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On the level-convergence and fuzzy integration.*

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Abstract.
In this paper we define the level-convergence of measurable functions on a fuzzy measure space. We study some of the properties of this convergence and give conditions for the continuity of the fuzzy integral in relation to the level-convergence.

1 Introduction

The theory of fuzzy integration with respect to a fuzzy measure was introduced by Sugeno in [8] as a model for the treatment of non- deterministic problems. In particular, the continuity of the fuzzy integral with respect to different kinds of convergence has been exhaustly studied in the last years. Ralescu and Adams [3] proved theorems of continuity of fuzzy integral with respect to measure convergence and pointwise convergence, for a continuous and subadditive fuzzy measure. Recently [5], we study different kinds of multivalued convergences for fuzzy sets on $\mathbb{R}^n$ and its relationships.

The aims of this is to analize the continuity of fuzzy integral with respect to multivalued convergences, more precisely we introduce the concept of level-convergence (L-convergence) on a fuzzy measure space X, study some of its properties and give conditions for the continuity of the fuzzy integral in relation to the L-convergence.

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2 Preliminaries

Definition 2.1 *Let X be a set and $\sum$ be a σ- algebra of subsets of X. By fuzzy measure we mean a positive, extended real-valued set function $\mu : \sum \to [0, \infty]$ with properties:*

(FM1) $\mu(\phi) = 0$

(FM2) $A, B \in \sum$ *and* $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$.

Furthermore, if

(FM3) $A_1 \subseteq A_2 \subseteq \cdots, A_n \in \sum \Rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$, *then μ is upper continuous.*

Analogously, we say that μ is lower continuous when

(FM4) *If $A_1 \supseteq A_2 \supseteq \cdots, A_n \in \sum$ and there exits n_0 such that $\mu(A_{n_0}) < \infty$, then $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.*

If μ satisfies (FM3) *and* (FM4) *we say that μ is continuous.*

Throughout this paper $(X, \sum, \mu)$ will be a fuzzy measure space and $M(X)$ the family of all measurable functions $f : X \to [0, \infty]$.

If $f \in M(X)$, then the fuzzy integral of f is defined in [9] as:

$$\int_A f d\mu = \bigvee_{\alpha \geq 0} [\alpha \bigwedge \mu(A \cap \{f \geq \alpha\})], A \in \sum \qquad (1)$$

where $\bigvee$, $\bigwedge$ denote the operations sup and inf in $[0, \infty]$.

The following properties of the fuzzy integral are well-known:

Theorem 2.2 [7,10]

P1) $\int_A f d\mu = \int \mathcal{X}_A f d\mu$ *($\mathcal{X}_A$ is the characteristic function of A).*

P2) $\int_A k d\mu = k \bigwedge \mu(A)$, *$k$ constant.*

P3) i) *If $A \subseteq B$ then $\int_A f d\mu \leq \int_B f d\mu$.*

ii) *If $f \leq g$ in A then $\int_A f d\mu \leq \int_A g d\mu$.*

P4) i) *If $\mu(A) = 0$ then $\int_A f d\mu = 0$*

ii) *If $\int_A f d\mu = 0$ and μ is upper continuous then $\mu(A \cap \{f > 0\}) = 0$.*

P5) $\int_A (f + k) d\mu \leq \int_A f d\mu + \int_A k d\mu$, *$k$ constant.*

P6) *If $| f - g | \leq \epsilon$ in A, then $| \int_A f d\mu - \int_A g d\mu | \leq \epsilon$.*

Remark 2.3

i) From definition (1) we conclude that $\int I_A d\mu = \mu(A)$ (I_A is the indicator

function of A, i.e. $I_A(x) = \infty$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$).
ii) From P6) we conclude that the fuzzy integral in continuous respect to uniform convergence.

Theorem 2.4 [3] *If $f : X \to [0, \infty]$ is a measurable function, then*

$$\int_X f d\mu = \int_0^\infty \mu\{f \geq \alpha\} d\alpha$$

where the integral in the right-side of the last equation is the fuzzy integral of $F(\alpha) = \mu(\{f \geq \alpha\})$ with respect to the Lebesgue measure in $[0, \infty]$.

Theorem 2.5 [3] *If μ is subadditive (i.e. $\mu(A \cup B) \leq \mu(A) + \mu(B)$) and $f_n \to f$ in measure, then $\int f_n d\mu \to \int f d\mu$.*

Theorem 2.6 [3] *If μ is subadditive , $\mu(X) < \infty$ and $f_n \to f$ pointwise then $\int f_n d\mu \to \int f d\mu$.*

Others interestings properties and applications of this integral were discussed in [6],[7] and [8].

