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Behavior of multiple solutions for systems of semilinear elliptic equations. *

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Abstract.
In this work we present some partial results that will appear in a completed form in a forthcoming paper, [7]. We discuss the existence and particularly the multiplicity of solutions for the nonlinear system of elliptic equations

$$\triangle_{u_i} + \lambda f_i(x, u_1, ..., u_m) = 0 \qquad in \;\; \Omega \tag{1.1}$$

$$u_i \mid_{\partial\Omega} = 0, \; i = 1, \cdots, m \tag{1.2}$$

where $f_i(x, 0, \cdots, 0) > 0$ for all $x \in \Omega$, $i = 1, 2, \cdots, m$. The functions f_i, $i = 1, \cdots, m$, satisfy the quasimonotone condition and a certain blow up rate as to be made precise in the assumptions (H1) and (H2) below. Then results similar to those of the scalar equation case (see [6]) can be established. It should be noted that unless $\frac{\partial f_i}{\partial u_j} = \frac{\partial f_j}{\partial u_i}$ for all $1 \leq i, j \leq m$, the problem cannot be formulated in a variational form, hence techniques associated with variational structure are not applicable.

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1 Introduction

We will now make our assumptions more precise.

For equations (1.1) and (1.2) defined in a smooth bounded domain Ω in R^n, $n \geq 2$, we assume

$(H1)$ For $\vec{x} \in \bar{\Omega}$, $1 \leq i,j \leq m$, $\vec{u} \geq \vec{0}$ (i.e. $u_i \geq 0$ componentwise), let $f_i \in C^3(\bar{\Omega} \times I\!R^m)$, $\frac{\partial f_i}{\partial u_i}(x,\vec{u}) > 0$ and $\vec{f}$ satisfies the quasimonotone condition $\frac{\partial f_i}{\partial u_j}(x,\vec{u}) \geq 0$ for $i \neq j$. For sufficiently large $M > 0$, there exist constants c_1, $c_2 > 0$, independent of $\vec{x}$, such that when $u_i \geq M$, $i = 1,..,m$, then

$$f_i(\vec{x},\vec{u}) \leq c_2\|\vec{u}\|_s^s \tag{1.3}$$

$$c_1 u_i^s \leq f_i(\vec{x},0,...,0,u_i,0,...,0) \text{ for all } \vec{x} \in \Omega, \tag{1.4}$$

where $1 < s < \frac{n}{n-2}$ for $n > 2$, and any $s \in (1,\infty)$ for $n = 2$.

Here $\|\vec{u}\|_s = \left(\sum_{i=1}^{m} u_i^s\right)^{\frac{1}{s}}$ and $\vec{f} = (f_1, \cdots f_m)^T$.

$(H2)$ Ω is convex , $\partial\Omega$ has positive curvature everywhere, and there exist $r, \delta > 0$ such that for all $\vec{u} \geq 0$ and all $\vec{x} \in \Omega_r \equiv \{x \in \Omega | \text{dist}(\vec{x},\partial\Omega) < r\}$, $i = 1,..,m$,

$$\nabla_x f_i(\cdot,\vec{u}) \cdot \vec{\mu} \leq 0,$$

where $\vec{\mu}$ is a unit vector satifying

$$|\vec{\mu} - \vec{n}(\vec{x})| < \delta, \tag{1.5}$$

and $\vec{n}(\vec{x})$ is defined for $\vec{x} \in \Omega$ to be $\vec{n}(\vec{y})$, which is the unit outward normal vector at $\vec{y} \in \partial\Omega$ with $\vec{y}$ defined by $|\vec{y}-\vec{x}| = \text{dist}(\vec{x}, \partial\Omega)$ (the question of well definedness of $\vec{n}(\vec{x})$ is discussed in [4] .)

