Nonlinear elliptic $p(u)-$ Laplacian problem with Fourier boundary condition

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ABSTRACT

We study a nonlinear elliptic $p(u)-$ Laplacian problem with Fourier boundary conditions and L^1- data. The existence and uniqueness results of entropy solutions are established.

RESUMEN

Estudiamos un problema $p(u)-$Laplaciano elíptico nolineal con condiciones de borde Fourier y datos L^1. Se establecen resultados de existencia y unicidad de soluciones de entropía.

Keywords and Phrases: variable exponent, $p(u)-$Laplacian, Young measure, Fourier boundary condition, entropy solution.

2010 AMS Mathematics Subject Classification: 35J60, 35D05, 76A05.

1   Introduction

In this paper, we consider the following nonlinear Fourier boundary value problem

$$\begin{cases} b(u) - \operatorname{div}a(x,u,\nabla u) = f & \text{in } \Omega \\ a(x,u,\nabla u).\eta + \lambda u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ ($N \geq 3$) is a bounded open domain with Lipschitz boundary $\partial\Omega$, η is the outer unit normal vector on $\partial\Omega$ and $\lambda > 0$.

The operator $\operatorname{div}a(x,u,\nabla u)$ is called $p(u)$-Laplacian. It is more complicated than $p(x)$-Laplacian in the term of nonlinearity. A prototype of this operator is $\operatorname{div}\big(|\nabla u|^{p(u)-2}.\nabla u\big)$. The variable exponent p depend both on the space variable x and on the unknown solution u. The problem (1.1) is a generalization of the following nonlinear problem

$$\begin{cases} b(u) - \operatorname{div}a(x,\nabla u) = f & \text{in } \Omega \\ a(x,\nabla u).\eta + \lambda u = g & \text{on } \partial\Omega, \end{cases}$$

studied in [15] by Nyanquini and Ouaro. The authors used an auxiliary result due to Le (see [16], Theorem 3.1) to prove the existence of the weak solution when $f \in L^\infty(\Omega)$, $g \in L^\infty(\partial\Omega)$ and by approximation methods they obtained the entropy solution when $f \in L^1(\Omega)$, $g \in L^1(\partial\Omega)$.

In the present paper, as the function $(x,z,\eta) \mapsto a(x,z,\eta)$ is more general than $(x,\eta) \mapsto a(x,\eta)$, it is difficult to use the sub-supersolution method, as in [16], to get the existence of the weak solution. Therefore, we use the technic of pseudo-monotone operators in Banach spaces, some a priori estimates and the convergence in term of Young measure to obtain the existence of entropy solutions of problem (1.1). Indeed, we define an approximation problem, and we prove that this problem has a solution u_n which converges to u, an entropy solution of problem (1.1).

In this paper, we consider the following basic assumptions on the data for the study of the problem (1.1).

(A_1) f and g are some functions such as $f \in L^1(\Omega)$, $g \in L^1(\partial\Omega)$ and $g \not\equiv 0$.

(A_2) b is nondecreasing surjective and continuous function defined on $\mathbb{R}$ such that $b(0) = 0$.
Problem (1.1) is adapted into a generalized Leray-Lions framework under the assumption that $a : \Omega \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}^N$ is a Carathéodory function with:

(A_3) $a(x,z,0) = 0$ for all $z \in \mathbb{R}$, and a.e. $x \in \Omega$;

(A_4) $\big(a(x,z,\xi) - a(x,z,\eta)\big).(\xi - \eta) > 0$ for all $\xi, \eta \in \mathbb{R}^N, \xi \neq \eta$, as well as the growth and the coercivity assumptions with variable exponent

(A_5) $\big|a(x,z,\xi)\big|^{p'(x,z)} \leq C_1\big(|\xi|^{p(x,z)} + \mathcal{M}(x)\big)$
and

(A_6) $a(x, z, \xi).\xi \geq \dfrac{1}{C_2} |\xi|^{p(x,z)}.$

Here, C_1 and C_2 are positive constants and $\mathcal{M}$ is a positive function such that $\mathcal{M} \in L^1(\Omega)$.

$p : \Omega \times \mathbb{R} \to [p_-, p_+]$ is a Carathéodory function, $1 < p_- \leq p_+ < +\infty$ and $p'(x,z) = \dfrac{p(x,z)}{p(x,z) - 1}$ is the conjugate exponent of $p(x,z)$, with

$$p_- := ess \inf_{(x,z) \in \Omega \times \mathbb{R}} p(x,z) \ \text{ and } \ p_+ := ess \sup_{(x,z) \in \Omega \times \mathbb{R}} p(x,z).$$

The study of $p(u)$-Laplacian problem was recently developped by Andreianov et al. (see [2]). These authors established the partial existence and uniqueness result to the weak solution in the cases of homogeneous Dirichlet boundary condition.

The interest of the study of this kind of problem is due to the fact that they can model various phenomena which arise in the study of elastic mechanic (see [6]), electrorheological fluids (see [20]) or image restoration (see [9]).

In this paper, we study the existence of the weak solution for approximation problem and we also establish the existence and uniqueness results of the entropy solution when the data are in L^1.

In this work, we use the Sobolev embedding results and the convergence in term of Young measure (see [10, 12]).

Here, we consider a Fourier boundary condition which bring some difficulties to treat the term at the boundary.

We were inspired by the work of Ouaro and Tchousso (see [15]), where the authors defined for the first time a new space by taking into account the boundary.

For the next part of the paper (section 2), we introduce some preliminary results. In section 3, we study the existence and uniqueness of entropy solution when the data are in L^1.

2   Preliminary

• We will use the so-called truncation function

$$T_k(s) := \begin{cases} s & \text{if } |s| \leq k \\ k sign_0(s) & \text{if } |s| > k \end{cases}, \quad \text{where } sign_0(s) := \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0. \end{cases}$$

The truncation function possesses the following properties.

$$T_k(-s) = -T_k(s), |T_k(s)| = \min\{|s|, k\},$$

$$\lim_{k \to +\infty} T_k(s) = s \ \text{ and } \ \lim_{k \to 0} \frac{1}{k} T_k(s) = sign_0(s).$$

We also need to truncate vector valued-function with the help of the mapping

$$h_m : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad h_m(\lambda) = \begin{cases} \lambda, & \text{if } |\lambda| \leq m \\ m\dfrac{\lambda}{|\lambda|} & \text{if } |\lambda| > m, \end{cases} \quad \text{where } m > 0.$$

For a Lebesgue measurable set $A \subset \Omega$, χ_A denotes its characteristic function and $meas(A)$ denotes its Lebesgue measure. Let $u : \Omega \to \mathbb{R}$ be a function and $k \in \mathbb{R}$, we write $\{|u| \leq k\}$ or $[|u| \leq k]$ for the set $\{x \in \Omega : |u(x)| \leq k\}$, (respectively, $\geq, =, <, >$).

The function $a(.,.,.)$ appearing in (1.1) satisfies a generalized Leray-Lions assumptions given in Introduction. View that, $a(.,.,.)$ satisfies (A_5) and (A_6), we must work in Lebesgue and Sobolev spaces with variable exponent, that depend on x and on $u(x)$. For the study of problem (1.1), we need the Sobolev spaces $W^{1,\pi(.)}(\Omega)$, where $\pi(.) = p(., u(.))$.

Definition 1. *Let $\pi : \Omega \longrightarrow [1, +\infty)$ be a measurable function for $\pi(.) = p(., u(.))$.*
•$L^{\pi(.)}(\Omega)$ is the space of all measurable function $f : \Omega \longrightarrow \mathbb{R}$ such that the modular

$$\rho_{\pi(.)}(f) := \int_\Omega |f|^{\pi(x)} dx < +\infty.$$

If p_+ is finite, this space is equipped with the Luxembourg norm

$$||f||_{L^{\pi(.)}(\Omega)} := \inf\left\{\lambda > 0; \quad \rho_{\pi(.)}\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

In the sequel, we will use the same notation $L^{\pi(.)}(\Omega)$ for the space $(L^{\pi(.)}(\Omega))^N$ of vector-valued functions.
•$W^{1,\pi(.)}(\Omega)$ is the space of all functions $f \in L^{\pi(.)}(\Omega)$ such that the gradient of f (taken in the sense of distributions) belongs to $L^{\pi(.)}(\Omega)$. The space $W^{1,\pi(.)}(\Omega)$ is equipped with the norm

$$||u||_{W^{1,\pi(.)}(\Omega)} := ||u||_{L^{\pi(.)}(\Omega)} + ||\nabla u||_{L^{\pi(.)}(\Omega)}.$$

When $1 < p_- \leq \pi(.) \leq p_+ < +\infty$, all the above spaces are separable and reflexive Banach spaces.

We denote $\pi_n(x) := p(x, u_n(x))$.

Proposition 1. *(See [1], Proposition 2.3)*
For all measurable function $\pi : \Omega \to [p_-, p_+]$, the following properties hold.

  i) $L^{\pi(.)}(\Omega)$ and $W^{1,\pi(.)}(\Omega)$ are separable and reflexive Banach spaces.

  ii) $L^{\pi'(.)}(\Omega)$ can be identified with the dual space of $L^{\pi(.)}(\Omega)$, and the following Hölder type inequality holds:

$$\forall f \in L^{\pi(.)}(\Omega), g \in L^{\pi'(.)}(\Omega), \quad \left|\int_\Omega fg\, dx\right| \leq 2||f||_{L^{\pi(.)}(\Omega)}||g||_{L^{\pi'(.)}(\Omega)}.$$

iii) One has $\rho_{\pi(.)}(f) = 1$ if and only if $||f||_{L^{\pi(\cdot)}(\Omega)} = 1$; further,

$$\text{if } \rho_{\pi(.)}(f) \leq 1, \text{ then } ||f||^{p_+}_{L^{\pi(\cdot)}(\Omega)} \leq \rho_{\pi(.)}(f) \leq ||f||^{p_-}_{L^{\pi(\cdot)}(\Omega)};$$

$$\text{if } \rho_{\pi(.)}(f) \geq 1, \text{ then } ||f||^{p_-}_{L^{\pi(\cdot)}(\Omega)} \leq \rho_{\pi(.)}(f) \leq ||f||^{p_+}_{L^{\pi(\cdot)}(\Omega)}.$$

In particular, if $(f_n)_{n \in \mathbb{N}}$ is a sequence in $L^{\pi(\cdot)}(\Omega)$, then $||f_n||_{L^{\pi(\cdot)}(\Omega)}$ tends to zero (resp., to infinity) if and only if $\rho_{\pi(.)}(f_n)$ tends to zero (resp., to infinity), as $n \to +\infty$.

For a measurable function $f \in W^{1,\pi(.)}(\Omega)$ we introduce the following notation:

$$\rho_{1,\pi(.)}(f) = \int_{\Omega} |f|^{\pi(\cdot)} dx + \int_{\Omega} |\nabla f|^{\pi(\cdot)} dx.$$

Replacing $p(x)$ by $\pi(x)$ in [8]-Proposition 2.2, we obtain the following result that is fundamental in this paper.

Proposition 2. *(See [23, 24]) If $f \in W^{1,\pi(.)}(\Omega)$, the following properties hold:*

i) $||f||_{W^{1,\pi(.)}(\Omega)} > 1 \Rightarrow ||f||^{p_-}_{W^{1,\pi(.)}(\Omega)} < \rho_{1,\pi(.)}(f) < ||f||^{p_+}_{W^{1,\pi(.)}(\Omega)};$

ii) $||f||_{W^{1,\pi(.)}(\Omega)} < 1 \Rightarrow ||f||^{p_+}_{W^{1,\pi(.)}(\Omega)} < \rho_{1,\pi(.)}(f) < ||f||^{p_-}_{W^{1,\pi(.)}(\Omega)};$

iii) $||f||_{W^{1,\pi(.)}(\Omega)} < 1$ (respectively $= 1; > 1$) $\Leftrightarrow \rho_{1,\pi(.)}(f) < 1$ (respectively $= 1; > 1$).

The following lemma prove that the space $W^{1,\pi(.)}(\Omega)$ is stable by truncation.

Lemma 2.1. *If $u \in W^{1,\pi(.)}(\Omega)$ then $T_k(u) \in W^{1,\pi(.)}(\Omega)$.*

Now, we give the embedding results.

Proposition 3. *(See [1], Proposition 2.4) Assume that $\pi : \Omega \to [p_-, p_+]$ has a representative which can be extended into a continuous function up to the boundary $\partial\Omega$ and satisfying the log-Hölder continuity assumption:*

$$\exists L > 0, \quad \forall x, y \in \overline{\Omega}, x \neq y, \quad -\big(\log |x-y|\big)|\pi(x) - \pi(y)| \leq L. \tag{2.1}$$

i) Then, $\mathcal{D}(\Omega)$ is dense in $W^{1,\pi(.)}(\Omega)$.

ii) $W^{1,\pi(.)}(\Omega)$ is embedded into $L^{\pi^*(\cdot)}(\Omega)$, where $\pi^*(.)$ is the Sobolev embedding exponent defined as in (2.2) below. If q is a measurable variable exponent such that $\underset{x \in \Omega}{\text{ess inf}} (\pi^*(.) - q(.)) > 0$, then the embedding of $W^{1,\pi(.)}(\Omega)$ into $L^{q(.)}(\Omega)$ is compact.

For a given $\pi(.)$, a function taking values in $[p_-, p_+]$, $\pi^*(.)$ denotes the optimal Sobolev embedding defined for any $x \in \Omega$ by

$$\pi^*(x) = \begin{cases} \dfrac{N\pi(x)}{N - \pi(x)} & if \quad \pi(x) < N \\ any\ real\ value & if \quad \pi(x) = N \\ +\infty & if \quad \pi(x) > N. \end{cases} \tag{2.2}$$

Put

$$\pi^\partial(x) := \left(\pi(x)\right)^\partial := \begin{cases} \dfrac{(N-1)\pi(x)}{N - \pi(x)} & if \quad \pi(x) < N \\ +\infty & if \quad \pi(x) \geq N. \end{cases} \tag{2.3}$$

Proposition 4. *(See [18], Proposition 2.3)*
Let $\pi(.) \in C(\overline{\Omega})$ and $p_- > 1$. If $q(x) \in C(\partial\Omega)$ satisfies the condition:

$$1 \leq q(x) < \pi^\partial(x), \quad \forall x \in \partial\Omega,$$

then, there is a compact embedding

$$W^{1,\pi(.)}(\Omega) \hookrightarrow L^{q(.)}(\partial\Omega).$$

In particular there is compact embedding

$$W^{1,\pi(.)}(\Omega) \hookrightarrow L^{\pi(.)}(\partial\Omega).$$

T**Young measures and nonlinear weak-* convergence.**
Throughout the paper, we denote by δ_c the Dirac measure on $\mathbb{R}^d$ ($d \in \mathbb{N}$), concentrated at the point $c \in \mathbb{R}^d$.
In the following theorem, we gather the results of Ball [7], Pedregal [19] and Hungerbühler [13] which will be needed for our purposes (we limit the statement to the case of a bounded domain Ω). Let us underline that the results of (ii),(iii), expressed in terms of the convergence in measure, are very convenient for the applications that we have in mind.

