3.2), one has

$$\begin{aligned} a \notin \langle A \rangle_{c,1} &\Longleftrightarrow \exists v \in V \ : \ i_A^c(v) < c(a,v) \\ &\Longleftrightarrow 0 < \sup_{v \in V} \{c(a,v) - i_A^c(v)\} \\ &\Longleftrightarrow 0 < i_A^{\Gamma_c}(a) \\ &\Longleftrightarrow 0 \notin \left[i_A^{\Gamma_c} \leq 0 \right]. \end{aligned}$$

Thus $\langle A \rangle_{c,1} = \left[i_A^{\Gamma_c} \leq 0 \right]$. □

The following result makes the link between hull of set and Γ-regularization of function by means of indicator function.

Theorem 3.8. *Assume that the coupling function c satisfies the condition:*

$$\forall \ (v, \beta) \in V \times \mathbb{R}_+^*, \quad \exists \bar{v} \in V \mid \beta c(.,v) = c(.,\bar{v}). \tag{3.9}$$

Then for each $A \subset U$ such that $\operatorname{dom} i_A^c \neq \emptyset$*, one has: $i_A^{\Gamma_c} = i_{\langle A \rangle_{c,1}}$.*

Proof Let $b \in U$.

(1) Assume that $b \notin \langle A \rangle_{c,1}$. By (3.3), there exists $(v, \epsilon) \in V \times \mathbb{R}_+^*$ such that $c(b,v) - \sup_{a \in A} c(a,v) \geq \epsilon$. From (3.9), one has:

$$\forall n \geq 1, \exists v_n \in V \ : \ nc(.,v) = c(.,v_n).$$

Consequently,

$$n\epsilon \leq c(b, v_n) - \sup_{a\in A} c(a, v_n) = c(b, v_n) - i_A^c(v_n) \leq i_A^{\Gamma_c}(b), \quad \forall n \geq 1.$$

Therefore $i_A^{\Gamma_c}(b) = +\infty$.

(2) Assume that $b \in \langle A\rangle_{c,1}$. By (3.3), one has

$$c(b, w) - \sup_{a\in A} c(a, w) \leq 0, \quad \forall v \in V.$$

Thus

$$i_A^{\Gamma_c}(b) = \sup_{v\in V}\left\{c(b, v) - \sup_{a\in A} c(a, v)\right\} \leq 0.$$

Let $v \in \text{dom } i_A^c$. By (3.9), one gets

$$\forall n \geq 1,\ \exists\ v_n \in V\ :\ \frac{1}{n}c(., v) = c(., v_n).$$

Consequently,

$$\frac{1}{n}\left(c(b, v) - \sup_{a\in A} c(a, v)\right) = c(b, v_n) - \sup_{a\in A} c(a, v_n) \leq i_A^{\Gamma_c}(b), \quad \forall n \geq 1.$$

Therefore,

$$0 = \lim_{n\to+\infty} \frac{1}{n}\left(c(b, v) - \sup_{a\in A} c(a, v)\right) \leq i_A^{\Gamma_c}(b),$$

and finally, $i_A^{\Gamma_c}(b) = 0$.

□

Remark 3.9. *Assumption* (3.9) *is satisfied by coupling functions (1), (3) and (5) of Example 2.2. Coupling functions (2) and (4) of the same example do not satisfy assumption* (3.9).

4 Level set regularization of functions

In this section, we introduce a notion of $(c, \mathbb{P})-$level set regularization of extended real-valued functions. We show that this level set regularization can be interpreted as bi-conjugacy relative to a couple of dual polarities by decomposition of the closure operator. We then give some other expressions of these regularizations.

4.1 Definitions and properties

Definition 4. *A function $h : U \to \overline{\mathbb{R}}$ is said to be $(c, \mathbb{P})-$level regular if all of its sub-level sets are $(c, \mathbb{P})-$regular, i.e $\langle[h \leq r]\rangle_{c,\mathbb{P}} = [h \leq r],\ \forall r \in \mathbb{R}$.*

We denote $\mathcal{N}_{c,\mathbb{P}}(U)$, the set of $(c,\mathbb{P})-$level regular functions defined on U to $\overline{\mathbb{R}}$. Observe that this set contains the constant function $-\infty$.

