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Common Fixed Point Results in C^*-Algebra Valued b-Metric Spaces Via Digraphs

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ABSTRACT

We discuss the existence and uniqueness of points of coincidence and common fixed points for a pair of self-mappings defined on a C^*-algebra valued b-metric space endowed with a graph. Our results extend and supplement several recent results in the literature. Strength of hypotheses made in the first result have been weighted through illustrative examples.

RESUMEN

Discutimos la existencia y unicidad de puntos de coincidencia y puntos fijos comumes para un par de aplicaciones definidas en un b-espacio métrico a valores en una álgebra C^* dotado de un grafo en sí mismo. Nuestros resultados extienden y suplementan diversos resultados recientes en la literatura. La fuerza de las hipótesis impuestas al primer resultado se evalúa a través de ejemplos ilustrativos.

Keywords and Phrases: C^*-algebra, C^*-algebra valued b-metric, directed graph, C^*-algebra valued G-contraction, common fixed point.

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1 Introduction

In 1922 [5], Polish mathematician S. Banach proved a very important result regarding a contraction mapping, known as the Banach contraction principle. This fundamental principle was largely applied in many branches of mathematics. Several authors generalized this interesting theorem in different ways(see [1, 2, 6, 13, 18, 25, 26, 27]). In this context, Bakhtin [4] and Czerwik [11] developed the notion of $\mathfrak{b}$-metric spaces and proved some fixed point theorems for single-valued and multi-valued mappings in the setting of $\mathfrak{b}$-metric spaces. In 2014, Z. Ma et.al.[22] introduced the concept of C^*-algebra valued metric spaces by using the set of all positive elements of an unital C^*-algebra instead of the set of real numbers. In [21], the authors introduced another new concept, known as C^*-algebra valued $\mathfrak{b}$-metric spaces as a generalization of C^*-algebra valued metric spaces and $\mathfrak{b}$-metric spaces.

In recent investigations, the study of fixed point theory endowed with a graph plays an important role in many aspects. In 2005, Echenique [15] studied fixed point theory by using graphs. After that, Espinola and Kirk [16] applied fixed point results in graph theory. Recently, combining fixed point theory and graph theory, a series of articles(see [3, 8, 9, 10, 20, 24] and references therein) have been dedicated to the improvement of fixed point theory.

The idea of common fixed point was initially given by Junck [19]. In fact, the author introduced the concept of weak compatibility and obtained a common fixed point result. Several authors have obtained coincidence points and common fixed points for various classes of mappings on a metric space by using this concept. Motivated by some recent works on the extension of Banach contraction principle to metric spaces with a graph, we reformulated some important common fixed point results in metric spaces to C^*-algebra valued $\mathfrak{b}$-metric spaces endowed with a graph. As some consequences of this study, we deduce several related results in fixed point theory. Finally, some examples are provided to illustrate the results.

2 Some basic concepts

We begin with some basic notations, definitions and properties of C^*-algebras. Let $\mathbb{A}$ be an unital algebra with the unit I. An involution on $\mathbb{A}$ is a conjugate linear map $\mathfrak{a} \mapsto \mathfrak{a}^*$ on $\mathbb{A}$ such that $\mathfrak{a}^{**} = \mathfrak{a}$ and $(\mathfrak{a}\mathfrak{b})^* = \mathfrak{b}^*\mathfrak{a}^*$ for all $\mathfrak{a}, \mathfrak{b} \in \mathbb{A}$. The pair $(\mathbb{A}, *)$ is called a $*$-algebra. A Banach $*$-algebra is a $*$-algebra $\mathbb{A}$ together with a complete submultiplicative norm such that $\| \mathfrak{a}^* \| = \| \mathfrak{a} \|$ for all $\mathfrak{a} \in \mathbb{A}$. A C^*-algebra is a Banach $*$-algebra such that $\| \mathfrak{a}^*\mathfrak{a} \| = \| \mathfrak{a} \|^2$ for all $\mathfrak{a} \in \mathbb{A}$. Let H be a Hilbert space and $\mathsf{B}(\mathsf{H})$, the set of all bounded linear operators on H. Then, under the norm topology, $\mathsf{B}(\mathsf{H})$ is a C^*-algebra.

Throughout this discussion, by $\mathbb{A}$ we always denote an unital C^*-algebra with the unit I and the zero element θ. Set $\mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$. We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$, if $x \in \mathbb{A}_h$ and $\sigma(x) \subset [0, \infty)$, where $\sigma(x)$ is the spectrum of x. Using positive elements, one can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows:

$$x \preceq y \text{ if and only if } y - x \succeq \theta.$$

We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$.

From now on, by $\mathbb{A}_+$, we denote the set $\{x \in \mathbb{A} : x \succeq \theta\}$ and by $\mathbb{A}'$, we denote the set $\{a \in \mathbb{A} : ab = ba, \forall b \in \mathbb{A}\}$.

Lemma 2.1. *[14, 23] Suppose that $\mathbb{A}$ is an unital C^*-algebra with a unit I.*

(i) For any $x \in \mathbb{A}_+$, we have $x \preceq I \Leftrightarrow \| x \| \leq 1$.

(ii) If $a \in \mathbb{A}_+$ with $\| a \| < \frac{1}{2}$, then $I - a$ is invertible and $\| a(I - a)^{-1} \| < 1$.

(iii) Suppose that $a, b \in \mathbb{A}$ with $a, b \succeq \theta$ and $ab = ba$, then $ab \succeq \theta$.

(iv) Let $a \in \mathbb{A}'$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq \theta$, and $I - a \in \mathbb{A}'_+$ is an invertible operator, then $(I - a)^{-1} b \succeq (I - a)^{-1} c$.

Remark 2.2. *It is worth mentioning that $x \preceq y \Rightarrow \| x \| \leq \| y \|$ for $x, y \in \mathbb{A}_+$. In fact, it follows from Lemma 2.1 (i).*

Definition 2.3. *[22] Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies:*

(i) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a C^*-algebra valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra valued metric space.

Definition 2.4. *[4] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:*

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq s\,(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

Definition 2.5. *[21] Let* X *be a nonempty set and* $A \in \mathbb{A}_+^{'}$ *such that* $A \succeq I$. *Suppose the mapping* $d : X \times X \to \mathbb{A}$ *satisfies:*

(i) $\theta \preceq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = \theta$ *if and only if* $x = y$*;*

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*

(iii) $d(x, y) \preceq A\,(d(x, z) + d(z, y))$ *for all* $x, y, z \in X$.

Then d *is called a* C^**-algebra valued* b*-metric on* X *and* $(X, \mathbb{A}, d)$ *is called a* C^**-algebra valued* b*-metric space.*

It seems important to note that if $\mathbb{A} = \mathbb{C}, A = 1$, then the C^*-algebra valued b-metric spaces are just the ordinary metric spaces. Moreover, it is obvious that C^*-algebra valued b-metric spaces generalize the concepts of C^*-algebra valued metric spaces and b-metric spaces.

Definition 2.6. *[26] Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space,* $x \in X$ *and* (x_n) *be a sequence in* X. *Then*

(i) (x_n) *converges to* x *with respect to* $\mathbb{A}$ *if for any* $\epsilon > 0$ *there is* n_0 *such that for all* $n > n_0$*,* $\| d(x_n, x) \| \leq \epsilon$. *We denote it by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x (n \to \infty)$.

(ii) (x_n) *is Cauchy with respect to* $\mathbb{A}$ *if for any* $\epsilon > 0$ *there is* n_0 *such that for all* $n, m > n_0$*,* $\| d(x_n, x_m) \| \leq \epsilon$.

(iii) $(X, \mathbb{A}, d)$ *is a complete* C^**-algebra valued* b*-metric space if every Cauchy sequence with respect to* $\mathbb{A}$ *is convergent.*

Example 2.7. *If* X *is a Banach space, then* $(X, \mathbb{A}, d)$ *is a complete* C^**-algebra valued* b*-metric space with* $A = 2^{p-1} I$ *if we set*

$$d(x, y) = \| x - y \|^p \, I$$

where $p > 1$ *is a real number. But* $(X, \mathbb{A}, d)$ *is not a* C^**-algebra valued metric space because if* $X = \mathbb{R}$*, then* $| x - y |^p \leq | x - z |^p + | z - y |^p$ *is impossible for all* $x > z > y$.

