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Geodesically compatible metrics. Existence of commutative conservation laws

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Abstract

We give a natural geometric condition called *geodesic compatibility* that implies the existence of integrals in involution of the geodesic flow of a (pseudo)Riemannian metric. We prove that if two metrics satisfy the condition of geodesic compatibility then we can produce a hierarchy of metrics that also satisfy this condition. A lot of metrics studed in Riemannian and Kählerian geometry satisfy such conditions. We apply our results for obtaining an infinite family (hierarchy) of completely integrable flows on the complex projective plane CP^n.

Contents

1 Introduction

The main purpose of the present paper is to prove that if a pair of (pseudo) Riemannian metrics g and $\bar{g}$ satisfies a natural geometric condition called *geodesic compatibility* (or *PQ^ϵ-projectivity*) then the geodesic flows of the metrics g and $\bar{g}$ admit integrals in involution of a special form. The integrals we find usually have singularities that can be localized out of every arbitrary taken open set D with compact closure (Theorem 1). If the manifold is compact then the integrals can be taken smooth. We prove a Kählerian analog of the results proved in [11].

In what follows we call (pseudo)Riemannian metrics simply *metrics*. Positively definite metrics are called *Riemannian metrics*. All tensor object and manifolds we consider are smooth (C^∞). If $E \to M$ is a vector bundle over a manifold M then $\Gamma(E)$ denotes the space of the smooth sections of $E \to M$.

Let us consider some examples of metrics that satisfy the condition of the geodesic compatibility.

(a) Geodesic equivalence

A classical example of geodesic compatibility is the so-called *geodesic equivalence*. Recall the main definitions.

Let g and $\bar{g}$ be (pseudo)Riemannian metrics given on the manifold M^n, $n = \dim M^n$.

Definition 1. *The metrics g and $\bar{g}$ are called* geodesically equivalent *iff they have the same geodesics (considered as unparametrized curves on M^n).*

We say that the metric g admits *non-trivial geodesic equivalence* iff there exists a metric $\bar{g} \neq \mathrm{const}\, g$ such that g and $\bar{g}$ are geodesically equivalent. The first theorems concerning the existence of integrals of the geodesic flows of the metrics admitting non-trivial geodesic equivalence were proved by U. Dini, P. Painlevé, T. Levi-Civita and R. Liouville (see [1]). The Liouville integrability of the corresponding geodesic flows in the case when one of the metrics g and $\bar{g}$ is Riemannian is proved in [4] (see also [3, 5, 6, 8, 9, 6, 10]). The (pseudo)Riemannian analogs of these theorems are proved in [11].

(b) h-projectivity

Another example of geodesic compatibility appears about a century later in the papers of Ōtsuki and Tashiro ([15, 16]).

Let M^{2n} be a $2n$-dimensional real manifold endowed with a complex structure J, $J^2 = -\mathbf{1}$, where $\mathbf{1}$ denotes the identity operator of the tangent bundle TM^{2n}. Recall that, a metric g on M^{2n} is called *Kählerian* iff the next two conditions are satisfied: 1) g is *hermitian* metric, i.e., $g(J\xi, J\eta) = g(\xi, \eta)$, $\xi, \eta \in T_xM^{2n}$, and 2) the *Kähler form* $\Omega(\xi, \eta) \stackrel{\text{def}}{=} g(J\xi, \eta)$ is a symplectic form (i.e., Ω is non-degenerate and closed, $d\Omega = 0$).

Denote by ∇ the Levi-Civita connection corresponding to the Kählerian metric g. Following [15] we give:

Definition 2. *A smooth curve $(t_1, t_2) \ni t \to \gamma(t) \in M^{2n}$, $t_1 < t_2$, is called* holomorphically planar *(with respect to the Kählerian metric g) iff*

$$\frac{\nabla \dot{\gamma}}{dt}(t) = a(t)\dot{\gamma}(t) + b(t)J\dot{\gamma}(t), \tag{1}$$

where a and b are smooth functions of the parameter t.

Holomorphically planar curves always exist. For example, the geodesic lines of the metric g are holomorphically planar curves. More generally, fixing any smooth functions $a, b \in C^\infty(t_1, t_2)$, a point $x_0 \in M^{2n}$ and a tangent vector $\gamma_0 \in T_{x_0}M^{2n}$, we can find a small $\epsilon_0 > 0$ and a unique solution $\gamma : (-\epsilon_0, \epsilon_0) \to M^{2n}$ of equation (1) with initial data $\gamma(0) = x_0$, $\dot{\gamma}(0) = \gamma_0$.

Let g and $\bar{g}$ be Kählerian metrics on M^{2n}.

Definition 3 (see [16]). *The Kählerian metrics g and $\bar{g}$ are called* holomorphically projective *(or h-projective) iff every holomorphically planar with respect to the metric g curve $\gamma(t)$ is holomorphically planar with respect to the metric $\bar{g}$ as well, and vice versa.*

It is not hard to prove (see for example [16, 17]) that the condition that the Kählerian metrics g and $\bar{g}$ are h-projective is equivalent to a non-linear partial differential equation on the "deformation" tensor of the Levi-Civita connections of the metrics g and $\bar{g}$. In coordinates $\{(x^1, ..., x^{2n})\}$, the equation is

$$\bar{\Gamma}^i_{jk} - \Gamma^i_{jk} = \delta^i_j \phi_k + \delta^i_k \phi_j - \phi_\alpha J^\alpha_j J^i_k - \phi_\alpha J^\alpha_k J^i_j, \tag{2}$$

where ϕ_l (l=1,...,2n) are the components of a globally defined on M^{2n} 1-form, Γ^i_{jk} and $\bar{\Gamma}^i_{jk}$ are the Christoffel symbols of the metrics g and $\bar{g}$ respectively, δ^i_j is the Kronecker delta and J^i_j are the components of the complex structure J. As usual, we use the standard tensor conventions and omit the summation symbols in the formulas. It can be easily seen that $\phi_k = \partial_k \phi$, where ∂_k stands for the partial derivative $\frac{\partial}{\partial x^k}$ and ϕ is a globally defined on M^{2n} function (see Sect.2.1). Remark that equation (2) cannot be obtained without using the condition that the metrics g and $\bar{g}$ are Kählerian.

(c) PQ^ϵ-projectivity

Here we give the definition of the geodesic compatibility that generalize the both previous notions.

Consider two metrics g and $\bar{g}$ given on the manifold M^m, $m = \dim M^m$. Let $P, Q \in \Gamma(\mathrm{End}(TM^m))$ satisfy the next properties:

(i) P and Q are antisymmetric with respect to the both metrics g and $\bar{g}$;

(ii) $PQ = \epsilon \mathbf{1}$, where ϵ is a fixed real constant such that $\epsilon \neq m + 1$ and $\epsilon \neq 1$.

Denote by χ the number $\chi \stackrel{\text{def}}{=} m + 1 - \epsilon$. It follows from (ii) that $\chi \neq 0$.

Definition 4. *The metrics g and $\bar{g}$ are called PQ^ϵ-*projective *(or* geodesically compatible*) iff their Christoffel symbols satisfy the equation*

$$\bar{\Gamma}^i_{jk} - \Gamma^i_{jk} = \phi_{(j}\delta^i_{k)} - \phi_\alpha P^\alpha_{(j} Q^i_{k)}, \tag{3}$$

where ϕ_k are the components of a globally defined on M^m 1-form and the operators P and Q satisfy relations (i) and (ii).

As usual, equation (3) is written in a fixed coordinate chart $\{(x^1, ..., x^m)\}$ and P^i_j and Q^i_j are the components of the operators P and Q. The brackets $(i...j)$ in formula (3) denote symmetrization with respect to the pointed indices. For example, the tensor field $T_{\alpha(ij)\beta}$ stands for $T_{\alpha ij\beta} + T_{\alpha ji\beta}$. It is not hard to give an invariant definition of PQ^ϵ-projectivity.

Remark 1. *If $\epsilon \neq 0$ then the operators P and Q are non-degenerate on M^m, $Q = \epsilon P^{-1}$ and the dimension m of the manifold is even.*

Remark 2. *Suppose that g and $\bar{g}$ are Kählerian metrics given on the complex manifold M^{2n} with complex structure J. Taking $P = Q = J$ and $\epsilon = -1$ we see that h-projectivity is a particular case of PQ^ϵ-projectivity. Taking $P = Q = 0$ and $\epsilon = 0$ we obtain the notion of the geodesic equivalence.*

Let us give a brief description of the results proved in the paper. Suppose that the metrics g and $\bar{g}$ given on the manifold M^m are PQ^ϵ-projective. Define the endomorphism $A \in \Gamma(\mathrm{End}(TM^m))$ of the tangent bundle TM^m by the formula

$$A^i_j(g,\bar{g}) \stackrel{\text{def}}{=} \left|\frac{\det \bar{g}}{\det g}\right|^{\frac{1}{x}} \bar{g}^{i\alpha} g_{\alpha j}. \qquad (4)$$

Consider the locally defined one-parameter family of quadratic forms

$$K_c(g,\bar{g})(\xi) \stackrel{\text{def}}{=} |\det(A + c\mathbf{1})|^{1/(1-\epsilon)} g((A + c\mathbf{1})^{-1}\xi, \xi), \qquad (5)$$

where $\xi \in TM^m$ and c is an appropriately chosen real parameter.

Following [11] define the *rank of the pair* g and $\bar{g}$. Denote by $r(g,\bar{g})(x)$ the degree of the minimal polynomial of the operator $A(g,\bar{g})|_x$.

Definition 5. *The number $r(g,\bar{g})(x)$ is called* rank of the pair g and $\bar{g}$ of PQ^ϵ-projective metrics at the point $x \in M^m$. *The number* $r(g,\bar{g}) \stackrel{\text{def}}{=} \max_{x \in M^m} r(g,\bar{g})(x)$ *is called* rank of the pair of PQ^ϵ-projective metrics.

Recall that the smooth functions $F_1, ..., F_k$ given on a smooth manifold V are called *functionally independent in* V iff the set of the points $x \in V$ where the differentials $d_x F_1, ..., d_x F_k$ are linearly independent is dense in V.

Theorem 1. *Suppose that the manifold M^m is connected and let the rank of the pair of PQ^ϵ-projective metrics g and $\bar{g}$ be r. Denote by ω_g the symplectic structure on TM^m given by the pull-back $\mathrm{FL}_g^*\omega$ where ω is the canonical symplectic structure on T^*M^m (the form "$dp \wedge dq$") and $FL_g : TM^m \to T^*M^m$ denotes the Legendre transformation corresponding to the metric g. Then for every open set $D \subset M^m$ with compact closure in M^m there exist r quadratic in velocities functions $B^D_1(\xi), ..., B^D_r(\xi)$, $\xi \in TD$, such that:*

(a) $B_1^D(\xi), ..., B_r^D(\xi)$ *are smooth functionally independent pairwise commuting integrals of the geodesic flow of the metric* g *on* D;

(b) *if the integral* $K_c(g, \bar{g})(\xi)$, $c = \text{const}$, *is correctly defined on some open set* $U \subset D$ *then there exist constants* $\alpha_1, ..., \alpha_r$ *such that* $K_c(\xi) = \sum_{k=1}^r \alpha_k B_k^D(\xi)$ *on* D.