3 Level-convergence

Definition 3.1 *We say that a sequence of sets (A_n), $A_n \in \sum$, converges to $A \in \sum$, denoted by $A = \lim A_n$ (shortly: $A_n \to A$), if $A = \lim inf A_n = \lim sup A_n$ where*

$$\lim sup A_n = \cap_{n=1}^\infty \left[\bigcup_{k=n}^\infty A_k\right] \text{ and } \lim inf A_n = \cup_{n=1}^\infty \left[\bigcap_{k=n}^\infty A_k\right]$$

Remark 3.2 *It is clear that $\lim sup A_n$ consists of all x which are in infinitely many of the A_n and $\lim inf A_n$ consists of all x wich are in all but finitely many of the A_n.*

Remark 3.3 *If (A_n) is an increasing sequence in $\sum$, then $\lim A_n$ exists and it is equal to $\cup A_n$.*

Analogously, if (B_n) is a decreasing sequence in $\sum$, then $\lim B_n$ exists and it is equal to $\cap B_n$.

Definition 3.4 *Let $f \in M(X)$ and $\alpha \in [0,\infty]$. Then, the α-level of f is defined by*

$$L_\alpha f = \{x \in X / f(x) \geq \alpha\}$$

The support of f is defined by:

$$supp(f) = L_0 f = \{x \in X / f(x) > 0\} = \bigcup_{\alpha>0} L_\alpha f$$

Definition 3.5 *We say that a sequence of functions $(f_n), f_n \in M(X)$, L-converges to $f \in M(X)$ (shortly: $f_n \xrightarrow{L} f$) if for every $\alpha \geq 0$, $L_\alpha f_n \to L_\alpha f$.*

The next proposition show that L-convergence is stronger than pointwise convergence.

Proposition 3.6 *If $f_n \xrightarrow{L} f$, then $f_n \to f$ pointwise.*

Proof. Suppose that $f_n \xrightarrow{L} f$ and let $x_0 \in X$ with $f(x_0) = \alpha_0$. Then

$$x_0 \in L_{\alpha_0} f = \lim inf L_{\alpha_0} f_n = \bigcup_{n=1}^{\infty}\left[\bigcap_{k=n}^{\infty} L_{\alpha_0} f_k\right]$$

Consequently, $\exists n_0 \in N$ such that $x_0 \in \bigcap_{k=n_0}^{\infty} L_{\alpha_0} f_k$. That is, $x_0 \in L_{\alpha_0} f_n = \{f_n \geq \alpha_0\}$ $\forall n \geq n_0$. Hence, $f_n(x_0) \geq \alpha_0 \forall n \geq n_0$. Thus, $\lim inf f_n(x_0) \geq \alpha_0$
Now suppose that $\beta_0 = \lim sup f_n(x_0) > \alpha_0$ and let $\epsilon > 0$ such that $\beta_0 - \epsilon > \alpha_0$. Then, $f_n(x_0) \geq \beta_0 - \epsilon$ for infinite values of n. Hence, $x_0 \in L_{\beta_0 - \epsilon} f_n$ for infinite values of n.
Consequently,

$$x_0 \in \bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty} L_{\beta_0-\epsilon} f_n\right] \subseteq \lim sup\ L_{\beta_0-\epsilon} f_n = L_{\beta_0-\epsilon} f$$

Thus $f(x_0) \geq \beta_0 - \epsilon > \alpha_0$. But this is impossible since $f(x_0) = \alpha_0$. This implies that $\alpha_0 \leq \lim inf f_n(x_0) \leq \lim sup f_n(x_0) \leq \alpha_0$.
Consequently, $\lim_{n\to\infty} f_n(x_0) = f(x_0)$, *i.e.* $f_n \to f$ pointwise. ∎

Corollary 3.7 *With the same conditions of Proposition 3.6, then: $f_n \xrightarrow{L} f$ and μ finite implies $f_n \xrightarrow{\mu} f$.*

∎

In the next example we show that pointwise convergence does not imply L-convergence.

Example 3.8 *Let $X = \mathbb{R}$, $\sum = \sigma$ - algebra of Lebesgue measurable sets on X. Define f_n, f by:*

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases} \quad \text{and} \quad f_n(x) = \begin{cases} \frac{x}{n} + 1 - \frac{1}{n} & \text{if } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Clearly $f_n \to f$ uniformly, but $L_1 f = [0,1]$ whereas $L_1 f_n = \{1\}, \forall n$.
Thus, $\lim inf L_1 f_n = \lim sup L_1 f_n = \{1\} \neq [0,1] = L_1 f$
Consequently, (f_n) does not converges levelwise to f.

4 L-convergence and fuzzy integral

Here we investigate the continuity of fuzzy integral with respect to L-convergence.

Lemma 4.1 *If $A_n \to A$, μ continuous, and there exists n_0 such that*

$$\mu\left[\cup_{k=n_0}^{\infty} A_k\right] < \infty, \quad \text{then} \quad \mu(A_n) \to \mu(A)$$

Proof. $A = \lim A_n$ implies $A = \lim inf A_n = \bigcup_{n=1}^{\infty}\left[\bigcap_{k=n}^{\infty} A_k\right]$.