Definition 2.8. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space with the coefficient* $A \succeq I$. *We call a mapping* $f : X \to X$ *a* C^**-algebra valued contraction mapping on* X *if there exists* $B \in \mathbb{A}$ *with* $\| B \|^2 < \frac{1}{\|A\|}$ *such that*

$$d(fx, fy) \preceq B^* \, d(x, y) B$$

for all $x, y \in X$.

Definition 2.9. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space with the coefficient* $A \succeq I$. *A mapping* $f : X \to X$ *is called a* C^**-algebra valued Fisher contraction if there exists* $B \in \mathbb{A}_+^{'}$ *with* $\| BA \| < \frac{1}{\|A\|+1}$ *such that*

$$d(fx, fy) \preceq B\,[d(fx, y) + d(fy, x)]$$

for all $x, y \in X$.

Definition 2.10. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space with the coefficient* $A \succeq I$*. A mapping* $f : X \to X$ *is called a* C^**-algebra valued Kannan operator if there exists* $B \in \mathbb{A}_+^{'}$ *with* $\| B \| < \frac{1}{\|A\|+1}$ *such that*

$$d(fx, fy) \preceq B\,[d(fx, x) + d(fy, y)]$$

for all $x, y \in X$*.*

Definition 2.11. *[2] Let* T *and* S *be self mappings of a set* X*. If* $y = Tx = Sx$ *for some* x *in* X*, then* x *is called a coincidence point of* T *and* S *and* y *is called a point of coincidence of* T *and* S*.*

Definition 2.12. *[19] The mappings* $T, S : X \to X$ *are weakly compatible, if for every* $x \in X$*, the following holds:*

$$T(Sx) = S(Tx) \text{ whenever } Sx = Tx.$$

Proposition 2.13. *[2] Let* S *and* T *be weakly compatible selfmaps of a nonempty set* X*. If* S *and* T *have a unique point of coincidence* $y = Sx = Tx$*, then* y *is the unique common fixed point of* S *and* T*.*

Definition 2.14. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space with the coefficient* $A \succeq I$*. A mapping* $f : X \to X$ *is called* C^**-algebra valued expansive if there exists* $B \in \mathbb{A}$ *with* $0 < \| B \|^2 < \frac{1}{\|A\|}$ *such that*

$$B^* d(fx, fy) B \succeq d(x, y)$$

for all $x, y \in X$*.*

We next review some basic notions in graph theory.

Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued b-metric space. Let G be a directed graph (digraph) with a set of vertices $V(G) = X$ and a set of edges $E(G)$ contains all the loops, i.e., $E(G) \supseteq \Delta$, where $\Delta = \{(x, x) : x \in X\}$. We also assume that G has no parallel edges and so we can identify G with the pair $(V(G), E(G))$. G may be considered as a weighted graph by assigning to each edge the distance between its vertices. By G^{-1} we denote the graph obtained from G by reversing the direction of edges i.e., $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Our graph theory notations and terminology are standard and can be found in all graph theory books, like [7, 12, 17]. If x, y are vertices of the digraph G, then a path in G from x to y of length n $(n \in \mathbb{N})$ is a sequence $(x_i)_{i=0}^{n}$ of $n + 1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \cdots, n$. A graph G is connected if there is a path between any two vertices of G. G is weakly connected if $\tilde{G}$ is connected.

Definition 2.15. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space with the coefficient* $A \succeq I$ *and let* $G = (V(G), E(G))$ *be a graph. A mapping* $f : X \to X$ *is called a* C^**-algebra valued* G*-contraction if there exists a* $B \in \mathbb{A}$ *with* $\| B \|^2 < \frac{1}{\|A\|}$ *such that*

$$d(fx, fy) \preceq B^* d(x, y)B,$$

for all $x, y \in X$ *with* $(x, y) \in E(G)$.

Any C^*-algebra valued contraction mapping on X is a G_0-contraction, where G_0 is the complete graph defined by $(X, X \times X)$. But it is worth mentioning that a C^*-algebra valued G-contraction need not be a C^*-algebra valued contraction (see Remark 3.23).

Definition 2.16. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space with the coefficient* $A \succeq I$ *and let* $G = (V(G), E(G))$ *be a graph. A mapping* $f : X \to X$ *is called* C^**-algebra valued Fisher* G*-contraction if there exists* $B \in \mathbb{A}_+^{'}$ *with* $\| BA \| < \frac{1}{\|A\|+1}$ *such that*

$$d(fx, fy) \preceq B \left[d(fx, y) + d(fy, x)\right]$$

for all $x, y \in X$ *with* $(x, y) \in E(G)$.

It is easy to observe that a C^*-algebra valued Fisher contraction is a C^*-algebra valued Fisher G_0-contraction. But it is important to note that a C^*-algebra valued Fisher G-contraction need not be a C^*-algebra valued Fisher contraction. The following example supports the above remark.

Example 2.17. *Let* $X = [0, \infty)$ *and* $B(H)$ *be the set of all bounded linear operators on a Hilbert space* H. *Define* $d : X \times X \to B(H)$ *by* $d(x, y) = | x - y |^2 \ I$ *for all* $x, y \in X$. *Then* $(X, B(H), d)$ *is a* C^**-algebra valued* b*-metric space with the coefficient* $A = 2I$. *Let* G *be a digraph such that* $V(G) = X$ *and* $E(G) = \Delta \cup \{(3^t x, 3^t(x+1)) : x \in X \text{ with } x \geq 2,\ t = 0,\ 1,\ 2,\ \cdots\}$.

Let $f : X \to X$ *be defined by* $fx = 3x$ *for all* $x \in X$.

For $x = 3^t z,\ y = 3^t(z+1),\ z \geq 2$, *we have*

$$\begin{aligned} d(fx, fy) &= d\left(3^{t+1}z, 3^{t+1}(z+1)\right) \\ &= 3^{2t+2} I \\ &\preceq \frac{9}{58} 3^{2t}(8z^2 + 8z + 10)I \\ &= B\left[d\left(3^{t+1}z, 3^t(z+1)\right) + d\left(3^{t+1}(z+1), 3^t z\right)\right] \\ &= B\left[d(fx, y) + d(fy, x)\right], \end{aligned}$$

where $B = \frac{9}{58} I \in B(H)_+^{'}$ *with* $\| BA \| < \frac{1}{\|A\|+1}$. *Thus,* f *is a* C^**-algebra valued Fisher* G*-contraction.*

We now verify that f *is not a* C^**-algebra valued Fisher contraction. In fact, if* $x = 3,\ y = 0$, *then for any arbitrary* $B \in B(H)_+^{'}$ *with* $\| BA \| < \frac{1}{\|A\|+1} = \frac{1}{3}$*(which implies* $3BA \prec I$*), we have*

$$\begin{aligned} B\left[d(fx,y)+d(fy,x)\right] &= B\left[d(f3,0)+d(f0,3)\right] \\ &= 90BI \\ &= 45BA \\ &= \frac{5}{27}(3BA)(81I) \\ &\prec 81I \\ &= d(fx,fy). \end{aligned}$$

Definition 2.18. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space with the coefficient* $A \succeq I$ *and let* $G = (V(G), E(G))$ *be a graph. A mapping* $f : X \to X$ *is called* C^**-algebra valued* G*-Kannan if there exists* $B \in \mathbb{A}_+^{'}$ *with* $\| B \| < \frac{1}{\|A\|+1}$ *such that*

$$d(fx,fy) \preceq B\left[d(fx,x)+d(fy,y)\right]$$

for all $x, y \in X$ *with* $(x, y) \in E(G)$.

Note that any C^*-algebra valued Kannan operator is C^*-algebra valued G_0-Kannan. However, a C^*-algebra valued G-Kannan operator need not be a C^*-algebra valued Kannan operator (see Remark 3.28).