The integrals $B_1^D(\xi), ..., B_r^D(\xi)$ *can be taken in the form* $B_k^D(\xi) \stackrel{\text{def}}{=} K_{c_k}(\xi)$, *where the constants* $c_1, ..., c_r$ *are appropriately chosen.*

Remark 3. *Actually, the constants* $c_1, ..., c_r$ *in Theorem 1 are taken "sufficiently big", i.e.* $|c_k| > m_D$, $m_D \stackrel{\text{def}}{=} \sup_{x \in D} \max_{\lambda \in \text{Spect}(A|_x)} |\lambda|$.

As a simple corollary of this theorem we obtain the next statement.

Corollary 1. *The set of the points* $x \in M^m$ *where* $r(g, \bar{g})(x) = r$ *is open and dense in* M^m.

We prove Theorem 1 in Sect.4.

A hermitian version of Theorem 1 is given in Sect.5 (see Theorem 4). An important corollary is the next Kählerian analog of the proved in [11] Theorem 2 . Suppose that the Kählerian metrics s and $\bar{s}$ are h-projective. Denote by r the *hermitian rank* of the pair s and $\bar{s}$ (see Definition 9).

Theorem 2. *If two Kählerian metrics* s *and* $\bar{s}$ *are* h*-projective then their geodesic flows admit* r *functionally independent integrals in involution.*

The paper is organized as follows.

In Sect.2 we prove that the quadratic forms given by formula (5) are integrals of the geodesic flow of the metric g (Proposition 3, Sect.2.3). In Sect.2.4 we prove Proposition 4. As in [11] the existence of hierarchies is a crucial point proving the commutativity of the integrals given by formula (5). We prove the commutativity of these integrals in Sect.3, Theorem 3. Sect.4 is devoted to the proof of Theorem 1. The hermitian analog of this theorem is proved in Sect.5. The last section is devoted to the applications of the results. A simple geometrical construction allows us to find an infinite family ("$\mathbf{CP}^n$-hierarchy") of completely integrable Hamiltonian systems on the complex projective plane $\mathbf{CP}^n$ (see Theorem 6). Our integrals don't coincide with the integrals obtained by Thimm in [12].

In what follows, if a coordinate chart is fixed, we usually identify the tensor fields we consider with their coordinate ("index") representations and denote the corresponding objects by the same letters. We use also the common tensor notations and conventions and omit the summation symbols in the formulas. An endomorphism $L \in \Gamma(\mathrm{End}(TM^n))$ is called *invertible* or *non-degenerate* on M^n iff $\det L \neq 0$ on M^n.

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2 Properties of PQ-projectivity

The present section establishes the main properties of the PQ^ϵ-projective metrics.

2.1 Existence of an integral

Suppose that the metrics g and $\bar{g}$ are PQ^ϵ-projective. Here we prove that the geodesic flow of the metric g admits an integral (Proposition 1).

Contracting the indices i and j in formula (3) and using that $\mathrm{trace}\, Q = 0$ we get $\bar{\Gamma}^i_{ik} - \Gamma^i_{ik} = \chi\phi_k$ where $\chi \stackrel{\mathrm{def}}{=} m+1-\epsilon$. Using that $\Gamma^i_{ik} = \partial_k \ln\sqrt{|\det g|}$ we obtain

$$2\chi\phi_k = \frac{\partial}{\partial x^k}\left(\ln\left|\frac{\det \bar{g}}{\det g}\right|\right). \tag{6}$$

Therefore, $\phi_k = \partial_k\phi$, where ϕ is a smooth function on M^m.

The next technical lemma is needed for the sequel.

Lemma 1. *The metrics g and $\bar{g}$ are PQ^ϵ-projective if and only if their components satisfy the next equation in covariant derivatives*

$$\nabla_k\bar{g}_{ij} = 2\phi_k\bar{g}_{ij} + \phi_{(i}\bar{g}_{j)k} - \phi_\alpha P^\alpha_{(i}\bar{g}_{j)\beta}Q^\beta_k, \tag{7}$$

where ∇ denotes the Levi-Civita connection of the metric g and the operators P and Q satisfy conditions (i) and (ii).

Proof of Lemma 1. Suppose that the metrics g and $\bar g$ are PQ^ϵ-projective. In coordinates, we obtain

$$\begin{aligned}\nabla_k \bar g_{ij} &= \partial_k \bar g_{ij} - \Gamma^\alpha_{ik}\bar g_{\alpha j} - \Gamma^\alpha_{jk}\bar g_{i\alpha}\\ &= \bar\nabla_k \bar g_{ij} + (\phi_{(i}\delta^\alpha_{k)} - \phi_l P^l_{(i}Q^\alpha_{k)})\bar g_{\alpha j} + (\phi_{(j}\delta^\alpha_{k)} - \phi_l P^l_{(j}Q^\alpha_{k)})\bar g_{i\alpha}\\ &= 2\phi_k \bar g_{ij} + \phi_{(i}\bar g_{j)k} - \phi_l P^l_{(i}\bar g_{j)\alpha}Q^\alpha_k - \phi_l P^l_k Q^\alpha_{(i}\bar g_{j)\alpha}\\ &= 2\phi_k \bar g_{ij} + \phi_{(i}\bar g_{j)k} - \phi_l P^l_{(i}\bar g_{j)\alpha}Q^\alpha_k.\end{aligned}$$

The inverse statement immediately follows from the classical fact that the Levi-Civita connection of some metric is the unique torsion free connection that preserves the considered metric. Lemma 1 is proved.

Equations (7) are equivalent to

$$2\mu \bar g_{ij,k} = -2\mu_k \bar g_{ij} - \mu_{(i}\bar g_{j)k} - \mu_l P^l_{(i}\bar\Omega^Q_{j)k}, \tag{8}$$

where $\mu \stackrel{\text{def}}{=} \left|\frac{\det g}{\det \bar g}\right|^{\frac{1}{\varkappa}}$, $\mu_k \stackrel{\text{def}}{=} \frac{\partial \mu}{\partial x^k}$ $(k = 1, ..., 2n)$, and $\bar\Omega^Q_{kj}$ are the components of the 2-form $\bar\Omega^Q(\xi,\eta) \stackrel{\text{def}}{=} \bar g(Q\xi,\eta)$, $\xi,\eta \in T_xM^m$. The equivalence of equations (7) and (8) easily follows from the relation $2\phi_k = -\frac{\mu_k}{\mu}$.

Proposition 1. *Suppose that the metrics g and $\bar g$ are PQ^ϵ-projective. Then the quadratic form*

$$K(\xi) \stackrel{\text{def}}{=} \left|\frac{\det g}{\det \bar g}\right|^{\frac{2}{\varkappa}} \bar g(\xi,\xi) \tag{9}$$

is an integral of the geodesic flow of the metric g.

Proof of Proposition 1. In coordinates, we have $K_{ij} \stackrel{\text{def}}{=} \mu^2 \bar g_{ij}$. Following Levi-Civita, a quadratic form $Q(\xi,\xi) = Q_{ij}\xi^i\xi^j$, $Q_{ij} = Q_{ji}$, is an integral of the geodesic flow of the metric g if and only if $Q_{(ijk)} \stackrel{\text{def}}{=} (Q_{ij,k} + Q_{jk,i} + Q_{ki,j})/3 = 0$, where $Q_{ij,k} \stackrel{\text{def}}{=} \nabla_k Q_{ij}$ and ∇ is the Levi-Civita connection of the metric g. Indeed, let $\gamma(s)$ $(\gamma(0) = x_0$, $\dot\gamma(0) = \xi)$ be a geodesic line of the metric g. We have $0 = \frac{d}{ds}|_{s=0}(Q_{ij}\dot\gamma^i\dot\gamma^j) = Q_{ij,k}(x_0)\xi^k\xi^i\xi^j = Q_{(ijk)}(x_0)\xi^k\xi^i\xi^j$ and therefore $Q_{ij,k} + Q_{jk,i} + Q_{ki,j} = 0$. Conversely, if $Q_{ij,k} + Q_{jk,i} + Q_{ki,j} = 0$ then Q is an integral of the geodesic flow of the metric g.

We have

$$\begin{aligned}2K_{ij,k} &= 4\mu\mu_k \bar g_{ij} + 2\mu^2 \bar g_{ij,k},\\ 2K_{jk,i} &= 4\mu\mu_i \bar g_{jk} + 2\mu^2 \bar g_{jk,i},\\ 2K_{ki,j} &= 4\mu\mu_j \bar g_{ki} + 2\mu^2 \bar g_{ki,j}.\end{aligned}$$

Summing these equations and using (8) we obtain that

$$\begin{aligned} 2(K_{ij,k}+K_{jk,i}+K_{ki,j}) &= -\mu\Big(\mu_\alpha P_i^\alpha \bar{\Omega}_{jk}^Q + \mu_\alpha P_j^\alpha \bar{\Omega}_{ik}^Q + \\ &+ \mu_\alpha P_j^\alpha \bar{\Omega}_{ki}^Q + \mu_\alpha P_k^\alpha \bar{\Omega}_{ji}^Q + \\ &+ \mu_\alpha P_k^\alpha \bar{\Omega}_{ij}^Q + \mu_\alpha P_i^\alpha \bar{\Omega}_{kj}^Q\Big) = 0. \end{aligned}$$

This completes the proof of Proposition 1.

Remark 4. *In the case of geodesically equivalent metrics the integral given by formula (9) coincides with the classical Painlevé integral (see [1, 11]). The existence of the integral (9) in the case of h-projective Kählerian metrics is a new fact.*

2.2 Existence of a family of PQ-projective metrics

Suppose that the metrics g and $\bar{g}$ are PQ^ϵ-projective. Consider the endomorphism A of the tangent bundle TM^m defined by formula (4). It is clear that A is self-adjoint with respect to the both metrics g and $\bar{g}$. Condition (i) imposed on the operators P and Q (see Sect. 1) yields that A commutes with P and Q. The next important technical lemma is needed for the sequel.