Under these hypothesis, it is proven in [7] that there exists a $\lambda^* > 0$ such that for $\lambda < \lambda^*$, there are at least two solutions, for $\lambda = \lambda^*$, there exists at least one solution, while for $\lambda > \lambda^*$, there is no solution. Here we discuss in detail the case that strict convexity of $\vec{f}$ is assumed, i.e., $\left(\frac{\partial^2 f_i}{\partial u_j \partial u_k}\right)_{j,k=1,\ldots,m}$ is a positive definite matrix for each $i = 1, \ldots, m$, then the previous statements can be made more precise for $\lambda = \lambda^*$: there exists exactly one solution for such λ, and it is a simple turning point.

Finally we remark that similar techniques can be applied to study the existence and multiplicity of the system of quasimonotone semilinear equations

$$\Delta u_i + \vec{b} \cdot \nabla u_i + \lambda f_i(x, u_1, \ldots, u_m) = 0 \quad \text{in } \Omega \tag{1.6}$$

$$u_i|_{\partial\Omega} = 0 \,, \quad i = 1, \ldots, m \tag{1.7}$$

where $\vec{b} \in C^1(\bar{\Omega})$ satifies the additional condition

$$\vec{b} \cdot \vec{\mu} \geq 0,$$

in assumption (H2).

2 Existence, multiplicity and *A-priori* Bounds of Solutions

For simplicity we use the vectors $\vec{1} = (1,1,...,1)$ and $\vec{0} = (0,0,...,0)$. When $\lambda = 0$, $\vec{u} \equiv \vec{0}$ is the unique solution. Let S denote the set of nonnegative λ for which equations (1.1)-(1.2) have a nonnegative solution. We shall begin by proving that S is a bounded interval. The following lemma is a condensation of three lemmas in [7].

Lemma 2.1. *i). If λ is sufficiently small, then $\lambda \in S$.*

ii)If a solution of the equations $(1.1) - (1.2)$ *exists for $\tilde{\lambda} > 0$, then solution exists for all λ such that* $0 \leq \lambda \leq \tilde{\lambda}$.

iii). S is bounded.

It should be noted that the monotone iterations methods used in [7] give minimal positive solution for each λ for which solution exists: There exists a minimal positive solution $\vec{u}$ of (1.1)-(1.2) that satisfies $0 < \vec{u} \leq \vec{v}$ in Ω (componentwise) for all positive solution $\vec{v}$ of (1.1)-(1.2). It is also clear from the monotone iteration technique that $\vec{u}^{\lambda_1}_{\min} < \vec{u}^{\lambda_2}_{\min}$ in Ω for $\lambda_1 < \lambda_2$, where $\vec{u}^{\lambda_j}_{\min}$ is the minimal positive solution for $\lambda = \lambda_j$, $j = 1,2$.

We establish that there is a second solution besides this minimal positive solution.

The next lemma is also a condensed form of some lemmas in [7].

Lemma 2.2. *For $\lambda \geq \varepsilon > 0$ there exists a $C_1, C_2, C_0, C_s > 0$ such that for all solutions of (1.1)-(1.2) ,*

$$i). \qquad \|\vec{u}\|_{L^1} \leq C_1$$

$$ii). \qquad \|\vec{u}\|_{L^s} \leq C_s.$$

$$iii). \qquad \|\vec{u}\|_{\mathcal{H}^1} = \sum_{i=1}^{m} \|u_i\|_{\mathcal{H}^1} \leq C_2.$$

$$iv). \qquad \|\vec{u}\|_{C^{2+\alpha}} = \sum_{i=1}^{m} \|u_i\|_{C^{2+\alpha}} \leq C_0$$

(where s is the blow up rate of $\vec{f}$ as defined in the assumption (H1).)

An immediate consequence of Lemma 2.1 is the existence of solution for $\lambda = \lambda^*$.