Remark 2.19. *If* f *is a* C^**-algebra valued* G*-contraction(resp.,* G*-Kannan or Fisher* G*-contraction), then* f *is both a* C^**-algebra valued* G^{-1}*-contraction(resp.,* G^{-1}*-Kannan or Fisher* G^{-1}*-contraction) and a* C^**-algebra valued* $\tilde{G}$*-contraction(resp.,* $\tilde{G}$*-Kannan or Fisher* $\tilde{G}$*-contraction).*

3 Main Results

In this section we always assume that $(X, \mathbb{A}, d)$ is a C^*-algebra valued b-metric space with the coefficient $A \succeq I$ and G is a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$.
Let $f,\ g : X \to X$ be such that $f(X) \subseteq g(X)$. If $x_0 \in X$ is arbitrary, then there exists an element $x_1 \in X$ such that $fx_0 = gx_1$, since $f(X) \subseteq g(X)$. Proceeding in this way, we can construct a sequence (gx_n) such that $gx_n = fx_{n-1},\ n = 1, 2, 3, \cdots$.

Definition 3.1. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space endowed with a graph* G *and* $f,\ g : X \to X$ *be such that* $f(X) \subseteq g(X)$*. We define* C_{gf} *the set of all elements* x_0 *of* X *such that* $(gx_n, gx_m) \in E(\tilde{G})$ *for* $m,\ n = 0, 1, 2, \cdots$ *and for every sequence* (gx_n) *such that* $gx_n = fx_{n-1}$.

If $g = I$, the identity map on X, then obviously C_{gf} becomes C_f which is the collection of all elements x of X such that $(f^n x, f^m x) \in E(\tilde{G})$ for $m,\ n = 0, 1, 2, \cdots$.

Theorem 3.2. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space endowed with a graph* G *and the mappings* $f,\ g : X \to X$ *be such that*

$$d(fx, fy) \preceq B^*\, d(gx, gy)\, B \tag{3.1}$$

for all $x, y \in X$ *with* $(gx, gy) \in E(\tilde{G})$, *where* $B \in \mathbb{A}$ *and* $\| B \|^2 < \frac{1}{\|A\|}$. *Suppose* $f(X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* X *with the following property:*

$(*)$ *If* (gx_n) *is a sequence in* X *such that* $gx_n \to x$ *and* $(gx_n, gx_{n+1}) \in E(\tilde{G})$ *for all* $n \geq 1$, *then there exists a subsequence* (gx_{n_i}) *of* (gx_n) *such that* $(gx_{n_i}, x) \in E(\tilde{G})$ *for all* $i \geq 1$.
Then f *and* g *have a point of coincidence in* X *if* $C_{gf} \neq \emptyset$. *Moreover,* f *and* g *have a unique point of coincidence in* X *if the graph* G *has the following property:*

$(**)$ *If* $x,\ y$ *are points of coincidence of* f *and* g *in* X, *then* $(x, y) \in E(\tilde{G})$.
Furthermore, if f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point in* X.

Proof. Suppose that $C_{gf} \neq \emptyset$. We choose an $x_0 \in C_{gf}$ and keep it fixed. Since $f(X) \subseteq g(X)$, there exists a sequence (gx_n) such that $gx_n = fx_{n-1},\ n = 1, 2, 3, \cdots$ and $(gx_n, gx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$.
It is a well known fact that in a C^*-algebra $\mathbb{A}$, if $a, b \in \mathbb{A}_+$ and $a \preceq b$, then for any $x \in \mathbb{A}$ both x^*ax and x^*bx are positive elements and $x^*ax \preceq x^*bx$[23].

For any $n \in \mathbb{N}$, we have by using condition (3.1) that

$$d(gx_n, gx_{n+1}) = d(fx_{n-1}, fx_n) \preceq B^* d(gx_{n-1}, gx_n) B. \tag{3.2}$$

By repeated use of condition (3.2), we get

$$d(gx_n, gx_{n+1}) \preceq (B^*)^n d(gx_0, gx_1) B^n = (B^n)^* B_0 B^n, \tag{3.3}$$

for all $n \in \mathbb{N}$, where $B_0 = d(gx_0, gx_1) \in \mathbb{A}_+$.

For any $m, n \in \mathbb{N}$ with $m > n$, we have by using condition (3.3) that

$$\begin{aligned}
d(gx_n, gx_m) &\preceq A[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)] \\
&\preceq Ad(gx_n, gx_{n+1}) + A^2 d(gx_{n+1}, gx_{n+2}) + \cdots \\
&\quad + A^{m-n-1} d(gx_{m-2}, gx_{m-1}) + A^{m-n-1} d(gx_{m-1}, gx_m) \\
&\preceq A(B^*)^n B_0 B^n + A^2 (B^*)^{n+1} B_0 B^{n+1} + A^3 (B^*)^{n+2} B_0 B^{n+2} + \cdots \\
&\quad + A^{m-n-1} (B^*)^{m-2} B_0 B^{m-2} + A^{m-n-1} (B^*)^{m-1} B_0 B^{m-1} \\
&\preceq \sum_{k=1}^{m-n-1} A^k (B^*)^{n+k-1} B_0 B^{n+k-1} + A^{m-n} (B^*)^{m-1} B_0 B^{m-1} \\
&= \sum_{k=1}^{m-n} A^k (B^*)^{n+k-1} B_0 B^{n+k-1} \\
&\preceq \sum_{k=1}^{m-n} \| A^k (B^*)^{n+k-1} B_0 B^{n+k-1} \| I \\
&\preceq \| B_0 \| \sum_{k=1}^{m-n} \| A \|^k \| B \|^{2(n+k-1)} I \\
&= \| B_0 \| \| B \|^{2n} \| A \| \sum_{k=1}^{m-n} \left(\| A \| \| B \|^2 \right)^{k-1} I \\
&\preceq \frac{\| B_0 \| \| B \|^{2n} \| A \|}{1 - \| A \| \| B \|^2} I, \ since \ \| B \|^2 < \frac{1}{\| A \|} \\
&\to \theta \ as \ n \to \infty.
\end{aligned}$$

Therefore, (gx_n) is a Cauchy sequence with respect to $\mathbb{A}$. Since $g(X)$ is complete, there exists an $u \in g(X)$ such that $\lim_{n\to\infty} gx_n = u = gv$ for some $v \in X$.

As $x_0 \in C_{gf}$, it follows that $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \geq 0$, and so by property $(*)$, there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, gv) \in E(\tilde{G})$ for all $i \geq 1$.

Using condition (3.1), we have

$$\begin{aligned}
d(fv, gv) &\preceq A[d(fv, fx_{n_i}) + d(fx_{n_i}, gv)] \\
&\preceq AB^* d(gv, gx_{n_i}) B + Ad(gx_{n_i+1}, gv) \\
&\to \theta \ as \ i \to \infty.
\end{aligned}$$

This implies that $d(fv, gv) = \theta$ and hence $fv = gv = u$. Therefore, u is a point of coincidence of f and g.

The next is to show that the point of coincidence is unique. Assume that there is another point of coincidence u^* in X such that $fx = gx = u^*$ for some $x \in X$. By property $(**)$, we have

$(u, u^*) \in E(\tilde{G})$. Then,

$$\begin{aligned} d(u, u^*) &= d(fv, fx) \\ &\preceq B^* d(gv, gx) B \\ &= B^* d(u, u^*) B, \end{aligned}$$

which implies that,

$$\begin{aligned} \| d(u, u^*) \| &\leq \| B^* d(u, u^*) B \| \\ &\leq \| B^* \| \| d(u, u^*) \| \| B \| \\ &= \| B \|^2 \| d(u, u^*) \| . \end{aligned}$$

Since $\| B \|^2 < \frac{1}{\|A\|} \leq 1$, it follows that $d(u, u^*) = \theta$ i.e., $u = u^*$. Therefore, f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 2.13, f and g have a unique common fixed point in X. □

The following corollary gives fixed point of Banach G-contraction in C^*-algebra valued b-metric spaces.