Theorem 2.1.   *(i) Let $\Omega \subset \mathbb{R}^N, N \in \mathbb{N}$, and a sequence $(v_n)_{n\in\mathbb{N}}$ of $\mathbb{R}^d$ -valued functions, $d \in \mathbb{N}$, such that $(v_n)_{n\in\mathbb{N}}$ is equi-integrable on Ω. Then, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and a parametrized family $(\nu_x)_{x\in\Omega}$ of probability measures on $\mathbb{R}^d$ ($d \in \mathbb{N}$) , weakly measurable in x with respect to the Lebesgue measure in Ω, such that for all Carathéodory function $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^t$, $t \in \mathbb{N}$, we have*

$$\lim_{k\to+\infty} \int_\Omega F(x, v_{n_k})dx = \int_\Omega \int_{\mathbb{R}^d} F(x, \lambda)d\nu_x(\lambda)dx, \tag{2.4}$$

whenever the sequence $(F(., v_n(.)))_{n\in\mathbb{N}}$ *is equi-integrable on* Ω*. In particular,*

$$v(x) := \int_{\mathbb{R}^d} \lambda d\nu_x(\lambda) \tag{2.5}$$

is the weak limit of the sequence $(v_{n_k})_{k\in\mathbb{N}}$ *in* $L^1(\Omega)$*.*

The family $(\nu_x)_{x\in\Omega}$ *is called the Young measure generated by the subsequence* $(v_{n_k})_{k\in\mathbb{N}}$*.*

(ii) *If* Ω *is of finite measure, and* $(\nu_x)_{x\in\Omega}$ *is the Young measure generated by a sequence* $(v_n)_{n\in\mathbb{N}}$*, then* $\nu_x = \delta_{v(x)}$ *for a.e.* $x \in \Omega \Leftrightarrow v_n$ *converges in measure in* Ω *to* v *as* $n \to +\infty$*.*

(iii) *If* Ω *is of finite measure,* $(u_n)_{n\in\mathbb{N}}$ *generates a Dirac Young measure* $(\delta_{u(x)})_{x\in\Omega}$ *on* $\mathbb{R}^{d_1}$*, and* $(v_n)_{n\in\mathbb{N}}$ *generates a Young measure* $(\nu_x)_{x\in\Omega}$ *on* $\mathbb{R}^{d_2}$*, then the sequence* $(u_n, v_n)_{n\in\mathbb{N}}$ *generates the Young measure* $(\delta_{u(x)} \otimes \nu_x)_{x\in\Omega}$ *on* $\mathbb{R}^{d_1+d_2}$*. Whenever a sequence* $(v_n)_{n\in\mathbb{N}}$ *generates a Young measure* $(\nu_x)_{x\in\Omega}$*, following the terminology of [11] we will say that* $(v_n)_{n\in\mathbb{N}}$ *nonlinear weak-* converges, and* $(\nu_x)_{x\in\Omega}$ *is the nonlinear weak-* limit of the sequence* $(v_n)_{n\in\mathbb{N}}$*. In the case where* $(v_n)_{n\in\mathbb{N}}$ *possesses a nonlinear weak-* convergent subsequence, we will say that it is nonlinear weak-* compact. ([1], Theorem 2.10(i)) It means that any equi-integrable sequence of measurable functions is nonlinear weak-* compact on* Ω*.*

Lemma 2.2. *(See [1], Theorem 3.11 and [2] Step 2 of proof of Theorem 2.6). Assume that* $(u_n)_{n\in\mathbb{N}}$ *converges a.e. on* Ω *to some function* u*, then*

$$|p(x, u_n(x)) - p(x, u(x))| \quad \text{converges in measure to } 0 \text{ on } \Omega,$$
$$\text{and for all bounded subset } K \text{ of } \mathbb{R}^N,$$
$$\sup_{\xi\in K} |a(x, u_n(x), \xi) - a(x, u(x), \xi)| \quad \text{converges in measure to } 0 \text{ on } \Omega. \tag{2.6}$$

For the sequel, we assume that $p(.,.)$ is log Hölder continuous uniformly on $\overline{\Omega} \times [-M, M]$ and $p_- > N$. We recall some notations.

For any $u \in W^{1,\pi(.)}(\Omega)$, we denote by $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense. We will identify at boundary u and $\tau(u)$.

Set

$$\mathcal{T}^{1,\pi(.)}(\Omega) = \big\{ u : \Omega \to \mathbb{R}, \ \text{measurable such that} \ T_k(u) \in W^{1,\pi(.)}(\Omega), \ \text{for any} \ k > 0 \big\}.$$

3 Entropy solution

In this part, we study the existence and uniqueness of the entropy solution to the problem (1.1). We give some notations.

We define $\mathcal{T}_{tr}^{1,\pi(.)}(\Omega)$ as the set of the functions $u \in \mathcal{T}^{1,\pi(.)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,p_+}(\Omega)$ satisfying the following conditions:

(C_1) $u_n \to u$ a.e. in Ω.

(C_2) $\nabla T_k(u_n) \to \nabla T_k(u)$ in $L^1(\Omega)$.

(C_3) There exists a measurable function v on $\partial\Omega$, such that $u_n \to v$ a.e. on $\partial\Omega$.

The function v is the trace of u in the generalized sense as introduced in [4, 5]. In the sequel the trace of $u \in \mathcal{T}_{tr}^{1,\pi(.)}(\Omega)$ on $\partial\Omega$ will be denoted $tr(u)$. If $u \in W^{1,\pi(.)}(\Omega)$, $tr(u)$ coincides with $\tau(u)$ in the usual sense. Moreover, for $u \in \mathcal{T}_{tr}^{1,\pi(.)}(\Omega)$ and for all $k > 0$, $tr(T_k(u)) = T_k(tr(u))$ and if $\varphi \in W^{1,\pi(.)}(\Omega)$ then $u - \varphi \in \mathcal{T}_{tr}^{1,\pi(.)}(\Omega)$ and $tr(u - \varphi) = tr(u) - tr(\varphi)$.

As in [1]-Proposition 3.5, we give the following result.

Proposition 5. *Let $u \in \mathcal{T}^{1,\pi(.)}(\Omega)$. There exists a unique measurable function $w : \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = w\chi_{\{|u|<k\}}$ for $k > 0$. The function w is denoted by ∇u. Moreover, if $u \in W^{1,\pi(.)}(\Omega)$ then $w \in L^{\pi(.)}(\Omega)$ and $w = \nabla u$ in the usual sense.*

Remark 3.1. *The space $\mathcal{T}_{tr}^{1,\pi(.)}(\Omega)$ in our context will be a subset of $\mathcal{T}^{1,\pi(.)}(\Omega)$ consisting to the function can be approximated by function of $W^{1,p_+}(\Omega)$. Since the weak solution of approximated problem (3.2) belongs to $W^{1,p_+}(\Omega)$.*

Now, we introduce the notion of entropy solution due to Ouaro and al. [14, Definition 3.1].

Definition 2. *A measurable function $u : \Omega \to \mathbb{R}$ for $\pi(.) = p(., u(.))$ is called entropy solution of the problem (1.1) if*

$$u \in \mathcal{T}_{tr}^{1,\pi(.)}(\Omega), \quad b(u) \in L^1(\Omega), \quad u \in L^1(\partial\Omega)$$

and for all $k > 0$,

$$\int_\Omega b(u)T_k(u-\varphi)dx + \int_\Omega a(x,u,\nabla u).\nabla T_k(u-\varphi)dx \quad + \quad \lambda \int_{\partial\Omega} uT_k(u-\varphi)d\sigma$$
$$\leq \quad \int_\Omega fT_k(u-\varphi)dx + \int_{\partial\Omega} gT_k(u-\varphi)d\sigma,$$

(3.1)

where $\varphi \in W^{1,\pi(.)}(\Omega) \cap L^\infty(\Omega)$.

The following theorem gives existence result of entropy solution.

Theorem 3.2. *Assume that $(A_3) - (A_6)$ hold and $f \in L^1(\Omega)$, $g \in L^1(\partial\Omega)$. Then, there exists at least one entropy solution to the problem (1.1).*

The proof of Theorem 3.2 is done in two parts.

Part 1**: The approximate problem.**

Let $f_n = T_n(f)$ and $g_n = T_n(g)$. Then, $f_n \in L^\infty(\Omega)$ and $g_n \in L^\infty(\partial\Omega)$. Moreover, $(f_n)_{n\in\mathbb{N}}$ strongly converges to f in $L^1(\Omega)$ and $(g_n)_{n\in\mathbb{N}}$ strongly converges to g in $L^1(\partial\Omega)$ such that $||f_n||_{L^1(\Omega)} \leq ||f||_{L^1(\Omega)}$ and $||g_n||_{L^1(\partial\Omega)} \leq ||g||_{L^1(\partial\Omega)}$.

We consider the following problem

$$\begin{cases} T_n(b(u_n)) - \mathrm{div}\, a(x, u_n, \nabla u_n) - \varepsilon \triangle_{p_+} u_n + \varepsilon |u_n|^{p_+ - 2} u_n = f_n & \text{in} \quad \Omega \\ \\ \left(a(x, u_n, \nabla u_n) + \varepsilon |\nabla u_n|^{p_+ - 2} \nabla u_n\right).\eta + \lambda T_n(u_n) = g_n & \text{on} \quad \partial\Omega, \end{cases} \quad (3.2)$$

where

$$-\triangle_{p_+} u_n := -\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_+ - 2} \frac{\partial u_n}{\partial x_i}\right).$$

In this part, we show that the problem (3.2) admits at least one weak solution u_n, for all $\varepsilon > 0$. We define the following reflexive space

$$E = W^{1,p_+}(\Omega) \times L^{p_+}(\partial\Omega).$$

Let

$$X_0 = \{(u, v) \in E : \quad v = \tau(u)\}.$$

In the sequel, we will identify an element $(u, v) \in X_0$ with its representative $u \in W^{1,p_+}(\Omega)$ (since $W^{1,p_+}(\Omega) \hookrightarrow\hookrightarrow L^{p_+}(\partial\Omega)$).

Theorem 3.3. *There exists at least one weak solution u_n for the problem* (3.2) *in the sense that $u_n \in X_0$ and for all $v \in X_0$,*

$$\int_\Omega T_n(b(u_n))v dx + \int_\Omega a(x, u_n, \nabla u_n)\nabla v dx \quad + \quad \int_{\partial\Omega} \lambda T_n(u_n)v d\sigma$$

$$+ \quad \varepsilon \int_\Omega \left[|u_n|^{p_+ - 2}u_n v + |\nabla u_n|^{p_+ - 2}\nabla u_n \nabla v\right] dx$$

$$= \quad \int_\Omega f_n v dx + \int_{\partial\Omega} g_n v d\sigma. \quad (3.3)$$

To prove the Theorem 3.3, we need the following result.

Lemma 3.1. *(See [22], Corollary 2.2). If an operator $\mathcal{A}$ is of type (M), bounded and coercive on a separable Banach space to its dual, then $\mathcal{A}$ is surjective.*

We define the operator A_n by

$$A_n u = Au + B_n u,$$

where

$$< Au, v >= \int_\Omega a(x, u, \nabla u) \nabla v dx$$

and

$$< B_n u, v >= \int_\Omega T_n(b(u)) v dx + \lambda \int_{\partial\Omega} T_n(u) v d\sigma + \varepsilon \int_\Omega \left[|u|^{p_+ - 2} uv + |\nabla u|^{p_+ - 2} \nabla u \nabla v \right] dx,$$

with $u, v \in X_0$.

Proof of the Theorem 3.3. The proof is organized in three Steps.

Step 1: A_n is bounded.

By using Hölder type inequality and (A_5) with constant exponent p_+, we deduce that A is bounded. Moreover, B_n is bounded. Indeed, let $u \in F$, where F is a bounded subset of X_0.

As b is onto, we have

$$
\begin{aligned}
< B_n u, v > &= \int_\Omega T_n(b(u)) v dx + \lambda \int_{\partial\Omega} T_n(u) v d\sigma + \varepsilon \int_\Omega \left[|u|^{p_+ - 2} uv + |\nabla u|^{p_+ - 2} \nabla u \nabla v \right] dx \\
&\leq \int_\Omega |b(u)||v| dx + \lambda \int_{\partial\Omega} |u||v| d\sigma + \varepsilon \int_\Omega \left[|u|^{p_+ - 1}|v| + |\nabla u|^{p_+ - 1}|\nabla v| \right] dx \\
&\leq C(\lambda) \left(||v||_{L^1(\Omega)} + ||v||_{L^1(\partial\Omega)} \right) + \varepsilon \left[||u||_{L^{p_+}(\Omega)}^{\frac{p_+}{(p_+)'}} ||v||_{L^{p_+}(\Omega)} + ||\nabla u||_{L^{p_+}(\Omega)}^{\frac{p_+}{(p_+)'}} ||\nabla v||_{L^{p_+}(\Omega)} \right] \\
&\leq C(\lambda) \left(||v||_{L^1(\Omega)} + ||v||_{L^1(\partial\Omega)} \right) + C(\varepsilon) ||v||_{W^{1, p_+}(\Omega)}.
\end{aligned}
$$

Therefore, A_n is bounded.

We recall the following notion:

Definition 3. *An operator $A : V \to V'$ is type of (M) if:*

$$
\left.
\begin{aligned}
&u_n \rightharpoonup u \;\; in \;\; V \\
&A(u_n) \rightharpoonup \chi \;\; in \;\; V' \\
&\limsup_{n\to\infty} < A(u_n), u_n > \leq < \chi, u >
\end{aligned}
\right\} \Rightarrow \chi = A(u).
$$

Step 2: A_n is pseudo-monotone.

Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in X_0 such that

$$
\begin{cases}
u_k \rightharpoonup u \;\; in \;\; X_0 \\
A_n u_k \rightharpoonup \chi \;\; in \;\; X_0' \\
\limsup_{k\to\infty} < A_n u_k, u_k >=< \chi, u > .
\end{cases}
$$

We will prove that $\chi = A_n u$.

As $T_n(b(u_k)) u_k \geq 0$ and $\lambda T_n(u_k) u_k \geq 0$, by Fatou's Lemma, we deduce that

$$\liminf_{k\to\infty} \left(\int_\Omega T_n(b(u_k)) u_k dx + \lambda \int_{\partial\Omega} T_n(u_k) u_k d\sigma \right) \geq \int_\Omega T_n(b(u)) u dx + \lambda \int_{\partial\Omega} T_n(u) u d\sigma.$$

One the other hand, thanks to the Lebesgue dominated convergence Theorem, we have

$$\lim_{k \to \infty} \left(\int_\Omega T_n(b(u_k))vdx + \lambda \int_{\partial\Omega} T_n(u_k)vd\sigma + \varepsilon \int_\Omega \left[|u_k|^{p_+-2}u_k v + |\nabla u_k|^{p_+-2}\nabla u_k \nabla v \right]dx \right)$$
$$= \int_\Omega T_n(b(u))vdx + \lambda \int_{\partial\Omega} T_n(u)vd\sigma + \varepsilon \int_\Omega \left[|u|^{p_+-2}uv + |\nabla u|^{p_+-2}\nabla u\nabla v \right]dx,$$

for any $v \in X_0$. Therefore, for k large enough,

$$T_n(b(u_k))+\lambda T_n(u_k)+\varepsilon\left[|u_k|^{p_+-2}u_k+|\nabla u_k|^{p_+-2}\nabla u_k\right] \rightharpoonup T_n(b(u))+\lambda T_n(u)+\varepsilon\left[|u|^{p_+-2}u+|\nabla u|^{p_+-2}\nabla u\right] \text{ in } X_0'.$$

Thus,

$$Au_k \rightharpoonup \chi - \left(T_n(b(u)) + \lambda T_n(u) + \varepsilon[|u|^{p_+-2}u + |\nabla u|^{p_+-2}\nabla u] \right) \text{ in } X_0', \text{ as } k \to +\infty.$$

Now, we are going to prove that A is of type (M).

Let us set

$$a_1(u,v,w) = \int_\Omega a(x,u,\nabla v)\nabla w dx.$$

Then, $w \mapsto a_1(u,v,w)$ is continuous on $W^{1,p_+}(\Omega)$, thus

$$a_1(u,v,w) = \langle A(u,v), w \rangle, \quad A(u,v) \in (W^{1,p_+}(\Omega))',$$

and verify

$$A(u,u) = Au, \text{ where } Au := -\mathrm{div}a(x,u,\nabla u).$$

Let us prove that A is of type of Calculus of variation.