Proposition 6. *The set $\mathcal{N}_{c,\mathbb{P}}(U)$ is closed under pointwise suprema, i.e given $(h_i)_{i\in I}$ a family of $(c,\mathbb{P})-$level regular functions,then $h := \sup\limits_{i\in I} h_i$ is $(c,\mathbb{P})-$level regular.*

Proof Let $r \in \mathbb{R}$. Since $[h \leq r] = \cap_{i\in I}[h_i \leq r]$, the conclusion follows from the fact that any intersection $(c,\mathbb{P})-$regular sets is $(c,\mathbb{P})-$regular. □

We define the $(c,\mathbb{P})-$level set regularization of an extended real-valued function as follows.

Definition 5. *The $(c,\mathbb{P})-$level set regularization of a function $h : U \to \overline{\mathbb{R}}$ is the greatest $(c,\mathbb{P})-$level regular minorant of h. This function is denoted by $h^{\langle\rangle_{c,\mathbb{P}}}$.*

Example 4.1. *Assume U is topological vector space with topological dual V and c the standard coupling function. $(c,\mathbb{P}_2)-$level regular functions are evenly quasi-convex functions. Moreover, if U is locally convex then $(c,\mathbb{P}_1)-$level regular functions are lower semi-continuous quasi-convex functions.*

Example 4.2. *Assume that U is a metric space, $V = \mathscr{C}(U,\mathbb{R})$ a space of continuous functions from U to $\mathbb{R}$, and $c : U \times V \to \mathbb{R}$ defined by $c(u,v) = v(u)$. A function $h : U \to \mathbb{R} \cup \{+\infty\}$ is $(c,\mathbb{P}_1)-$level regular if and only if h is lower semi-continuous ([3], corollary 11).*

Proposition 7. *Any $c-$elementary function is $(c,\mathbb{P}_i)-$level regular for $i = 1,2,3$. More precisely, one has*

$$\Gamma_c(U) \subset \mathcal{N}_{c,\mathbb{P}_1}(U) \subset \mathcal{N}_{c,\mathbb{P}_2}(U) \subset \mathcal{N}_{c,\mathbb{P}_3}(U) \text{ and } \mathcal{N}_{c,\mathbb{P}_4}(U) \subset \mathcal{N}_{c,\mathbb{P}_3}(U).$$

Proof Let $h := c(.,v) - r$ an $c-$elementary function. For any $t \in \mathbb{R}$, we have

$$[h \leq t] = \{u \in U \mid t + r - c(u,v) \geq 0\},$$

which is obviously $(c,\mathbb{P}_1)-$elementary set. Therefore $\Gamma_c(U) \subset \mathcal{N}_{c,\mathbb{P}_1}(U)$. The other inclusions follow from (3.6). □

Remark 4.3. *$c-$elementary functions are not necessary $(c,\mathbb{P}_4)-$level set regular functions. For example, in the topological case, one cannot write a half space as an intersection of affine hyperplanes.*

Example 4.4. *Let $n \geq 1$, an integer number. Assume that $U = V = \mathbb{R}^n$ and c a standard scalar product of $\mathbb{R}^n$. Let $h_1, h_2 : \mathbb{R}^n \to [\![0;n]\!]$ two functions defined by*

$$h_1(x) = \begin{cases} 0 & \text{if } x = 0 \\ \max\{i \in [\![1,n]\!] \mid x_i \neq 0\} & \text{if } x \neq 0, \end{cases}$$

$$h_2(x) = \begin{cases} 0 & \text{if } x_i \neq 0, \ \forall i \in [\![1,n]\!] \\ \max\{i \in [\![1,n]\!] \mid x_i = 0\} & \text{else }. \end{cases}$$

For any $r \in \mathbb{R}$, we have

$$[h_1 \leq r] = \begin{cases} \emptyset & \text{if } \ r < 0 \\ \{x \in \mathbb{R}^n \mid x_{i+1} = \ldots = x_n = 0\} & \text{if } \ i \leq r < i+1, \quad i = 0, 1, \ldots, n-1 \\ \mathbb{R}^n & \text{if } \ n \leq r, \end{cases}$$

$$[h_2 \leq r] = \begin{cases} \emptyset & \text{if } r < 0 \\ \{x \in \mathbb{R}^n \ \mid x_{i+1} \neq 0, \ldots, x_n \neq 0\} & \text{if } i \leq r < i+1, \quad i = 0, 1, \ldots, n-1 \\ \mathbb{R}^n & \text{if } \ n \leq r. \end{cases}$$

It is clear that:

(1) h_1 is $(c, \mathbb{P}_4)-$level regular. In particular, $h_1 \in \mathcal{N}_{c,\mathbb{P}_i}(U)$, for $i = 1, 2, 3, 4$.