Lemma 2. *If the metrics g and $\bar{g}$ are PQ^ϵ-projective then the metric $a_{ij} \stackrel{\text{def}}{=} g_{i\alpha}A_j^\alpha$ satisfies the equation*

$$a_{ij,k} = \lambda_{(i}g_{j)k} - \lambda_l P_{(i}^l g_{j)q} Q_k^q, \tag{10}$$

where $\lambda_i \stackrel{\text{def}}{=} -\phi_l A_i^l$. The operators P and Q are antisymmetric with respect to the metric a. Inversely, consider a metric g and two antisymmetric with respect to g operators P and Q such that $PQ = \epsilon\mathbf{1}$. Suppose that P and Q are antisymmetric with respect to a non-degenerate symmetric form a_{ij} and let a_{ij} satisfies equation (10) for some globally defined on M^{2n} 1-form λ_k. Then the metrics g and $\bar{g} \stackrel{\text{def}}{=} \left|\frac{\det \check{g}}{\det g}\right|^{\frac{1}{1-\epsilon}} \check{g}$, are PQ^ϵ-projective, where $\check{g}_{ij} \stackrel{\text{def}}{=} g_{i\alpha}a^{\alpha\beta}g_{\beta j}$, $a_{i\alpha}a^{\alpha j} = \delta_i^j$.

Proof of Lemma 2. Suppose that g and $\bar{g}$ are PQ^ϵ-projective. Denote $\breve{g}_{ij} \stackrel{\text{def}}{=} \exp(-2\phi)\bar{g}_{ij}$. Using formula (7) we obtain

$$\begin{aligned}\breve{g}_{ij,k} &= -2\phi_k\breve{g}_{ij} + \exp(-2\phi)\bar{g}_{ij,k} \\ &= \phi_{(i}\breve{g}_{j)k} - \phi_l P^l_{(i}\breve{g}_{j)\alpha}Q^\alpha_k.\end{aligned}$$

For the inverse tensor $\breve{g}^{ij}$ ($\breve{g}_{i\alpha}\breve{g}^{\alpha j} = \delta^j_i$) we have

$$\begin{aligned}\breve{g}^{ij}_{,k} &= -\breve{g}^{i\alpha}\breve{g}_{\alpha\beta,k}\breve{g}^{\beta j} \\ &= -\phi_l\breve{g}^{l(i}\delta^{j)}_k + \phi_l P^l_s\breve{g}^{s(i}Q^{j)}_k\end{aligned}$$

Finally, taking $a_{\alpha\beta} \stackrel{\text{def}}{=} g_{\alpha i}\breve{g}^{ij}g_{j\beta}$ obtain

$$\begin{aligned}a_{\alpha\beta,k} &= -\phi_l\breve{g}^{li}g_{i(\alpha}g_{\beta)k} + \phi_l P^l_r A^r_{(\alpha}g_{\beta)j}Q^j_k \\ &= \lambda_{(\alpha}g_{\beta)k} - \lambda_l P^l_{(\alpha}g_{\beta)q}Q^q_k,\end{aligned}$$

where $\lambda_\alpha \stackrel{\text{def}}{=} -\phi_l A^l_\alpha$.

Let us prove the inverse part of the proposition. Suppose that the non-degenerate symmetric tensor a_{ij} satisfies equation (10) where P and Q are antisymmetric with respect to g and a, $PQ = \epsilon\mathbf{1}$, and λ_k are the components of a globally defined on M^m 1-form. Consider the tensor $\breve{g}^{ij} \stackrel{\text{def}}{=} g^{i\alpha}a_{\alpha\beta}g^{\beta j}$. For the inverse tensor $\breve{g}_{pq}$ ($\breve{g}_{p\alpha}\breve{g}^{\alpha q} = \delta^q_p$) we have

$$\begin{aligned}\breve{g}_{pq,k} &= -\breve{g}_{pi}\breve{g}^{ij}_{,k}\breve{g}_{jq} \\ &= -\lambda_\alpha g^{\alpha i}\breve{g}_{i(p}\breve{g}_{q)k} + \lambda_l P^l_\alpha g^{\alpha i}\breve{g}_{i(p}\breve{g}_{q)j}Q^j_k\end{aligned}$$

Denoting $\phi_p \stackrel{\text{def}}{=} -\lambda_\alpha g^{\alpha i}\breve{g}_{ip}$ we obtain

$$\breve{g}_{pq,k} = \phi_{(p}\breve{g}_{q)k} - \phi_l P^l_{(p}\breve{g}_{q)j}Q^j_k. \tag{11}$$

Lemma 3. $2\phi_k = \partial_k \ln\left|\frac{\det\breve{g}}{\det g}\right|^{\frac{1}{1-\epsilon}}$.

Proof of Lemma 3. Denote by $\breve{\Gamma}^i_{jk}$ the Christoffel symbols of the metric $\breve{g}$. Using formula (11) we obtain

$$\breve{\Gamma}^\alpha_{\alpha k} = \frac{1}{2}\breve{g}^{\alpha\beta}\frac{\partial\breve{g}_{\alpha\beta}}{\partial x^k}$$

$$= \frac{1}{2}\breve{g}^{\alpha\beta}(\breve{g}_{\alpha\beta,k} + \breve{g}_{l(\alpha}\Gamma^l_{\beta)k})$$
$$= \frac{1}{2}\breve{g}^{\alpha\beta}((\phi_{(\alpha}\breve{g}_{\beta)k} - \phi_l P^l_{(\alpha}\breve{g}_{\beta)j}Q^j_k) + \breve{g}_{l(\alpha}\Gamma^l_{\beta)k})$$
$$= (1-\epsilon)\phi_k + \Gamma^\alpha_{\alpha k}.$$

Therefore, $(1-\epsilon)\phi_k = \breve{\Gamma}^\alpha_{\alpha k} - \Gamma^\alpha_{\alpha k} = \partial_k \ln\left|\frac{\det \breve{g}}{\det g}\right|^{\frac{1}{2}}$. Lemma 3 is proved.

Finally, taking $\bar{g}_{ij} \stackrel{\text{def}}{=} \exp(2\phi)\breve{g}_{ij}$ we obtain that $g_{ij,k} = 2\phi_k\bar{g}_{ij} + \phi_{(i}\bar{g}_{j)k} - \phi_l P^l_{(i}\bar{g}_{j)\alpha}Q^\alpha_k$. Using Lemma 1 we complete the proof of Lemma 2.

Proposition 2. *Suppose that the metrics g and $\bar{g}$ are PQ^ϵ-projective and let c be a real constant such that the operator $A + c\mathbf{1}$ is invertible. Then the metrics g and*

$$\bar{g}_c(g,\bar{g}) \stackrel{\text{def}}{=} |\det(A + c\mathbf{1})|^{-1/(1-\epsilon)} g(A + c\mathbf{1})^{-1}, \tag{12}$$

where $A = A(g,\bar{g})$ is given by formula (4), are PQ^ϵ-projective.

Proof of Proposition 2. Suppose that the metrics g and $\bar{g}$ are PQ^ϵ-projective and let c be the constant given in the statement of the corollary. It follows from Lemma 2 that the metric $a \stackrel{\text{def}}{=} gA$ satisfies equation (10). Hence, the metric $a + cg = g(A + c)$ also satisfies equation (10). Using the inverse part of Lemma 2 we obtain that the metrics g and $\bar{g}_c \stackrel{\text{def}}{=} \left|\frac{\det \breve{g}_c}{\det g}\right|^{\frac{1}{1-\epsilon}} \breve{g}_c$, $\breve{g}_c \stackrel{\text{def}}{=} g(a + cg)^{-1}g = g(A + c\mathbf{1})^{-1}$. This completes the proof of Proposition 2.

2.3 Existence of a family of integrals

Here we prove that the geodesic flows of a pair of PQ^ϵ-projective metric locally admit a family of integrals.

Proposition 3. *Suppose that the metrics g and $\bar{g}$ are PQ^ϵ-projective and let c be a real constant such that the operator $A + c\mathbf{1}$ is non-degenerate on M^m. Then the quadratic form*

$$K_c(g,\bar{g}) \stackrel{\text{def}}{=} |\det(A + c\mathbf{1})|^{1/(1-\epsilon)} g(A + c\mathbf{1})^{-1}, \tag{13}$$

is an integral of the geodesic flow of the metric g.

Remark 5. *Let $D \subset M^m$ be an open set with compact closure in M^m. Taking $|c| \geq m_D$, $m_D \stackrel{\text{def}}{=} \sup_{x\in D} \max_{\lambda\in \text{Spect}\, A|_x} |\lambda|$, we obtain a one-parameter family of integrals of the geodesic flow of the metric g.*

Proof of Proposition 3. Suppose that the metrics g and $\bar{g}$ are PQ^ϵ-projective. It follows from Proposition 2 that g and $\bar{g}_c$ (given by formula (12)) are PQ^ϵ-projective. Applying Proposition 1 to these metrics we obtain that the quadratic form

$$\begin{aligned} K_c &\stackrel{\text{def}}{=} \left|\frac{\det g}{\det \bar{g}_c}\right|^{\frac{2}{\chi}} \bar{g}_c \\ &= |\det(A + c\mathbf{1})|^{1/(1-\epsilon)} g(A + c\mathbf{1})^{-1} \end{aligned}$$

is an integral of the geodesic flow of the metric g. Proposition 3 is proved.

2.4 Existence of hierarchies

In the present section we prove the next proposition.

Proposition 4. *If the metrics g and $\bar{g}$ are PQ^ϵ-projective then the metrics $a \stackrel{\text{def}}{=} gA$ and $\bar{a} \stackrel{\text{def}}{=} \bar{g}A$ are $P_1Q_1^\epsilon$-projective, where $P_1 \stackrel{\text{def}}{=} PA$, $Q_1 \stackrel{\text{def}}{=} A^{-1}Q$, and $A = A(g, \bar{g})$ is given by formula (4).*

Proof of Proposition 4. It follows from Lemma 2 that $a_{\alpha\beta,k} = \lambda_{(\alpha}g_{\beta)m} - \lambda_l P^l_{(\alpha}g_{\beta)q}Q^q_k$, where $\lambda_\alpha \stackrel{\text{def}}{=} -\phi_l A^l_\alpha$. We have

$$\begin{aligned} -\frac{\partial a_{\alpha\beta}}{\partial x^k} + \Gamma^l_{k(\alpha}a_{\beta)l} &= -\lambda_{(\alpha}g_{\beta)k} + \lambda_l P^l_{(\alpha}g_{\beta)q}Q^q_k, \\ \frac{\partial a_{\alpha k}}{\partial x^\beta} - \Gamma^l_{\beta(\alpha}a_{k)l} &= \lambda_{(\alpha}g_{k)\beta} - \lambda_l P^l_{(\alpha}g_{k)q}Q^q_\beta, \\ \frac{\partial a_{\beta k}}{\partial x^\alpha} - \Gamma^l_{\alpha(\beta}a_{k)l} &= \lambda_{(\beta}g_{k)\alpha} - \lambda_l P^l_{(\beta}g_{k)q}Q^q_\alpha. \end{aligned}$$

Summing these equations, and using the relations $\lambda_l \stackrel{\text{def}}{=} -\phi_\alpha A^\alpha_l$, $PA = AP$, $g(Q\xi, \eta) = -g(\xi, Q\eta)$, we obtain the next formula for the Christoffel symbols A^i_{jk} of the metric a_{ij}

$$\begin{aligned} A^k_{\alpha\beta} &= \Gamma^k_{\alpha\beta} + \lambda_l a^{lk} g_{\alpha\beta} + \lambda_l P^l_{(\alpha}g_{\beta)q}Q^q_r a^{rk} \\ &= \Gamma^k_{\alpha\beta} - \phi_l g^{lk} g_{\alpha\beta} + \phi_s P^s_l A^l_{(\alpha}Q^q_{\beta)}\bar{A}^k_q, \end{aligned} \quad (14)$$

where $\tilde{A}^k_q$ are the components of the inverse operator A^{-1}.