Corollary 3.3. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued b-metric space endowed with a graph G and the mapping $f : X \to X$ be such that*

$$d(fx, fy) \preceq B^* d(x, y) B \tag{3.4}$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$, where $B \in \mathbb{A}$ with $\| B \|^2 < \frac{1}{\|A\|}$. Suppose $(X, \mathbb{A}, d, G)$ has the following property:

$(*')$ *If (x_n) is a sequence in X such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that $(x_{n_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.*
Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, f has a unique fixed point in X if the graph G has the following property:

$(**')$ *If x, y are fixed points of f in X, then $(x, y) \in E(\tilde{G})$.*

Proof. The proof can be obtained from Theorem 3.2 by considering $g = I$, the identity map on X. □

Corollary 3.4. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued b-metric space and the mappings f, $g : X \to X$ be such that (3.1) holds for all $x, y \in X$, where $B \in \mathbb{A}$ with $\| B \|^2 < \frac{1}{\|A\|}$. If $f(X) \subseteq g(X)$ and*

$g(X)$ *is a complete subspace of* X*, then* f *and* g *have a unique point of coincidence in* X*. Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point in* X*.*

Proof. The proof follows from Theorem 3.2 by taking $G = G_0$, where G_0 is the complete graph $(X, X \times X)$.

□

The following corollary is analogue of Banach Contraction Principle.

Corollary 3.5. *Let* $(X, \mathbb{A}, d)$ *be a complete* C^**-algebra valued* b*-metric space and the mapping* $f : X \to X$ *be such that (3.4) holds for all* $x, y \in X$*, where* $B \in \mathbb{A}$ *with* $\| B \|^2 < \frac{1}{\|A\|}$*. Then* f *has a unique fixed point* u *in* X *and* $f^n x \to u$ *for all* $x \in X$*.*

Proof. It follows from Theorem 3.2 by putting $G = G_0$ and $g = I$. □

Remark 3.6. *We observe that Banach contraction theorem in a complete metric space can be obtained from Corollary 3.5 by taking* $\mathbb{A} = \mathbb{C}, A = 1$*. Thus, Theorem 3.2 is a generalization of Banach contraction theorem in metric spaces to* C^**-algebra valued* b*-metric spaces.*

From Theorem 3.2, we obtain the following corollary concerning the fixed point of expansive mapping in C^*-algebra valued b-metric spaces.

Corollary 3.7. *Let* $(X, \mathbb{A}, d)$ *be a complete* C^**-algebra valued* b*-metric space and let* $g : X \to X$ *be an onto mapping satisfying*

$$B^* d(gx, gy) B \succeq d(x, y)$$

for all $x, y \in X$*, where* $B \in \mathbb{A}$ *with* $\| B \|^2 < \frac{1}{\|A\|}$*. Then* g *has a unique fixed point in* X*.*

Proof. The conclusion of the corollary follows from Theorem 3.2 by taking $G = G_0$ and $f = I$. □

Corollary 3.8. *Let* $(X, \mathbb{A}, d)$ *be a complete* C^**-algebra valued* b*-metric space endowed with a partial ordering* $\sqsubseteq$ *and the mapping* $f : X \to X$ *be such that (3.4) holds for all* $x, y \in X$ *with* $x \sqsubseteq y$ *or,* $y \sqsubseteq x$*, where* $B \in \mathbb{A}$ *and* $\| B \|^2 < \frac{1}{\|A\|}$*. Suppose* $(X, \mathbb{A}, d, \sqsubseteq)$ *has the following property:*

(†) *If* (x_n) *is a sequence in* X *such that* $x_n \to x$ *and* x_n*,* x_{n+1} *are comparable for all* $n \geq 1$*, then there exists a subsequence* (x_{n_i}) *of* (x_n) *such that* x_{n_i}*,* x *are comparable for all* $i \geq 1$*.*
If there exists $x_0 \in X$ *such that* $f^n x_0, f^m x_0$ *are comparable for* $m, n = 0, 1, 2, \cdots$*, then* f *has a fixed point in* X*. Moreover,* f *has a unique fixed point in* X *if the following property holds:*

(††) *If* x*,* y *are fixed points of* f *in* X*, then* x, y *are comparable.*

Proof. The proof can be obtained from Theorem 3.2 by taking $g = I$ and $G = G_2$, where the graph G_2 is defined by $E(G_2) = \{(x, y) \in X \times X : x \sqsubseteq y \text{ or } y \sqsubseteq x\}$. □

Theorem 3.9. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space endowed with a graph* G *and the mappings* $f,\ g : X \to X$ *be such that*

$$d(fx, fy) \preceq B\,[d(fx, gy) + d(fy, gx)] \tag{3.5}$$

for all $x, y \in X$ *with* $(gx, gy) \in E(\tilde{G})$*, where* $B \in \mathbb{A}_+^{'}$ *and* $\| BA \| < \frac{1}{\|A\|+1}$. *Suppose* $f(X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* X *with the property* $(*)$. *Then* f *and* g *have a point of coincidence in* X *if* $C_{gf} \neq \emptyset$. *Moreover,* f *and* g *have a unique point of coincidence in* X *if the graph* G *has the property* $(**)$. *Furthermore, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point in* X.

Proof. It follows from condition (3.5) that $B(d(fx, gy) + d(fy, gx))$ is a positive element.

Suppose that $C_{gf} \neq \emptyset$. We choose an $x_0 \in C_{gf}$ and keep it fixed. We can construct a sequence (gx_n) such that $gx_n = fx_{n-1},\ n = 1, 2, 3, \cdots$. Evidently, $(gx_n, gx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$.

For any $n \in \mathbb{N}$, we have by using condition (3.5) and Lemma 2.1(iii) that

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\preceq B[d(fx_{n-1}, gx_n) + d(fx_n, gx_{n-1})] \\ &= B[d(fx_{n-1}, fx_{n-1}) + d(fx_n, fx_{n-2})] \\ &\preceq BA[d(fx_n, fx_{n-1}) + d(fx_{n-1}, fx_{n-2})] \\ &= BA\, d(gx_{n+1}, gx_n) + BA\, d(gx_n, gx_{n-1})] \end{aligned}$$

which implies that,

$$(I - BA)d(gx_n, gx_{n+1}) \preceq BA d(gx_n, gx_{n-1}). \tag{3.6}$$

Now, $A,\ B \in \mathbb{A}_+^{'}$ implies that $BA \in \mathbb{A}_+^{'}$. Since $\| BA \| < \frac{1}{2}$, by Lemma 2.1, it follows that $(I - BA)$ is invertible and $\| BA(I - BA)^{-1} \| = \| (I - BA)^{-1}BA \| < 1$. Moreover, by Lemma 2.1, $BA \preceq I$ i.e., $I - BA \succeq \theta$. Since $BA \in \mathbb{A}_+^{'}$, we have $(I - BA) \in \mathbb{A}_+^{'}$. Furthermore, $(I - BA)^{-1} \in \mathbb{A}_+^{'}$. By using Lemma 2.1(iv), it follows from (3.6) that

$$d(gx_n, gx_{n+1}) \preceq (I - BA)^{-1}BA\, d(gx_n, gx_{n-1}) = t d(gx_{n-1}, gx_n), \tag{3.7}$$

where $t = (I - BA)^{-1}BA \in \mathbb{A}_+^{'}$.

By repeated use of condition (3.7), we get

$$d(gx_n, gx_{n+1}) \preceq t^n d(gx_0, gx_1) = t^n B_0, \tag{3.8}$$

for all $n \in \mathbb{N}$, where $B_0 = d(gx_0, gx_1) \in \mathbb{A}_+$.