Using the previous *a-priri* bounds for the solutions together with some Degree Theory methods it is shown in [7] the following multiplicity result:

Lemma 2.3 *There exists at least 2 solutions of the equation (1.1)-(1.2), for λ in the range (ε, λ^*).*

3 Behavior of the Solution for $\lambda = \lambda^*$ for Strictly Convex $\vec{f}$

In this section we assume that for each $i = 1, ..., m$, the matrix $\left(\frac{\partial^2 f_i}{\partial u_j \partial u_k}\right)_{j,k=1,...,m}$ is positive definite and that $\frac{\partial f_i}{\partial u_j} > 0$ for all $1 \le i, j \le m$. Then as in the case of a scalar equation (see [7]) we can prove the following properties of the solution set,

(i) The solution for $\lambda = \lambda^*$ is unique.

(ii) Around a neighborhood of $\lambda = \lambda^*$, the solution set can be parametrized by $\lambda = \lambda(s)$ and $u = u(s)$ for $-\delta < s < \delta$ for some $\delta > 0$, with $\lambda(0) = \lambda^*$. Further, $\lambda(s) < \lambda^*$ for $s \neq 0$ in that neighborhood. Hence $\lambda = \lambda^*$ corresponds to a simple turning point.

To prove these claims, we define

$$\vec{F}(\vec{x}, \vec{u}, \lambda) \equiv \Delta \vec{u} + \lambda \vec{f}(x, \vec{u}) \tag{3.1}$$

for $\vec{u}$ in $(C^{2+\alpha}(\bar{\Omega}))^m$ with zero Dirichlet boundary condition.

Thus solutions of the equations (1.1)-(1.2) correspond to

$$\vec{F}(\vec{x},\vec{u},\lambda) = 0. \tag{3.2}$$

Denote the minimal positive solution for λ by $\vec{u}^{\lambda}_{\min}$. The Frechet derivative of $\vec{F}$ evaluated at the minimal solution is given by

$$D_{\vec{u}}\vec{F}(\vec{x},\vec{u}^{\lambda}_{\min},\lambda)\vec{v} \equiv \Delta\vec{v} + \lambda D_{\vec{u}}\vec{f}(\vec{x},\vec{u}^{\lambda}_{\min})\vec{v} \tag{3.3}$$

for any $\vec{v}$ in $(C^{2+\alpha}(\bar{\Omega}))^m$, where

$$D_{\vec{u}}\vec{f}(\vec{x},\vec{u}^{\lambda}_{\min}) = \left(\frac{\partial f_i}{\partial u_j}(\vec{x},\vec{u}^{\lambda}_{\min})\right)_{i,j=1,\dots,m}.$$

First we establish a lemma:

Lemma 3.1 *Let* $A(\vec{x}) = (a_{ij})_{i,j=1,\dots,m} > 0$, *i.e.* $a_{ij}(\vec{x}) > 0$ *for all* $1 \le i,j \le m$. *Then there exists a positive eigenvalue* η_1 *and a positive vector eigenfunction* $\vec{\psi}$ *such that*

$$\Delta\vec{\psi} + \eta_1 A(\vec{x})\vec{\psi} = 0, \tag{3.4}$$

$$\vec{\psi}|_{\partial\Omega} = 0. \tag{3.5}$$

Proof: Define $\tilde{T} : (C(\bar{\Omega}))^m \to (C(\bar{\Omega}))^m$ by $\tilde{T}\vec{\varphi} \equiv -\Delta^{-1}(A\vec{\varphi})$ subject to zero Direchlet boundary conditions. Then $\tilde{T}$ is a positive operator: If $\vec{\varphi} \ge 0$ with at least one component not identically zero in Ω , then if $\vec{\phi} = \tilde{T}\vec{\varphi}$ we have $-\Delta\vec{\phi} = A\vec{\varphi} \geq \vec{0}$, with no component being identically zero in Ω. Hence by the Maximum Principle $\vec{\phi} > \vec{0}$, i.e. each component of ϕ is positive on Ω.