By definition, $\bar{a}_{ij} \stackrel{\text{def}}{=} \bar{g}_{ik} A^k_j = \exp(2\phi) g_{ij}$. Using this relation we immediately obtain the next relation between the Christoffel symbols of the metrics $\bar{a}$ and g

$$\bar{A}^k_{ij} = \Gamma^k_{ij} + \phi_{(i}\delta^k_{j)} - \phi_l g^{lk} g_{ij}. \tag{15}$$

Finally, (14) and (15) yield

$$\bar{A}^k_{ij} - A^k_{ij} = \phi_{(i}\delta^k_{j)} - \phi_s P^s_l A^l_{(i} Q^q_{j)} \tilde{A}^k_q. \tag{16}$$

This completes the proof of Proposition 4.

Suppose that a pair of PQ^ϵ-projective metrics g and $\bar{g}$ is given. It follows from Proposition 4 that *for every integer k the metrics $g^{(k)} \stackrel{\text{def}}{=} gA^k$ and $\bar{g}^{(k)} \stackrel{\text{def}}{=} gA^k$ are P_kQ_k-projective, where $P_k \stackrel{\text{def}}{=} PA^k$ and $Q_k \stackrel{\text{def}}{=} A^{-k}Q$.* Indeed, suppose that $g^{(l)}$ and $\bar{g}^{(l)}$ are P_lQ_l-projective. It follows from formula (4) that $A(g^{(l)}, \bar{g}^{(l)}) = A$ and $A(\bar{g}^{(l)}, g^{(l)}) = A^{-1}$. Applying Proposition 4 to the pair $g^{(l)}$ and $\bar{g}^{(l)}$ we obtain that $g^{(l+1)}$ and $\bar{g}^{(l+1)}$ are $P_{l+1}Q_{l+1}$-projective. Similarly, applying Proposition 4 to the pair $\bar{g}^{(l)}$ and $g^{(l)}$ we obtain that $g^{(l-1)}$ and $\bar{g}^{(l-1)}$ are $P_{l-1}Q_{l-1}$-projective.

Definition 6. *The sequence of pairs $g^{(k)}$ and $\bar{g}^{(k)}$ $(k = 0, \pm 1, \ldots)$ are called* PQ^ϵ-hierarchy *corresponding to the pair g and $\bar{g}$ of PQ^ϵ-projective metrics.*

The PQ^ϵ-hierarchy is an analog of the *geodesic hierarchy* considered in [11]. Proposition 4 is an analog of the Sinyukov transformation in the theory of geodesically equivalent metrics (see [19, 20]). We will describe the PQ^ϵ-hierarchy by the next formal scheme

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
g^{(-1)} & \longleftrightarrow & \bar{g}^{(-1)} \\
\downarrow & & \downarrow \\
g & \longleftrightarrow & \bar{g} \\
\downarrow & & \downarrow \\
g^{(1)} & \longleftrightarrow & \bar{g}^{(1)} \\
\downarrow & & \downarrow
\end{array}$$

where the horizontal arrows mean that the metrics $g^{(k)}$ and $\bar{g}^{(k)}$ are P_kQ_k-projective.

3 Involutivity of the family of the integrals

In the present section we prove that the integrals given by Proposition 3 are in involution.

Theorem 3. *Suppose that the metrics g and $\bar{g}$ are PQ^ϵ-projective. Suppose in addition that the operators $A + c_1\mathbf{1}$ and $A + c_2\mathbf{1}$ are non-degenerate on M^m where c_1 and c_2 are some fixed real numbers. Then the quadratic forms $K_{c_1}(g, \bar{g})$ and $K_{c_2}(g, \bar{g})$, given by formula (5), are in involution with respect to the symplectic structure $\omega_g \stackrel{\text{def}}{=} \mathrm{FL}_g^*\omega$, where ω denotes the canonical symplectic structure on the cotangent bundle T^*M^m and $FL_g : TM^m \to T^*M^m$ is the Legendre transformation corresponding to the metric g.*

Proof of Theorem 3. Suppose that the metrics g and $\bar{g}$ are PQ^ϵ-projective. Proposition 2 shows that the metrics g and $\bar{g}_{c_1} \stackrel{\text{def}}{=} |\det(A+c_1\mathbf{1})|^{-1/(1-\epsilon)}g(A+c_1\mathbf{1})^{-1}$ are PQ^ϵ-projective as well. Consider the corresponding PQ^ϵ-hierarchy

$$\begin{array}{ccc} \downarrow & & \downarrow \\ g & \longleftrightarrow & \bar{g}_{c_1} \\ \downarrow & & \downarrow \\ g_{c_1}^{(1)} & \longleftrightarrow & \bar{g}_{c_1}^{(1)} \\ \downarrow & & \downarrow \\ g_{c_1}^{(2)} & \longleftrightarrow & \bar{g}_{c_1}^{(2)} \\ \downarrow & & \downarrow \end{array}$$

A simple calculation shows that $A(g, \bar{g}_{c_1}) = A + c_1\mathbf{1}$, and $A(\bar{g}_{c_1}, g) = (A + c_1\mathbf{1})^{-1}$. Hence, $\bar{g}_{c_1}^{(2)} \stackrel{\text{def}}{=} \bar{g}_{c_1}A(g, \bar{g}_{c_1})^2 = |\det(A + c_1\mathbf{1})|^{-1/(1-\epsilon)}g(A + c_1\mathbf{1})$. It is clear that $A(\bar{g}_{c_1}^{(2)}, g_{c_1}^{(2)}) = (A + c_1\mathbf{1})^{-1}$. Suppose that the non-zero real number $(-\alpha)$ is not an eigenvalue of the operator $(A + c_1\mathbf{1})^{-1}$. It follows from Proposition 3 that the quadratic form $K_\alpha(\bar{g}_{c_1}^{(2)}, g_{c_1}^{(2)})$ is an integral of the geodesic flow of the metric $\bar{g}_{c_1}^{(2)}$. Applying the inverse to the Legendre transformation corresponding to the metric $\bar{g}_{c_1}^{(2)}$, we obtain that the forms

$$(\bar{g}_{c_1}^{(2)})^{-1} = |\det(A + c_1\mathbf{1})|^{1/(1-\epsilon)}(A + c_1\mathbf{1})^{-1}g^{-1} \tag{17}$$

and

$$\begin{aligned} (\mathrm{FL}^{-1}_{\bar{g}_{c_1}^{(2)}})^*(K_\alpha) &\stackrel{\text{def}}{=} |\det((A + c_1)^{-1} + \alpha)|^{1/(1-\epsilon)}((A + c_1)^{-1} + \alpha)^{-1}(\bar{g}_{c_1}^{(2)})^{-1} \\ &= c(\alpha)|\det(A + (c_1 + 1/\alpha))|^{1/(1-\epsilon)}(A + (c_1 + 1/\alpha))^{-1}g^{-1}, \end{aligned}$$

considered as functions on the cotangent bundle T^*M^m, are in involution with respect to the canonical symplectic structure ω on T^*M^m. Finally, applying the Legendre transformation corresponding to the metric g, we obtain that the forms $K_{c_1}(g,\bar{g})$ and $K_{c_1+\frac{1}{\alpha}}(g,\bar{g})$ are in involution with respect to ω_g. Theorem 3 is proved.

4 Functional independence of the integrals

Here we prove Theorem 1 formulated in the Introduction.

Proof of Theorem 1. We follow the idea of the proof of Theorem 2 in [11].

In what follows we use the next convention. If V is a complex vector space we assume that all linear maps and subspaces are complex linear, all tensor products are over $\mathbf{C}$, and the word "dimension" means the complex dimension.

Let V be a real (or complex) vector space of dimension m. Consider a non-degenerate symmetric bilinear form $s \in \mathrm{Symm}(V^* \otimes V^*)$ and a linear self-adjoint with respect to s operator $L \in \mathrm{End}(V)$. Denote by $r(L)$ *the degree of the minimal polynomial of the operator* L. Let δ be a fixed real number. Consider the curves

$$\gamma : t \to |\det(L - t\mathbf{1})|^{\delta} s(L - t\mathbf{1})^{-1} \in \mathrm{Symm}(V^* \otimes V^*)$$

and

$$\gamma_\xi : t \to |\det(L - t\mathbf{1})|^{\delta} s((L - t\mathbf{1})^{-1}\xi, .) \in V^*,$$

where $\xi \in V$ is a fixed vector in V, $\mathbf{1}$ is the identity operator, and the parameter t is defined in an open set U in $\mathbf{R}$ (or $\mathbf{C}$), $U \cap \mathrm{Spect}(L) = \emptyset$. Remark that we admit three possibilities: V is real and t is real, V is complex and t is complex or real. Given a curve $l : D \to W$, where W is a vector space and D is an open domain in $\mathbf{R}$ or $\mathbf{C}$, denote by $\dim_D l$ the dimension of the linear subspace spanned on the vectors $\{l(t) | t \in D\}$. Following our convention, if W is a complex vector space, $\dim_D l$ denotes the complex dimension of the complex linear subspace spanned on $\{l(t) | t \in D\}$.

Lemma 4.