• As $A(u,.)$ is bounded, we prove that $v \mapsto A(u,v)$ is hemi-continuous from $W^{1,p_+}(\Omega) \to (W^{1,p_+}(\Omega))'$. Since $a(x,u,\nabla(v_1 + tv_2)) \rightharpoonup a(x,u,\nabla v_1)$ in $L^{p'_+}(\Omega)$ as $t \to 0$ and $u, v_1, v_2 \in W^{1,p_+}(\Omega)$ then, $a_1(u, v_1 + tv_2, w) \to a_1(u, v_1, w)$ as $t \to 0$.
In the same manner we prove that $u \mapsto A(u,v)$ is hemi-continuous from $W^{1,p_+}(\Omega) \to (W^{1,p_+}(\Omega))'$.

Moreover, for all $u, v \in W^{1,p_+}(\Omega)$, we have

$$
\begin{aligned}
< A(u,u) - A(u,v), u - v > &= < A(u,u), u - v > - < A(u,v), u - v > \\
&= a_1(u,u,u-v) - a_1(u,v,u-v) \\
&= \int_\Omega a(x,u,\nabla u)\nabla(u-v)dx - \int_\Omega a(x,u,\nabla v)\nabla(u-v)dx \\
&= \int_\Omega \big(a(x,u,\nabla u) - a(x,u,\nabla v)\big)\nabla(u-v)dx \geq 0.
\end{aligned}
$$

• Let us suppose that $u_k \rightharpoonup u$ in $W^{1,p_+}(\Omega)$ and $< A(u_k, u_k) - A(u_k, u), u_k - u > \to 0$. We prove that

$$\forall v \in W^{1,p_+}(\Omega), \quad A(u_k, v) \rightharpoonup A(u, v) \text{ in } (W^{1,p_+}(\Omega))'.$$

Let's set

$$\int_\Omega F_k dx = \langle A(u_k, u_k) - A(u_k, u), u_k - u \rangle, \quad \text{then} \quad F_k \to 0.$$

As $u_k \rightharpoonup u$, we have

$$a(x, u_k, \nabla v) \rightharpoonup a(x, u, \nabla v) \text{ in } L^{p'_+}(\Omega)$$

(see [17], Lemma 2.2 with $m = 1$). Therefore, $A(u_k, v) \rightharpoonup A(u, v)$ in $(W^{1,p_+}(\Omega))'$.

• Now, we suppose that $u_k \rightharpoonup u$ in $W^{1,p_+}(\Omega)$ and $A(u_k, v) \rightharpoonup \Theta$ in $(W^{1,p_+}(\Omega))'$. We prove that

$$\langle A(u_k, v), u_k \rangle \to \langle \Theta, u \rangle.$$

Then, by using ([17], Lemma 2.1), we obtain that $a(x, u_k, \nabla v) \to a(x, u, \nabla v)$ in $L^{p'_+}(\Omega)$ and thus, $a_1(u_k, v, u_k) \to a_1(u, v, u)$.

Therefore,

$$< A(u_k, v), u_k >= a_1(u_k, v, u_k) \to < A(u, v), u > \quad \text{and} \quad \Theta = A(u, v).$$

Hence, A is of type of Calculus of variation. Finally, by using ([17], Proposition 2.6 and Proposition 2.5), we prove that A is of type (M).

As the operator A is of type (M), so we have immediately

$$Au = \chi - \left(T_n(b(u)) + \lambda T_n(u) + \varepsilon \left[|u|^{p_+ - 2}u + |\nabla u|^{p_+ - 2}\nabla u \right] \right).$$

Therefore, we deduce that $A_n u = \chi$.

Step 3: A_n is coercive.

$$
\begin{aligned}
< A_n u, u > \quad &= \quad \int_\Omega a(x, u, \nabla u).\nabla u \, dx + \int_\Omega T_n(b(u)) u \, dx \\
&+ \quad \lambda \int_{\partial\Omega} T_n(u) u \, dx + \varepsilon \int_\Omega \left[|u|^{p_+} + |\nabla u|^{p_+} \right] dx \\
&\geq \quad \varepsilon \int_\Omega \left[|u|^{p_+} + |\nabla u|^{p_+} \right] dx \\
&\geq \quad \varepsilon \|u\|_{W^{1,p_+}(\Omega)}^{p_+}.
\end{aligned}
$$

We deduce that

$$\frac{< A_n u, u >}{\|u\|_{W^{1,p_+}(\Omega)}} \to +\infty \quad \text{as} \quad \|u\|_{W^{1,p_+}(\Omega)} \to +\infty.$$

Hence, A_n is coercive.

Then, according to Lemma 3.1, A_n is surjective.

Thus, for any $F_n = < T_n(f), T_n(g) > \subset E' \subset X_0'$, there exists at least one solution $u_n \in X_0$ of the problem

$$< A_n u_n, v > = < F_n, v > \quad \text{for all } v \in X_0.$$

Therefore, u_n is a weak solution of the problem (3.2). This ends the proof of Theorem 3.3.

Remark 3.4. *If u_n is a weak solution of the problem* (3.2), *then $u_n \in W^{1,\pi_n(\cdot)}(\Omega)$, since $W^{1,p_+}(\Omega) \hookrightarrow W^{1,\pi_n(\cdot)}(\Omega)$ continuously. Moreover, $a(x, u_n, \nabla u_n)$ satisfies $(A_3) - (A_6)$ with variable exponent $\pi_n(x) := p(x, u_n(x))$.*

Part 2: A priori estimates and convergence results.

This part is done in three steps, we make a priori estimates, some convergence results and other based on the Young measure and nonlinear weak$-^*$ convergence.

Step 1: A priori estimates

Lemma 3.2. *Suppose that $(A_3) - (A_6)$ hold with variable exponent $\pi_n(.)$ and $f_n \in L^\infty(\Omega)$, $g_n \in L^\infty(\partial\Omega)$. Let u_n be a weak solution of* (3.2). *Then, for all $k > 0$,*

$$\int_\Omega \left|\nabla T_k(u_n)\right|^{\pi_n(\cdot)} dx \leq C_2 k\left(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\right), \tag{3.4}$$

$$\int_\Omega \left|T_n(b(u_n))\right| dx \leq ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}, \tag{3.5}$$

$$\int_{\partial\Omega} \left|T_n(u_n)\right| dx \leq \frac{1}{\lambda}\left(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\right). \tag{3.6}$$

Proof of Lemma 3.2. By taking $v = T_k(u_n)$ in the weak formulation (3.3), we obtain

$$\begin{aligned}
\int_\Omega T_n(b(u_n)) T_k(u_n) dx \quad &+ \quad \int_\Omega a(x, u_n, \nabla u_n).\nabla T_k(u_n) dx + \int_{\partial\Omega} \lambda T_n(u_n) T_k(u_n) d\sigma \\
&+ \quad \varepsilon \int_\Omega \left[|u_n|^{p_+-2} u_n T_k(u_n) + |\nabla u_n|^{p_+-2}\nabla u_n \nabla T_k(u_n)\right] dx \\
&= \quad \int_\Omega f_n T_k(u_n) dx + \int_{\partial\Omega} g_n T_k(u_n) d\sigma.
\end{aligned} \tag{3.7}$$

Since all the terms of the left hand side of (3.7) are nonnegative, we deduce that

$$\int_\Omega a(x, u_n, \nabla u_n).\nabla T_k(u_n) dx \leq \int_\Omega f_n T_k(u_n) dx + \int_{\partial\Omega} g_n T_k(u_n) d\sigma. \tag{3.8}$$

By using (A_6) and (3.8), we get

$$
\begin{aligned}
\int_\Omega \big|\nabla T_k(u_n)\big|^{\pi_n(.)}dx & \le C_2\bigg(\int_\Omega f_n T_k(u_n)dx + \int_{\partial\Omega} g_n T_k(u_n)d\sigma \bigg) \\
& \le C_2 k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big).
\end{aligned}
$$

From (3.7), we deduce that

$$
\begin{aligned}
\int_\Omega T_n(b(u_n))T_k(u_n)dx & \le \int_\Omega f_n T_k(u_n)dx + \int_{\partial\Omega} g_n T_k(u_n)d\sigma \\
& \le k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big)
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
\lambda \int_{\partial\Omega} T_n(u_n)T_k(u_n)dx & \le \int_\Omega f_n T_k(u_n)dx + \int_{\partial\Omega} g_n T_k(u_n)d\sigma \\
& \le k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big).
\end{aligned}
\tag{3.10}
$$

Dividing (3.9) and (3.10) by k and letting k goes to 0, we obtain

$$
\int_\Omega T_n(b(u_n))sign_0(u_n)dx \le ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}
$$

and

$$
\lambda \int_{\partial\Omega} T_n(u_n)sign_0(u_n)dx \le ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}.
$$

Hence,

$$
\int_\Omega \big|T_n(b(u_n))\big|dx \le ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}
$$

and

$$
\int_{\partial\Omega} \big|T_n(u_n)\big|dx \le \frac{1}{\lambda}\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big).
$$

Lemma 3.3. *Assume that (A_3)-(A_6) hold. If u_n is a weak solution of the problem (3.2), $f_n \in L^\infty(\Omega)$ and $g_n \in L^\infty(\partial\Omega)$, then for all $k > 0$*

$$
\int_\Omega \big|\nabla T_k(u_n)\big|^{p^-}dx \le C\big(||f||_{L^1(\Omega)}, ||g||_{L^1(\partial\Omega)}, meas(\Omega)\big)(k+1)
\tag{3.11}
$$

and

$$
\int_{\partial\Omega} \big|T_k(u_n)\big|d\sigma \le \frac{1}{\lambda}\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big),
\tag{3.12}
$$

for all $n \ge k > 0$.

Proof of Lemma 3.3. Firstly, we prove (3.11). We know that

$$\int_{\Omega}\big|\nabla T_k(u_n)\big|^{\pi_n(.)}dx \leq C_2 k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big). \tag{3.13}$$

Let us note that

$$
\begin{aligned}
\int_{\Omega}\big|\nabla T_k(u_n)\big|^{p^-}dx &= \int_{\{|\nabla T_k(u_n)|>1\}}\big|\nabla T_k(u_n)\big|^{p^-}dx + \int_{\{|\nabla T_k(u_n)|\leq 1\}}\big|\nabla T_k(u_n)\big|^{p^-}dx \\
&\leq \int_{\{|\nabla T_k(u_n)|>1\}}\big|\nabla T_k(u_n)\big|^{p^-}dx + meas(\Omega) \\
&\leq \int_{\Omega}\big|\nabla T_k(u_n)\big|^{\pi_n(.)}dx + meas(\Omega).
\end{aligned}
\tag{3.14}
$$

By using (3.13) and (3.14), we get

$$
\begin{aligned}
\int_{\Omega}\big|\nabla T_k(u_n)\big|^{p^-}dx &\leq \max\big(C_2\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big), meas(\Omega)\big)(k+1) \\
&:= C\big(||f||_{L^1(\Omega)}, ||g||_{L^1(\partial\Omega)}, meas(\Omega)\big)(k+1).
\end{aligned}
\tag{3.15}
$$

Now, from the formula (3.6), we obtain $||T_n(u_n)||_{L^1(\partial\Omega)} \leq \dfrac{1}{\lambda}\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big)$ and as $|T_k(u_n)| \leq |T_n(u_n)|$ for all $n \geq k > 0$, one deduces that

$$\int_{\partial\Omega}|T_k(u_n)|d\sigma \leq \frac{1}{\lambda}\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big).$$

Lemma 3.4. *For any $k > 0$, we have*

$$||T_k(u_n)||_{W^{1,\pi_n(.)}(\Omega)} \leq 1 + C\big(k, f, g, p_-, p_+, meas(\Omega)\big)$$

and for all $k \geq 1$,

$$meas\big(\{|u_n| > k\}\big) \leq \frac{C}{\min\big(b(k), |b(-k)|\big)}.$$

Proof of Lemma 3.4. By using (3.4), we have

$$\int_{\Omega}\big|\nabla T_k(u_n)\big|^{\pi_n(.)}dx \leq C_2 k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big). \tag{3.16}$$

We also have

$$\int_{\Omega}|T_k(u_n)|^{\pi_n(.)}dx = \int_{\{|u_n|\leq k\}}|T_k(u_n)|^{\pi_n(.)}dx + \int_{\{|u_n|>k\}}|T_k(u_n)|^{\pi_n(.)}dx.$$

Furthermore,

$$
\begin{aligned}
\int_{\{|u_n|>k\}}|T_k(u_n)|^{\pi_n(.)}dx &= \int_{\{|u_n|>k\}}k^{\pi_n(.)}dx \\
&\leq \begin{cases} k^{p_+}meas(\Omega) & \text{if } \quad k \geq 1 \\ meas(\Omega) & \text{if } \quad k < 1 \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\{|u_n|\leq k\}} |T_k(u_n)|^{\pi_n(.)}dx &\leq \int_{\{|u_n|\leq k\}} k^{\pi_n(.)}dx \\
&\leq \begin{cases} k^{p+}meas(\Omega) & \text{if } k \geq 1 \\ meas(\Omega) & \text{if } k < 1. \end{cases}
\end{aligned}
$$

This allow us to write

$$
\int_{\Omega} |T_k(u_n)|^{\pi_n(.)}dx \leq 2(1+k^{p+})meas(\Omega). \tag{3.17}
$$

Hence, adding (3.16) and (3.17) one gets

$$
\rho_{1,\pi_n(.)}(T_k(u_n)) \leq C_2 k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big) + 2(1+k^{p+})meas(\Omega).
$$

For $||T_k(u_n)||_{W^{1,\pi_n(.)}(\Omega)} \geq 1$, we have according to Proposition 2 that

$$
||T_k(u_n)||^{p_-}_{W^{1,\pi_n(.)}(\Omega)} \leq \rho_{1,\pi_n(.)}(T_k(u_n)) \leq \big[C_2 k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big) + 2(1+k^{p+})meas(\Omega)\big],
$$

which implies that

$$
\begin{aligned}
||T_k(u_n)||_{W^{1,\pi_n(.)}(\Omega)} &\leq \big[C_2 k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big) + 2(1+k^{p+})meas(\Omega)\big]^{\frac{1}{p_-}} \\
&:= C(k,f,g,p_+,p_-,meas(\Omega)).
\end{aligned}
$$

Thus,

$$
||T_k(u_n)||_{W^{1,\pi_n(.)}(\Omega)} < 1 + C(k,f,g,p_+,p_-,meas(\Omega)).
$$

Moreover, from (3.5), we have

$$
\int_{\partial\Omega} \big|T_n(b(u_n))\big|dx \leq ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}.
$$

We deduce that the sequence $(T_n(b(u_n)))_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^1(\Omega)$. Thus, $(b(u_n))_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^1(\Omega)$. So, there exists a positive constant C such that

$$
\int_{\Omega} |b(u_n)|dx \leq C.
$$

Furthermore, for all $k \geq 1$, we have

$$
\int_{\{|u_n|>k\}} |b(u_n)|dx \leq \int_{\Omega} |b(u_n)|dx \leq C.
$$

As b is continuous, nondecreasing and surjective, we infer

$$
\int_{\{|u_n|>k\}} \min\big(b(k),|b(-k)|\big)dx \leq \int_{\{|u_n|>k\}} |b(u_n)|dx \leq C.
$$

Therefore,

$$
meas\big(\{|u_n|>k\}\big) \leq \frac{C}{\min\big(b(k),|b(-k)|\big)}, \quad \forall k \geq 1.
$$

Then, the proof of Lemma 3.4 is complete.

From the Lemma 3.4, we deduce that for any $k > 0$, the sequence $\left(T_k(u_n)\right)_{n \in \mathbb{N}}$ is uniformly bounded in $W^{1, \pi_n(\cdot)}(\Omega)$ and also in $W^{1, p-}(\Omega)$.

Then, up to a subsequence still denoted $T_k(u_n)$, we can assume that for any $k > 0$, $T_k(u_n)$ weakly converges to s_k in $W^{1, p-}(\Omega)$ and also $T_k(u_n)$ strongly converges to s_k in $L^{p-}(\Omega)$.