We now prove that if $\| BA \| < \frac{1}{\|A\|+1}$, then $\| t \| < \frac{1}{\|A\|}$.
We have,

$$\begin{aligned}
\| t \| &= \| (I-BA)^{-1}BA \| \\
&\leq \| (I-BA)^{-1} \| \| BA \| \\
&\leq \frac{1}{1-\| BA \|} \| BA \| \\
&< \frac{1}{\| A \|}, \text{since } \| BA \| < \frac{1}{\| A \| +1}.
\end{aligned}$$

For any $m, n \in \mathbb{N}$ with $m > n$, we have by using condition (3.8) that

$$\begin{aligned}
d(gx_n, gx_m) &\preceq A[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)] \\
&\preceq Ad(gx_n, gx_{n+1}) + A^2d(gx_{n+1}, gx_{n+2}) + \cdots \\
&\quad + A^{m-n-1}d(gx_{m-2}, gx_{m-1}) + A^{m-n-1}d(gx_{m-1}, gx_m) \\
&\preceq At^nB_0 + A^2t^{n+1}B_0 + A^3t^{n+2}B_0 + \cdots \\
&\quad + A^{m-n-1}t^{m-2}B_0 + A^{m-n-1}t^{m-1}B_0 \\
&\preceq \sum_{k=1}^{m-n} A^k t^{n+k-1} B_0, \text{ since } A \succeq I \text{ and } A \in \mathbb{A}'_+ \\
&\preceq \sum_{k=1}^{m-n} \| A^k t^{n+k-1} B_0 \| I \\
&\preceq \| B_0 \| \| A \| \| t \|^n \sum_{k=1}^{m-n} (\| A \| \| t \|)^{k-1} I \\
&\preceq \| B_0 \| \| A \| \| t \|^n \frac{1}{1-\| A \| \| t \|} I \\
&\to \theta \text{ as } n \to \infty.
\end{aligned}$$

Therefore, (gx_n) is a Cauchy sequence with respect to $\mathbb{A}$. As $g(X)$ is complete, there exists an $u \in g(X)$ such that $\lim_{n\to\infty} gx_n = u = gv$ for some $v \in X$. By property $(*)$, there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, gv) \in E(\tilde{G})$ for all $i \geq 1$.

Using condition (3.5), we have

$$\begin{aligned}
d(fv, gv) &\preceq A[d(fv, fx_{n_i}) + d(fx_{n_i}, gv)] \\
&\preceq AB[d(fv, gx_{n_i}) + d(fx_{n_i}, gv)] + Ad(gx_{n_i+1}, gv) \\
&\preceq ABA[d(fv, gv) + d(gv, gx_{n_i})] + ABd(gx_{n_i+1}, gv) + Ad(gx_{n_i+1}, gv)
\end{aligned}$$

which implies that,

$$(I - BA^2)d(fv, gv) \preceq BA^2d(gv, gx_{n_i}) + ABd(gx_{n_i+1}, gv) + Ad(gx_{n_i+1}, gv).$$

Since $\| BA^2 \| < \frac{\|A\|}{\|A\|+1} < 1$, we have $(I - BA^2)^{-1}$ exists. By using Lemma 2.1, it follows that

$$\begin{aligned} d(fv, gv) &\preceq (I - BA^2)^{-1} BA^2 d(gv, gx_{n_i}) + (I - BA^2)^{-1} ABd(gx_{n_i+1}, gv) \\ &\quad + (I - BA^2)^{-1} Ad(gx_{n_i+1}, gv) \\ &\to \theta \text{ as } i \to \infty. \end{aligned}$$

This implies that $d(fv, gv) = \theta$ i.e., $fv = gv = u$ and hence u is a point of coincidence of f and g.

Finally, to prove the uniqueness of point of coincidence, suppose that there is another point of coincidence u^* in X such that $fx = gx = u^*$ for some $x \in X$. By property $(**)$, we have $(u, u^*) \in E(\tilde{G})$. Then,

$$\begin{aligned} d(u, u^*) &= d(fv, fx) \\ &\preceq B[d(fv, gx) + d(fx, gv)] \\ &= B[d(u, u^*) + d(u, u^*)] \\ &\preceq AB[d(u, u^*) + d(u, u^*)] \end{aligned}$$

which implies that,

$$d(u, u^*) \preceq (I - AB)^{-1} AB\, d(u, u^*).$$

So, it must be the case that

$$\begin{aligned} \| d(u, u^*) \| &\leq \| (I - AB)^{-1} AB\, d(u, u^*) \| \\ &\leq \| (I - AB)^{-1} AB \| \| d(u, u^*) \| . \end{aligned}$$

Since $\| (I - AB)^{-1} AB \| < 1$, we have $\| d(u, u^*) \| = 0$ i.e., $u = u^*$. Therefore, f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 2.13, f and g have a unique common fixed point in X. □

Corollary 3.10. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued b-metric space endowed with a graph G and the mapping $f : X \to X$ be such that*

$$d(fx, fy) \preceq B\,[d(fx, y) + d(fy, x)] \tag{3.9}$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$, where $B \in \mathbb{A}_+'$ and $\| BA \| < \frac{1}{\|A\|+1}$. Suppose $(X, \mathbb{A}, d, G)$ has the property $(\acute{*})$. Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, f has a unique fixed point in X if the graph G has the property $(*\acute{*})$.

Proof. The proof can be obtained from Theorem 3.9 by putting $g = I$. □

Corollary 3.11. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued b-metric space and the mappings $f, g : X \to X$ be such that (3.5) holds for all $x, y \in X$, where $B \in \mathbb{A}'_+$ and $\| BA \| < \frac{1}{\|A\|+1}$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.*

Proof. The proof can be obtained from Theorem 3.9 by taking $G = G_0$. □

Corollary 3.12. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued b-metric space and the mapping $f : X \to X$ be such that (3.9) holds for all $x, y \in X$, where $B \in \mathbb{A}'_+$ with $\| BA \| < \frac{1}{\|A\|+1}$. Then f has a unique fixed point in X.*

Proof. The proof follows from Theorem 3.9 by taking $G = G_0$ and $g = I$.

□

Remark 3.13. *We observe that Brian Fisher's theorem in a complete metric space can be obtained from Corollary 3.12 by taking $\mathbb{A} = \mathbb{C}, A = 1$. Thus, Theorem 3.9 is a generalization of Brian Fisher's theorem in metric spaces to C^*-algebra valued b-metric spaces.*

Corollary 3.14. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued b-metric space endowed with a partial ordering $\sqsubseteq$ and the mapping $f : X \to X$ be such that (3.9) holds for all $x, y \in X$ with $x \sqsubseteq y$ or, $y \sqsubseteq x$, where $B \in \mathbb{A}'_+$ with $\| BA \| < \frac{1}{\|A\|+1}$. Suppose $(X, \mathbb{A}, d, \sqsubseteq)$ has the property $(\dagger)$. If there exists $x_0 \in X$ such that $f^n x_0, f^m x_0$ are comparable for $m, n = 0, 1, 2, \cdots$, then f has a fixed point in X. Moreover, f has a unique fixed point in X if the property $(\dagger\dagger)$ holds.*

Proof. The proof can be obtained from Theorem 3.9 by taking $G = G_2$ and $g = I$. □

Theorem 3.15. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued b-metric space endowed with a graph G and the mappings $f, g : X \to X$ be such that*

$$d(fx, fy) \preceq B\,[d(fx, gx) + d(fy, gy)] \tag{3.10}$$

for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$, where $B \in \mathbb{A}'_+$ and $\| B \| < \frac{1}{\|A\|+1}$. Suppose $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X with the property $(*)$. Then f and g have a point of coincidence in X if $C_{gf} \neq \emptyset$. Moreover, f and g have a unique point of coincidence in X if the graph G has the property $(**)$. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. We observe that $B(d(fx, gx) + d(fy, gy))$ is a positive element.

Suppose that $C_{gf} \neq \emptyset$. We choose an $x_0 \in C_{gf}$ and keep it fixed. We can construct a sequence (gx_n) such that $gx_n = fx_{n-1},\ n = 1, 2, 3, \cdots$. Evidently, $(gx_n, gx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$.