(2) h_2 is $(c, \mathbb{P}_3)-$level regular but not $(c, \mathbb{P}_2)-$level regular since $[h_2 \leq n-1] = \{x \in \mathbb{R}^n \mid x_n \neq 0\}$ is not convex.

Example 4.5. *Let $U = V = \mathbb{R}$, c the standard product of $\mathbb{R}$. The indicator function of $\mathbb{R}^*$,$i_{\mathbb{R}^*}$ is $(c, \mathbb{P}_3)-$level regular but not quasi-convex.*

4.2 Decomposition of $\langle\rangle_{c,\mathbb{P}}$

Let us consider a map $\Delta_{c,\mathbb{P}} : 2^U \to 2^{V \times \mathbb{R}}$ defined by:

$$\Delta_{c,\mathbb{P}}(A) := \{(v,r) \in V \times \mathbb{R} \mid A \subset E^{\mathbb{P}}_{v,r}\}, \tag{4.1}$$

which, we simply denote Δ in the sequel. Given $(A_i)_{i \in I}$ a family of subsets of U, we have

$$\begin{aligned} \Delta \bigcup_{i \in I} A_i &:= \left\{(v,r) \in V \times \mathbb{R} \mid \bigcup_{i \in I} A_i \subset E^{\mathbb{P}}_{v,r}\right\} \\ &= \left\{(v,r) \in V \times \mathbb{R} \mid A_i \subset E^{\mathbb{P}}_{v,r}, \quad \forall i \in I\right\} \\ &= \bigcap_{i \in I} \left\{(v,r) \in V \times \mathbb{R} \mid A_i \subset E^{\mathbb{P}}_{v,r}\right\} \\ &= \bigcap_{i \in I} \Delta A_i. \end{aligned}$$

Therefore Δ is said to be a polarity ([16]). We associate to Δ, its dual polarity $\Delta^* : 2^{V \times \mathbb{R}} \to 2^U$ defined by

$$\Delta^*(B) = \bigcup \{A \in 2^U \mid B \subset \Delta(A)\}. \tag{4.2}$$

Observe that for each $(v,r) \in V \times \mathbb{R}$ and for each $u \in U$, one has

$$u \in \Delta^*(v,r) \iff (v,r) \in \Delta(u) \iff u \in E^{\mathbb{P}}_{v,r}, \tag{4.3}$$

therefore $\Delta^*(v,r) = E^{\mathbb{P}}_{v,r}$. Since Δ^* is a polarity, then we have for each $B \subset V \times \mathbb{R}$,

$$\Delta^*(B) = \Delta^*\big(\bigcup_{(v,r)\in B} \{(v,r)\}\big) = \bigcap_{(v,r)\in B} \Delta^*(v,r) = \bigcap_{(v,r)\in B} E^{\mathbb{P}}_{v,r}. \tag{4.4}$$

The operator $\langle\rangle_{c,\mathbb{P}}$ can be decomposed as follows.

Proposition 8. *For any $A \subset U$, we have $(\Delta^* \circ \Delta)(A) = \langle A\rangle_{c,\mathbb{P}}$.*

Proof Given $A \subset U$, one has

$$(\Delta^* \circ \Delta)(A) = \Delta^*(\{(v,r) \in V \times \mathbb{R} \mid A \subset E^{\mathbb{P}}_{v,r}\}) = \bigcap_{A\subset E^{\mathbb{P}}_{v,r}} E^{\mathbb{P}}_{v,r} = \langle A\rangle_{c,\mathbb{P}}.$$

□

4.3 Conjugacy associated to polarities Δ and Δ^* ([16, 17])

The conjugate of a function $h : U \to \overline{\mathbb{R}}$ relative to the polarity Δ is the function $h^{\Delta} : V \times \mathbb{R} \to \overline{\mathbb{R}}$ given by

$$h^{\Delta}(v,r) := \sup_{u\notin\Delta^*(v,r)} -h(u) = \sup_{u\notin E^{\mathbb{P}}_{v,r}} -h(u). \tag{4.5}$$