(a) $\dim_U \gamma = r(L)$;

(b) $\dim_U \gamma_\xi \leq r(L)$ *and there exists an open and dense subset* $\Sigma \subset V$ *such that for every* $\xi \in \Sigma$, $\dim_U \gamma_\xi = r(L)$;

(c) if $t_1, ..., t_{r(L)} \in U$, $t_i \neq t_j$ $(i \neq j)$, *then* $\gamma(t_1), ..., \gamma(t_{r(L)})$ *are linearly independent.*

Proof of Lemma 4. Consider the case when V is a real vector space and the parameter t is real, $t \in U \subset \mathbf{R}$, $U \cap \mathrm{Spect}(L) \neq \emptyset$. The case of complex vector space is considered similarly. It follows from the non-degeneracy of the metric s that $\dim_U \gamma = \dim_U \tilde\gamma$ and $\dim_U \gamma_\xi = \dim_U \tilde\gamma_\xi$, where $\tilde\gamma(t) \stackrel{\mathrm{def}}{=} \det(L - t\mathbf{1})(L - t\mathbf{1})^{-1}$ and $\tilde\gamma_\xi(t) \stackrel{\mathrm{def}}{=} \det(L - t\mathbf{1})(L - t\mathbf{1})^{-1}\xi$. It is clear that $\tilde\gamma(t) = L_{m-1}t^{m-1} + ... + L_0$, $L_k \in \mathrm{End}(V)$, $L_{m-1} = (-1)^{m-1}\mathbf{1}$. Using the non-degeneracy of the Vandermonde determinant we obtain that $\dim_U \tilde\gamma = \mathrm{rk}\{L_{m-1}, ..., L_0\}$ and $\dim_U \tilde\gamma_\xi = \mathrm{rk}\{L_{m-1}\xi, ..., L_0\xi\}$. Denote by $V^{\mathbf{C}}$ the complexification of V. Let $L^{\mathbf{C}}, L_k^{\mathbf{C}} \in \mathrm{End}(V^{\mathbf{C}})$ be the complexifications of the real operators L and L_k $(k = 0, ..., m-1)$. Denote by r_0 the number $r_0 \stackrel{\mathrm{def}}{=} \max_{\xi \in V} \mathrm{rk}\{L_{m-1}\xi, ..., L_0\xi\}$.

Lemma 5.

(i) $\mathrm{rk}\{L_{m-1}, ..., L_0\} = \mathrm{rk}_{\mathbf{C}}\{L^{\mathbf{C}}_{m-1}, ..., L^{\mathbf{C}}_0\}$;

(ii) $r_0 = \max_{\xi \in V^{\mathbf{C}}} \mathrm{rk}_{\mathbf{C}}\{L^{\mathbf{C}}_{m-1}\xi, ..., L^{\mathbf{C}}_0\xi\}$;

(iii) there exists an open dense subset $\Sigma \subset V$ *such that for every* $\xi \in \Sigma$, $r_0 = \mathrm{rk}\{L_{m-1}\xi, ..., L_0\xi\}$.

Proof of Lemma 5. Item (i) of the lemma is obvious. Let us prove (iii). Taking a basis in V, denote by $(\xi_1, ..., \xi_m)$ the coordinates of the vectors of V. Consider the $m \times m$ matrix T formed of the coordinates of the vectors $L_{m-1}\xi, ..., L_0\xi$. The elements of this matrix are linear polynomials of the variables $\xi_1, ..., \xi_m$. It is clear that there exists a non-zero minor $T_{r_0} \in \mathbf{R}^{r_0}[\xi_1, ..., \xi_m]$ of the matrix T. The set $\Sigma \stackrel{\mathrm{def}}{=} \{\xi \in V | T_{r_0}(\xi) \neq 0\}$ satisfies the statement of item (iii). The assumption that the variables $\xi_1, ..., \xi_m$ take complex values doesn't change the rank of the matrix T. This proves item (ii). Lemma 5 is proved.

Consider the curves $\tilde\gamma^{\mathbf{C}}(\lambda) \stackrel{\mathrm{def}}{=} (L^{\mathbf{C}} - \lambda\mathbf{1})^{-1}$ and $\tilde\gamma^{\mathbf{C}}_\xi(\lambda) \stackrel{\mathrm{def}}{=} \det(L^{\mathbf{C}} - \lambda\mathbf{1})^{-1}\xi$, where $\lambda \in \mathbf{C} \setminus \mathrm{Spect}(L)$, $\xi \in V^{\mathbf{C}}$, and $\mathbf{1}$ denotes the identity operator in

$V^{\mathbb{C}}$. As above, using the non-degeneracy of the Vandermonde determinant, we obtain that $\dim_{\mathbb{C}\setminus\mathrm{Spect}(L)}\tilde{\gamma}^{\mathbb{C}} = \mathrm{rk}_{\mathbb{C}}\{L_{m-1}^{\mathbb{C}}, ..., L_0^{\mathbb{C}}\}$ and $\dim_{\mathbb{C}\setminus\mathrm{Spect}(L)}\tilde{\gamma}_\xi^{\mathbb{C}} = \mathrm{rk}_{\mathbb{C}}\{L_{m-1}^{\mathbb{C}}\xi, ..., L_0^{\mathbb{C}}\xi\}$. Lemma 5 shows that $\dim_U\gamma = \dim_{\mathbb{C}\setminus\mathrm{Spect}(L)}\tilde{\gamma}^{\mathbb{C}}$ and $r_0 = \max_{\xi\in V^{\mathbb{C}}}\dim_{\mathbb{C}\setminus\mathrm{Spect}(L)}\tilde{\gamma}_\xi^{\mathbb{C}}$. Without loss of generality we can suppose that the operator $L^{\mathbb{C}}$ is given in a Jordan's basis. Using the explicit form of $L^{\mathbb{C}}$, it is not hard to see that

$$\dim_{\mathbb{C}\setminus\mathrm{Spect}(L)}\tilde{\gamma}^{\mathbb{C}} = r(L)$$

and

$$\max_{\xi\in V^{\mathbb{C}}}\dim_{\mathbb{C}\setminus\mathrm{Spect}(L)}\tilde{\gamma}_\xi^{\mathbb{C}} = r(L).$$

The first equality proves item (a). The second one shows that $r_0 = r(L)$. Applying Lemma 5(iii) we prove (b). To prove item (c) remark that instead of the curves $\gamma(t_1), ..., \gamma(t_{r(L)})$ it is sufficient to prove the linear independence over $\mathbb{C}$ of the curves $\tilde{\gamma}^{\mathbb{C}}(t_1), ..., \tilde{\gamma}^{\mathbb{C}}(t_{r(L)})$, where the operator $L^{\mathbb{C}}$ is given in a Jordan's basis. Using the simple form of these curves we prove item (c). Lemma 4 is proved.

Let us return to the proof of Theorem 1. Suppose that g and $\bar{g}$ are PQ^ϵ-projective metrics given on the connected manifold M^m. Denote by r the rank of the pair g and $\bar{g}$, and let $D \subset M^m$ be an open set with compact closure in M^m. Without loss of generality we can suppose that D is connected. Taking r different real numbers $c_1, ..., c_r$ such that $|c_k| > m_D$, $m_D \stackrel{\mathrm{def}}{=} \sup_{x\in D}\max_{\lambda\in\mathrm{Spect}(A|_x)}|\lambda|$, consider the quadratic forms $B_i^D \stackrel{\mathrm{def}}{=} K_{c_i}(g,\bar{g})$ $(i = 1, ..., r)$. The forms B_i^D are well-defined on D. Considered as functions of the tangent bundle TD, B_i^D $(i = 1, ..., r)$ are pairwise commuting integrals of the geodesic flow of the metric g (Theorem 3).

Definition 7. *A point $x_0 \in M^m$ is called* stable *iff the rank $r(g,\bar{g})(x)$ of the pair of PQ^ϵ-projective metrics g and $\bar{g}$ is equal to some constant q in an open neighborhood of the point x_0. We say that x_0 is a* stable point of rank q.

Definition 8. *A point $x_0 \in M^m$ is called* singular *iff it is not stable.*

Denote the set of stable points by $\mathcal{M}(g,\bar{g})$ and the set of singular points by $S(g,\bar{g})$.

Lemma 6. *The set of stable points $\mathcal{M}(g,\bar{g})$ is open and dense in M^m.*

Proof of Lemma 6. The set $\mathcal{M}(g,\bar{g})$ is open by its definition. It follows from Lemma 4 that *for every $x \in M^m$ there exists an open neighborhood $U(x)$ such that if $y \in U(x)$ then $r(g,\bar{g})(y) \geq r(g,\bar{g})(x)$*. Indeed, taking $p = r(g,\bar{g})(x)$ different real numbers $t_1, ..., t_p$, $(-t_i) \notin \mathrm{Spect}(A|_x)$, we see (item (c), Lemma 4) that the forms $K_{t_1}|_x, ..., K_{t_p}|_x \in \mathrm{Symm}(T^*_x M^m \otimes T^*_x M^m)$ are linearly independent. There exists an open neighborhood $U(x)$ such that if $y \in U(x)$ then $K_{t_1}|_y, ..., K_{t_p}|_y$ are linearly independent. Finally, applying item (a) of Lemma 4 we obtain that $r(g,\bar{g})(y) \geq r(g,\bar{g})(x)$.

Therefore, *if x_0 is a singular point, then every open neighborhood of x_0 contains a point y such that $r(g,\bar{g})(y) > r(g,\bar{g})(x_0)$.*

Let us prove that the set of stable points $\mathcal{M}(g,\bar{g})$ is dense in M^m. Suppose that there exists an open set $Y \subset M^m$ that consists of singular points. Take a point $y_1 \in Y$, $r(g,\bar{g})(y_1) = r_1$. The point y_1 is singular, and therefore there exists a point $y_2 \in Y$ such that $r(g,\bar{g})(y_2) = r_2 > r_1$. Applying this argument several times we find a (singular!) point $y_l \in Y$ of maximal rank m. From another side, the points of maximal rank m are stable. This contradiction proves Lemma 6.

Lemma 7. *Let x_0 be a stable point of rank $q \leq r$, $x_0 \in D$. Then there exists an open neighborhood $U(x_0) \subset D$ such that:*

(a) the quadratic in velocities functions $B^D_1(\xi), ..., B^D_q(\xi)$ are functionally independent on $TU(x_0)$. For every fixed $x \in U(x_0)$ the set of the points $\xi \in T_x M^m$ where the differentials $d_\xi B^D_1, ..., d_\xi B^D_q$ are linearly independent is open and dense in $T_x M^m$;

(b) if c is a real constant such that $(A + c\mathbf{1})$ is non-degenerate on $U(x_0)$, then there exist constants $\alpha_1, ..., \alpha_q$ such that

$$K_c(g,\bar{g})(\xi) = \sum_{k=1}^{q} \alpha_k B^D_k(\xi), \quad \xi \in TU(x_0).$$

Proof of Lemma 7. Lemma 4 (c) shows that the forms $B^D_1|_{x_0}, ...B^D_q|_{x_0}$, considered as elements of $\mathrm{Symm}(T^*_{x_0} M^m \otimes T^*_{x_0} M^m)$, are linearly independent. There exists an open neighborhood $U(x_0)$ of the point x_0 such that for every

$y \in U(x_0)$ the forms $B_1^D|_y, ...B_q^D|_y$ are linearly independent and $r(g,\bar{g})(y) = q$. Hence, there exist smooth functions $\alpha_1, ..., \alpha_q \in C^\infty(U(x_0))$ such that

$$K_c(g,\bar{g})(\xi) = \sum_{k=1}^{q} \alpha_k(y) B_k^D(\xi),$$

where $\xi \in U(x_0)$, $y = \pi(\xi)$, and $\pi : TM^m \to M^m$ is the projection on the base M^m. Denote by E_g the "energy" integral $E_g(\xi) \stackrel{\text{def}}{=} \frac{1}{2} g(\xi,\xi)$ and let $\{.,.\}_g$ be the Poisson bracket corresponding to the symplectic structure ω_g. Using that $B_1^D(\xi), ..., B_q^D(\xi)$ and $K_c(\xi)$ are integrals of the geodesic flow of the metric g, we obtain $0 = \{E_g, K_c\}_g = \sum_{k=1}^q \{E_g, \alpha_k\}_g B_k^D(\xi)$ for every $\xi \in TU(x_0)$. The linear independence of the forms $B_1^D|_y, ..., B_q^D|_y$ for every fixed $y \in U(x_0)$ shows that $\{E_g, \alpha_k\} \equiv 0$ on $TU(x_0)$, and therefore α_k are constants. Item (b) of Lemma 7 is proved.