Since $\tilde{T}$ is also compact, the Krein-Rutman Theorem implies the existence of a positive eigenvalue μ_1 and a positive vector eigenfunction $\vec{\psi}$ such that

$$\tilde{T}\vec{\psi} = \mu_1\vec{\psi}$$

so

$$A\vec{\psi} = -\mu_1\Delta\vec{\psi}.$$

The lemma follows with $\eta_1 = \frac{1}{\mu_1}$.

For a fixed $\lambda_0 < \lambda^*$, by the previous lemma, there exists a positive eigenvalue κ_1 and a positive vector eigenfunction $\vec{\varphi}_1$ such that

$$\Delta\vec{\varphi}_1 + \kappa_1\lambda_0 D_{\vec{u}}\vec{f}(\vec{x}, \vec{u}^{\lambda_0}_{\min})\varphi_1 = 0.$$

hence for $\lambda < \min\{\kappa_1\lambda_0, \lambda_0\}$ there cannot be a nontrivial solution for the problem

$$\Delta\vec{\phi} + \lambda D_{\vec{u}}\vec{f}(\vec{x}, \vec{u}^{\lambda}_{\min})\vec{\phi} = 0$$

because of the comparison theorem 1.13 in [2] and $D_{\vec{u}}\vec{f}(\vec{x}, \vec{u}^{\lambda}_{\min})$ being an increasing function in λ for each entry in the matrix. Thus for sufficiently small λ the Frechet derivative in (3.3) is non-singular. On the other hand there should be a first $\lambda = \bar{\lambda} \leq \lambda^*$ at which $D_{\vec{u}}\vec{F}$ becomes singular. Otherwise we can continue the minimal solution branch to $\lambda = \lambda^*$ since the solution can never blow up due to the *a-priori* bound that we have established . Using Implicit Function theorem at $\lambda = \lambda^*$, we can obtain a solution of (3.2) with $\lambda > \lambda^*$, which is a contradiction.

We claim this $\bar{\lambda}$ is the first eigenvalue for

$$\Delta\vec{\psi} + \lambda D_{\vec{u}}\vec{f}(\vec{x}, \vec{u}^{\lambda}_{\text{min}})\vec{\psi} = 0 \tag{3.6}$$

subject to Dirichlet boundary condition, and therefore its corresponding vector eigenfunction $\vec{\psi}$ is positive. If not, there exists a first eigenvalue $\nu_1 < \bar{\lambda}$ and a corresponding vector eigenfunction $\vec{\psi}_1 > 0$ to the problem

$$\Delta\vec{\psi} + \nu D_{\vec{u}}\vec{f}(\vec{x}, \vec{u}^{\lambda}_{\text{min}})\vec{\psi} = 0. \tag{3.7}$$

However

$$\nu_1 D_{\vec{u}}\vec{f}(\vec{x}, \vec{u}^{\bar{\lambda}}_{\text{min}}) < \beta D_{\vec{u}}\vec{f}(\vec{x}, \vec{u}^{\beta}_{\text{min}}) < \bar{\lambda} D_{\vec{u}}\vec{f}(\vec{x}, \vec{u}^{\bar{\lambda}}_{\text{min}})$$

for some $\beta < \bar{\lambda}$ and close to $\bar{\lambda}$ by simple continuity.

Since

$$\nu_1 D_{\vec{u}}\vec{f}(\vec{x},\vec{u}^{\nu_1}_{\min}) < \nu_1 D_{\vec{u}}\vec{f}(\vec{x},\vec{u}^{\lambda}_{\min}) < \beta D_{\vec{u}}\vec{f}(\vec{x},\vec{u}^{\beta}_{\min})$$

there exist positive first eigenvalues ρ_1, ρ_2, ρ_3 to the problems

$$\Delta Z_1 + \rho_1\nu_1 D_{\vec{u}}\vec{f}(\vec{x},\vec{u}^{\nu_1}_{\min})Z_1 = 0$$

$$\Delta Z_2 + \rho_2\nu_1 D_{\vec{u}}\vec{f}(\vec{x},\vec{u}^{\lambda}_{\min})Z_2 = 0$$

$$\Delta Z_3 + \rho_3\beta D_{\vec{u}}\vec{f}(\vec{x},\vec{u}^{\beta}_{\min})Z_3 = 0$$

with $\rho_2 = 1$ because of equation (3.7) and $\rho_1 \geq \rho_2 \geq \rho_3$ by the comparison theorem 1.13 in [2].