By using the above a priori estimates, we obtain the following convergence results .

Step 2**: The convergence results**
The proof of the following proposition use the Lemma 3.4.

Proposition 6. *Assume that* $(A_3) - (A_6)$ *hold and let* u_n *be a weak solution of the problem* (3.2), *then the sequence* $(u_n)_{n \in \mathbb{N}}$ *is Cauchy in measure.*
In particular, there exists a measurable function u *and a subsequence still denoted* u_n *such that* $u_n \to u$ *in measure, as* $n \to +\infty$.

As $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in measure, so (up to a subsequence) it converges almost everywhere to some measurable function u.

As for any $k > 0$, T_k is continuous; then $T_k(u_n) \to T_k(u)$ a.e. $x \in \Omega$, so $s_k = T_k(u)$.
Therefore,
$$T_k(u_n) \rightharpoonup T_k(u) \text{ in } W^{1, p-}(\Omega)$$
and by compact embedding Theorem, we have

$T_k(u_n) \to T_k(u)$ in $L^{p-}(\Omega)$ (respectively in $L^{p-}(\partial\Omega)$) and a.e. in Ω (respectively a.e. on $\partial\Omega$).

Lemma 3.5. u_n *converges a.e. on* $\partial\Omega$ *to some function* v.

Proof of Lemma 3.5
Since $T_k(u_n) \rightharpoonup T_k(u)$ in $W^{1, p-}(\Omega)$ and $W^{1, p-}(\Omega) \hookrightarrow L^{p-}(\partial\Omega)$ (compact embedding), then $T_k(u_n) \to T_k(u)$ in $L^{p-}(\partial\Omega)$ and a.e. on $\partial\Omega$. Therefore, $T_k(u_n) \to T_k(u)$ in $L^1(\partial\Omega)$ and a.e. in $\partial\Omega$. We deduce that there exists $E \subset \partial\Omega$ such that $T_k(u_n) \to T_k(u)$ on $\partial\Omega \setminus E$ with $\mu(E) = 0$, where μ is area measure on $\partial\Omega$.
For every $k > 0$, let $E_k = \{x \in \partial\Omega \text{ such that } |T_k(u)| < k\}$ and $F = \partial\Omega \setminus \bigcup_{k>0} E_k$. By using Fatou's Lemma, we have

$$
\begin{aligned}
\int_{\partial\Omega} |T_k(u)| d\sigma &\leq \liminf_{n \to +\infty} \int_{\partial\Omega} |T_k(u_n)| d\sigma \\
&\leq \frac{||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}}{\lambda}.
\end{aligned} \tag{3.18}
$$

Now, we use (3.18) to get

$$
\begin{aligned}
\mu(F) = \frac{1}{k}\int_F \big|T_k(u)\big|d\sigma \ &\leq \ \frac{1}{k}\int_{\partial\Omega}\big|T_k(u)\big|d\sigma \\
&\leq \ \frac{||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}}{k\lambda}.
\end{aligned}
$$

We obtain $\mu(F) = 0$, as k goes to ∞. Let's now define on $\partial\Omega$ the function v by

$$
v(x) = T_k(u(x)), \quad x \in E_k.
$$

We take $x \in \partial\Omega \setminus (E \cup F)$, then there exists $k > 0$ such that $x \in E_k$ and we have

$$
u_n(x) - v(x) = \big(u_n(x) - T_k(u_n(x))\big) + \big(T_k(u_n(x)) - T_k(u(x))\big).
$$

Since $x \in E_k$, we have $|T_k(u(x))| < k$ and so $|T_k(u_n(x))| < k$, from which we deduce that $|u_n(x)| < k$. Therefore,

$$
u_n(x) - v(x) = T_k(u_n(x)) - T_k(u(x)) \to 0, \quad \text{as } n \to +\infty.
$$

This means that u_n converges to v a.e. on $\partial\Omega$, but for all $x \in E_k$, $T_k(u(x)) = u(x)$. Thus, $v = u$ a.e. on $\partial\Omega$. Therefore,

$$
u_n \to u \text{ a.e. on } \partial\Omega.
$$

The following assertions are based on the Young measure and nonlinear weak $-^*$ convergence results (see [7, 19, 13]).

Step 3: The convergence in term of Young measure
Assertion 1
The sequence $(\nabla T_k(u_n))_{n\in\mathbb{N}}$ converges to a Young measure $\nu_x^k(\lambda)$ on $\mathbb{R}^N$ in the sense of the nonlinear weak-* convergence and

$$
\nabla T_k(u) = \int_{\mathbb{R}^N} \lambda d\nu_x^k(\lambda). \tag{3.19}
$$

Proof. Using Lemma 3.3, $\nabla T_k(u_n)$ is uniformly bounded in $L^{p-}(\Omega)$, so, equi-integrable on Ω. Moreover, $\nabla T_k(u_n)$ weakly converges to $\nabla T_k(u)$ in $L^{p-}(\Omega)$. Therefore, using the representation of weakly convergence sequences in $L^1(\Omega)$ in terms of Young measures (see Theorem 2.1 and formula (2.5)), we can write

$$
\nabla T_k(u) = \int_{\mathbb{R}^N} \lambda d\nu_x^k(\lambda) \ \square
$$

Assertion 2. $|\lambda|^{\pi(.)}$ is integrable with respect to the measure $\nu_x^k(\lambda)dx$ on $\mathbb{R}^N \times \Omega$, moreover, $T_k(u) \in W^{1,\pi(.)}(\Omega)$.

Proof. We know that $p(.,u_n(.)) \to p(.,u(.))$ in measure on Ω. Now, using Theorem 2.1 (ii),

(iii) $(p(.,u_n(.)),\nabla T_k(u_n))_{n\in\mathbb{N}}$ converges on $\mathbb{R}\times\mathbb{R}^N$ to Young measure $\mu_x^k=\delta_{\pi(x)}\otimes\nu_x^k$.

Thus, we can apply the weak convergence properties (2.4) to the Carathéodory function $F_m(x,\lambda_0,\lambda)\in\Omega\times(\mathbb{R}\times\mathbb{R}^N)\mapsto|h_m(\lambda)|^{\lambda_0}$ with $m\in\mathbb{N}$, where h_m is defined in the preliminaries. Then, we obtain

$$
\begin{aligned}
\int_{\Omega\times\mathbb{R}^N}|h_m(\lambda)|^{\pi(x)}d\nu_x^k(\lambda)dx &= \int_{\Omega\times(\mathbb{R}\times\mathbb{R}^N)}|h_m(\lambda)|^{\lambda_0}d\mu_x^k(\lambda_0,\lambda)dx\\
&= \int_\Omega\int_{\mathbb{R}\times\mathbb{R}^N}F_m(x,\lambda_0,\lambda)d\mu_x^k(\lambda_0,\lambda)dx\\
&= \lim_{n\to+\infty}\int_\Omega F_m(x,p(x,u_n(x)),\nabla T_k(u_n(x)))dx\\
&= \lim_{n\to+\infty}\int_\Omega|h_m(\nabla T_k(u_n))|^{p(.,u_n(.))}dx\\
&\leq \lim_{n\to+\infty}\int_\Omega|\nabla T_k(u_n)|^{p(.,u_n(.))}dx\\
&\leq C_2k\big(||f||_{L^1(\Omega)}+||g||_{L^1(\partial\Omega)}\big)\quad\text{(using (3.4))}.
\end{aligned}
$$

$h_m(\lambda)\to\lambda$, as $m\to+\infty$ and $m\mapsto h_m(\lambda)$ is increasing. Then, using Lebesgue convergence Theorem , we deduce from last inequality that

$$\int_{\Omega\times\mathbb{R}^N}|\lambda|^{\pi(x)}d\nu_x^k(\lambda)dx\leq C_2k\big(||f||_{L^1(\Omega)}+||g||_{L^1(\partial\Omega)}\big).$$

Hence, $|\lambda|^{\pi(.)}$ is integrable with respect to the measure $\nu_x^k(\lambda)dx$ on $\mathbb{R}^N\times\Omega$.

From (3.19), the last inequality and Jensen inequality, we get

$$\int_\Omega|\nabla T_k(u)|^{\pi(x)}dx=\int_\Omega\left|\int_{\mathbb{R}^N}\lambda d\nu_x^k(\lambda)\right|^{\pi(x)}dx\leq\int_{\Omega\times\mathbb{R}^N}|\lambda|^{\pi(x)}d\nu_x^kdx<\infty.$$

Thus, $\nabla T_k(u)\in L^{\pi(.)}(\Omega)$. Moreover, $\displaystyle\int_\Omega|T_k(u)|^{\pi(.)}dx\leq\max\big(k^{p+},k^{p-}\big)meas(\Omega)$. Hence, $T_k(u)\in L^{\pi(.)}(\Omega)$ and we conclude that $T_k(u)\in W^{1,\pi(.)}(\Omega)$. $\qquad\square$

Assertion 3.

i) The sequence $\big(\Phi_n^k\big)_{n\in\mathbb{N}}$ defined by $\Phi_n^k:=a(x,u_n,\nabla T_k(u_n))$ is equi-integrable on Ω.

ii) The sequence $\big(\Phi_n^k\big)_{n\in\mathbb{N}}$ weakly converges to Φ^k in $L^1(\Omega)$ and we have

$$\Phi^k(x)=\int_{\mathbb{R}^N}a(x,u,\lambda)d\nu_x^k(\lambda).\tag{3.20}$$

Proof. i) Using the growth assumption (A_5) with variable exponent $p(.,u_n(.))$ and relation (3.4), we deduce that (Φ_n^k) is bounded in $L^{\pi_n'(.)}(\Omega)$, so, $L^{\pi_n'(.)}-$ equi-integrable on Ω.

Moreover, as $\pi_n'(.)>1$, we obtain

$$|a(x,u_n,\nabla T_k(u_n))|\leq 1+|a(x,u_n,\nabla T_k(u_n))|^{\pi_n'(.)}.$$

Thus, for all subset $E \subset \Omega$, we have

$$\int_E |a(x, u_n, \nabla T_k(u_n))| dx \leq meas(E) + \int_E |a(x, u_n, \nabla T_k(u_n))|^{\pi'_n(.)} dx.$$

Therefore, for $meas(E)$ small enough, (Φ_n^k) is equi-integrable on Ω.

ii) Set $\tilde{\Phi}_n^k = a(x, u(x), \nabla v_n)$ with $\nabla v_n = \nabla T_k(u_n).\chi_{S_n}$ where $S_n = \{x \in \Omega, \ |\pi(x) - \pi_n(x)| < \frac{1}{2}\}$. Applying (A_5) with variable exponent $\pi(.)$ on $a(x, u(x), \nabla v_n)$, we have for all subset $E \subset \Omega$,

$$\begin{aligned}
\int_E |a(x, u(x), \nabla v_n)| dx &\leq C \int_E \left(1 + \mathcal{M}(x) + |\nabla v_n|^{\pi(.)-1}\right) dx \\
&\leq C \int_E \left(1 + \mathcal{M}(x)\right) dx + \int_{E \cap S_n} |\nabla T_k(u_n)|^{\pi(.)-1} dx.
\end{aligned}$$

The first term of the right hand side of the last inequality is small for $meas(E)$ small enough. For $x \in S_n$, $\pi(x) < \pi_n(x) + \frac{1}{2}$, thus

$$\int_{E \cap S_n} |\nabla T_k(u_n)|^{\pi(.)-1} dx \leq \int_{E \cap S_n} \left(1 + |\nabla T_k(u_n)|^{\pi_n(.)-\frac{1}{2}}\right) dx$$

and

$$\int_\Omega |\nabla T_k(u_n)|^{(\pi_n(.)-\frac{1}{2})(2\pi_n(.))'} dx = \int_\Omega |\nabla T_k(u_n)|^{\pi_n(.)} dx < \infty,$$

which is equivalent to saying $|\nabla T_k(u_n)|^{\pi_n(.)-\frac{1}{2}} \in L^{(2\pi_n(.))'}(\Omega)$. Now, using Hölder type inequality,

$$\begin{aligned}
\int_{E \cap S_n} |\nabla T_k(u_n)|^{\pi(.)-1} dx &\leq \int_E \left(1 + |\nabla T_k(u_n)|^{\pi_n(.)-\frac{1}{2}}\right) dx \\
&\leq meas(E) + 2\big\|\nabla T_k(u_n)\big\|_{L^{\pi_n(.)}(\Omega)} \|\chi_E\|_{L^{2\pi_n(.)}(\Omega)}. \quad (3.21)
\end{aligned}$$

According to Proposition 1,

$$\begin{aligned}
\|\chi_E\|_{L^{2\pi_n(.)}(\Omega)} &\leq \max\left\{\left(\rho_{2\pi_n(.)}(\chi_E)\right)^{\frac{1}{2p_-}}, \left(\rho_{2\pi_n(.)}(\chi_E)\right)^{\frac{1}{2p_+}}\right\} \\
&= \max\left\{\left(meas(E)\right)^{\frac{1}{2p_-}}, \left(meas(E)\right)^{\frac{1}{2p_+}}\right\}.
\end{aligned}$$

The right-hand side of (3.21) is uniformly small for $meas(E)$ small, and the equi-integrability of $\tilde{\Phi}_n^k$ follows. Therefore, (up to a subsequence) $\tilde{\Phi}_n^k$ weakly converges in $L^1(\Omega)$ to $\tilde{\Phi}^k$, as $n \to +\infty$. Now, we prove that $\tilde{\Phi}^k = \Phi^k$; more precisely, we show that $\tilde{\Phi}_n^k - \Phi_n^k$ strongly converges in $L^1(\Omega)$ to 0.

Let $\beta > 0$, by (3.4), $\int_\Omega |\nabla T_k(u_n)|^{\pi_n(.)} dx$ is uniformly bounded, which implies that $\int_\Omega |\nabla T_k(u_n)| dx$ is finite, since

$$\int_\Omega |\nabla T_k(u_n)| dx \leq \int_\Omega (1 + |\nabla T_k(u_n)|^{\pi_n(x)}) dx.$$

By Chebyschev Inequality, we have

$$meas(\{|\nabla T_k(u_n)| > L\}) \leq \frac{\int_\Omega |\nabla T_k(u_n)| dx}{L}.$$

Therefore, $\sup\limits_{n\in\mathbb{N}} meas(\{|\nabla T_k(u_n)| > L\})$ tends to 0 for L large enough. Since $\tilde{\Phi}_n^k - \Phi_n^k$ is equi-integrable, there exists $\delta = \delta(\beta)$ such that for all $A \subset \Omega$, $meas(A) < \delta$ and $\int_A |\tilde{\Phi}_n^k - \Phi_n^k| dx < \dfrac{\beta}{4}$.

Therefore, if we choose L large enough, we get $\dfrac{\int_\Omega |\nabla T_k(u_n)| dx}{L} < \delta$, so $meas(\{|\nabla T_k(u_n)| > L\}) < \delta$. Hence,

$$\int_{\{|\nabla T_k(u_n)|>L\}} |\tilde{\Phi}_n^k - \Phi_n^k| dx < \frac{\beta}{4}.$$

By Lemma 2.2, we also have

$$meas\left(\left\{x \in \Omega; \ \sup_{\lambda\in K} |a(x, u_n(x), \lambda) - a(x, u(x), \lambda)| \geq \sigma\right\}\right) \longrightarrow 0,$$

as $n \to +\infty$.