Analogously, the conjugate of a function $k : V \times \mathbb{R} \to \overline{\mathbb{R}}$ relative to the polarity Δ^* is defined by

$$k^{\Delta^*}(u) := \sup_{(v,r)\notin\Delta(u)} -k(v,r) = \sup_{u\notin E^{\mathbb{P}}_{v,r}} -k(v,r). \tag{4.6}$$

Thus, the bi-conjugacy relative to polarities Δ, Δ^* of a function $h : U \to \overline{\mathbb{R}}$ is the function $h^{\Delta\Delta^*} : U \to \overline{\mathbb{R}}$ given by

$$h^{\Delta\Delta^*}(a) := \sup_{(v,r)\notin\Delta(a)} \inf_{u\notin\Delta^*(v,r)} h(u) = \sup_{a\notin E^{\mathbb{P}}_{v,r}} \inf_{u\notin E^{\mathbb{P}}_{v,r}} h(u). \tag{4.7}$$

It is well known ([16]) that this conjugacy can be interpreted by means of coupling function $\delta : U \times (V \times \mathbb{R}) \to \overline{\mathbb{R}}$ defined by

$$\delta(u,(v,r)) = \begin{cases} 0 & \text{si} \quad u \notin E^{\mathbb{P}}_{v,r} \\ -\infty & \text{si} \quad u \in E^{\mathbb{P}}_{v,r}. \end{cases}$$

More precisely, given a function $h : U \to \overline{\mathbb{R}}$, we have $h^{\Delta} = h^{\delta}$ and $h^{\Delta\Delta^*} = h^{\delta\delta}$.

Theorem 4.6 ([16])**.** *The $(c,\mathbb{P})-$level regularization of a function $h : U \to \overline{\mathbb{R}}$ coincides with bi-conjugacy relative to polarities Δ, Δ^*: $h^{\langle\rangle_{c,\mathbb{P}}} = h^{\Delta\Delta^*}$.*

Corollary 4.7. *For any subset A of U, we have*

$$i_A^{\langle\rangle_{c,\mathbb{P}}} = i_{\langle A\rangle_{c,\mathbb{P}}} \quad \textit{and} \quad v_A^{\langle\rangle_{c,\mathbb{P}}} = v_{\langle A\rangle_{c,\mathbb{P}}}.$$

Proof Let $A \subset U$. Let $a \in A$. It follows from Theorem 4.6 that

$$i_A^{\langle\rangle_{c,\mathbb{P}}}(a) = \sup_{a \notin E_{v,r}^{\mathbb{P}}} \inf_{u \notin E_{v,r}^{\mathbb{P}}} i_A(u).$$

We distinguish two cases:

- We first assume that $a \notin \langle A \rangle_{c,\mathbb{P}}$. There exists $(v,r) \in V \times \mathbb{R}$ such that $a \notin E_{v,r}^{\mathbb{P}}$ and $A \subset E_{v,r}^{\mathbb{P}}$. Consequently,

$$\inf_{u \notin E_{v,r}^{\mathbb{P}}} i_A(u) = +\infty.$$

- Secondly, assume that $a \in \langle A \rangle_{c,\mathbb{P}}$. For all $(v,r) \in V \times \mathbb{R}$ such that $a \notin E_{v,r}^{\mathbb{P}}$, there exists $u \in A$ such that $u \notin E_{v,r}^{\mathbb{P}}$. Consequently,

$$\begin{aligned} i_A^{\langle\rangle_{c,\mathbb{P}}}(a) &= \sup_{a \notin E_{v,r}^{\mathbb{P}}} \inf_{u \notin E_{v,r}^{\mathbb{P}}} i_A(u) \\ &= \begin{cases} 0 & \text{if } \ a \in \langle A \rangle_{c,\mathbb{P}} \\ +\infty & \text{if } \ a \notin \langle A \rangle_{c,\mathbb{P}} \end{cases} \\ &= i_{\langle A \rangle_{c,\mathbb{P}}}(a). \end{aligned}$$