Since $\nu D_{\vec{u}}\vec{f}(\vec{x},\vec{u}^{\nu}_{\min})$ is a continuous function of ν, and eigenvalues depend continuously on the coefficients, hence ρ is a continuous function of ν. By the Intermediate Value Theorem, there exist a $\rho = 1$ and a $\nu_1 < \nu < \beta$ such that

$$\Delta\vec{Z} + \rho\nu D_{\vec{u}}\vec{f}(\vec{x},\vec{u}^{\nu}_{\min})\vec{Z} = 0.$$

But this contradicts our assumption that $\lambda = \lambda^*$ is the first value where the Frechet derivative (3.3) becomes singular. Hence we have proved that there exists a first eigenvalue $\bar{\lambda} \leq \lambda^*$ and a positive vector eigenfunction $\vec{\psi}$ such that

$$\Delta\vec{\psi} + \bar{\lambda} D_{\vec{u}}\vec{f}(\vec{x},\vec{u}^{\lambda}_{\min})\vec{\psi} = 0. \tag{3.8}$$

By lemma 3.1, we have the existence of a first eigenvalue $\bar{\bar{\lambda}}$ and a positive vector eigenfunction $\vec{\psi}^* > 0$ such that

$$\Delta\vec{\psi}^* + \bar{\bar{\lambda}}\left(D_{\vec{u}}\vec{f}(\vec{x}, \vec{u}^{\bar{\lambda}}_{\min})\right)^T \vec{\psi}^* = 0. \tag{3.9}$$

Let $\langle \vec{f}, \vec{g}\rangle = \sum_{i=1}^m f_i g_i$ denote the usual inner product in $\mathbb{R}^m$. Take inner product of (3.8) with $\vec{\psi}^*$ and (3.9) with $\vec{\psi}$. Integrating by parts and substracting, since $\int_\Omega \langle D_{\vec{u}}\vec{f}, \vec{\psi}\rangle d\vec{x}$ is positive it follows that $\bar{\lambda} = \bar{\bar{\lambda}}$.

Now it can be checked that

$$\vec{F}_\lambda(\vec{x}, \vec{u}^{\bar{\lambda}}_{\min}, \bar{\lambda}) = \vec{f}(\vec{x}, \vec{u}^{\bar{\lambda}}_{\min}) > \vec{0}.$$

Since $\int_\Omega \langle \vec{\psi}^*, \vec{f}(\vec{x}, \vec{u}^{\bar{\lambda}}_{\min})\rangle d\vec{x}$ is positive, it follows that

$$\vec{F}_\lambda \notin \text{Range}\left(D_{\vec{u}}\vec{f}(\vec{x}, \vec{u}^{\bar{\lambda}}_{\min})\right).$$

So $\lambda = \bar{\lambda}$ is not a bifurcation point [3].

We can therefore parametrize the solution set in a neighborhood around $\bar{\lambda}$ by: $\vec{u} = \vec{u}(s)$, $\lambda = \lambda(s)$, for some sufficient small $\delta > 0$ and $-\delta < s < \delta$ with $\lambda(0) = \bar{\lambda}$ as a consequence of implicit function theorem.