For any $n \in \mathbb{N}$, we have by using condition (3.10) that

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\preceq B[d(fx_{n-1}, gx_{n-1}) + d(fx_n, gx_n)] \\ &= B\, d(gx_n, gx_{n-1}) + B\, d(gx_n, gx_{n+1}) \end{aligned}$$

which implies that,

$$(I - B)d(gx_n, gx_{n+1}) \preceq Bd(gx_n, gx_{n-1}). \tag{3.11}$$

Since $B \in \mathbb{A}'_+$ and $\| B \| < \frac{1}{2}$, by Lemma 2.1, it follows that $B \preceq I$ and $(I - B)$ is invertible with $\| B(I-B)^{-1} \| = \| (I-B)^{-1}B \| < 1$. Furthermore, $(I-B)$, $(I-B)^{-1} \in \mathbb{A}'_+$ and so, $(I-B)^{-1}B \in \mathbb{A}'_+$. Again, by using Lemma 2.1(iv), it follows from condition (3.11) that

$$d(gx_n, gx_{n+1}) \preceq (I - B)^{-1}B\, d(gx_n, gx_{n-1}) = td(gx_{n-1}, gx_n), \tag{3.12}$$

where $t = (I - B)^{-1}B \in \mathbb{A}'_+$.

By repeated use of condition (3.12), we get

$$d(gx_n, gx_{n+1}) \preceq t^n d(gx_0, gx_1) = t^n B_0, \tag{3.13}$$

for all $n \in \mathbb{N}$, where $B_0 = d(gx_0, gx_1) \in \mathbb{A}_+$.

We now prove that if $\| B \| < \frac{1}{\|A\|+1}$, then $\| t \| < \frac{1}{\|A\|}$.
We have,

$$\begin{aligned} \| t \| &= \| (I - B)^{-1}B \| \\ &\leq \| (I - B)^{-1} \| \| B \| \\ &\leq \frac{1}{1 - \| B \|} \| B \| \\ &< \frac{1}{\| A \|}, \text{since } \| B \| < \frac{1}{\| A \| + 1}. \end{aligned}$$

For any $m, n \in \mathbb{N}$ with $m > n$, we have by using condition (3.13) that

$$\begin{aligned}
d(gx_n, gx_m) &\preceq A[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)] \\
&\preceq Ad(gx_n, gx_{n+1}) + A^2 d(gx_{n+1}, gx_{n+2}) + \cdots \\
&\quad + A^{m-n-1} d(gx_{m-2}, gx_{m-1}) + A^{m-n-1} d(gx_{m-1}, gx_m) \\
&\preceq At^n B_0 + A^2 t^{n+1} B_0 + A^3 t^{n+2} B_0 + \cdots \\
&\quad + A^{m-n-1} t^{m-2} B_0 + A^{m-n-1} t^{m-1} B_0 \\
&\preceq \sum_{k=1}^{m-n} A^k t^{n+k-1} B_0, \text{ since } A \succeq I \text{ and } A \in \mathbb{A}'_+ \\
&\preceq \sum_{k=1}^{m-n} \| A^k t^{n+k-1} B_0 \| I \\
&\preceq \| B_0 \| \| A \| \| t \|^n \sum_{k=1}^{m-n} (\| A \| \| t \|)^{k-1} I \\
&\preceq \| B_0 \| \| A \| \| t \|^n \frac{1}{1 - \| A \| \| t \|} I \\
&\to \theta \text{ as } n \to \infty.
\end{aligned}$$

Therefore, (gx_n) is a Cauchy sequence with respect to $\mathbb{A}$. By completeness of $g(X)$, there exists an $u \in g(X)$ such that $\lim_{n\to\infty} gx_n = u = gv$ for some $v \in X$. By property $(*)$, there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, gv) \in E(\tilde{G})$ for all $i \geq 1$.

Using condition (3.10), we have

$$\begin{aligned}
d(fv, gv) &\preceq A[d(fv, fx_{n_i}) + d(fx_{n_i}, gv)] \\
&\preceq AB[d(fv, gv) + d(fx_{n_i}, gx_{n_i})] + Ad(gx_{n_i+1}, gv)
\end{aligned}$$

which implies that,

$$(I - AB)d(fv, gv) \preceq ABd(gx_{n_i+1}, gx_{n_i}) + Ad(gx_{n_i+1}, gv).$$

Since $\| AB \| < \frac{\|A\|}{\|A\|+1} < 1$, we have $(I - AB)^{-1}$ exists and $(I - AB) \in \mathbb{A}'_+$. By using Lemma 2.1, it follows that

$$d(fv, gv) \preceq (I - AB)^{-1} ABd(gx_{n_i+1}, gx_{n_i}) + (I - AB)^{-1} Ad(gx_{n_i+1}, gv).$$

Then,

$$\begin{aligned}
\| d(fv, gv) \| &\leq \| (I - AB)^{-1} AB \| \| d(gx_{n_i+1}, gx_{n_i}) \| \\
&\quad + \| (I - AB)^{-1} A \| \| d(gx_{n_i+1}, gv) \| \\
&\leq \| (I - AB)^{-1} AB \| \| t \|^{n_i} \| B_0 \| \\
&\quad + \| (I - AB)^{-1} A \| \| d(gx_{n_i+1}, gv) \| \\
&\to 0 \text{ as } i \to \infty.
\end{aligned}$$

This implies that $d(fv, gv) = \theta$ i.e., $fv = gv = u$ and hence u is a point of coincidence of f and g.

Finally, to prove the uniqueness of point of coincidence, suppose that there is another point of coincidence u^* in X such that $fx = gx = u^*$ for some $x \in X$. By property $(**)$, we have $(u, u^*) \in E(\tilde{G})$. Then,

$$\begin{aligned} d(u, u^*) &= d(fv, fx) \\ &\preceq B[d(fv, gv) + d(fx, gx)] \\ &= \theta \end{aligned}$$

which implies that, $d(u, u^*) = \theta$ i.e., $u = u^*$. Therefore, f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 2.13, f and g have a unique common fixed point in X. □

Corollary 3.16. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued b-metric space endowed with a graph G and the mapping $f : X \to X$ be such that*

$$d(fx, fy) \preceq B\,[d(fx, x) + d(fy, y)] \tag{3.14}$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$, where $B \in \mathbb{A}_+^{'}$ and $\| B \| < \frac{1}{\|A\|+1}$. Suppose $(X, \mathbb{A}, d, G)$ has the property $(\acute{*})$. Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, f has a unique fixed point in X if the graph G has the property $(\acute{**})$.

Proof. The proof can be obtained from Theorem 3.15 by putting $g = I$. □

Corollary 3.17. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued b-metric space and the mappings f, $g : X \to X$ be such that (3.10) holds for all $x, y \in X$, where $B \in \mathbb{A}_+^{'}$ and $\| B \| < \frac{1}{\|A\|+1}$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.*

Proof. The proof can be obtained from Theorem 3.15 by taking $G = G_0$. □

Corollary 3.18. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued b-metric space and the mapping $f : X \to X$ be such that (3.14) holds for all $x, y \in X$, where $B \in \mathbb{A}_+^{'}$ with $\| B \| < \frac{1}{\|A\|+1}$. Then f has a unique fixed point in X.*

Proof. The proof follows from Theorem 3.15 by taking $G = G_0$ and $g = I$.

□

Remark 3.19. *We observe that Kannan's fixed point theorem in a complete metric space can be obtained from Corollary 3.18 by taking* $\mathbb{A} = \mathbb{C}, A = 1$. *Thus, Theorem 3.15 is a generalization of Kannan's fixed point theorem in metric spaces to* C^**-algebra valued* b*-metric spaces.*

Corollary 3.20. *Let* $(X, \mathbb{A}, d)$ *be a complete* C^**-algebra valued* b*-metric space endowed with a partial ordering* $\sqsubseteq$ *and the mapping* $f : X \to X$ *be such that (3.14) holds for all* $x, y \in X$ *with* $x \sqsubseteq y$ *or,* $y \sqsubseteq x$*, where* $B \in \mathbb{A}_+^{'}$ *with* $\| B \| < \frac{1}{\|A\|+1}$. *Suppose* $(X, \mathbb{A}, d, \sqsubseteq)$ *has the property* (†). *If there exists* $x_0 \in X$ *such that* $f^n x_0, f^m x_0$ *are comparable for* $m, n = 0, 1, 2, \cdots$, *then* f *has a fixed point in* X. *Moreover,* f *has a unique fixed point in* X *if the property* (††) *holds.*

Proof. The proof can be obtained from Theorem 3.15 by taking $G = G_2$ and $g = I$. □

We furnish some examples in favour of our results.

