Cubo Matemática Educacional
Vol., 4. Nº 2, Junio 2002

INTEGRAL BORDISMS AND GREEN KERNELS IN PDEs *

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ABSTRACT - We characterize integral bordisms of (nonlinear) PDEs by means of geometric Green kernels and prove that these are invariant for the classic limit of statistical sets of formally integrable PDEs. Such geometric characterization of Green kernels is related to the geometric approach of canonical quantization of (nonlinear) PDEs, previously introduced by us [3,4,5,6]. Some applications are given where particle fields on curved space-times having physical or unphysical masses, (i.e., bradions, luxons and massive neutrinos) are canonically quantized respecting microscopic causality.

1 - GREEN KERNELS AND CLASSIC-LIMIT STATISTICAL SETS OF NONLINEAR PDEs

In this section we shall relate integral and quantum bordisms to Green kernels of nonlinear PDEs. In particular we will prove that to any classic-

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* Work partially supported by Italian grants MURST , GNFM/INDAM and University of Rome "La Sapienza".

limit-statistical set of PDEs [1] we can associate a Green kernel.

Set, for any vector fiber bundle $\pi : F \to M$, over a n-dimensional manifold M, $F' \equiv F^* \otimes \Lambda^0_n M$, where F^* is the dual of the fiber bundle F and $\Lambda^0_n M$ is the fiber bundle of n-forms on M. We call F' the **formal adjoint** of F. If $\bar{\pi} : E \to N$ is another vector fiber bundle over N we set $E \boxtimes F \equiv \bigcup_{(p,q)\in N\times M} E_p \bigotimes F_q$. This is a vector fiber bundle over $N \times M$ of dimension $n+m+r.s$, if $\dim E_p = r$, $\forall p \in N$, and $\dim F_q = s$, $\forall q \in M$. For any vector fiber bundle $E \to N$ we denote by $C^\infty_0(E)$ the space of smooth sections with compact support over N.

THEOREM 1.1 - GREEN KERNELS AND DISTRIBUTION SOLUTIONS OF AFFINE PDEs. *Let $\kappa : J\mathcal{D}^r(E) \to F$ be a morphism of vector fiber bundles on a manifold M of dimension n such that it defines a linear differential operator of order r, $\kappa. : C^\infty(E) \to C^\infty(F)$. Let $E_r \equiv \ker_f \kappa \subset J\mathcal{D}^r(E)$ be an affine equation, where $f \in C^\infty(F)$. Assume that $R \subset J\mathcal{D}^r(E) \subset J^r_n(E)$ is an integral manifold of dimension n, such that in some neighborhood of all its points $q \in R$, (except for a nowhere dense subset $\Sigma(R) \subset R$), it can be represented as the r-holonomic prolongation $N^{(r)}$ of some n-dimensional submanifold $N \subset E$. Furthermore, we assume that $\pi_{r,0}|_R : R \to E$ is a proper application, and ${\omega_1}^{(r)}(R) = 0$, where ${\omega_1}^{(r)}(R) = i^*_R \omega_1^{(r)}$, with $i_R : R \to J^r_n(W)$ the natural inclusion mapping and $\omega_1^{(r)}$ is the first generator of the cohomology algebra $H^\bullet(I_r(W); \mathbf{Z}_2) \cong \mathbf{Z}_2[\omega_1^{(r)}, \dots, \omega_m^{(r)}]$, where $I_r(W)$ is the Grassmannian bundle over $J^r_n(W)$ of oriented integral planes of $J^r_n(W)$ [2]. Then, we can associate to R a distribution $F[R] \in (C^\infty_0(E'))'$. Therefore, $F[R]$ is a*

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[1] Recall that for any two admissible $(n-1)$-dimensional closed compact integral manifolds N_1, N_2 contained into a k-order PDE $E_k \subset J^k_n(W)$ of n-dimensional submanifolds of the $(n+m)$-dimensional manifold W, the classic-limit statistical set $\Omega_c(N_1, N_2)$ is the set of all solutions V of E_k such that $\partial V = N_1 \bigcup N_2$. The admissibility is in the sense of Theorem 4.5 in ref.[12].