Thus, by the above equi-integrability, for all $\sigma > 0$, there exists $n_0 = n_0(\sigma, L) \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\int_{\left\{x\in\Omega; \ \sup_{|\lambda|\leq L} |a(x,u_n(x),\lambda)-a(x,u(x),\lambda)|\geq\sigma\right\}} |\tilde{\Phi}_n^k - \Phi_n^k| dx < \frac{\beta}{4}.$$

Using the definition of Φ_n^k and $\tilde{\Phi}_n^k$, we have

$$\Phi_n^k - \tilde{\Phi}_n^k = a(x, u_n(x), \nabla T_k(u_n)) - a(x, u(x), \nabla T_k(u_n)) \quad \text{on} \ \ S_n.$$

Now, we reason on

$$S_{n,L,\sigma} := \left\{x \in \Omega; \ \sup_{|\lambda|\leq L} |a(x, u_n(x), \lambda) - a(x, u(x), \lambda)| < \sigma, |\nabla T_k(u_n)| \leq L\right\}.$$

We get

$$\begin{aligned}\int_{S_{n,L,\sigma}} |\tilde{\Phi}_n^k - \Phi_n^k| dx \ &\leq \ \int_{S_{n,L,\sigma}} \sup_{|\lambda|\leq L} |a(x, u_n(x), \lambda) - a(x, u(x), \lambda)| dx \\ &\leq \ \sigma meas(\Omega).\end{aligned}$$

We observe that

$$\int_{S_n} |\tilde{\Phi}_n^k - \Phi_n^k| dx = \int_{S_n\cap S_{n,L,\sigma}} |\tilde{\Phi}_n^k - \Phi_n^k| dx + \int_{S_n\setminus S_{n,L,\sigma}} |\tilde{\Phi}_n^k - \Phi_n^k| dx$$

and

$$S_n \setminus S_{n,L,\sigma} \subset \left\{x \in \Omega; \ \sup_{|\lambda|\leq L} |a(x, u_n(x), \lambda) - a(x, u(x), \lambda)| \geq \sigma\right\} \cup \left\{|\nabla T_k(u_n)| > L\right\}.$$

Consequently, by choosing $\sigma = \sigma(\beta) < \dfrac{\beta}{4meas(\Omega)}$, we get

$$\int_{S_n} |\tilde{\Phi}_n^k - \Phi_n| dx < \frac{\beta}{4} + \frac{\beta}{4} + \frac{\beta}{4} = \frac{3\beta}{4},$$

for all $n \geq n_0(\sigma, L)$. By Lemma 2.2, we also have $meas(\{x \in \Omega, |\pi(x) - \pi_n(x)| \geq \frac{1}{2}\}) \to 0$ for n large enough; which means that $meas(\Omega \setminus S_n)$ converges to 0 for n large enough. Thus,

$$\int_{\Omega \setminus S_n} |\tilde{\Phi}_n^k - \Phi_n^k| dx = \int_{\Omega \setminus S_n} |\Phi_n^k| dx \leq \frac{\beta}{4}.$$

Therefore, for all $\beta > 0$ there exists $n_0 = n_0(\beta)$ such that for all $n \geq n_0$, $\int_\Omega |\tilde{\Phi}_n^k - \Phi_n^k| dx \leq \beta$. Hence, $\tilde{\Phi}_n^k - \Phi_n^k$ strongly converges to 0 in $L^1(\Omega)$. We prove that

$$\Phi^k(x) = \int_{\mathbb{R}^N} a(x, u(x), \lambda) d\nu_x^k(\lambda) \quad a.e. \ x \in \Omega \ \text{and} \ \Phi^k \in L^{\pi'(.)}(\Omega).$$

Notice that

$$\lim_{n \to +\infty} \int_\Omega |\nabla T_k(u_n)|(1 - \chi_{S_n}) dx = \lim_{n \to +\infty} \int_{\Omega \setminus S_n} |\nabla T_k(u_n)| dx = 0,$$

since $(\nabla T_k(u_n))_{n \in \mathbb{N}}$ is equi-integrable and $meas(\Omega \setminus S_n)$ converges to 0 for n large enough. Therefore, $(\nabla T_k(u_n))_{n \in \mathbb{N}}$ and $\nabla T_k(u_n)\chi_{S_n}$ converge to the same Young measure $\nu_x^k(\lambda)$. Moreover, by applying Theorem 2.1 i) to the Carathéodory function $F(x, (\lambda_0, \lambda)) := a(x, \lambda_0, \lambda)$, we infer that

$$\tilde{\Phi}(x) = \Phi(x) = \int_{\mathbb{R}^N} a(x, u(x), \lambda) d\nu_x^k(\lambda) \ \text{a.e.} \ x \in \Omega.$$

Using (A_5), it follows that $|a(x, u(x), \lambda)|^{\pi'(.)} \leq C(\mathcal{M}(x) + |\lambda|^{\pi(.)})$. Thus, with Jensen Inequality, it follows that

$$
\begin{aligned}
\int_\Omega |\Phi^k(x)|^{\pi'(.)} dx &= \int_\Omega \left| \int_{\mathbb{R}^N} a(x, u(x), \lambda) d\nu_x^k(\lambda) \right|^{\pi'(.)} dx \\
&\leq \int_{\Omega \times \mathbb{R}^N} |a(x, u(x), \lambda)|^{\pi'(.)} d\nu_x^k(\lambda) dx \\
&\leq C \int_{\Omega \times \mathbb{R}^N} \left(\mathcal{M}(x) + |\lambda|^{\pi(.)} \right) d\nu_x^k(\lambda) dx < \infty.
\end{aligned}
$$

Hence, $\Phi^k \in L^{\pi'(.)}(\Omega)$. □

Assertion 4

(a) For all $k' > k > 0$, we have $\Phi^k = \Phi^{k'} \chi_{\{|u| < k\}}$.

(b) For all $k > 0$,

$$\int_\Omega \Phi^k . \nabla T_k(u) dx \geq \int_{\Omega \times \mathbb{R}^N} a(x, u(x), \lambda).\lambda d\nu_x^k(\lambda) dx. \tag{3.22}$$

(c) The "div-curl" inequality holds:

$$\int_{\Omega \times \mathbb{R}^N} \big(a(x, u(x), \lambda) - a(x, u(x), \nabla T_k(u(x))) \big)(\lambda - \nabla T_k(u(x))) d\nu_x^k(\lambda) dx \leq 0. \tag{3.23}$$

(d) For all $k > 0$,

$$\Phi^k = a(x, u(x), \nabla T_k(u)) \text{ for a.e. } x \in \Omega$$

and $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ in measure on Ω, as $n \to +\infty$.

Proof.

(a) Let $k' > k > 0$ and $g_n^k := a(x, u_n, \nabla T_{k'}(u_n))\chi_{[|u|<k]}$. By Assertion 3-ii), $(g_n^k)_{n\in\mathbb{N}}$ weakly converges to $\Phi^{k'}\chi_{[|u|<k]}$ in $L^1(\Omega)$. If we prove that $(g_n^k)_{n\in\mathbb{N}}$ weakly converges to Φ^k in $L^1(\Omega)$, then the wished result will come of the uniqueness of the limit. Let us put

$$h_n^k := a(x, u_n, \nabla T_{k'}(u_n))\chi_{[|u_n|<k]}.$$

As $\nabla T_k(u_n) \equiv \nabla T_{k'}(u_n)\chi_{[|u_n|<k]}$, for all $k' > k > 0$, then, we get

$$h_n^k := a(x, u_n, \nabla T_{k'}(u_n))\chi_{[|u_n|<k]} \equiv a(x, u_n, \nabla T_k(u_n)),$$

so, $(h_n^k)_{n\in\mathbb{N}}$ weakly converges to Φ^k in $L^1(\Omega)$ by Assertion 3-ii). Set

$$d_n^k := g_n^k - h_n^k = a(x, u_n, \nabla T_{k'}(u_n))(\chi_{[|u|<k]} - \chi_{[|u_n|<k]}).$$

On the one hand, thanks to Assertion 3-i), $(d_n^k)_{n\in\mathbb{N}}$ is equi-integrable. On the other hand $d_n^k \to 0$ a.e. on Ω. Indeed, $\chi_{[|u_n|<k]} = \chi_{(-k,k)}(u_n)$ and if $|u_n| \neq k$ a.e. on Ω, $\chi_{(-k,k)}(.)$ is continuous on $\mathbb{R}$. In other words $\chi_{(-k,k)}(.)$ is continuous on the image of Ω by u a.e. $k > 0$. Moreover, $u_n \to u$ a.e. on Ω, then $\chi_{[|u_n|<k]} \to \chi_{[|u|<k]}$ a.e. in Ω. Now, using Vitali's Theorem $(d_n^k)_{n\in\mathbb{N}}$ strongly converges to 0 in $L^1(\Omega)$, so it weakly converges in $L^1(\Omega)$. Hence, $(g_n^k)_{n\in\mathbb{N}}$ and $(h_n^k)_{n\in\mathbb{N}}$ weakly converge to the same limit Φ^k in $L^1(\Omega)$.

(b) Let $\mathcal{S}$ be a set of $W^{2,\infty}$ functions $S : \mathbb{R} \to \mathbb{R}$ such that $S'(.)$ has a compact support.
We construct a sequence $(S_M)_{M\in\mathbb{N}} \subset \mathcal{S}$ such that
• S'_M and S''_M are uniformly bounded;
• for all $M \in \mathbb{N}$, $S'_M = 1$ on $[-M + 1, M - 1]$, supp$S' \subset [-M, M]$;
• the sequence $(b(z)S'_M(z))_{M\in\mathbb{N}}$ is non-decreasing for all $z \in \mathbb{R}$.
For all $\varphi \in C^\infty(\overline{\Omega})$, $v = \varphi S'_M(u_n)$ is an admissible test function in the weak formulation (3.3). We have

$$\int_\Omega T_n(b(u_n))S'_M(u_n)\varphi dx + \int_\Omega S'_M(u_n)a(x, u_n, \nabla T_M(u_n)).\nabla\varphi dx$$

$$+ \int_\Omega S''_M(u_n)a(x, u_n, \nabla T_M(u_n)).\nabla T_M(u_n)\varphi dx + \int_{\partial\Omega} \lambda T_n(u_n)S'_M(u_n)\varphi d\sigma$$

$$+ \varepsilon \int_\Omega [|\nabla u_n|^{p_+-2}\nabla u_n\nabla(\varphi S'_M(u_n)) + |u_n|^{p_+-2}u_n S'_M(u_n)\varphi]dx$$

$$= \int_\Omega f_n S'_M(u_n)\varphi dx + \int_{\partial\Omega} g_n S'_M(u_n)\varphi d\sigma. \qquad (3.24)$$

Since u_n converges to u a.e. in Ω and a.e. on $\partial\Omega$, by continuity of b, S'_M and the compacteness of $\text{supp} S'_M$, we obtain

$$\int_\Omega T_n(b(u_n))S'_M(u_n)\varphi dx \to \int_\Omega b(u)S'_M(u)\varphi dx, \ \text{ as } \ n \to +\infty \qquad (3.25)$$

and

$$\lambda\int_\Omega T_n(u_n)S'_M(u_n)\varphi d\sigma \to \lambda\int_\Omega uS'_M(u)\varphi d\sigma, \ \text{ as } \ n \to +\infty. \qquad (3.26)$$

Moreover, we have $|f_nS'_M(u_n)\varphi| \leq ||S'_M||_{L^\infty(\mathbb{R})}|f||\varphi| \in L^1(\Omega)$, $f_nS'_M(u_n)\varphi \to fS'_M(u)\varphi$ a.e. in Ω. and $|g_nS'_M(u_n)\varphi| \leq ||S'_M||_{L^\infty(\mathbb{R})}|g||\varphi| \in L^1(\mathbb{R})$, $g_nS'_M(u_n)\varphi \to gS'_M(u)\varphi$ a.e. on $\partial\Omega$. Thus, by Lebesgue dominated convergence Theorem

$$\int_\Omega f_nS'_M(u_n)\varphi dx \to \int_\Omega fS'_M(u)\varphi dx, \ \text{ as } \ n \to +\infty \qquad (3.27)$$

and

$$\int_{\partial\Omega} g_nS'_M(u_n)\varphi d\sigma \to \int_{\partial\Omega} gS'_M(u)\varphi d\sigma, \ \text{ as } \ n \to +\infty. \qquad (3.28)$$

Let us prove now, that

$$\int_\Omega S'_M(u_n)a(x,u_n,\nabla T_M(u_n)).\nabla\varphi dx \to \int_\Omega S'_M(u)\Phi^M.\nabla\varphi dx, \ \text{ as } \ n \to +\infty. \qquad (3.29)$$

For all $L > 0$, we have

$$\int_\Omega S'_M(u_n)a(x,u_n,\nabla T_M(u_n)).\nabla\varphi dx \ = \ \int_{\{|\nabla\varphi|\leq L\}} S'_M(u_n)\Phi^M_n.\nabla\varphi dx$$
$$+ \ \int_{\{|\nabla\varphi|>L\}} S'_M(u_n)\Phi^M_n.\nabla\varphi dx. \qquad (3.30)$$

For the first term of the right-hand side of (3.30), we have

$$\int_{\{|\nabla\varphi|\leq L\}} S'_M(u_n)\Phi^M_n.\nabla\varphi dx \to \int_{\{|\nabla\varphi|\leq L\}} S'_M(u)\Phi^M.\nabla\varphi dx, \ \text{ as } \ n \to +\infty. \qquad (3.31)$$

Thanks $\Phi^M_n \rightharpoonup \Phi^M$ in $L^1(\Omega)$ and $\nabla\varphi S'_M(u_n)\chi_{\{|\nabla\varphi|\leq L\}} \to^* \nabla\varphi S'_M(u)\chi_{\{|\nabla\varphi|\leq L\}}$ in $L^\infty(\Omega)$. Furthermore, the second term of the right hand-side of (3.30) converges to zero for L large enough, uniformly in n. Indeed, using Hölder type inequality and the fact that $L^{p_+}(\Omega) \hookrightarrow L^{\pi_n(\cdot)}(\Omega)$, we get

$$\left|\int_{\{|\nabla\varphi|>L\}} \Phi^M_n\nabla\varphi S'_M(u_n)dx\right|$$
$$\leq \ C||S'_M||_{L^\infty(\mathbb{R})}||\Phi^M_n||_{L^{\pi'_n(\cdot)}(\Omega)}||\nabla\varphi\chi_{\{|\nabla\varphi|>L\}}||_{L^{\pi_n(\cdot)}(\Omega)}$$
$$\leq \ C\big(p_-,||S'_M||_{L^\infty(\mathbb{R})},meas(\Omega)\big)||\Phi^M_n||_{L^{\pi'_n(\cdot)}(\Omega)}||\nabla\varphi||_{L^{p_+}(\Omega)}meas\big(\{|\nabla\varphi|>L\}\big).$$

From (A_5), (3.4) and Proposition 2, we obtain

$$||\Phi^n_M||_{L^{\pi'_n(\cdot)}(\Omega)} < C.$$

Moreover, $\varphi \in C^\infty(\overline{\Omega})$ and $C^\infty(\overline{\Omega})$ is dense in the space $W^{1,p_+}(\Omega)$. Then, by Proposition 2 and the fact that $\lim\limits_{L \to +\infty} meas(\{|\nabla \varphi| > L\}) = 0$, we get

$$meas(\{|\nabla \varphi| > L\})\|\Phi_n^M\|_{L^{\pi'_n(\cdot)}(\Omega)}\|\nabla \varphi\|_{L^{p_+}(\Omega)} \to 0, \quad \text{as } L \to +\infty.$$

Hence, the second term of the right hand-side of (3.30) converges to zero, as L tends to infinity. Thus, as $n \to +\infty$ and $L \to +\infty$ in (3.30), we deduce (3.29).