Example 3.21. *Let* $X = \mathbb{R}$ *and* $B(H)$ *be the set of all bounded linear operators on a Hilbert space* H. *Define* $d : X \times X \to B(H)$ *by* $d(x, y) = \mid x - y \mid^3 I$ *for all* $x, y \in X$*, where* I *is the identity operator on* H. *Then* $(X, B(H), d)$ *is a complete* C^**-algebra valued* b*-metric space with the coefficient* $A = 4I$. *Let* G *be a digraph such that* $V(G) = X$ *and* $E(G) = \Delta \cup \{(\frac{1}{n}, 0) : n = 1, 2, 3 \cdots\}$.

Let $f, g : X \to X$ *be defined by*

$$\begin{aligned} fx &= \frac{x}{5}, \text{ if } x \neq \frac{4}{5} \\ &= 1, \text{ if } x = \frac{4}{5} \end{aligned}$$

and $gx = 2x$ *for all* $x \in X$. *Obviously,* $f(X) \subseteq g(X) = X$.

If $x = 0, \ y = \frac{1}{2n}, n = 1, 2, 3, \cdots$, *then* $gx = 0, \ gy = \frac{1}{n}$ *and so* $(gx, gy) \in E(\tilde{G})$.

For $x = 0, \ y = \frac{1}{2n}$, *we have*

$$\begin{aligned} d(fx, fy) &= d\left(0, \frac{1}{10n}\right) \\ &= \frac{1}{10^3.n^3} I \\ &\prec \frac{1}{25n^3} I \\ &= \frac{1}{25} d(gx, gy) \\ &= B^* d(gx, gy) B, \end{aligned}$$

where $B = \frac{1}{5} I \in B(H)$.

Therefore,

$$d(fx, fy) \preceq B^* \, d(gx, gy)B$$

for all $x, y \in X$ *with* $(gx, gy) \in E(\tilde{G})$*, where* $B \in B(H)$ *and* $\| B \|^2 < \frac{1}{\|A\|}$*. We can verify that* $0 \in C_{gf}$*. In fact,* $gx_n = fx_{n-1}$*,* $n = 1, 2, 3, \cdots$ *gives that* $gx_1 = f0 = 0 \Rightarrow x_1 = 0$ *and so* $gx_2 = fx_1 = 0 \Rightarrow x_2 = 0$*. Proceeding in this way, we get* $gx_n = 0$ *for* $n = 0, 1, 2, \cdots$ *and hence* $(gx_n, gx_m) = (0, 0) \in E(\tilde{G})$ *for* $m, n = 0, 1, 2, \cdots$*.*

Also, any sequence (gx_n) *with the property* $(gx_n, gx_{n+1}) \in E(\tilde{G})$ *must be either a constant sequence or a sequence of the following form*

$$\begin{aligned} gx_n &= 0, \text{ if } n \text{ is odd} \\ &= \frac{1}{n}, \text{ if } n \text{ is even} \end{aligned}$$

where the words 'odd' and 'even' are interchangeable. Consequently it follows that property $(*)$ *holds. Furthermore,* f *and* g *are weakly compatible. Thus, we have all the conditions of Theorem 3.2 and* 0 *is the unique common fixed point of* f *and* g *in* X*.*

Remark 3.22. *It is worth mentioning that weak compatibility condition in Theorem 3.2 cannot be relaxed. In Example 3.21, if we take* $gx = 2x - 9$ *for all* $x \in X$ *instead of* $gx = 2x$*, then* $5 \in C_{gf}$ *and* $f(5) = g(5) = 1$ *but* $g(f(5)) \neq f(g(5))$ *i.e.,* f *and* g *are not weakly compatible. However, all other conditions of Theorem 3.2 are satisfied. We observe that* 1 *is the unique point of coincidence of* f *and* g *without being any common fixed point.*

Remark 3.23. *In Example 3.21,* f *is a* C^**-algebra valued* G*-contraction but it is not a* C^**-algebra valued contraction. In fact, for* $x = \frac{4}{5}$*,* $y = 0$*, we have*

$$\begin{aligned} d(fx, fy) &= d(1, 0) \\ &= I \\ &= \frac{125}{64} \cdot \frac{64}{125} I \\ &= \frac{125}{64} d(x, y) \\ &\succ B^* \, d(x, y) \, B, \end{aligned}$$

for any $B \in B(H)$ *with* $\| B \|^2 < \frac{1}{\|A\|}$*. This implies that* f *is not a* C^**-algebra valued contraction.*

The following example shows that property $(*)$ is necessary in Theorem 3.2.

Example 3.24. *Let* $X = [0, \infty)$ *and* $B(H)$ *be the set of all bounded linear operators on a Hilbert space* H*. Define* $d : X \times X \to B(H)$ *by* $d(x, y) = | x - y |^3 \, I$ *for all* $x, y \in X$*, where* I *is the identity operator on* H*. Then* $(X, B(H), d)$ *is a complete* C^**-algebra valued* b*-metric space with the coefficient*

$A = 4I$. *Let* G *be a digraph such that* $V(G) = X$ *and* $E(G) = \Delta\cup\{(x, y) : (x, y) \in (0, 1]\times(0, 1],\ x \geq y\}$.

Let $f,\ g : X \to X$ *be defined by*

$$\begin{aligned} fx &= \frac{x}{6},\ \text{ if } x \neq 0 \\ &= 1,\ \text{ if } x = 0 \end{aligned}$$

and $gx = \frac{x}{2}$ *for all* $x \in X$. *Obviously,* $f(X) \subseteq g(X) = X$.

For $x, y \in X$ *with* $(gx, gy) \in E(\tilde{G})$, *we have*

$$\begin{aligned} d\,(fx, fy) &= \frac{1}{27}d\,(gx, gy) \\ &\preceq \frac{1}{9}d\,(gx, gy) \\ &= B^*\,d(gx, gy)\,B, \end{aligned}$$

where $B = \frac{1}{3}I \in B(H)$ *with* $\| B \|^2 < \frac{1}{\|A\|}$.

We see that f *and* g *have no point of coincidence in* X. *We now verify that the property* $(*)$ *does not hold. In fact,* (gx_n) *is a sequence in* X *with* $gx_n \to 0$ *and* $(gx_n, gx_{n+1}) \in E(\tilde{G})$ *for all* $n \in \mathbb{N}$ *where* $x_n = \frac{2}{n}$. *But there exists no subsequence* (gx_{n_i}) *of* (gx_n) *such that* $(gx_{n_i}, 0) \in E(\tilde{G})$.

Example 3.25. *Let* $X = \mathbb{R}$ *and* $B(H)$ *be the set of all bounded linear operators on a Hilbert space* H. *Choose a positive operator* $T \in B(H)$. *Define* $d : X\times X \to B(H)$ *by* $d(x, y) =| x-y |^5\ T$ *for all* $x,\ y \in X$. *Then* $(X, B(H), d)$ *is a complete* C^**-algebra valued* b*-metric space with the coefficient* $A = 16I$. *Let* $f,\ g : X \to X$ *be defined by*

$$\begin{aligned} fx &= 2,\ \text{if } x \neq 5 \\ &= 3,\ \text{if } x = 5 \end{aligned}$$

and $gx = 3x - 4$ *for all* $x \in X$. *Obviously,* $f(X) \subseteq g(X) = X$.
Let G *be a digraph such that* $V(G) = X$ *and* $E(G) = \Delta \cup \{(2, 3),\ (3, 5)\}$. *If* $x = 2,\ y = \frac{7}{3}$, *then* $gx = 2,\ gy = 3$ *and so* $(gx, gy) \in E(\tilde{G})$.
Again, if $x = \frac{7}{3},\ y = 3$, *then* $gx = 3,\ gy = 5$ *and so* $(gx, gy) \in E(\tilde{G})$.
It is easy to verify that condition (3.5) of Theorem 3.9 holds for all $x, y \in X$ *with* $(gx, gy) \in E(\tilde{G})$. *Furthermore,* $2 \in C_{gf}$ *i.e.,* $C_{gf} \neq \emptyset$, f *and* g *are weakly compatible, and* $(X, B(H), d, G)$ *has the property* $(*)$. *Thus, all the conditions of Theorem 3.9 are satisfied and* 2 *is the unique common fixed point of* f *and* g *in* X.