[2] If r is orientable then $\omega_1^{(r)}(R)=0$, as it coincides with the first Stiefel-Whitney characteristic class of R. This is a direct consequence of the following short exact sequence: $0 \to <\omega_1> \to H^\bullet(BO(n);\mathbf{Z}_2) \xrightarrow{j^\bullet} H^\bullet(BSO(n);\mathbf{Z}_2) \to 0$, where $<\omega_1>$ is generated as

distributive solution of E_r, i.e., a solution of the distributional extension $\widetilde{E}_r$ of E_r, iff R is a multivalued solution of $E_r \subset J\mathcal{D}^r(E)$. Furthermore, if R is such that $\partial R = R_1 \dot{\bigcup} R_2$ with $R_1 \in [R_2]_{E_r} \in \Omega^{E_r}_{n-1}$ [3] *, then we say that $F[R]$ satisfies the boundary condition $\partial R = R_1 \bigcup R_2$ and it can be represented in the following form: $F[R] = {}_f\mathbb{G} + \omega_0$, with $\omega_0 \in Sol(\overline{\overline{E}}_r)$, where $Sol(\overline{\overline{E}}_r)$ is the space of solutions of the linear equation $\overline{\overline{E}}_r$ associated to $\widetilde{E}_r$, and $\mathbb{G}$ is the Green kernel of κ. Then, ${}_f\mathbb{G}$ can be considered an invariant of the classic-limit-statistical set $\Omega_c(R_1, R_2)$.*

PROOF. We shall introduce some definitions and lemmas. Let $E \xrightarrow{\pi} M \xleftarrow{\bar{\pi}} F$ be two vector fiber bundles over a n-dimensional manifold M. We call **formal adjoint** of a r-order linear differential operator $\kappa : C^\infty(E) \to C^\infty(F)$ the linear operator $\kappa^* : C^\infty(F') \to C^\infty(E')$ defined by $<\kappa^*(\alpha), e> = <\alpha, \kappa(e)>$, $\forall e \in C^\infty(E)$, $\alpha \in C^\infty(F')$. The restriction $\kappa^* : C^\infty_0(F') \to C^\infty_0(E')$ satisfies the following formula: $\int_M <e, \kappa^*(\alpha)> = \int_M <\kappa(e), \alpha>$, $\forall e \in C^\infty(E)$, $\alpha \in C^\infty_0(F')$. The **Dirac kernel** $\mathbb{D}$ of the vector fiber bundle F on a n-dimensional manifold M, $\dim F = n + m$, is the distributive kernel of local character $\mathbb{D} \in (C^\infty_0(F \boxtimes F'))' \cong (C^\infty_0(F' \boxtimes F))'$ defined by $\mathbb{D}(f \otimes \alpha) = \int_M <f, \alpha>$, $\forall f \in C^\infty_0(F)$, $\alpha \in C^\infty_0(F')$. The Dirac kernel admits the following local representation: $\mathbb{D}^j_{U,i}(x, x') = \delta^j_i \delta(x, x')$, where $\delta(x, x')$ is the Dirac function. In fact we can write:

$$\begin{aligned}\mathbb{D}(f \otimes \alpha) &= \sum_U \int_U g_U < f_U, \alpha_U \\ &>= \sum_U \int_{U \times U} g_{U \times U} f^i_U(x) \alpha_{U,j}(x') \delta^j_i \delta(x, x') d\mu(x')\end{aligned}$$

where g_U is a partition of unity on M, subordined to the open covering $\{U\}$ of M. Furthermore, the **Green kernel** of κ is a kernel $\mathbb{G} \in (C^\infty_0(E' \boxtimes F))' \cong (C^\infty_0(F \boxtimes E'))'$, such that it satisfies the following

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ideal by the first Stiefel-Whitney class $\omega_1 \in H^1(BSO(n); \mathbb{Z}_2)$ and $j^\bullet$ is induced by the natural surjection $j: BSO(n) \to BO(n)$ which forgets the orientation on the oriented n-dimensional planes representing the points $\hat{G}_{\infty,n} = BSO(n)$.