Let us consider the third term of left hand-side of (3.24), we obtain

$$
\begin{aligned}
\int_\Omega |S_M''(u_n)|a(x,u_n,\nabla T_M(u_n)).\nabla T_M(u_n)\varphi dx &\leq C \int_{\{|u_n|<M\}} |S_M''(u_n)|a(x,u_n,\nabla T_M(u_n)).\nabla T_M(u_n)dx \\
&\leq C' \int_{\{M-1<|u_n|<M\}} a(x,u_n,\nabla T_M(u_n))\nabla T_M(u_n)dx \\
&+ C \int_{\{|u_n|\leq M-1\}} \underbrace{|S_M''(u_n)|}_{=0}a(x,u_n,\nabla T_M(u_n)).\nabla T_M(u_n)dx,
\end{aligned}
\tag{3.32}
$$

where $C = C(\|\varphi\|_{L^\infty(\Omega)})$, $C' = C(\|S_M''\|_{L^\infty(\mathbb{R})}, \|\varphi\|_{L^\infty(\Omega)})$ and $|a(x,u_n,\nabla T_M(u_n)).\nabla T_M(u_n)$ is finite. Otherwise,

$$\int_{\{M-1<|u_n|<M\}} a(x,u_n,\nabla T_M(u_n))\nabla T_M(u_n)dx \to 0, \quad \text{as } M \to +\infty.$$

Since, thanks to Lemma 3.4, $\lim\limits_{M \to +\infty} meas(\{M-1 < |u_n| < M\}) = 0$ and $a(x,u_n,\nabla T_M(u_n))\nabla T_M(u_n)$ is equi-integrable.

Finally, using (3.25), (3.26) (3.27), (3.28), (3.29), (3.32) and passing to the limit in (3.24), as n tends to infinty and as ε goes to 0, we obtain

$$\int_\Omega b(u)S_M'(u)\varphi dx + \int_\Omega S_M'(u)\Phi^M.\nabla \varphi dx + \lambda \int_{\partial\Omega} uS_M'(u)\varphi d\sigma = \int_\Omega fS_M'(u)\varphi dx + \int_{\partial\Omega} gS_M'(u)\varphi d\sigma. \tag{3.33}$$

For $k > 0$ fixed, $T_k(u) \in W^{1,\pi(\cdot)}(\Omega)$ and the exponent $\pi(.)$ verify (2.1). Therefore, $C^\infty(\overline{\Omega})$ is dense in $W^{1,\pi(\cdot)}(\Omega)$, so, we replace φ by $T_k(u)$. Now, for $M > k$, thanks to (a), we replace $\Phi^M.\nabla T_k(u)$ by $\Phi^k.\nabla T_k(u)$ in (3.33).

S_M' converges a.e. to 1 on $\mathbb{R}$, as $M \to +\infty$, then using the monotone convergence theorem in the first term of left hand-side of (3.33) and dominated convergence theorem in the other term of (3.33), we get

$$\int_\Omega [b(u)T_k(u) + \Phi^k.\nabla T_k(u)]dx + \lambda \int_{\partial\Omega} uT_k(u)d\sigma = \int_\Omega fT_k(u)dx + \int_{\partial\Omega} gT_k(u)d\sigma. \tag{3.34}$$

The relation (3.7) is equivalent to

$$
\begin{aligned}
\int_\Omega T_n(b(u_n))T_k(u_n)dx \quad & + \quad \int_\Omega a(x,u_n,\nabla T_k(u_n)).\nabla T_k(u_n)dx + \int_{\partial\Omega} \lambda T_n(u_n)T_k(u_n)d\sigma \\
& + \quad \varepsilon\int_\Omega \big[|u_n|^{p_+-2}u_n T_k(u_n) + |\nabla u_n|^{p_+-2}\nabla u_n\nabla T_k(u_n)\big]dx \\
& = \quad \int_\Omega f_n T_k(u_n)dx + \int_{\partial\Omega} g_n T_k(u_n)d\sigma.
\end{aligned}
\tag{3.35}
$$

The sequences $\big(T_n(b(u_n))T_k(u_n)\big)_{n\in\mathbb{N}}$, $\big(T_n(u_n)T_k(u_n)\big)_{n\in\mathbb{N}}$ are nonnegative and converge a.e. in Ω to $b(u)T_k(u)$ and a.e. on $\partial\Omega$ to $uT_k(u)$. By Fatou's Lemma, we have

$$
\liminf_{n\to+\infty}\int_\Omega T_n(b(u_n))T_k(u_n)dx \geq \int_\Omega b(u)T_k(u)dx
\tag{3.36}
$$

and

$$
\lambda\liminf_{n\to+\infty}\int_{\partial\Omega} T_n(u_n)T_k(u_n)dx \geq \lambda\int_{\partial\Omega} uT_k(u)d\sigma.
\tag{3.37}
$$

Now, we consider the right hand side of (3.35). We have
$|f_n T_k(u_n)| \leq k|f| \in L^1(\Omega)$, $f_n T_k(u_n) \to fT_k(u)$ a.e. in Ω and $|g_n T_k(u_n)| \leq k|g| \in L^1(\partial\Omega)$, $g_n T_k(u_n) \to gT_k(u)$ a.e. on $\partial\Omega$. Thus, by Lebesgue dominated convergence Theorem

$$
\int_\Omega f_n T_k(u_n)dx \to \int_\Omega fT_k(u)dx, \;\; \text{as } n\to+\infty
\tag{3.38}
$$

and

$$
\int_{\partial\Omega} g_n T_k(u_n)d\sigma \to \int_{\partial\Omega} gT_k(u)d\sigma, \;\; \text{as } n\to+\infty.
\tag{3.39}
$$

Combining (3.36),(3.37), (3.38), (3.39) and using (3.35), we get

$$
\begin{aligned}
& \liminf_{n\to+\infty}\left(\int_\Omega f_n T_k(u_n)dx + \int_{\partial\Omega} g_n T_k(u_n)d\sigma\right) - \left(\int_\Omega b(u)T_k(u)dx + \lambda\int_{\partial\Omega} uT_k(u)d\sigma\right) \\
\geq \;\; & \liminf_{n\to+\infty}\left(\int_\Omega f_n T_k(u_n)dx + \int_{\partial\Omega} g_n T_k(u_n)d\sigma - \int_\Omega T_n(b(u_n))T_k(u_n)dx - \lambda\int_{\partial\Omega} T_n(u_n)T_k(u_n)d\sigma\right),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
& \int_\Omega fT_k(u)dx + \int_{\partial\Omega} gT_k(u)d\sigma - \left(\int_\Omega b(u)T_k(u)dx + \lambda\int_{\partial\Omega} uT_k(u)d\sigma\right) \\
\geq \;\; & \liminf_{n\to+\infty}\int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u_n)dx + \varepsilon\int_\Omega \big[|u_n|^{p_+-2}u_n T_k(u_n) + |\nabla u_n|^{p_+-2}\nabla u_n\nabla T_k(u_n)\big]dx \\
\geq \;\; & \liminf_{n\to+\infty}\int_\Omega a(x,u_n,\nabla T_k(u_n)\nabla T_k(u_n)dx.
\end{aligned}
$$

Thus, by using the relation (3.34), we obtain

$$
\int_\Omega \Phi^k\nabla T_k(u)dx \geq \liminf_{n\to+\infty}\int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u_n)dx.
\tag{3.40}
$$

(c) From [1]-Lemma 2.1, $m \mapsto a(x, u_n, h_m(\nabla T_k(u_n))).h_m(\nabla T_k(u_n))$ is increasing and converges to $a(x, u_n, \nabla T_k(u_n)).\nabla T_k(u_n)$ for m large enough. Thus, we deduce that

$$a(x, u_n, h_m(\nabla T_k(u_n))).h_m(\nabla T_k(u_n)) \leq a(x, u_n, \nabla T_k(u_n)).\nabla T_k(u_n) = \Phi_n^k.\nabla T_k(u_n).$$

Therefore, using (b) and Theorem 2.1, we get

$$\begin{aligned}
\int_\Omega \Phi^k.\nabla T_k(u)dx &\geq \liminf_{n \to +\infty} \int_\Omega \Phi_n^k.\nabla T_k(u_n)dx \\
&\geq \lim_{n \to +\infty} \int_\Omega a(x, u_n, h_m(\nabla T_k(u_n))).h_m(\nabla T_k(u_n))dx \\
&= \int_{\Omega \times \mathbb{R}^N} a(x, u, h_m(\lambda)).h_m(\lambda)d\nu_x^k(\lambda)dx.
\end{aligned} \tag{3.41}$$

Using Lebesgue convergence Theorem in (3.41), we get for m large enough

$$\int_\Omega \Phi^k.\nabla T_k(u)dx \geq \int_{\Omega \times \mathbb{R}^N} a(x, u, \lambda).\lambda d\nu_x^k(\lambda)dx. \tag{3.42}$$

We have

$$\begin{aligned}
&\int_{\Omega \times \mathbb{R}^N} (a(x, u(x), \lambda) - a(x, u(x), \nabla T_k(u(x))))(\lambda - \nabla T_k(u(x)))d\nu_x^k(\lambda)dx \\
&= \int_{\Omega \times \mathbb{R}^N} a(x, u(x), \lambda).\lambda d\nu_x^k(\lambda)dx - \int_{\Omega \times \mathbb{R}^N} a(x, u(x), \lambda).\nabla T_k(u(x))d\nu_x^k(\lambda)dx \\
&\quad - \int_{\Omega \times \mathbb{R}^N} a(x, u(x), \nabla T_k(u(x))).\lambda d\nu_x^k(\lambda)dx + \int_{\Omega \times \mathbb{R}^N} a(x, u(x), \nabla T_k(u(x))).\nabla T_k(u(x))d\nu_x^k(\lambda)dx \\
&= \int_{\Omega \times \mathbb{R}^N} a(x, u(x), \lambda).\lambda d\nu_x^k(\lambda)dx - \int_\Omega \left(\int_{\mathbb{R}^N} a(x, u(x), \lambda)d\nu_x^k(\lambda) \right) \nabla T_k(u(x))dx \\
&\quad - \int_\Omega a(x, u(x), \nabla T_k(u(x))).\left(\int_{\mathbb{R}^N} \lambda d\nu_x^k \right)dx + \int_\Omega a(x, u(x), \nabla T_k(u(x))).\nabla T_k(u(x))\left(\int_{\mathbb{R}^N} d\nu_x^k \right)dx \\
&= \int_{\Omega \times \mathbb{R}^N} a(x, u(x), \lambda).\lambda d\nu_x^k(\lambda)dx - \int_\Omega \Phi^k.\nabla T_k(u(x))dx \leq 0.
\end{aligned}$$

We pass from the first equality to the second equality by using Fubini-Tonelli Theorem and from the second inequality to the third one by using (3.19), (3.20) and the fact that ν_x is probability measures on $\mathbb{R}^N$. Finally (3.42) give us the desired inequality.

(d) Using (3.23) and the strict monotonicity assumption (A_4), we deduce that

$$\big(a(x, u(x), \lambda) - a(x, u(x), \nabla T_k(u(x)))\big)\big(\lambda - \nabla T_k(u(x))\big) = 0 \text{ a.e. } x \in \Omega, \quad \lambda \in \mathbb{R}^N.$$

Thus, $\lambda = \nabla T_k(u(x))$ a.e. $x \in \Omega$ with respect to the measure ν_x^k on $\mathbb{R}^N$. Therefore, the measure ν_x^k reduces to the Dirac measure $\delta_{\nabla T_k(u(x))}$. Using (3.20), we obtain

$$\Phi^k = \int_{\mathbb{R}^N} a(x, u(x), \lambda)d\nu_x^k(\lambda) = a(x, u(x), \nabla T_k(u(x))) \text{ a.e. } x \in \Omega.$$

Now, by using Theorem 2.1 (ii), we deduce that $\nabla T_k(u_n)$ converges in measure to $\nabla T_k(u)$.
$\square$

Lemma 3.6. *u is an entropy solution of (1.1).*

Proof of the Lemma 3.6.

Let u_n be a weak solution of the problem (3.2). Then, by Assertion $4-(d)$, $(\nabla T_k(u_n))_{n \in \mathbb{N}}$ converges to $\nabla T_k(u)$ in measure, thus (up to a subsequence still denoted $(\nabla T_k(u_n))_{n \in \mathbb{N}}$), $(\nabla T_k(u_n))_{n \in \mathbb{N}}$ converges to $\nabla T_k(u)$ a.e. Ω. Moreover, we deduce from Lemma 3.4 that $\nabla T_k(u_n)$ is uniformly bounded in $L^{p_-}(\Omega)$, so, p_-−equi-integrable on Ω. Then, by using Vitali's Theorem

$$\nabla T_k(u_n) \to \nabla T_k(u) \text{ in } L^{p_-}(\Omega), \text{ which implies that } \nabla T_k(u_n) \to \nabla T_k(u) \text{ in } L^1(\Omega).$$

Furthermore, thanks to Assertion 2, $u \in \mathcal{T}^{1,\pi(\cdot)}(\Omega)$ and it follows from Lemma 3.5 that

$$u_n \to u \text{ a.e on } \partial\Omega.$$

Therefore, $u \in \mathcal{T}_{tr}^{1,\pi(\cdot)}(\Omega)$. Now, using Lemma 3.2, the fact that $T_n(b(u_n)) \to b(u)$ a.e. in Ω and $u_n \to u$ a.e. on $\partial\Omega$, it follows from Fatou's Lemma that

$$\int_\Omega |b(u)| \leq \liminf_{n \to +\infty} \int_\Omega |T_n(b(u_n))| dx \leq ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}$$

and

$$\int_{\partial\Omega} |u| \leq \liminf_{n \to +\infty} \int_{\partial\Omega} |T_n(u_n)| dx \leq \frac{1}{\lambda} \big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)} \big).$$

Hence, $b(u) \in L^1(\Omega)$ and $u \in L^1(\partial\Omega)$.

Let $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$, then we can choose $T_k(u_n - \varphi)$ as a test function in (3.3) ($C^\infty(\overline{\Omega})$ is dense in the space $W^{1,p_+}(\Omega)$ and $T_k(u_n - \varphi) \in L^\infty(\partial\Omega)$) to get

$$
\begin{aligned}
\int_\Omega T_n(b(u_n)) T_k(u_n - \varphi) dx \quad &+ \quad \int_\Omega a(x, u_n, \nabla u_n) . \nabla T_k(u_n - \varphi) dx + \int_{\partial\Omega} \lambda T_n(u_n) T_k(u_n - \varphi) d\sigma \\
&+ \quad \varepsilon \int_\Omega \big[|\nabla u_n|^{p_+ - 2} \nabla u_n \nabla T_k(u_n - \varphi) + |u_n|^{p_+ - 2} u_n T_k(u_n - \varphi) \big] dx \\
&= \quad \int_\Omega f_n T_k(u_n - \varphi) dx + \int_{\partial\Omega} g_n T_k(u_n - \varphi) d\sigma.
\end{aligned}
\tag{3.43}
$$

For the first term of the left hand side of (3.43), we have

$$
\begin{aligned}
\int_\Omega T_n(b(u_n)) T_k(u_n - \varphi) dx \quad &= \quad \int_\Omega \big[T_n(b(u_n)) - T_n(b(\varphi)) \big] T_k(u_n - \varphi) dx \\
&+ \quad \int_\Omega T_n(b(\varphi)) T_k(u_n - \varphi) dx.
\end{aligned}
$$

By Fatou's Lemma, we infer

$$\liminf_{n \to +\infty} \int_\Omega T_n(b(u_n)) T_k(u_n - \varphi) dx \geq \int_\Omega b(u) T_k(u - \varphi) dx, \tag{3.44}$$

since,

$$\big[T_n(b(u_n)) - T_n(b(\varphi))\big]T_k(u_n - \varphi) \to \big(b(u - b(\varphi))T_k(u - \varphi) \ \text{ a.e.}$$

with

$$\big[T_n(b(u_n)) - T_n(b(\varphi))\big]T_k(u_n - \varphi) \geq 0$$

and

$$T_n(b(\varphi))T_k(u_n - \varphi) \to b(\varphi)T_k(u - \varphi) \ \text{ in } \ L^1(\Omega).$$

In the same manner

$$\liminf_{n \to +\infty} \lambda \int_{\partial\Omega} T_n(u_n)T_k(u_n - \varphi)d\sigma \geq \lambda \int_{\partial\Omega} uT_k(u - \varphi)d\sigma. \tag{3.45}$$