Analogously, we have

$$\begin{aligned} v_A^{\langle\rangle_{c,\mathbb{P}}}(a) &= \sup_{a \notin E_{v,r}^{\mathbb{P}}} \inf_{u \notin E_{v,r}^{\mathbb{P}}} v_A(u) \\ &= \begin{cases} -\infty & \text{if } \ a \in \langle A \rangle_{c,\mathbb{P}} \\ +\infty & \text{if } \ a \notin \langle A \rangle_{c,\mathbb{P}} \end{cases} \\ &= v_{\langle A \rangle_{c,\mathbb{P}}}(a). \end{aligned}$$

$\square$

Proposition 9. *For any function $h : U \to \overline{\mathbb{R}}$ and for any real number t, one has*

$$[h^{\langle\rangle_{c,\mathbb{P}}} \leq t] = \bigcap_{s > t} \langle [h \leq s] \rangle_{c,\mathbb{P}}.$$

Proof Let $s > t$ and $a \notin \langle [h \leq s] \rangle_{c,\mathbb{P}}$. There exists $(\bar{v}, \bar{r}) \in V \times \mathbb{R}$ such that $a \notin E_{\bar{v},\bar{r}}^{\mathbb{P}}$ and $[h \leq s] \subset E_{\bar{v},\bar{r}}^{\mathbb{P}}$. We deduce that

$$h^{\langle\rangle_{c,\mathbb{P}}}(a) = \sup_{a \notin E_{v,r}^{\mathbb{P}}} \inf_{u \notin E_{v,r}^{\mathbb{P}}} h(u) \geq \inf_{u \notin E_{\bar{v},\bar{r}}^{\mathbb{P}}} h(u) \geq s > t.$$

Let $a \notin \left\langle [h^{\langle\rangle_{c,\mathbb{P}}} \leq s] \right\rangle_{c,\mathbb{P}}$. There exists $s \in \mathbb{R}$ such that $h^{\langle\rangle_{c,\mathbb{P}}}(a) > s > t$. By Theorem 4.6, there exists $(v,r) \in V \times \mathbb{R}$ such that

$$a \notin E_{v,r}^{\mathbb{P}} \quad \text{and} \quad \inf_{u \notin E_{v,r}^{\mathbb{P}}} h(u) > s,$$

thus $[h \leq s] \subset E_{v,r}^{\mathbb{P}}$, and finally $a \notin \langle [h \leq s] \rangle_{c,\mathbb{P}}$. $\square$

4.4 Other expressions of $(c, \mathbb{P})-$level regularizations

We now give another expression of the $(c, \mathbb{P})-$level regularization of an extended real-valued function h. These expressions give the value of the $(c, \mathbb{P})-$level regularization of h at a given point. Given $h : U \to \overline{\mathbb{R}}$ and $a \in U$, we define sets $\mathcal{I}_h(a)$ and $\mathcal{J}_h(a)$ by:

$$\mathcal{I}_h(a) := \{t \in \mathbb{R} \mid a \notin \langle [h \le t] \rangle_{c,\mathbb{P}}\} \quad \text{and} \quad \mathcal{J}_h(a) = \{t \in \mathbb{R} \mid a \in \langle [h \le t] \rangle_{c,\mathbb{P}}\}. \tag{4.8}$$

Sets $\mathcal{I}_h(a)$ and $\mathcal{J}_h(a)$ are two intervals of $\mathbb{R}$ such that $\mathcal{I}_h(a) \cap \mathcal{J}_h(a) = \emptyset$ and $\mathcal{I}_h(a) \cup \mathcal{J}_h(a) = \mathbb{R}$. Moreover, for any $(r, s) \in \mathcal{I}_h(a) \times \mathcal{J}_h(a)$, we have $r < s$. We deduce that $\sup \mathcal{I}_h(a) = \inf \mathcal{J}_h(a)$.