Denote by $B_k^D|_{T_x M^m}$ the restriction of the function $B_k^D(\xi)$ on the fiber $T_x M^m \hookrightarrow TM^m$. Taking a point $\xi \in T_x M^m$, we obtain $d_\xi(B_k^D|_{T_x M^m})(\eta) = 2B_k^D|_x(\xi,\eta)$, where $\eta \in T_x M^m \cong T_\xi(T_x M^m)$. It follows from Lemma 4 (b) that for every point $x \in U(x_0)$ there exists an open and dense in $T_x M^m$ subset $\Sigma_x \subset T_x M^m$ such that if $\xi \in \Sigma_x$ then $B_1^D|_x(\xi,.), ..., B_q^D|_x(\xi,.) \in T_x^* M^m$ are linearly independent. Therefore, the functions $B_1^D(\xi), ..., B_q^D(\xi)$ are functionally independent in $TU(x_0)$. Lemma 7 is proved.

Let us prove that the functions $B_1^D(\xi), ..., B_r^D(\xi)$ are functionally independent in TD. According to Lemma 6 and Lemma 7, it is sufficient to prove that the stable points in D have rank r. Assume that there exists a stable point $y_0 \in D$ having rank $r_0 < r$. Without loss of generality we can suppose that the open set D contains a stable point $x_0 \in D$ of rank r. Assume for simplicity that the points x_0 and y_0 can be connected by a geodesic line $t \to \gamma(t) \in D$, $\gamma(0) = x_0$, $\gamma(1) = y_0$. Let us take neighborhoods $U(x_0)$ and $V(y_0)$ of the points x_0 and y_0 respectively, such that the conditions of Lemma 7 are satisfied. We can suppose that the differentials $d_w B_1^D, ..., d_w B_r^D$ are linearly independent at the point $w \stackrel{\text{def}}{=} \dot{\gamma}(0)$ (If not, we take $w' \in T_{x_0} M^m$ such that $d_{w'} B_1^D, ..., d_{w'} B_r^D$ are linearly independent and $\exp w' \in V(y_0)$ (see Lemma 7 (a))). Denote by $\zeta_t(\xi)$, $\xi \in TM^m$, the one-parameter family of local diffeomorphisms of TM^m corresponding to the geodesic flow of the metric g. Using the fact that the functions $B_1^D(\xi), ..., B_r^D(\xi)$ are integrals of the geodesic flow of g, we obtain that the differentials $d_v B_1^D, ..., d_v B_r^D$, $v \stackrel{\text{def}}{=} \zeta_1(w)$, are linearly independent. ¿From another side, Lemma 7 (*b*) shows that $B^D_{r_0+1}(\xi) = \sum_{k=1}^{r_0} \alpha_k B^D_k(\xi)$ where ξ lies in $TV(y_0)$ and α_k are some constants. Hence, for every $\xi \in TV(y_0)$, $d_\xi B^D_{r_0+1} = \sum_{k=1}^{r_0} \alpha_k d_\xi B^D_k$. This contradiction proves that *the stable points $x \in D$, that can be connected by a geodesic line $\gamma(t)$ lying in D with another stable point $y \in D$ of rank r, are also of rank r.* By assumption D is connected. Therefore, the stable points in D have rank r. Item (*a*) of Theorem 1 is proved. Item (*b*) easily follows from Lemma 7 (*b*). Theorem 1 is proved.

5 $PQ^{(-1)}$-projective hermitian metric

Let M^n be a complex manifold of complex dimension n. Denote by J the complex structure of M^n, $(M^{2n}, J) \cong M^n$. Consider a pair of $PQ^{(-1)}$-projective hermitian metrics s and $\bar{s}$ ($\epsilon = -1$ and $PQ = -\mathbf{1}$). Denote by $K_c(s, \bar{s})$ the local family of pairwise commuting integrals of the geodesic flow of the metric s (Theorem 3). In our case $\epsilon = -1$ and we obtain that $K_c(s, \bar{s}) \overset{\text{def}}{=} |\det(A + c\mathbf{1})|^{1/2} s(A + c\mathbf{1})^{-1}$, where $A = A(s, \bar{s})$.

Fixing a complex chart $\{(z^1, ..., z^n)\}$ consider the hermitian matrices $S \overset{\text{def}}{=} (s_{\alpha\bar{\beta}})$ and $\bar{S} \overset{\text{def}}{=} (\bar{s}_{\alpha\bar{\beta}})$ related to the metrics $ds^2 = 2s_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$ and $d\bar{s}^2 = 2\bar{s}_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$ respectively. As usual, we omit the summation symbols in the formulas. Define the operator $\mathcal{A} \overset{\text{def}}{=} \mathcal{A}^\alpha_\beta \frac{\partial}{\partial z^\alpha} \otimes dz^\beta$, where

$$\mathcal{A}^\alpha_\beta \overset{\text{def}}{=} \left| \frac{\det \bar{S}}{\det S} \right|^{\frac{1}{n+1}} \bar{s}^{\alpha\bar{\nu}} s_{\bar{\nu}\beta}, \tag{18}$$

and $\bar{s}^{\alpha\bar{\nu}} \bar{s}_{\bar{\nu}\beta} = \delta^\alpha_\beta$. By definition $\mathcal{A}$ is a smooth section of the vector bundle $\mathrm{End}_{\mathbb{C}}(T^{(1,0)} M^n)$, where $T^{(1,0)} M^n$ denotes the bundle of the *holomorphic* tangent vectors to M^n. Fixing a point $x \in M^n$, denote by $\rho(s, \bar{s})(x)$ the minimal polynomial of the operator $\mathcal{A}|_x$.

Definition 9. *The number* $\rho(s, \bar{s}) \overset{\text{def}}{=} \max_{x \in M^n} \rho(s, \bar{s})(x)$ *is called* hermitian rank of the pair of $PQ^{(-1)}$-projective hermitian metrics.

Theorem 4. *Suppose that the complex manifold M^n is connected and let the hermitian rank of the pair of $PQ^{(-1)}$-projective hermitian metrics s and $\bar{s}$ be r. Then there exist r hermitian forms $B_1, ..., B_r$ such that:*

(a) *considered as functions of the tangent bundle TM^n, the quadratic forms $B_1(\xi), ..., B_r(\xi)$ are functionally independent pairwise commuting integrals of the geodesic flow of the metric s;*

(b) *for every fixed real constant c, the integral $K_c(s,\tilde{s})(\xi)$ is well-defined and*

$$K_c(s,\tilde{s})(\xi) = \sum_{k=1}^{r} \alpha_k B_k(\xi),$$

where α_k are some constants.

The functions $B_1(\xi), ..., B_r(\xi)$ can be taken in the form $B_k = K_{c_k}(s,\tilde{s})$, where c_k are appropriately chosen constants.

Proof of Theorem 4. Fixing a complex chart $\{(z^1, ...z^n)\}$ we have $s = s_{\alpha\bar{\beta}}(dz^\alpha \otimes d\bar{z}^\beta + d\bar{z}^\beta \otimes dz^\alpha)$ and $\tilde{s} = \tilde{s}_{\alpha\bar{\beta}}(dz^\alpha \otimes d\bar{z}^\beta + d\bar{z}^\beta \otimes dz^\alpha)$, where $S \stackrel{\text{def}}{=} (s_{\alpha\bar{\beta}})$ and $\tilde{S} \stackrel{\text{def}}{=} (\tilde{s}_{\alpha\bar{\beta}})$ are hermitian matrices. Denote by $[s]$ and $[\tilde{s}]$ the Gramians of the metrics s and $\tilde{s}$ respectively, i.e.

$$[s] = \begin{bmatrix} 0 & S \\ \bar{S} & 0 \end{bmatrix}$$

and

$$[\tilde{s}] = \begin{bmatrix} 0 & \tilde{S} \\ \bar{\tilde{S}} & 0 \end{bmatrix}.$$

It follows from the definition of the operator $A(s,\tilde{s})$ (formula (4)) that in the fixed complex chart we have

$$A(s,\tilde{s}) \stackrel{\text{def}}{=} \left|\frac{\det[\tilde{s}]}{\det[s]}\right|^{\frac{1}{2(n+1)}} [\tilde{s}]^{-1}[s] = \left|\frac{\det \tilde{S}}{\det S}\right|^{\frac{1}{n+1}} [\tilde{s}]^{-1}[s].$$

Hence, $A(s,\tilde{s}) = \mathcal{A}^\alpha_\beta \frac{\partial}{\partial z^\alpha} \otimes dz^\beta + \mathcal{A}^{\bar{\alpha}}_{\bar{\beta}} \frac{\partial}{\partial \bar{z}^\alpha} \otimes d\bar{z}^\beta$, where $\mathcal{A}^\alpha_\beta$ is given by formula (18) and $\mathcal{A}^{\bar{\alpha}}_{\bar{\beta}}$ coincides with the complex conjugation of $\mathcal{A}^\alpha_\beta$ (i.e. $\mathcal{A}^{\bar{\alpha}}_{\bar{\beta}} = \bar{\mathcal{A}}^\alpha_\beta$). As usual, we identify the operator $\mathcal{A}$ with the square matrix $(\mathcal{A}^\alpha_\beta)$.

Lemma 8. *The degree of the minimal polynomial of the operator $\mathcal{A}(s,\tilde{s})$ coincides with the degree of the minimal polynomial of the operator $A(s,\tilde{s})$.*

Proof of Lemma 8. A simple calculation shows that

$$\mathcal{A} - \bar{\lambda}\mathbf{1} = \bar{S}^{-1}(\mathcal{A} - \lambda)^{*}\bar{S},$$

where $(.)^{*}$ denotes the hermitian conjugation of a matrix. Hence, for every integer k we have $(\mathcal{A} - \bar{\lambda}\mathbf{1})^{k} = \bar{S}^{-1}[(\mathcal{A} - \lambda)^{k}]^{*}\bar{S}$. Consider the Jordan's normal form of the operator $\mathcal{A}$. The last equalities show that the Jordan's decompositions corresponding to the eigenvalues λ and $\bar{\lambda}$ coincide. Finally, we conclude the statement of the lemma from the formula $A(s,\bar{s}) = \mathcal{A}^{\alpha}_{\beta}\frac{\partial}{\partial z^{\alpha}} \otimes dz^{\beta} + \bar{\mathcal{A}}^{\alpha}_{\beta}\frac{\partial}{\partial \bar{z}^{\alpha}} \otimes d\bar{z}^{\beta}$. Lemma 8 is proved.