For the fourth term of the left hand side of (3.43), we prove that

$$\lim_{n \to +\infty} \varepsilon \int_\Omega \big[|\nabla u_n|^{p_+ - 2}\nabla u_n \nabla T_k(u_n - \varphi) + |u_n|^{p_+ - 2}u_n T_k(u_n - \varphi)\big]dx \geq 0 \ \text{ as } \ \varepsilon \to 0. \tag{3.46}$$

Setting $l = k + ||\varphi||_{L^\infty(\Omega)}$ we have,

$$\varepsilon \int_\Omega |\nabla u_n|^{p_+ - 2}\nabla u_n \nabla T_k(u_n - \varphi)dx = \varepsilon \int_{\{|u_n - \varphi| < k\}} |\nabla T_l(u_n)|^{p_+ - 2}\nabla T_l(u_n)\nabla\big(T_l(u_n) - \varphi\big)dx$$

$$= \ \varepsilon \int_{\{|u_n - \varphi| < k\}} |\nabla T_l(u_n)|^{p_+}dx - \varepsilon \int_{\{|u_n - \varphi| < k\}} |\nabla T_l(u_n)|^{p_+ - 2}\nabla T_l(u_n)\nabla\varphi dx$$

$$\geq \ -\varepsilon \int_{\{|u_n - \varphi| < k\}} |\nabla T_l(u_n)|^{p_+ - 2}\nabla T_l(u_n)\nabla\varphi dx. \tag{3.47}$$

Moreover, by taking $v = T_l(u_n)$ in (3.3), we deduce that

$$\varepsilon \int_\Omega \big[|\nabla u_n|^{p_+ - 2}\nabla u_n \nabla T_l(u_n) + |u_n|^{p_+ - 2}u_n T_l(u_n)\big]dx \leq l\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big),$$

which implies that

$$\varepsilon \int_\Omega |\nabla T_l(u_n)|^{p_+}dx \leq l\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big).$$

Therefore, $\varepsilon\nabla T_l(u_n)$ is uniformly bounded in $L^{p_+}(\Omega)$. From, Assertion $4-(d)$, $\nabla T_l(u_n)$ converges a.e. in Ω (up to a subsequence) to $\nabla T_l(u)$. So, by Vitali's Theorem, $\varepsilon\nabla T_l(u_n)$ converges to $\varepsilon\nabla T_l(u)$ in $L^{p_+}(\Omega)$. Thus, $\varepsilon|\nabla T_l(u_n)|^{p_+ - 2}\nabla T_l(u_n)\chi_{\{|u_n - \varphi| < k\}}$ converges to $\varepsilon|\nabla T_l(u)|^{p_+ - 2}\nabla T_l(u)\chi_{\{|u - \varphi| < k\}}$ in $L^{p'_+}(\Omega)$. Using (3.47), we obtain

$$\lim_{n \to +\infty} \varepsilon \int_\Omega |\nabla u_n|^{p_+ - 2}\nabla u_n \nabla T_k(u_n - \varphi)dx \geq -\varepsilon \int_{\{|u - \varphi| < k\}} |\nabla T_l(u)|^{p_+ - 2}\nabla T_l(u)\nabla\varphi dx.$$

Therefore,

$$\lim_{n \to +\infty} \varepsilon \int_\Omega |\nabla u_n|^{p_+ - 2}\nabla u_n \nabla T_k(u_n - \varphi)dx \geq 0, \ \text{ as } \ \varepsilon \to 0. \tag{3.48}$$

Now, we prove that

$$\lim_{n\to+\infty} \varepsilon \int_\Omega |u_n|^{p_+-2}u_n T_k(u_n-\varphi)dx \geq 0, \quad \text{as } \varepsilon \to 0.$$

We have

$$\int_\Omega |u_n|^{p_+-2}u_n T_k(u_n-\varphi)dx = \int_\Omega \left(|u_n|^{p_+-2}u_n - |\varphi|^{p_+-2}\varphi\right)T_k(u_n-\varphi)dx$$
$$+ \int_\Omega |\varphi|^{p_+-2}\varphi T_k(u_n-\varphi)dx$$
$$\geq \int_\Omega |\varphi|^{p_+-2}\varphi T_k(u_n-\varphi)dx, \tag{3.49}$$

since $\left(|u_n|^{p_+-2}u_n - |\varphi|^{p_+-2}\varphi\right)T_k(u_n-\varphi)$ is nonnegative.

Furthermore, $T_k(u_n-\varphi)$ converges weakly* to $T_k(u-\varphi)$ in $L^\infty(\Omega)$ and $|\varphi|^{p_+-2}\varphi \in L^{p'_+}(\Omega)$, so

$$\lim_{n\to+\infty} \int_\Omega |\varphi|^{p_+-2}\varphi T_k(u_n-\varphi)dx = \int_\Omega |\varphi|^{p_+-2}\varphi T_k(u-\varphi)dx. \tag{3.50}$$

Combining (3.49) and (3.50), we obtain

$$\lim_{n\to+\infty} \varepsilon \int_\Omega |u_n|^{p_+-2}u_n T_k(u_n-\varphi)dx \geq 0, \quad \text{as } \varepsilon \to 0. \tag{3.51}$$

The relations (3.48) and (3.51) give us (3.46).

For the second term of the left hand side of (3.43), we recall that $l = k + ||\varphi||_{L^\infty(\Omega)}$ and we get

$$\int_\Omega a(x,u_n,\nabla u_n).\nabla T_k(u_n-\varphi)dx = \int_\Omega a(x,u_n,\nabla T_l(u_n)).\nabla(T_l(u_n)-\varphi)\chi_{\{|u_n-\varphi|<k\}}dx$$
$$= \int_\Omega a(x,u_n,\nabla T_l(u_n)).\nabla T_l(u_n)\chi_{\{|u_n-\varphi|<k\}}dx - \int_\Omega a(x,u_n,\nabla T_l(u_n)).\nabla\varphi\chi_{\{|u_n-\varphi|<k\}}dx.$$
$$\tag{3.52}$$

Moreover, $a(x,u_n,\nabla T_l(u_n)).\nabla T_l(u_n)\chi_{\{|u_n-\varphi|<k\}}$ is nonnegative and converges a.e. in Ω to $a(x,u,\nabla T_l(u))\nabla T_l(u)\chi_{\{|u-\varphi|<k\}}$. Thanks to Fatou's Lemma, we get

$$\liminf_{n\to+\infty} \int_\Omega a(x,u_n,\nabla T_l(u_n)).\nabla T_l(u_n)\chi_{\{|u_n-\varphi|<k\}}dx \geq \int_\Omega a(x,u,\nabla T_l(u)).\nabla T_l(u)\chi_{\{|u-\varphi|<k\}}dx.$$
$$\tag{3.53}$$

We now focus our attention on $\int_\Omega a(x,u_n,\nabla T_l(u_n))\nabla\varphi\chi_{\{|u_n-\varphi|<k\}}$.

Let us prove that $a(x,u_n,\nabla T_l(u_n)).\nabla\varphi\chi_{\{|u_n-\varphi|<k\}}$ is equi-integrable. Let E be a subset of Ω.

$$\int_E a(x,u_n,\nabla T_l(u_n)).\nabla\varphi\chi_{\{|u_n-\varphi|<k\}}dx \leq \int_E |a(x,u_n,\nabla T_l(u_n))||\nabla\varphi|dx$$
$$\leq \int_E \frac{1}{\pi'_n(.)}|a(x,u_n,\nabla T_l(u_n))|^{\pi'_n(.)}dx + \int_E \frac{1}{\pi_n(.)}|\nabla\varphi|^{\pi_n(.)}dx$$
$$\leq C_1 \int_E \left(\mathcal{M}(x)+|\nabla T_l(u_n)|^{\pi_n(.)}\right)dx + \int_E |\nabla\varphi|^{\pi_n(.)}dx.$$

Moreover,

$$
\begin{aligned}
\int_E |\nabla\varphi|^{\pi_n(.)}dx &= \int_{E\cap\{|\nabla\varphi|\leq 1\}} |\nabla\varphi|^{\pi_n(.)}dx + \int_{E\cap\{|\nabla\varphi|>1\}} |\nabla\varphi|^{\pi_n(.)}dx \\
&\leq meas(E) + \int_E |\nabla\varphi|^{p_+}dx,
\end{aligned}
$$

since $|\nabla\varphi|^{p_+}, \mathcal{M} \in L^1(\Omega)$ and $|\nabla T_l(u_n)|^{\pi_n(.)}$ is equi-integrable $\big($using density argument for $C^\infty(\overline{\Omega})$ and (3.4)$\big)$. Then, we obtain

$$
\lim_{meas(E)\to 0} \int_E a(x, u_n, \nabla T_l(u_n))\nabla\varphi\chi_{\{|u_n-\varphi|<k\}}dx = 0.
$$

Furthermore,

$$
a(x, u_n, \nabla T_l(u_n))\nabla\varphi\chi_{\{|u_n-\varphi|<k\}} \to a(x, u, \nabla T_l(u)).\nabla\varphi\chi_{\{|u-\varphi|<k\}} \quad \text{a.e. in} \quad \Omega.
$$

By applying Vitali's Theorem, we obtain

$$
\lim_{n\to+\infty} \int_\Omega a(x, u_n, \nabla T_l(u_n))\nabla\varphi\chi_{\{|u_n-\varphi|<k\}}dx = \int_\Omega a(x, u, \nabla T_l(u)).\nabla\varphi\chi_{\{|u-\varphi|<k\}}dx. \quad (3.54)
$$

Using (3.52), (3.53) and (3.54) we get

$$
\begin{aligned}
\liminf_{n\to+\infty} \int_\Omega a(x, u_n, \nabla u_n).\nabla T_k(u_n-\varphi)dx &\geq \int_\Omega a(x, u, \nabla T_l(u))\nabla(T_l(u)-\varphi)\chi_{\{|u-\varphi|<k\}}dx \\
&= \int_\Omega a(x, u, \nabla u)\nabla T_k(u-\varphi)dx. \quad (3.55)
\end{aligned}
$$

Now, we consider the right hand side of (3.43). For the first term of the right hand side of (3.43), since $f_n \to f$ in $L^1(\Omega)$ and $T_k(u_n-\varphi) \rightharpoonup^* T_k(u-\varphi)$ in $L^\infty(\Omega)$, we have

$$
\lim_{n\to+\infty} \int_\Omega f_n T_k(u_n-\varphi)dx = \int_\Omega f T_k(u-\varphi)dx. \quad (3.56)
$$

For the second term of the right hand side of (3.43), by using the fact that g_n strongly converges to g in $L^1(\partial\Omega)$, we obtain

$$
\lim_{n\to+\infty} \int_{\partial\Omega} g_n T_k(u_n-\varphi)d\sigma = \int_{\partial\Omega} g T_k(u-\varphi)d\sigma, \quad (3.57)
$$

since

$$
T_k(u_n-\varphi) \rightharpoonup^* T_k(u-\varphi) \text{ in } L^\infty(\partial\Omega), \quad (3.58)
$$

because $u_n \to u$ a.e. on $\partial\Omega$.

Using (3.44), (3.45), (3.46), (3.55), (3.56), (3.57) and (3.43), we get

$$\int_\Omega b(u)T_k(u-\varphi)dx + \int_\Omega a(x,u,\nabla u).\nabla T_k(u-\varphi)dx + \int_{\partial\Omega} \lambda u T_k(u-\varphi)d\sigma$$

$$\leq \quad \liminf_{n\to+\infty} \left(\int_\Omega T_n(b(u_n))T_k(u_n-\varphi)dx + \int_\Omega a(x,u_n,\nabla u_n).\nabla T_k(u_n-\varphi)dx \right.$$

$$+ \quad \int_{\partial\Omega} \lambda T_n(u_n)T_k(u_n-\varphi)d\sigma + \varepsilon \int_\Omega \left[|\nabla u_n|^{p_+-2}\nabla u_n \nabla T_k(u_n-\varphi) + |u_n|^{p_+-2}u_n T_k(u_n-\varphi) \right]dx \Bigg)$$

$$= \quad \int_\Omega f T_k(u-\varphi)dx + \int_{\partial\Omega} g T_k(u-\varphi)d\sigma, \quad \text{as } \varepsilon \to 0,$$

which is equivalent to

$$\int_\Omega b(u)T_k(u-\varphi)dx + \int_\Omega a(x,u,\nabla u).\nabla T_k(u-\varphi)dx \quad + \quad \int_{\partial\Omega} \lambda u T_k(u-\varphi)d\sigma$$

$$\leq \quad \int_\Omega f T_k(u-\varphi)dx + \int_{\partial\Omega} g T_k(u-\varphi)d\sigma,$$

$$(3.59)$$

for $\varphi \in C^\infty(\overline\Omega)$.

As $\pi(.)$ verifies the log-Hölder condition (2.1), $C^\infty(\overline\Omega)$ is dense in the space $W^{1,\pi(.)}(\Omega)$. Moreover, $W^{1,\pi(.)}(\Omega) \hookrightarrow W^{1,p_-}(\Omega) \hookrightarrow L^\infty(\Omega)$, since $\pi(.) \geq p_- > N$ and Ω is a bounded open domain with Lipschitz boundary $\partial\Omega$. Therefore, the inequality (3.59) holds true for $\varphi \in W^{1,\pi(.)}(\Omega) \cap L^\infty(\Omega)$. Hence, u is an entropy solution of (1.1). $\square$

Now, we state the uniqueness result of entropy solution. This result uses the same arguments as [2]-Theorem 2.8.

Theorem 3.5. *Assume that b is strictly increasing. Assume that $a = a(x,z,\eta)$ satisfies $(A_3)-(A_6)$ and $\mathcal{M}$ constant. Moreover, a satisfies:*
for all bounded subset K of $\mathbb{R} \times \mathbb{R}^N$, there exists a constant $C(K)$ such that

$$a.e. \ \ x \in \Omega, \ \ for \ all \ \ (z,\eta),(\tilde z,\eta) \in K,$$

$$|a(x,z,\eta) - a(x,\tilde z,\eta)| \leq C(K)|z - \tilde z|. \tag{3.60}$$

Finally, suppose the following regularity property:

$$for \ all \ \ f \in L^\infty(\Omega) \ \ and \ \ g \in L^\infty(\partial\Omega)$$

$$there \ exists \ an \ entropy \ solution \ of \ (1.1),$$

$$which \ is \ Lipchitz \ continuous \ on \ \overline\Omega. \tag{3.61}$$

Then, for all $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$ the problem (1.1) admits a unique entropy solution.

Remark 3.6. *As in [2, Theorem 2.8], the condition (3.61) goes back to idea of [3]. Moreover, in the Theorem 3.5 the relation (3.60) is used to obtain the inequality (3.69) below.*

Proof. The proof of this theorem is done in two steps.

Step 1. A priori estimates.

Lemma 3.7. *If v is an entropy solution of (1.1), there exists a positive constant C such that*

$$\rho_{p(.,v(.))}\big(|\nabla v|\chi_F\big) \leq Ck,$$

where $F = \{h - k < |v| < h\}$, $h > k > 0$.

Proof. Let $\varphi = T_{h-k}(v)$ as test function in the entropy inequality (3.1), we get

$$\int_\Omega a(x, v, \nabla v).\nabla T_k(v - T_{h-k}(v))dx \quad + \quad \int_\Omega b(v)T_k(v - T_{h-k}(v))dx + \lambda \int_{\partial\Omega} vT_k(v - T_{h-k}(v))d\sigma$$

$$\leq \quad \int_\Omega fT_k(v - T_{h-k}(v))dx + \int_{\partial\Omega} gT_k(v - T_{h-k}(v))d\sigma.$$

Thus,

$$\int_{\{h-k<|v|<h\}} a(x, v, \nabla v).\nabla T_k(v - T_{h-k}(v))dx \leq k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big)$$

and using (A_6), we have

$$\int_F |\nabla v|^{p(x,v(x))}dx \leq kC_2\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big).$$

Consequently,

$$\rho_{p(.,v(.))}\big(|\nabla v|\chi_F\big) \leq Ck, \quad \text{where} \quad C = C_2\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big). \qquad \square$$

We give the following lemma.