Proposition 10.

$$h^{\langle\rangle_{c,\mathbb{P}}}(a) = \sup \left\{ t \in \mathbb{R} : a \notin \langle [h \le t] \rangle_{c,\mathbb{P}} \right\} = \inf \left\{ t \in \mathbb{R} : a \in \langle [h \le t] \rangle_{c,\mathbb{P}} \right\}.$$

Proof Let $t \in \mathcal{I}_h(a)$. There exists $(v, r) \in V \times \mathbb{R}$ such that $a \notin E^{\mathbb{P}}_{v,r}$ and $[h \le t] \subset E^{\mathbb{P}}_{v,r}$. Therefore

$$\inf_{u \notin E^{\mathbb{P}}_{v,r}} h(u) \ge t \quad \text{and so} \quad \sup_{a \notin E^{\mathbb{P}}_{v,r}} \inf_{u \notin E^{\mathbb{P}}_{v,r}} h(u) \ge t.$$

By Theorem 4.6, one gets $h^{\langle\rangle_{c,\mathbb{P}}}(a) \ge t$. Hence $h^{\langle\rangle_{c,\mathbb{P}}}(a) \ge \sup \mathcal{I}_h(a)$. Conversely, let $t < h^{\langle\rangle_{c,\mathbb{P}}}(a)$, then $a \notin [h^{\langle\rangle_{c,\mathbb{P}}} \le t]$ and by Proposition 9, there exists $s > t$ such that $a \notin \langle [h \le s] \rangle_{c,\mathbb{P}}$. Consequently, $\sup \mathcal{I}_h(a) \ge s > t$. Hence $\sup \mathcal{I}_h(a) \ge h^{\langle\rangle_{c,\mathbb{P}}}(a)$. $\square$

5 Applications to an optimization problem: sub-level set duality

Let us consider the following minimization problem:

$$\min_x f(x), \qquad s.t \quad x \in X, \tag{$\mathscr{P}$}$$

where X is a nonempty set and $f : X \to \mathbb{R} \cup \{+\infty\}$ is an extended real-valued function.

5.1 Level set perturbational duality

We consider a perturbation function $F : X \times U \to \overline{\mathbb{R}}$ satisfying

$$\exists a \in U \ : \ F(., a) = f(.). \tag{5.1}$$

We associate to F a valued function $h : U \to \overline{\mathbb{R}}$ defined by

$$h(u) := \inf_{x \in X} F(x, u). \tag{5.2}$$

We denote by α the optimal value of $(\mathscr{P})$. It is obvious that $\alpha = h(a)$.

The perturbational dual of $(\mathscr{P})$ ([16]) is given by

$$\max_{(v,r)} -h^{\Delta}(v,r) \quad s.t \quad a \notin E^{\mathbb{P}}_{v,r}. \tag{$\mathscr{D}$}$$

We denote by β the optimal value of $(\mathscr{D})$. By definition, one has

$$-\infty \leq \beta := \sup(\mathscr{D}) = h^{\Delta\Delta^*}(a) \leq h(a) =: \alpha = \inf(\mathscr{P}) \leq +\infty. \tag{5.3}$$

Thus, the weak duality holds. The following theorem gives a necessary and sufficient condition to assure the strong duality.

Theorem 5.1. *The following statements are equivalent:*

(1) The strong duality holds for $(\mathscr{P})$ *i.e* $\inf(\mathscr{P}) = \max(\mathscr{D})$,

(2) $a \notin \langle [h < \alpha] \rangle_{c,\mathbb{P}}$.

Proof. Assume that (1) holds. There exists $(\bar{v}, \bar{r}) \in V \times \mathbb{R}$ such that

$$a \notin E^{\mathbb{P}}_{\bar{v},\bar{r}} \text{ and } \alpha = h(a) = -h^{\Delta}(\bar{v},\bar{r}) := \inf_{u \notin E^{\mathbb{P}}_{\bar{v},\bar{r}}} h(u).$$

We deduce that $[h < \alpha] \subset E^{\mathbb{P}}_{\bar{v},\bar{r}}$. Thus $a \notin \langle [h < \alpha] \rangle_{c,\mathbb{P}}$. Conversely, assume that (2) holds. There exists $(\bar{v}, \bar{r}) \in V \times \mathbb{R}$ such that $a \notin E^{\mathbb{P}}_{\bar{v},\bar{r}}$ and $[h < \alpha] \subset E^{\mathbb{P}}_{\bar{v},\bar{r}}$. Therefore

$$\inf_{u \notin E^{\mathbb{P}}_{\bar{v},\bar{r}}} h(u) \geq \alpha \geq \beta.$$

Remember that

$$\beta := \sup_{a \notin E^{\mathbb{P}}_{v,r}} -h^{\Delta}(v,r) = \sup_{a \notin E^{\mathbb{P}}_{v,r}} \inf_{u \notin E^{\mathbb{P}}_{v,r}} h(u).$$

Thus

$$\beta \geq \inf_{u \notin E^{\mathbb{P}}_{\bar{v},\bar{r}}} h(u) = -h^{\Delta}(\bar{v},\bar{r}) \geq \alpha \geq \beta.$$

Hence $\beta = -h^{\Delta}(\bar{v},\bar{r}) = \alpha$. □

Theorem 5.1 is interesting in evenly convex case which is used in economic theory.