It follows from Lemma 8 that the rank $r(s,\bar{s})$ of the pair s and $\bar{s}$ coincides with their hermitian rank, i.e. $r = r(s,\bar{s}) = \rho(s,\bar{s})$.

Let us fix an arbitrary real constant c. Denote by $\mathcal{K}_c(s,\bar{s})$ the hermitian matrix of the form $K_c(s,\bar{s})$. A simple calculation shows that

$$\begin{aligned}\mathcal{K}_c(s,\bar{s}) &= \det(\bar{A} + c\mathbf{1})S(\bar{A} + c\mathbf{1})^{-1}\\ &= \mathcal{I}_{n-1}(s,\bar{s})c^{n-1} + \ldots + \mathcal{I}_0(s,\bar{s}).\end{aligned}$$

The last formula is valid in every fixed complex chart. Therefore, there exist well-defined on the whole M^n hermitian forms $I_{n-1},\ldots,I_0$ such that $K_c = I_{n-1}c^{n-1} + \ldots + I_0$. Denote by $I_{n-1}(\xi),\ldots,I_0(\xi)$ the corresponding smooth functions of the tangent bundle TM^n. We obviously have

$$K_c(\xi) = I_{n-1}(\xi)c^{n-1} + \ldots + I_0(\xi), \tag{19}$$

where $I_j \in C^{\infty}(TM^n)$. Denote by $\mathcal{I}(s,\bar{s})$ the vector space spanned on the functions $I_j \in C^{\infty}(TM^n)$ $(j = 0,\ldots,n-1)$. Take a basis $B_1(\xi),\ldots,B_{r'}(\xi)$ of the linear space $\mathcal{I}(s,\bar{s})$. Let us fix an open set $D \subset M^n$ with compact closure in M^n and consider the quadratic forms $B^D_1(\xi),\ldots,B^D_r(\xi)$ given by Theorem 1. Item (b) of Theorem 1 and formula (19) show that $B_k|_{TD} \in \mathrm{Span}(B^D_1|_{TD},\ldots,B^D_r|_{TD})$ $(k = 1,\ldots,r')$ where $\mathrm{Span}(B^D_1|_{TD},\ldots,B^D_r|_{TD})$ denotes the vector space spanned on the restrictions of the functions $B^D_k(\xi)$ on TD. ¿From another side, $B^D_k(\xi) \stackrel{\text{def}}{=} K_{c_k}(s,\bar{s})(\xi)$. Using formula (19) again we obtain that $B^D_k|_{TD} \in \mathrm{Span}(B_1|_{TD},\ldots,B_r|_{TD})$ $(k = 1,\ldots,r)$. Therefore, $r = r'$. The functions $B^D_1(\xi),\ldots,B^D_r(\xi)$ are functionally independent on TD. Hence, the functions $B_1(\xi),\ldots,B_r(\xi)$ are functionally independent as well. Finally, recall that the set D was taken arbitrary. Theorem 4 is proved.

Remark 6. *In the present section we don't use essentially the integrability of the complex structure J. Therefore, Theorem 4 still holds if (M^{2n}, J) be an almost complex manifold of real dimension 2n.*

6 Examples. Integrable systems on $\mathbf{CP}^n$

The aim of the present section is to find a family of completely integrable systems on the complex projective space $\mathbf{CP}^n$. Denote by $\{(z_0 : ... : z_n)\}$ the homogeneous coordinates of $\mathbf{CP}^n$ and consider the affine chart $\mathbf{C}^n \ni (z_1, ..., z_n) \mapsto (1 : z_1 : ... : z_n) \in \mathbf{CP}^n$. In coordinates $\{(z_1, ..., z_n)\}$, the Fubini metrics are given by the formula

$$dg^2 \stackrel{\text{def}}{=} 2\frac{\sum_{\alpha=1}^{n} \epsilon_\alpha|dz_\alpha|^2 + \frac{K}{2}\left(\left(\sum_{\alpha=1}^{n} \epsilon_\alpha|z_\alpha|^2\right)\left(\sum_{\alpha=1}^{n} \epsilon_\alpha|dz_\alpha|^2\right) - \left|\sum_{\alpha=1}^{n} \epsilon_\alpha\bar z_\alpha dz_\alpha\right|^2\right)}{\left(1 + \frac{K}{2}\sum_{\alpha=1}^{n} \epsilon_\alpha|z_\alpha|^2\right)^2}, \tag{20}$$

where $K \neq 0$ and the "signs" $\epsilon_\alpha = \pm 1$ $(\alpha = 1, ..., n)$ are fixed. Consider the hermitian form $Q(\xi, \bar\eta) \stackrel{\text{def}}{=} \xi_0\bar\eta_0 + \frac{K}{2}\sum_{\alpha=1}^{n} \epsilon_\alpha\xi_\alpha\bar\eta_\alpha$, $\xi, \eta \in \mathbf{C}^{n+1}$. The hypersurface $Abs \hookrightarrow \mathbf{CP}^n$ given in homogeneous coordinates by $Abs \stackrel{\text{def}}{=} \{Q(z) \stackrel{\text{def}}{=} Q(z, \bar z) = 0\}$ is called *absolute* of the corresponding Fubini metric. The Fubini metrics are smoothly defined on $\mathbf{CP}^n \setminus Abs$. If $K > 0$ and $\epsilon_\alpha = 1$ then $Abs = \emptyset$, and the corresponding Fubini metric is a smooth Riemannian metric on $\mathbf{CP}^n$. The Fubini metrics are usually considered only on the subset $F \stackrel{\text{def}}{=} \{Q(z) > 0\} \subset \mathbf{CP}^n$. Nevertheless, it will be more convenient for us to think of the Fubini metrics as metrics defined on the whole $\mathbf{CP}^n$ and having "singularities" in Abs. The Fubini metrics are hermitian and the corresponding hermitian matrices are $g_{\alpha\bar\beta} = \frac{\epsilon_\alpha\delta_{\alpha\beta}}{Q(z)} - \frac{K}{2}\frac{(\epsilon_\alpha\bar z_\alpha)(\epsilon_\beta z_\beta)}{Q(z)^2}$. It is well-known that the Fubini metrics are Kählerian metrics.

Denote by ∇ the Levi-Civita connection corresponding to the Fubini metric g. The connection ∇ can be extended in a natural way to a connection on the complexification of the tangent bundle. It can be easily seen that the corresponding Christoffel symbols are given by the formula $\Gamma^\alpha_{pq} = -\frac{K}{2Q(z)}(\delta^\alpha_p\epsilon_q\bar z_q + \delta^\alpha_q\epsilon_p\bar z_p)$, $\Gamma^{\bar\alpha}_{\bar p\bar q} = \bar\Gamma^\alpha_{pq}$, and the other components of the Christoffel symbols vanish (see [15],§5, for the case $\epsilon_\alpha = 1$).

Let $\mu_L : \mathbf{CP}^n \to \mathbf{CP}^n$ be a projective transformation induced by some (complex)linear transformation $L : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ given in matrix form by the non-degenerate complex matrix $L \in GL_{n+1}(\mathbf{C})$. Denote by $\tilde{g}$ the pull-back $\mu_L^* g$.

Proposition 5. *The metrics g and $\tilde{g}$ are h-projective Kählerian metrics.*

Proof of Proposition 5. Let L_2 be a complex 2-plane in $\mathbf{C}^{n+1}$. The set of complex lines lying in L_2 gives a natural embedding of the complex projective line $\mathbf{CP}^1$ in $\mathbf{CP}^n$. We call such embeddings *projective lines*. Proposition 5 easily follows from the next simple lemma.

Lemma 9. *A smooth curve $\gamma(t) \in \mathbf{CP}^n \setminus$ Abs is holomorphically planar with respect to a fixed Fubini metric iff $\gamma(t)$ lies in a projective line.*

Proof of Lemma 9. Consider the affine chart $\{(z_1, ..., z_n)\}$ and suppose that $\gamma(t) = (\gamma^1(t), ..., \gamma^n(t))$. The condition that $\gamma(t)$ is holomorphically planar with respect to a fixed Fubini metric g is equivalent to the equation

$$\frac{\nabla \dot{\gamma}^\alpha}{dt}(t) = \rho(t)\dot{\gamma}^\alpha(t),\ \alpha = 1, ..., n,$$

where $\rho(t)$ is a smooth complex-valued function of the real parameter t and $\frac{\nabla \dot{\gamma}^\alpha}{dt}$ are the components of the "holomorphic" part of the real vector $\frac{\nabla \dot{\gamma}}{dt}$. Using the explicit form of the Christoffel symbols of the Fubini metric g we obtain

$$\begin{aligned} \frac{\nabla \dot{\gamma}^\alpha}{dt}(t) &= \frac{d^2\gamma^\alpha}{dt^2} + \Gamma^\alpha_{pq}\dot{\gamma}^p\dot{\gamma}^q \\ &= \frac{d^2\gamma^\alpha}{dt^2} - \frac{K}{Q}(\epsilon_p \bar{z}_p \dot{\gamma}^p)\dot{\gamma}^\alpha. \end{aligned}$$

Therefore, the curve $\gamma(t)$ is holomorphically planar if and only if $\frac{d^2\gamma^\alpha}{dt^2} = \rho_1(t)\frac{d\gamma^\alpha}{dt}$, where $\rho_1(t)$ is a smooth complex-valued function of t. This completes the proof of Lemma 9.
Finally, Proposition 5 follows from the fact that $\mu_L : \mathbf{CP}^n \to \mathbf{CP}^n$ maps projective lines to projective lines. Proposition 5 is proved.

An analog of Lemma 9 in the case of positive definite Fubini metrics is proved in [15], §6.