Remark 3.26. *It is observed that in Example 3.25,* f *is not a Fisher* G*-contraction. In fact,* *for* $x = 3,\ y = 5$*, we have*

$$\begin{aligned} B\left[d(fx,y)+d(fy,x)\right] &= B\left[d(2,5)+d(3,3)\right] \\ &= 243BT \\ &= \frac{243}{16}BAT \\ &= \frac{243}{16\times 17}17BAT \\ &\prec T \\ &= d(fx,fy), \end{aligned}$$

for any $B \in B(H)_+^{'}$ *with* $\| BA \| < \frac{1}{\|A\|+1}$*. This implies that* f *is not a Fisher* G*-contraction.*

The following example supports our Theorem 3.15.

Example 3.27. *Let* $X = [0,\infty)$ *and* $B(H)$ *be the set of all bounded linear operators on a Hilbert space* H*. Choose a positive operator* $T \in B(H)$*. Define* $d : X\times X \to B(H)$ *by* $d(x,y) = |x-y|^2\, T$ *for all* $x,\ y \in X$*. Then* $(X, B(H), d)$ *is a complete* C^**-algebra valued* b*-metric space with the coefficient* $A = 2I$*. Let* G *be a digraph such that* $V(G) = X$ *and* $E(G) = \Delta \cup \{(4^t x, 4^t(x+1)) : x \in X \text{ with } x \geq 2,\ t = 0,\ 1,\ 2,\ \cdots\}$*.*

Let $f,\ g : X \to X$ *be defined by* $fx = 4x$ *and* $gx = 16x$ *for all* $x \in X$*. Clearly,* $f(X) = g(X) = X$*.*

If $x = 4^{t-2}z,\ y = 4^{t-2}(z+1)$*, then* $gx = 4^t z,\ gy = 4^t(z+1)$ *and so* $(gx, gy) \in E(\tilde{G})$ *for all* $z \geq 2$*.*

For $x = 4^{t-2}z,\ y = 4^{t-2}(z+1),\ z \geq 2$ *with* $B = \frac{1}{117}I$*, we have*

$$\begin{aligned} d(fx,fy) &= d\left(4^{t-1}z, 4^{t-1}(z+1)\right) \\ &= 4^{2t-2}T \\ &\preceq \frac{1}{117}\,4^{2t-2}(18z^2+18z+9)T \\ &= \frac{1}{117}\left[d\left(4^{t-1}z, 4^t z\right) + d\left(4^{t-1}(z+1), 4^t(z+1)\right)\right] \\ &= B\left[d(fx,gx)+d(fy,gy)\right]. \end{aligned}$$

Thus, condition (3.10) is satisfied for all $x, y \in X$ *with* $(gx, gy) \in E(\tilde{G})$*. It is easy to verify that* $0 \in C_{gf}$*. Also, any sequence* (gx_n) *with* $gx_n \to x$ *and* $(gx_n, gx_{n+1}) \in E(\tilde{G})$ *must be a constant sequence and hence property* $(*)$ *holds. Furthermore,* f *and* g *are weakly compatible. Thus, we have all the conditions of Theorem 3.15 and* 0 *is the unique common fixed point of* f *and* g *in* X*.*

Remark 3.28. *It is easy to observe that in Example 3.27,* f *is a* C^**-algebra valued* G*-Kannan operator with* $B = \frac{16}{117}I$*. But* f *is not a* C^**-algebra valued Kannan operator because, if* $x = 4,\ y = 0$*,* *then for any arbitrary* $B \in B(H)'_+$ *with* $\| B \| < \frac{1}{\|A\|+1} = \frac{1}{3}$ *(which implies* $3B \prec I$*), we have*

$$\begin{aligned} B\,[d(fx,x)+d(fy,y)] &= B\,[d(f4,4)+d(f0,0)] \\ &= 144BT \\ &= \frac{144}{3\times 256}(3B)(256T) \\ &\prec 256T \\ &= d(fx,fy). \end{aligned}$$

References

[1] A. Aghajani, M. Abbas and J. R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, Math. Slovaca, 64, 2014, 941-960.

[2] M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl., 341, 2008, 416-420.

[3] M. R. Alfuraidan, M. A. Khamsi, Caristi fixed point theorem in metric spaces with a graph, Abstract and Applied Analysis, vol. 2014, Article ID 303484.

[4] I.A.Bakhtin, The contraction mapping principle in almost metric spaces, Funct. Anal.,Gos. Ped. Inst. Unianowsk, 30, 1989, 26-37.

[5] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math., 3, 1922, 133-181.

[6] M. Boriceanu, Strict fixed point theorems for multivalued operators in b-metric spaces, Int. J. Mod. Math., 4, 2009, 285-301.

[7] J. A. Bondy and U. S. R. Murty, Graph theory with applications, American Elsevier Publishing Co., Inc., New York, 1976.

[8] I. Beg, A. R. Butt, S. Radojevic, The contraction principle for set valued mappings on a metric space with a graph, Comput. Math. Appl., 60, 2010, 1214-1219.

[9] F. Bojor, Fixed point of φ-contraction in metric spaces endowed with a graph, Annala of the University of Cralova, Mathematics and Computer Science Series, 37, 2010, 85-92.

[10] F. Bojor, Fixed points of Kannan mappings in metric spaces endowed with a graph, An. St. Univ. Ovidius Constanta, 20, 2012, 31-40.

[11] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostrav, 1, 1993, 5-11.

[12] G. Chartrand, L. Lesniak, and P. Zhang, Graph and digraph, CRC Press, New York, NY, USA, 2011.

[13] M. Cosentino, P. Salimi, P. Vetro, Fixed point results on metric-type spaces, Acta Math. Sci. Ser. B Engl. Ed., 34, 2014, 1237-1253.

[14] R. Douglas, Banach algebra techniques in operator theory, Springer, Berlin, 1998.

[15] F. Echenique, A short and constructive proof of Tarski's fixed point theorem, Internat. J. Game Theory, 33, 2005, 215-218.

[16] R. Espinola and W. A. Kirk, Fixed point theorems in R-trees with applications to graph theory, Topology Appl., 153, 2006, 1046-1055.

[17] J. I. Gross and J. Yellen, Graph theory and its applications, CRC Press, New York, NY, USA, 1999.

[18] N. Hussain, D. Dorić, Z. Kadelburg, S. Radenović, Suzuki-type fixed point results in metric type spaces, Fixed Point Theory Appl., 2012, 2012:126, doi:10.1186/1687-1812-2012-126.

[19] G. Jungck, Common fixed points for noncontinuous nonself maps on non-metric spaces, Far East J. Math. Sci., 4, 1996, 199-215.

[20] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Amer. Math. Soc., 136, 2008, 1359-1373.

[21] Z. Ma, L. Jiang, C^*-algebra-valued $\mathfrak{b}$-metric spaces and related fixed point theorems, Fixed Point Theory and Applications, 2015, 2015:222.

[22] Z. Ma, L. Jiang and H. Sun, C^*-algebra-valued metric spaces and related fixed point theorems, Fixed Point Theory and Applications, 2014, 2014:206.

[23] G. Murphy, C^*-Algebra and operator theory, Academic Press, London, 1990.

[24] S. K. Mohanta, Some Fixed Point Theorems in Cone Modular Spaces with a Graph, Bolletino dellUnione Matematica Italiana, 2016, DOI 10.1007/s40574-016-0086-9.

[25] S. K. Mohanta, Some fixed point theorems using wt-distance in $\mathfrak{b}$-metric spaces, Fasciculi Mathematici, no. 54, 2015, 125-140.

[26] J. J. Nieto and R. Rodríguez-López, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, Acta Math. Sinica, Englosh Ser., 2007, 2205-2212.

[27] D. Reem, S. Reich, A. J. Zaslavski, Two results in metric fixed point theory, J. Fixed Point Theory Appl., 1, 2007, 149-157.