[3] $\Omega^{E_r}_{n-1}$ denotes the integral bordism group for $(n-1)$-dimensional closed admissible manifolds of $E_r \subset J^r_n(E)$. (See refs.[8-12] for more informations about.)

equation: $(\widetilde{\kappa}\otimes\mathbf{1})(\mathbb{G}) = (\mathbf{1}\otimes\widetilde{\kappa})(\mathbb{G}) = \mathbb{D}$, where $\widetilde{\kappa} : (C_0^\infty(E'))' \to (C_0^\infty(F'))'$ is the linear mapping such that the following diagram is commutative:

$$\begin{array}{ccccc} 0 & \to & C^\infty(F) & \xrightarrow{j} & (C_0^\infty(F'))' \\ & & \kappa\uparrow & & \uparrow\widetilde{\kappa} \\ 0 & \to & C^\infty(E) & \underset{j}{\to} & (C_0^\infty(E'))' \end{array}$$

where with the same symbol j we denote the canonical inclusions. We call $\widetilde{\kappa}$ the **distributional extension** of κ. More precisely $\widetilde{\kappa}$ is given by $\widetilde{\kappa}(\omega)(\alpha) =< \omega, \kappa^*(\alpha) >$, $\forall\omega \in (C_0^\infty(E'))'$, $\alpha \in C_0^\infty(F')$ where κ^* is the formal adjoint of κ. Furthermore, one has:

$$\mathbf{1}\otimes\widetilde{\kappa} : (C_0^\infty(F\boxtimes E'))' \to (C_0^\infty(F\boxtimes F'))', (\mathbf{1}\otimes\widetilde{\kappa})(\Xi)(f\otimes\alpha) ={}_f\,\Xi(\kappa^*(\alpha))$$

$$\widetilde{\kappa}\otimes\mathbf{1} : (C_0^\infty(E'\boxtimes F))' \to (C_0^\infty(F'\boxtimes F))', (\widetilde{\kappa}\otimes\mathbf{1})(\Xi)(\alpha\otimes f) = \Xi_f(\kappa^*(\alpha)).$$

LEMMA 1.1 - SOLUTIONS OF AFFINE PDEs AND GREEN KERNELS. 1) *Let $E_r \equiv \ker_f \kappa \subset J\mathcal{D}^r(E)$, be an affine PDE identified by a linear differential operator $\kappa : C^\infty(E) \to C^\infty(F)$, of order r, and a section $f \in C^\infty(F)$. Let us denote by $Sol(E_r)$ the set of solutions of E_r. Let $(\widetilde{E}_r)$: $\widetilde{\kappa}(\omega) = \widetilde{f}$ be the distributional equation corresponding to E_r. Then, $Sol(\widetilde{E}_r)$ is related to $Sol(E_r)$ by means of the following exact sequence: $0 \to Sol(E_r) \xrightarrow{j} Sol(\widetilde{E}_r)$, where $j : C^\infty(E) \to (C_0^\infty(E'))'$ is the canonical inclusion.*

2) *Let us denote by $Sol(\overline{E}_r)$ and $Sol(\overline{\widetilde{E}}_r)$ respectively the set of solutions of the following linear equations: $\kappa(e) = 0$, $e \in C^\infty(E)$, and $\widetilde{\kappa}(\omega) = 0$, $\omega \in (C_0^\infty(E'))'$. Then one has: $j(Sol(\overline{E}_r)) \subset Sol(\overline{\widetilde{E}}_r)$.*[4]

3) *Any solution $\omega \in Sol(\widetilde{E}_r)$ can be represented by a distribution $\omega \in (C_0^\infty(E'))'$ written in the following form: $(\heartsuit)$: $\omega = {}_f\mathbb{G} + \omega_0$, where $\omega_0 \in \ker(\widetilde{\kappa})$, and $\mathbb{G}$ is the Green kernel of κ.*

4) *Let us assume that ${}_f\mathbb{G}$ can be identified with a section ${}_fG \in C^\infty(E)$ by means of the canonical embedding $j : C^\infty(E) \to (C_0^\infty(E'))'$. Then, any solution $e \in Sol(E_r)$ can be written in the form: $e = {}_fG \bmod Sol(\overline{E}_r)$.*

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[4] In general $j(Sol(\overline{E}_r))$ is properly contained into $Sol(\overline{\widetilde{E}}_r)$. However, there are some equations, (e.g., elliptic equations), for which $j(Sol(\overline{E}_r))=Sol(\overline{\widetilde{E}}_r)$.