Lemma 3.8. *If u is an entropy solution of (1.1), then*

$$meas\big(\{|u| > h\}\big) \leq \frac{||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}}{\min\big(b(h), |b(-h)|\big)}, \quad \forall h \geq 1.$$

Proof. Let us take $\varphi = 0$ and $k = h$ in entropy inequality (3.1).
Since

$$\int_\Omega a(x, u, \nabla u).\nabla T_h(u)dx + \lambda \int_{\partial\Omega} uT_h(u)d\sigma \geq 0,$$

the relation (3.1) gives

$$\int_\Omega b(u)T_h(u)dx \leq \int_\Omega fT_h(u)dx + \int_{\partial\Omega} gT_h(u)d\sigma.$$

Then,

$$\int_{\{|u|\leq h\}} b(u)T_h(u)dx + \int_{\{|u|>h\}} b(u)T_h(u)dx \leq h\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big),$$

or

$$\int_{\{|u|>h\}} \frac{b(u)T_h(u)}{h}dx = \int_{\{u>h\}} b(u)dx + \int_{\{u<-h\}} -b(u)dx \leq \big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big).$$

Therefore,

$$\int_{\{|u|>h\}} |b(u)|dx \leq ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}.$$

Since b is nondecreasing, we deduce

$$\int_{\{|u|>h\}} \min(b(h),|b(-h)|)dx \leq \int_{\{|u|>h\}} |b(u)| \leq ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}, \quad \forall h \geq 1.$$

So,

$$meas\big(\{|u|>h\}\big) \leq \frac{||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}}{\min\big(b(h),|b(-h)|\big)}, \quad \forall h \geq 1. \qquad \square$$

Step 2. Uniqueness.

The existence has already been proved. Now, we show the uniqueness. For more details see [2]-Proof of Theorem 2.8.

Let u be a Lipschitz continuous entropy solution of (1.1) with $f \in L^\infty(\Omega), g \in L^\infty(\partial\Omega)$ and v be an entropy solution, with $\hat{f} \in L^1(\Omega), \hat{g} \in L^1(\partial\Omega)$.

Since Ω is open bounded domain with smooth boundary $\partial\Omega$, the space of Lipschitz functions $C^{0,1}(\overline{\Omega})$ and $W^{1,\infty}(\Omega)$ are homeomorphic and they can be identified. Therefore, u belongs to $W^{1,\infty}(\Omega)$. Thus, for all $h > 0$, we can write the entropy inequality corresponding to the solution u, with $T_h(v)$ as a test function and to the solution v, with $T_h(u)$ as a test function. For all $k > 0$, we get

$$\begin{cases} \displaystyle\int_\Omega a(x,u,\nabla u).\nabla T_k(u-T_h(v))dx + \int_\Omega b(u)T_k(u-T_h(v))dx \\ +\lambda \displaystyle\int_{\partial\Omega} uT_k(u-T_h(v))d\sigma \leq \int_\Omega fT_k(u-T_h(v))dx + \int_{\partial\Omega} gT_k(u-T_h(v))d\sigma \end{cases} \tag{3.62}$$

and

$$\begin{cases} \displaystyle\int_\Omega a(x,v,\nabla v).\nabla T_k(v-T_h(u))dx + \int_\Omega b(v)T_k(v-T_h(u))dx \\ +\lambda \displaystyle\int_{\partial\Omega} vT_k(v-T_h(u))d\sigma \leq \int_\Omega \hat{f}T_k(v-T_h(u))dx + \int_{\partial\Omega} \hat{g}T_k(v-T_h(u))d\sigma. \end{cases} \tag{3.63}$$

Adding (3.62) and (3.63) we obtain

$$
\begin{cases}
\displaystyle\int_\Omega a(x,u,\nabla u).\nabla T_k(u-T_h(v))dx + \int_\Omega a(x,v,\nabla v).\nabla T_k(v-T_h(u))dx \\
\displaystyle+ \int_\Omega b(u)T_k(u-T_h(v))dx + \int_\Omega b(v)T_k(v-T_h(u))dx \\
\displaystyle+\lambda \int_{\partial\Omega} uT_k(u-T_h(v))d\sigma + \lambda \int_{\partial\Omega} vT_k(v-T_h(u))d\sigma \\
\displaystyle\leq \int_\Omega \big[fT_k(u-T_h(v)) + \hat{f}T_k(v-T_h(u))\big]dx \\
\displaystyle+ \int_{\partial\Omega} \big[gT_k(u-T_h(v)) + \hat{g}T_k(v-T_h(u))\big]d\sigma.
\end{cases}
\tag{3.64}
$$

Set $A = \{0 < |u-v| < k, |v| \leq h\}$; $B = A \cap \{|u| \leq h\}$; $C = A \cap \{|u| > h\}$ and $A' = \{0 < |v-u| < k, |u| \leq h\}$; $B' = A' \cap \{|v| \leq h\}$; $C' = A' \cap \{|v| > h\}$. We start with the first integral in (3.64). We have

$$
\int_{\{0<|u-T_h(v)|<k\}} a(x,u,\nabla u).\nabla T_k(u-T_h(v))dx
$$

$$
= \int_{\{0<|u-T_h(v)|<k\}\cap\{|v|\leq h\}} a(x,u,\nabla u).\nabla T_k(u-T_h(v))dx
$$

$$
+ \int_{\{0<|u-T_h(v)|<k\}\cap\{|v|>h\}} a(x,u,\nabla u).\nabla T_k(u-T_h(v))dx
$$

$$
= \int_{\{0<|u-v|<k\}\cap\{|v|\leq h\}} a(x,u,\nabla u).\nabla(u-v)dx
$$

$$
+ \int_{\{0<|u-hsign(v)|<k\}\cap\{|v|>h\}} a(x,u,\nabla u).\nabla u dx
$$

$$
\geq \int_A a(x,u,\nabla u)\nabla(u-v)dx
$$

$$
= \int_B a(x,u,\nabla u)\nabla(u-v)dx + \int_C a(x,u,\nabla u)\nabla(u-v)dx.
$$

Then, we get

$$
\begin{cases}
\displaystyle\int\!\!\int_{\{0<|u-T_h(v)|<k\}} a(x,u,\nabla u).\nabla T_k(u-T_h(v))dx \\
\displaystyle\geq \int_B a(x,u,\nabla u)\nabla(u-v)dx - \int_C a(x,u,\nabla u)\nabla v dx.
\end{cases}
\tag{3.65}
$$

Now we use the fact that ∇u is bounded. By assumption of the theorem ($\mathcal{M}$ is constant), $|a(x,u,\nabla u)| \leq C(|\nabla u|^{p(x,u(x))} + 1) \in L^\infty(\Omega)$. Therefore, there exists a constant K such that

$$
\left| \int_C a(x,u,\nabla u)\nabla v dx \right| \leq \int_C |a(x,u,\nabla u)||\nabla v| dx
$$

$$
\leq K \int_C |\nabla v| dx \leq K \int_{\{h-k<|v|<h\}} |\nabla v| dx,
\tag{3.66}
$$

since $C \subset \{h-k < |v| < h\}$.

Thanks to Lemma 3.8, $\displaystyle\lim_{h\to+\infty} meas(\{h-k < |v| < h\}) = 0$ and by Lemma 3.7, $|\nabla v|\chi_F \in L^1(\Omega)$.

So, the right hand side of (3.66) converges to zero, as h goes to infinity.

Consequently, the second integral of the right hand side of (3.65) converges to zero, as h goes to infinity. Then, we can write that

$$\int_{\{0<|u-T_h(v)|<k\}} a(x,u,\nabla u).\nabla T_k(u-T_h(v))dx \geq \int_B a(x,u,\nabla u).\nabla(u-v)dx$$
$$+ \ I_h, \ \text{with} \ \lim_{h\to+\infty} I_h = 0.$$

As $B = B'$, we may adopt the same procedure to treat the second integral of (3.64) to obtain

$$\int_{\{0<|v-T_h(u)|<k\}} a(x,v,\nabla v).\nabla T_k(v-T_h(u))dx \geq -\int_B a(x,v,\nabla v).\nabla(u-v)dx$$
$$+ \ J_h, \ \text{with} \ \lim_{h\to+\infty} J_h = 0.$$

For the other terms in the left hand side of (3.64), we denote by

$$K_h = \int_\Omega b(u)T_k(u-T_h(v))dx + \int_\Omega b(v)T_k(v-T_h(u))dx$$

and

$$L_h = \lambda \int_{\partial\Omega} uT_k(u-T_h(v))d\sigma + \lambda \int_{\partial\Omega} vT_k(v-T_h(u))d\sigma.$$

We have

$$b(u)T_k(u-T_h(v)) \to b(u)T_k(u-v) \ \text{a.e. in} \ \Omega \ \text{as} \ h \to +\infty$$

and

$$|b(u)T_k(u-T_h(v))| \leq k|b(u)| \in L^1(\Omega).$$

Then, by Lebesgue dominated convergence Theorem, we get

$$\lim_{h\to+\infty} \int_\Omega b(u)T_k(u-T_h(v))dx = \int_\Omega b(u)T_k(u-v)dx$$

and

$$\lim_{h\to+\infty} \int_\Omega b(v)T_k(v-T_h(u))dx = \int_\Omega b(v)T_k(v-u)dx.$$

Then,

$$\lim_{h\to+\infty} K_h = \int_\Omega (b(u)-b(v))T_k(u-v)dx.$$

Similarly, we obtain

$$\lim_{h\to+\infty} L_h = \lambda \int_{\partial\Omega} (u-v)T_k(u-v)d\sigma.$$

Now, we consider the right hand side of (3.64), we have

$$\lim_{h\to+\infty} \left[fT_k(u-T_h(v)) + \hat{f}T_k(v-T_h(u)) \right] = (f-\hat{f})T_k(u-v) \ \text{a.e. in} \ \Omega$$

and

$$\left| fT_k(u - T_h(v)) + \hat{f}T_k(v - T_h(u)) \right| \leq k(|f| + |\hat{f}|) \in L^1(\Omega).$$

By Lebesgue dominated convergence Theorem, we get

$$\lim_{h \to +\infty} \int_\Omega f\big[T_k(u - T_h(v)) + T_k(v - T_h(u))\big]dx = \int_\Omega (f - \hat{f})T_k(u - v)dx.$$

Similarly, we have

$$\lim_{h \to +\infty} \int_{\partial\Omega} g\big[T_k(u - T_h(v)) + \hat{g}T_k(v - T_h(u))\big]d\sigma = \int_{\partial\Omega} (g - \hat{g})T_k(u - v)d\sigma.$$

After passing to the limit as h goes to $+\infty$ in (3.64), we get

$$\begin{cases} \displaystyle\int_{\{0<|u-v|<k\}} \big(a(x,u,\nabla u) - a(x,v,\nabla v)\big)\nabla(u - v)dx \\ \displaystyle + \int_\Omega (b(u) - b(v))T_k(u - v)dx + \lambda \int_{\partial\Omega} (u - v)T_k(u - v)d\sigma \\ \displaystyle \leq \int_\Omega (f - \hat{f})T_k(u - v)dx + \int_{\partial\Omega} (g - \hat{g})T_k(u - v)d\sigma, \end{cases} \tag{3.67}$$

which is equivalent to

$$\begin{cases} \displaystyle\int_{\{0<|u-v|<k\}} \big(a(x,u,\nabla v) - a(x,v,\nabla v)\big)\nabla(u - v)dx \\ \displaystyle + \int_{\{0<|u-v|<k\}} \big(a(x,u,\nabla u) - a(x,u,\nabla v)\big)\nabla(u - v)dx \\ \displaystyle + \int_\Omega (b(u) - b(v))T_k(u - v)dx + \lambda \int_{\partial\Omega} (u - v)T_k(u - v)d\sigma \\ \displaystyle \leq \int_\Omega (f - \hat{f})T_k(u - v)dx + \int_{\partial\Omega} (g - \hat{g})T_k(u - v)d\sigma. \end{cases} \tag{3.68}$$

Dividing (3.68) by k and letting k goes to 0, we have

$$\begin{cases} \displaystyle\lim_{k \to 0} \frac{1}{k} \int_{\{0<|u-v|<k\}} \big(a(x,u,\nabla v) - a(x,v,\nabla v)\big)\nabla(u - v)dx \\ \displaystyle + \lim_{k \to 0} \frac{1}{k} \int_{\{0<|u-v|<k\}} \big(a(x,u,\nabla u) - a(x,u,\nabla v)\big)\nabla(u - v)dx \\ \displaystyle + \int_\Omega |b(u) - b(v)|dx + \lambda \int_{\partial\Omega} |u - v|d\sigma \\ \displaystyle \leq \int_\Omega (f - \hat{f})sign(u - v)dx + \int_{\partial\Omega} (g - \hat{g})sign(u - v)d\sigma. \end{cases} \tag{3.69}$$

Thanks to the relation (3.60), the first integral of (3.69) goes to 0 as $k \to 0$ (See [2], proof of Theorem 2.8-Step 2). Thus, we obtain

$$\begin{cases} \displaystyle\lim_{k \to 0} \frac{1}{k} \int_{\{0<|u-v|<k\}} \big(a(x,u,\nabla u) - a(x,u,\nabla v)\big)\nabla(u - v)dx \\ \displaystyle + \int_\Omega |b(u) - b(v)|dx + \lambda \int_{\partial\Omega} |u - v|d\sigma \\ \displaystyle \leq \int_\Omega |f - \hat{f}|dx + \int_{\partial\Omega} |g - \hat{g}|d\sigma. \end{cases} \tag{3.70}$$

Since, the three integral of left-hand in (3.70) are positive, we deduce that

$$\int_\Omega |b(u) - b(v)|dx + \lambda \int_{\partial\Omega} |u - v|d\sigma \le \int_\Omega |f - \hat{f}|dx + \int_{\partial\Omega} |g - \hat{g}|d\sigma. \qquad (3.71)$$

Let us take a sequence $(f_i)_{i\in\mathbb{N}} \subset L^\infty(\Omega)$ and $(g_i)_{i\in\mathbb{N}} \subset L^\infty(\partial\Omega)$ and let $(u_i)_{i\in\mathbb{N}}$ be the corresponding sequence of Lipschitz continuous entropy solutions. By (3.71), we have

$$\begin{aligned}
\int_\Omega |b(u) - b(v)|dx \quad &+\quad \lambda \int_{\partial\Omega} |u - v|d\sigma \le \int_\Omega \big[|b(u) - b(u_i)| + |b(v) - b(u_i)|\big]dx \\
&+\quad \lambda \int_{\partial\Omega} \big[|u - u_i| + |v - u_i|\big]d\sigma \\
&\le\quad \int_\Omega \big[|f - f_i| + |\hat{f} - f_i|\big]dx + \int_{\partial\Omega} \big[|g - g_i| + |\hat{g} - g_i|\big]d\sigma, \qquad (3.72)
\end{aligned}$$

so that at the limit as $i \to \infty$ in left hand-side of (3.72) and using the density argument ($L^\infty(\Omega)$ and $L^\infty(\partial\Omega)$ are dense (respectively) in $L^1(\Omega)$ and $L^1(\partial\Omega)$), we infer that

$$b(u) = b(v) \ \text{ a.e. in } \ \Omega \ \text{ and } \ u = v \ \text{ a.e. on } \ \partial\Omega.$$

Hence,

$$u = v \ \text{ a.e. in } \ \Omega \ \text{ and } \ u = v \ \text{ a.e. on } \ \partial\Omega.$$

Since b is assumed strictly increasing. $\square$

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