5.2 Evenly quasi-convex duality ([1],[7],[11],[12],[17])

We assume X and U are topological vector spaces, $V = U^*$ the topological dual of U, $c = \langle , \rangle$ the standard coupling function between U and U^*.

Corollary 5.2. *Assume that function $F : X \times U \to \overline{\mathbb{R}}$ is quasi-convex and for each $x \in X$, $F(x,.) : U \to \overline{\mathbb{R}}$ is upper semi-continuous. One has:*

$$\inf(\mathscr{P}) = \max_{u^* \in U^*} \inf_{\substack{(x,u) \in X \times U \\ \langle u-a,u^* \rangle \geq 0}} F(x,u).$$

Proof Since F is quasi-convex and for each $x \in X$, $F(x,.)$ is upper semi-continuous then h is quasi-convex and upper semi-continuous. Consequently, $[h < \alpha]$ is open convex set and so it is evenly convex. As $a \notin [h < \alpha]$, it results from Theorem 5.1 that

$$\inf(\mathscr{P}) = \max(\mathscr{D}) = \max_{r-\langle a,u^* \rangle \leq 0} \inf_{r-\langle u,u^* \rangle \leq 0} h(u) = \max_{u^* \in U^*} \inf_{\langle u,u^* \rangle \geq \langle a,u^* \rangle} h(u),$$

where the last equality follows from the fact that for each $u^* \in U^*$, function $k_{u^*} : \mathbb{R} \to \overline{\mathbb{R}}$ defined by $k_{u^*}(r) = \inf_{r-\langle u,u^* \rangle \leq 0} h(u)$ is not decreasing. □

Corollary 5.3. *Assume that function $F : X \times U \to \overline{\mathbb{R}}$ is quasi-convex and for each $x \in X$, $F(x,.) : U \to \overline{\mathbb{R}}$ is upper semi-continuous. One has:*

$$\inf(\mathscr{P}) = \max_{u^* \in U^*} \inf_{\substack{(x,u) \in X \times U \\ \langle u-a,u^* \rangle = 0}} F(x,u).$$

Proof We know that under these assumptions on F, $[h < \alpha]$ is convex open set, therefore it is $(\langle,\rangle, \mathbb{R}^*)-$regular. Since $a \notin [h < \alpha]$, it results from Theorem 5.1 that

$$\begin{aligned}
\inf(\mathscr{P}) &= \max(\mathscr{D}) \\
&= \max_{\substack{(u^*,r) \in U^* \times \mathbb{R} \\ \langle a,u^* \rangle = r}} \inf_{\substack{u \in U \\ \langle u,u^* \rangle = r}} h(u) \\
&= \max_{u^* \in U^*} \max_{\substack{r \in \mathbb{R} \\ \langle a,u^* \rangle = r}} \inf_{\substack{u \in U \\ \langle u,u^* \rangle = r}} h(u) \\
&= \max_{u^* \in U^*} \inf_{\substack{u \in U \\ \langle u-a,u^* \rangle = 0}} h(u) \\
&= \max_{u^* \in U^*} \inf_{\substack{(x,u) \in X \times U \\ \langle u-a,u^* \rangle = 0}} F(x,u) \quad \text{by definition of } h.
\end{aligned}$$

□

6 Conclusion

In this work, we introduced a closure operator by means of coupling function and a subset of $\mathbb{R}$. This operator allowed us to define a hull of sets and level set regularization of extended real-valued functions. By decomposition of closure operator, we showed that a level set regularization of any extended real-valued function coincides with its bi-conjugacy relative to a couple of dual polarities. We derive an analytic expression of a level set regularization of extended real-valued function. Our results are applied to derive a strong duality for a minimization problem.

Acknowledgements The authors are grateful to the anonymous referees and the editor for their constructive comments which have contributed to the final presentation of the paper.

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