$u\,(uv) = 0$

$u\,v^2 = 0$

$(uv)^2 = u^2\,v^2 = 0$

$u_1\,(u_2\,u_3) + u_2\,(u_1\,u_3) + u_3\,(u_1u_2) = 0$

$x^2\,y^2 + 2\,(xy)^2 = 0$

$x^2\,(xy) = 0$

It follows that $UV \subseteq U$; $U^2 \subseteq V$; $V^2 \subseteq U$; $UV^2 = <0>$; $U^2V^2=<0>$.

With the above hypothesis the idempotents of $A = ke \oplus U \oplus V$ have the form $e + u + u^2$, with $u \in U$ arbitrary. From this it follows that dim U is independent of the idempotent e and so also is dim V. They are invariants of A. The core $C = ke \oplus U \oplus U^2$ of A is also a Bernstein algebra and $A^2 = C$. So, the dimension of U^2 is also an invariant of A. Another invariant is dim $(UV + V^2)$, see [5]; Prop. 9.19.

If we have $A = ke \oplus U \oplus V$ and $e_0 = e + u_0 + u_0^2$ is another idempotent then, the decomposition of A relative to e_0 is $Ke_0 \oplus U_0 \oplus V_0$ where $U_0 = \{u + 2uu_0 \;/\; u \in U\}$ and
$V_0 = \{v - 2(u_0 + u_0^2)v \;/\; v \in V\}$.

2.- Orthogonality. Holgate introduced the concept of orthogonal Bernstein algebras in [2]. He defined $A = Ke \oplus U \oplus V$ to be orthogonal when $U^3 = <0>$. This definition depends on the idempotent e. The following example shows that with another idempotent e_0, we may have $U_0^3 \neq <0>$.

Take $A = <e, u_1, u_2, u_3, v_1, v_2, v_3>$ and the following multiplication table:

$e^2 = e$ $\qquad eu_i = \frac{1}{2}u_i$ $\qquad ev_i = 0$ $\quad (i=1,2,3)$

$u_1^2 = 2v_1$ $\qquad u_2^2 = v_2$ $\qquad u_1u_2 = v_3$

$v_1v_2 = u_3 \quad v_3^2 = -u_3$; other products are zero.

We see that A is Bernstein and in the decomposition determined by e, $U = \langle u_1, u_2, u_3\rangle$, $V = \langle v_1, v_2, v_3\rangle$, $U^3 = \langle 0\rangle$.

Consider now $e_0 = e + u_1 + u_1^2$. In the corresponding decomposition, we have $U_0^3 \neq \langle 0\rangle$ because

$$(u_1 + 2u_1u_1)(u_2 + 2u_2u_1)^2 = (u_1 + 2u_1^2)(u_2^2 + 4u_2(u_2u_1) + 4(u_2u_1)^2)$$

$$= (u_1 + 4v_1)(v_2 - 4u_3) = 4u_3 \neq 0$$

Considering this fact, the orthogonality notion is redefined as follows:

Definition: Let A a K-Bernstein algebra. If idempotent $e \in A$ exists such that in the decomposition $A = Ke \oplus U \oplus V$ we have $U^3 = \langle 0\rangle$, then A is said to be *orthogonal*. Such idempotent e will be called *pivotal*.

Definition: A K-Bernstein algebra A is said to be *totally orthogonal*, if all non zero idempotent is pivotal.

Thus, three Bernstein algebra classes can be distinguished: those orthogonal but not totally ortoghonal, those totally orthogonal and those non orthogonal.

Pivotal idempotents are characterized in the following

Theorem: Let $A = Ke \oplus U \oplus V$ a K-Bernstein algebra. An idempotent $e_0 = e + u_0 + u_0^2$, with $u_0 \in U$, is pivotal if and only if

$$ux^2 + 2(uu_0)x^2 = 0 \quad \text{for all } u, x \text{ in } U.$$

Proof : Let $e_0 = e + u_0 + u_0^2$ be pivotal idempotent then, by definition in the decomposition $A = Ke \oplus U_0 \oplus V_0$ we have $U_0^3 = \langle 0\rangle$. Then for all u, $x \in U$ we have $(u + 2uu_0)(x + 2xu_0)^2 \in U_0^3 = \langle 0\rangle$. On the other hand

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$(uv)^2 = u^2 v^2 = 0$

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It follows that $UV \subseteq U$; $U^2 \subseteq V$; $V^2 \subseteq U$; $UV^2 = \langle 0\rangle$; $U^2V^2 = \langle 0\rangle$.
With the above hypothesis the idempotents of $A = ke \oplus U \oplus V$ have the form $e + u + u^2$, with $u \in U$ arbritrary. From this it follows that dim U is independent of the idempotent e and so also is dim V. They are invariants of A. The core $C = ke \oplus U \oplus U^2$ of A is also a Bernstein algebra and $A^2 = C$. So, the dimension of U^2 is also an invariant of A. Another invariant is dim $(UV + V^2)$, see [5]; Prop. 9.19.