With the assumed smoothness in f, we can differentiate the equation (3.2) with respect to s, which gives

$$\Delta\vec{v} + \lambda(s)D_{\vec{u}}\vec{f}(\vec{x}, \vec{u}^{\lambda}_{\min})\vec{v} + \lambda'(s)\vec{f}(\vec{x}, \vec{u}^{\lambda}_{\min}) = 0 \tag{3.10}$$

where $\vec{v} \equiv \frac{d\vec{u}}{ds}(s)$. We evaluate the equation at $s = 0$, take inner product with $\vec{\psi}^*$ and integrate over Ω, which results in

$$\lambda'(0) = 0. \tag{3.11}$$

Differentiate equation (3.10) once more. With $\vec{w} = \frac{d^2\vec{u}}{ds^2}(s)$ we have, after evaluating at $s = 0$ and using the equation (3.11),

$$\Delta w_i + \bar{\lambda} \sum_{j=1}^{m} \frac{\partial f_i}{\partial u_j} w_j + \bar{\lambda} \sum_{j,k=1}^{m} \frac{\partial^2 f_i}{\partial u_j \partial u_k} v_j v_k + \lambda''(0) f_i = 0$$

for $i = 1, ..., m$. Again take inner product with $\vec{\psi}^*$ and integrate to get $\lambda''(0) < 0$ after employing the assumption that $\left(\frac{\partial^2 f_i}{\partial u_j \partial u_k}\right)_{j,k=1,...,m}$ are positive definite for $i = 1, \ldots, m$.

Thus around a neighborhood of λ^*,

$$\lambda = \bar{\lambda} + \lambda''(0)s^2 + O(s^3). \tag{3.12}$$

So $\bar{\lambda}$ is a simple turning point.

Finally we show that $\bar{\lambda} = \lambda^*$, and there is only one solution for $\lambda = \lambda^*$. This will finish the proof of our claims.

Corresponding to $\lambda = \bar{\lambda}$ and $\lambda = \lambda^*$ we have

$$\Delta \vec{u}^{\bar{\lambda}}_{\min} + \bar{\lambda} \vec{f}(\vec{x}, \vec{u}^{\bar{\lambda}}_{\min}) = 0 \tag{3.13}$$

$$\Delta \vec{u}^{\lambda^*}_{\min} + \lambda^* \vec{f}(\vec{x}, \vec{u}^{\lambda^*}_{\min}) = 0 \tag{3.14}$$

Substract equation (3.14) from (3.13), take inner product with $\vec{\psi}^*$ which is the first eigenfunction for $\lambda = \bar{\lambda}$, and integrate to obtain

$$\int_\Omega \langle -\bar{\lambda} D_{\vec{u}} \vec{f}(\vec{x}, \vec{u}^{\bar{\lambda}}_{\min})(\vec{u}^{\bar{\lambda}}_{\min} - \vec{u}^{\lambda^*}_{\min}) + \bar{\lambda}\vec{f}(\vec{x}, \vec{u}^{\bar{\lambda}}_{\min}) - \lambda^* \vec{f}(\vec{x}, \vec{u}^{\lambda^*}_{\min})\ ,\ \vec{\psi}^* \rangle d\vec{x}$$

$$= 0$$

which can be written as

$$\bar{\lambda} \int_\Omega \langle -\vec{f}(\vec{x}, \vec{u}^{\lambda^*}_{\min}) + \vec{f}(\vec{x}, \vec{u}^{\bar{\lambda}}_{\min}) + D_{\vec{u}}\vec{f}(\vec{x}, \vec{u}^{\bar{\lambda}}_{\min})(\vec{u}^{\lambda^*}_{\min} - \vec{u}^{\bar{\lambda}}_{\min}), \vec{\psi}^* \rangle d\vec{x} =$$

$$\int_\Omega \langle (\lambda^* - \bar{\lambda})\vec{f}(\vec{x}, \vec{u}^{\lambda^*}_{\min}), \vec{\psi}^* \rangle d\vec{x}$$

By the convexness assumption on f, the left hand side is negative unless $u_* = u_0$ when it is zero. The right hand side is non-negative since $\lambda^* \geq \bar{\lambda}$, and can only be zero when $\lambda^* = \bar{\lambda}$. Hence the above equation holds only when $\lambda^* = \bar{\lambda}$, and $\vec{u}^{\bar{\lambda}}_{\min} = \vec{u}^{\bar{\lambda}}_{\min}$.

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