Since $\left(\bigcap_{k=n}^{\infty} A_k\right)_{n\in\mathbb{N}}$ is a increasing sequence then

$$\bigcap_{k=n}^{\infty} A_k \nearrow \bigcup_{n=1}^{\infty}\left[\bigcap_{k=n}^{\infty} A_k\right] = A = \lim inf A_n$$

Now, looking to the fact that $\bigcap_{k=n}^{\infty} A_k \subseteq A_n$ and the continuity of μ on monotone sequences, we obtain

$$\mu(A) = \lim inf(A_n) = \lim_{n\to\infty} \mu\left[\bigcap_{k=n}^{\infty} A_k\right] = \lim inf \mu\left[\bigcap_{k=n}^{\infty} A_k\right] \le \lim inf \mu(A_n)$$

Thus, we obtain $\mu(A) \le \lim inf \mu(A_n)$

Analogously, since $\left(\bigcup_{k=n}^{\infty} A_k\right)_{n\in\mathbb{N}}$ is a decreasing sequence, then

$$\bigcup_{k=n}^{\infty} A_k \searrow \bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty} A_k\right] = A = \lim sup A_n$$

So, since $\bigcup_{k=n}^{\infty} A_k \supseteq A_n$ then, by (FM4) we obtain:

$$\mu(A) = \mu(\lim supA_n) = \lim_{n\to\infty} \mu\left[\bigcap_{k=n}^{\infty} A_k\right] = \lim sup\mu\left[\bigcup_{k=n}^{\infty} A_k\right] \geq \lim sup\mu(A_n)$$

Consequently, $\mu(A) \geq \lim sup\mu(A_n)$.
Thus, $0 \leq \mu(A) \leq \lim inf\mu(A_n) \leq \lim sup\mu(A_n) \leq \mu(A)$, i.e. $\mu(A_n) \to \mu(A)$. ■

Theorem 4.2 *Let f_n, $f \in M(X)$ with L_α f_n s-measurables,*

$$\mu\left[\bigcup_{n=1}^{\infty} supp(f_n)\right] < \infty \quad \textit{and } \mu \textit{ continuous}$$

Then: $f_n \xrightarrow{L} f \Longrightarrow \int f_n d\mu \to \int f d\mu$.

Proof. If $f_n \xrightarrow{L} f$ then, by definition of L-convergence, it follow that

$$L_\alpha f_n \to L_\alpha f, \forall \alpha \geq 0$$

Hence, by Lemma 5.1, $\mu(L_\alpha f_n) \to \mu(L_\alpha f), \forall \alpha \geq 0$.

So, making use of Theorem 2.6 we obtain that $\int \mu(L_\alpha f_n)\, d\alpha \to \int \mu(L_\alpha f) d\alpha$. Thus, by Theorem 2.4, we conclude that $\int f_n d\mu \to \int f d\mu$. ■

The hypothesis $\mu[\bigcup_{n=1}^{\infty} supp(f_n)] < \infty$ in Theorem 4.2, cannot avoid as show the following example:

Example 4.3 *Let $(X, \sum, \mu)$ be as in example 3.8, μ the usual Lebesgue measure on X. Define f_n, f by*

$$f_n(x) = \begin{cases} \frac{|x|}{n} & \textit{if } -n \leq x \leq n \\ 1 & \textit{elsewhere} \end{cases} \qquad \textit{and } f(x) = 0, \forall x$$

Then $supp(f_n) = \mathbb{R} - \{0\}, \forall n$. So, $\mu\left[\bigcup_{n=1}^{\infty} supp(f_n)\right] = \infty$. On the other hand, $f_n \xrightarrow{L} f$ and $\int f_n d\mu = 1 \quad \forall n$, whereas $\int f d\mu = 0$. ■

Lemma 4.4 *Let $A_n, A \in \sum$, then $A_n \to A$ if and only if $I_{A_n} \xrightarrow{L} I_A$.*

Proof. Direct consequence of fact that $L_\alpha I_{A_n} = A_n, \forall \alpha$. ■

Theorem 4.5 *Let* $(X, \sum, \mu)$ *a finite fuzzy measure space. Then the following properties are equivalent:*
i) μ *is continous*
ii) $f_n \xrightarrow{L} f \Longrightarrow \int f_n d\mu \to \int f d\mu$.

Proof.
i) $\Longrightarrow$ ii) Direct consequence of Theorem 4.2.
ii) $\Longrightarrow$ i) Let (A_n) a monotone sequence in $\sum$ and $A = \lim A_n$. Then, by Lemma 4.4, $A_n \to A$ implies $I_{A_n} \xrightarrow{L} I_A$. Thus, by hypothesis, $\int I_{A_n} d\mu \to \int I_A d\mu$. That is $\mu(A_n) \to \mu(A)$ (see Remark 2.3.) Therefore, μ is continuous. ∎

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