If we have $A = ke \oplus U \oplus V$ and $e_o = e + u_o + u^2$ is another idempotent then, the decomposition of A relative to e_0 is $Ke_0 \oplus U_0 \oplus V_0$ where $U_0 = \{u + 2uu_0 \ / \ u \in U\}$ and

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We see that A is Bernstein and in the decomposition determined by e, $U = \langle u_1, u_2, u_3\rangle$, $V = \langle v_1, v_2, v_3\rangle$, $U^3 = \langle 0\rangle$. Consider now $e_0 = e + u_1 + u_1^2$. In the corresponding decomposition, we have $U_0^3 \neq \langle 0\rangle$ because

$$(u_1 + 2u_1u_1)(u_2 + 2u_2u_1)^2 = (u_1 + 2u_1^2)(u_2^2 + 4u_2(u_2u_1) + 4(u_2u_1)^2)$$

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Considering this fact, the orthogonality notion is redefined as follows:

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Proof : Let $e_0 = e + u_0 + u_0^2$ be pivotal idempotent then, by definition in the decomposition $A = Ke \oplus U_0 \oplus V_0$ we have $U_0^3 = \langle 0\rangle$. Then for all u, $x \in U$ we have $(u+2uu_0)(x+2xu_0)^2 \in U_0^3 = \langle 0\rangle$. On the other hand

$$(u+2uu_0)(x+2xu_0)^2 = (u+2uu_0)[x^2+4x(xu_0)+4(xu_0)^2] =$$

$[ux^2+2(uu_0)x^2+8(uu_0)(x(xu_0))] + 4u(x(xu_0))$. Since this expression is nule and belongs to $U \oplus V$, then we have

$$ux^2 + 2(uu_0)x^2 + 8(uu_0)(x(xu_0)) = 0 \ \forall\ u,\ x \in U \quad (*)$$

Replacing $u = u_0$, then for all x in U we have

$$u_0x^2 + 2u_0^2x^2 + u_0^2(x(xu_0)) = u_0x^2 + 2u_0^2x^2 - 4u_0^2(u_0x^2)$$
$$= u_0x^2 + 2u_0^2x^2$$
$$= 0$$

i. e. $u_0x^2 = -2u_0^2x^2$, then

$$(uu_0)(x(xu_0)) =- \tfrac{1}{2}(uu_0)(u_0x^2) =(uu_0)(u_0^2x^2) \in U^2V^2 =<0>.$$

Then in (*) we have $ux^2 + 2(uu_0)x^2 = 0$

Conversely, let us suppose that for all u, x in U we have

$$ux^2 + 2(uu_0)x^2 = 0 \quad (**)$$

Then for all u, x in U,

$(u+2uu_0)(x+2xu_0)^2= [ux^2+2(uu_0)x^2+8(uu_0)(x(xu_0))]+4u(x(xu_0))$

From (**) we have $u_0x^2 + 2u_0^2x^2 = 0$. Then: $(uu_0)(x(xu_0)) = (uu_0)(u_0^2x^2) =0$ and $u(x(xu_0))= - \tfrac{1}{2}u(u_0x^2) =u(u_0^2x^2) \in UV^2=<0>$.

We have $(u+2uu_0)(x+2xu_0)^2=ux^2+2(uu_0)x^2=0$; i.e. $U_0^3 =<0>$ ■

Corollary: Let $A = Ke \oplus U \oplus V$ be a Bernstein algebra with any e idempotent. If A is orthogonal then U^3 is a subspace of $(U^2)^2$ and dim $(U^2)^2$ is an invariant of A.

Proof. A is orthogonal, then pivotal idempotent exist, $e_0 = e + u_0 + u_0^2$. For all $u,x \in U$ we have $ux^2 = 2(uu_0)x^2$, then U^3 is a subspace of $(U^2)^2$. Besides, in the core of Bernstein's algebra ($C=Ke\oplus U\oplus U^2$) we have $\dim(UU^2+(U^2)^2)= \dim(U^3+(U^2)^2)= \dim(U^2)^2$ is invariant ■

Definition. A Bernstein algebra $A = Ke \oplus U \oplus V$ is said to be *normal if* $x^2y = \omega(x)xy$ for all $x,\ y \in A$.

It is known that the normallity condition is equivalent to $UV = V^2 = \langle 0\rangle$. Then, we have that all normal Bernstein algebra is totally orthogonal. It is remarked that the converse of this statement is not true. One example is $A=\langle e,u,v\rangle$ with $e^2 = e$, $eu = \frac{1}{2}u$, $ev = 0$, $u^2=0$, $uv = u$, $v^2 = u$.

Definition: A commutative K-algebra A is Jordan if $x^2(yx)=(x^2y)x$ for all $x, y \in A$.

It is known ([1], [6]) that a Bernstein algebra $A=Ke\oplus U\oplus V$ is Jordan if and only if $V^2=\langle 0\rangle$ and $v(vu)=0$ for all $u\in U$, $v\in V$. This is independent of the choice of the idempotent e.

Corollary: Let $A = Ke \oplus U \oplus V$ be a Bernstein algebra which is also Jordan. Then is totally orthogonal if and only if $U^3= \langle 0\rangle$ ■

The following theorem stablishes the invariance of dim U^3. This number may be used for the classification of Bernstein's algebras.

Theorem: Let $A = Ke \oplus U \oplus V$ be a Bernstein algebra which is also Jordan. Then dim U^3 is an invariant of A.

Proof: Let $e_0= e+u+u^2$ be an idempotent of A. As A is Jordan we have $(U^2)^2\subseteq V^2= \langle 0\rangle$ and $(U_0^2)^2\subseteq V_0^2= \langle 0\rangle$. Then the core $C= Ke\oplus U\oplus U^2= Ke_0\oplus U_0\oplus U_0^2$ is Bernstein, so $\dim (UU^2+(U^2)^2)=\dim(U_0U_0^2+(U_0^2)^2)$ so $\dim U^3= \dim U_0^3$. ■

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