Let us take $L = \mathrm{diag}(1, \lambda_1, ..., \lambda_n)$ where λ_α are fixed constants. In the chart $\{(z_1, ..., z_n)\}$, the induced projective transformation $\mu_L : \mathbf{CP}^n \to \mathbf{CP}^n$ is given by the formula $(z_1, ..., z_n) \mapsto (\lambda_1 z_1, ..., \lambda_n z_n)$. We have

$$d\tilde{g}^2 = 2\frac{\tilde{Q}(z)(\sum_{\alpha=1}^{n} \epsilon_\alpha \rho_\alpha |dz_\alpha|^2) - \frac{K}{2}|\sum_{\alpha=1}^{n} \epsilon_\alpha \rho_\alpha \bar{z}_\alpha dz_\alpha|^2}{\tilde{Q}(z)^2}, \quad (21)$$

where $\rho_\alpha \stackrel{\mathrm{def}}{=} |\lambda_\alpha|^2$ and $\tilde{Q}(z) \stackrel{\mathrm{def}}{=} 1 + \frac{K}{2}\sum_{\alpha=1}^{n} \epsilon_\alpha \rho_\alpha |z_\alpha|^2$. The components of the corresponding hermitian matrix are $\tilde{g}_{\alpha\bar{\beta}} = \frac{\epsilon_\alpha \rho_\alpha \delta_{\alpha\beta}}{\tilde{Q}(z)} - \frac{K}{2}\frac{(\epsilon_\alpha \rho_\alpha \bar{z}_\alpha)(\epsilon_\beta \rho_\beta z_\beta)}{\tilde{Q}^2(z)}$. Let us remark that the metric $\tilde{g}$ is smoothly defined on $\mathbf{CP}^n \setminus \widetilde{Abs}$, where $\widetilde{Abs} \stackrel{\mathrm{def}}{=} \{\tilde{Q}(z) = 0\}$.

Lemma 10. *Let B be a non-degenerate symmetric $n \times n$-matrix and $a, b \in \mathbf{C}^n$ are complex vectors considered as $n \times 1$-matrices. Denote by $(.)'$ the transposition of a matrix and $\langle x, y\rangle \stackrel{\mathrm{def}}{=} \sum_{\alpha=1}^{n} x_\alpha y_\alpha$. Then*

(*i*) $(B + ab')^{-1} = B^{-1} - \frac{1}{1+\langle B^{-1}a, b\rangle}(B^{-1}a)(B^{-1}b)'$;

(*ii*) $\det(B + ab') = (1 + \langle B^{-1}a, b\rangle)\det B$.

The proof of Lemma 10 is straightforward.

Using Lemma 10 we obtain $\tilde{g}^{\alpha\bar{\beta}} = \tilde{Q}(\epsilon_\alpha \rho_\alpha^{-1}\delta_{\alpha\beta} + \frac{K}{2} z_\alpha \bar{z}_\beta)$. Hence, the operator $A(g, \tilde{g})$ is given by the formula $A(g, \tilde{g}) = \mathcal{A}^\alpha_\beta \frac{\partial}{\partial z_\alpha} \otimes dz_\beta + \mathcal{A}^{\bar{\alpha}}_{\bar{\beta}} \frac{\partial}{\partial \bar{z}_\alpha} \otimes d\bar{z}_\beta$, where $\mathcal{A}^{\bar{\alpha}}_{\bar{\beta}} = \bar{\mathcal{A}}^\alpha_\beta$ and $\mathcal{A}^\alpha_\beta = \left|\frac{\det \tilde{G}}{\det G}\right|^{\frac{1}{n+1}} \tilde{g}^{\alpha\bar{\nu}} g_{\bar{\nu}\beta}$, $\tilde{G} \stackrel{\mathrm{def}}{=} (\tilde{g}_{\alpha\bar{\beta}})$, $G \stackrel{\mathrm{def}}{=} (g_{\alpha\bar{\beta}})$ (see Sect.5). Applying Lemma 10 we obtain

$$\mathcal{A}^\alpha_\beta = \rho_\beta^{-1}\delta^\alpha_\beta - \frac{K}{2Q}((1 - \rho_\alpha)z_\alpha/\rho_\alpha)(\epsilon_\beta \bar{z}_\beta). \quad (22)$$

Denote by $\mathcal{A}$ the matrix with elements $\mathcal{A}^\alpha_\beta$. Consider the sequences of hermitian matrices $\{G^{(l)}\}_{l\in\mathbf{Z}}$ and $\{\tilde{G}^{(l)}\}_{l\in\mathbf{Z}}$ defined by the formulas $G^{(l)} \stackrel{\mathrm{def}}{=} G\bar{\mathcal{A}}^l$ and $\tilde{G}^{(l)} \stackrel{\mathrm{def}}{=} \tilde{G}\bar{\mathcal{A}}^l$. Denote by $g^{(l)}$ and $\tilde{g}^{(l)}$ the hermitian metrics

$$dg^{(l)2} \stackrel{\mathrm{def}}{=} 2g^{(l)}_{\alpha\bar{\beta}} dz_\alpha d\bar{z}_\beta \quad (23)$$

and

$$d\bar{g}^{(l)2} \stackrel{\text{def}}{=} 2\bar{g}^{(l)}_{\alpha\beta} dz_\alpha d\bar{z}_\beta, \tag{24}$$

where $g^{(l)}_{\alpha\beta}$ and $\bar{g}^{(l)}_{\alpha\beta}$ are the elements of the hermitian matrices $G^{(l)}$ and $\bar{G}^{(l)}$ respectively.

Consider the operators $\mathcal{P}_l, \mathcal{Q}_l \in \Gamma(\mathrm{End}_{\mathbf{C}}(T^{(1,0)}(\mathbf{CP}^n \setminus Abs)))$ given in coordinates by the matrices $i\mathcal{A}^l$ and $i\mathcal{A}^{-l}$ respectively and define the "real" operators $P_l \stackrel{\text{def}}{=} \mathcal{P}_l + \bar{\mathcal{P}}_l$ and $Q_l \stackrel{\text{def}}{=} \mathcal{Q}_l + \bar{\mathcal{Q}}_l$. The next theorem follows from the results proved in Sect.2.4.

Theorem 5. *For every fixed integer $l \in \mathbf{Z}$ the hermitian metrics $g^{(l)}$ and $\bar{g}^{(l)}$ given by formulas (23) and (24) are $P_l Q_l^{(-1)}$-projective hermitian metrics.*

The sequence of metrics $g^{(l)}$ and $\bar{g}^{(l)}$ ($l \in \mathbf{Z}$) given by Theorem 5 is called $\mathbf{CP}^n$*-hierarchy.*

Let us consider the one-parameter groups of transformations of $\mathbf{CP}^n$ given in coordinates by the formulas

$$T_k(\phi) : (z_1, ..., z_k, ..., z_n) \mapsto (z_1, ..., \exp(i\phi) z_k, ..., z_n).$$

It follows from (20) and (21) that $T_k(\phi)$ preserve the metrics g and $\bar{g}$, and therefore they preserve the whole $\mathbf{CP}^n$-hierarchy. Denote by T_k the corresponding Killing symmetries,

$$T_k \stackrel{\text{def}}{=} i\left(z_k \frac{\partial}{\partial z_k} - \bar{z}_k \frac{\partial}{\partial \bar{z}_k}\right). \tag{25}$$

Consider the "complex" impulses $p_k \stackrel{\text{def}}{=} \frac{1}{2}(p_{x_k} - i p_{y_k})$ and $\bar{p}_k \stackrel{\text{def}}{=} \frac{1}{2}(p_{x_k} + i p_{y_k})$ $(k = 1, ..., n)$, where p_{x_k} and p_{y_k} are the impulses corresponding to the chart $\{(x_1, y_1, ..., x_n, y_n)\}$, $z_k \stackrel{\text{def}}{=} x_k + i y_k$. The chart $\{(p_1, ..., p_n; z_1, ..., z_n)\}$ is a complex chart of $T^*\mathbf{CP}^n$. The canonical symplectic structure ω on $T^*\mathbf{CP}^n$ is given by the formula $\omega = \sum_{k=1}^n dp_k \wedge dz_k + \sum_{k=1}^n d\bar{p}_k \wedge d\bar{z}_k$.

Theorem 6. *For every fixed integer $l \in \mathbf{Z}$ the functions $\mathcal{I}^{(l)}_{n-1}(p), ..., \mathcal{I}^{(l)}_0(p)$ given by the expansion*

$$\begin{aligned} \mathcal{K}^{(l)}_c(p) &\stackrel{\text{def}}{=} \det(\mathcal{A} + c\mathbf{1}) \left\langle (\mathcal{A} + c\mathbf{1})^{-1} \mathcal{A}^l \bar{G}^{-1} \bar{p}, p \right\rangle \\ &= \mathcal{I}^{(l)}_{n-1}(p) c^{n-1} + ... + \mathcal{I}^{(l)}_0(p), \end{aligned} \tag{26}$$

where $p \stackrel{\rm def}{=} (p_1, ..., p_n)$, $\bar p \stackrel{\rm def}{=} (\bar p_1, ..., \bar p_n)$ *and* $\langle X, Y\rangle \stackrel{\rm def}{=} \sum_{k=1}^n X_k Y_k$, *are in involution with respect to the canonical symplectic structure* ω *on the cotangent bundle* $T^*\mathbf{CP}^n$. *If* $\rho_\alpha \neq \rho_\beta$ $(\alpha \neq \beta)$ *then the functions* $\mathcal{I}^{(l)}_{n-1}(p), ..., \mathcal{I}^{(l)}_0(p)$ *are functionally independent on* $T^*(\mathbf{CP}^n \setminus \mathrm{Abs})$. *Adding to the functions* $\mathcal{I}^{(l)}_{n-1}(p), ..., \mathcal{I}^{(l)}_0(p)$ *the Noether integrals* $T_k(p) \stackrel{\rm def}{=} i(z_k p_k - \bar z_k \bar p_k)$ $(k = 1, ..., n)$ *corresponding to the Killing symmetries (25), we obtain a complete system of functionally independent functions in involutions on* $T^*(\mathbf{CP}^n \setminus \mathrm{Abs})$.

Proof of Theorem 6. It follows from formula (22) that $\mathcal{A}^\alpha_\beta(0) = \rho_\beta^{-1}\delta^\alpha_\beta$. Hence, if $\rho_\alpha \neq \rho_\beta$ $(\alpha \neq \beta)$ then the hermitian rank of the pair $g^{(l)}$ and $\tilde g^{(l)}$ is n. Finally, the statement of the theorem follows from Theorem 4, Sect.5, applied to the pair $g^{(-l)}$ and $\tilde g^{(-l)}$ from the **CP**-hierarchy. Theorem 6 is proved.

Corollary 2. *Provided* $\rho_\alpha \neq \rho_\beta$ $(\alpha \neq \beta)$, *the geodesic flows of the metrics* $g^{(l)}$ *and* $\tilde g^{(l)}$ *from the* $\mathbf{CP}^n$*-hierarchy are completely integrable.*

Remark 7 *Taking* $l = 0$ *and* $\epsilon_\alpha = 1$, *we obtain a complete family of pairwise commuting integrals of the geodesic flow of the standard Fubini metric on* $\mathbf{CP}^n$ *(see [12, 13]). Our theorem gives an infinite family of metrics on* $\mathbf{CP}^n$ *with completely integrable geodesic flows.*

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