PROOF OF LEMMA 1.1 - 1) If $\kappa(e) = f$, one has also

$$\widetilde{\kappa}(j(e))(\alpha) =< j(e), \kappa^*(\alpha) >= \int_M < \kappa^*(\alpha), e >= \int_M < \kappa(e), \alpha > = \int_M < f, \alpha >= \widetilde{j}(\alpha)$$

$\forall \alpha \in C_0^\infty(F')$. Therefore, we conclude that $\widetilde{\kappa}(j(e)) = \widetilde{f}$.

2) One has:

$$\omega \in Sol(\overline{\overline{E}}_r) \Leftrightarrow \widetilde{\kappa}(\omega)(\alpha) = 0, \forall \alpha \in C_0^\infty(F') \Leftrightarrow \omega(\kappa^*(\alpha)) = 0, \forall \alpha \in C_0^\infty(F').$$

Then, if $e \in Sol(\overline{E}_r)$, and $\omega = j(e)$, one has also, $\forall \alpha \in C_0^\infty(F')$:

$$\omega(\kappa^*(\alpha)) = \int_M < e, \kappa^*(\alpha) >= \int_M < \kappa(e), \alpha >= 0 \Rightarrow j(Sol(\overline{E}_r) \subset Sol(\overline{\overline{E}}_r).$$

Vice versa, if $\omega = j(e) \in j(C^\infty(E))$ and $\omega(\kappa^*(\alpha)) = 0$, $\forall \alpha \in C_0^\infty(F')$, one has also

$$0 = \int_M < \kappa(e), \alpha >\Rightarrow \kappa(e) = 0 \Rightarrow e \in Sol(\overline{E}_r).$$

Therefore, $Sol(\overline{\overline{E}}_r) \bigcap j(C^\infty(E)) \cong Sol(\overline{E}_r)$.

3) We must prove that $\widetilde{\kappa}({}_f\mathbb{G}) = \widetilde{f}$, where $\widetilde{f} = j(f) \in (C_0^\infty(F'))'$. In fact, for any $\alpha \in C_0^\infty(F')$ we have: $\widetilde{\kappa}({}_f\mathbb{G})(\alpha) = {}_f\mathbb{G}(\kappa^*(\alpha)) = (\kappa \otimes \mathbf{1})(\mathbb{G})(f \otimes \alpha) = \mathbb{D}(f \otimes \alpha) = \int_M < f, \alpha >= \widetilde{f}(\alpha)$. Vice versa, let $\omega \in Sol(\widetilde{E}_r)$. Then, one has: $\widetilde{\kappa}(\omega)(\alpha) = \widetilde{f}(\alpha) = \int_M < f, \alpha >= \mathbb{D}(f \otimes \alpha) = \widetilde{\kappa}({}_f\mathbb{G})(\alpha) = {}_f\mathbb{G}(\kappa^*(\alpha))$, $\forall \alpha \in C_0^\infty(F')$. On the other hand $\widetilde{\kappa}(\omega)(\alpha) = \omega(\kappa^*(\alpha))$, $\forall \alpha \in C_0^\infty(F')$. Then, as $\widetilde{E}_r$ is an affine equation, we have $\omega = {}_f\mathbb{G} + \omega_0$, where $\omega_0 \in \ker(\widetilde{\kappa})$, i.e., any distribution $\omega_0 \in (C_0^\infty(E'))'$ such that $\omega_0|_{\mathrm{im}\,(\kappa^*)} = 0$. In other words $\omega_0 \in Sol(\overline{\overline{E}}_r)$.

4) In fact as $j(e) \in Sol(\widetilde{E}_r)$, we can write $j(e) = {}_f\mathbb{G} + \omega_0$, $\forall \omega_0 \in Sol(\overline{\overline{E}}_r)$. On the other hand, if ${}_f\mathbb{G} = j({}_fG)$ and $\omega_0 = j(e_0)$, where $e_0 \in Sol(\overline{E}_r)$, (from point (2)), it follows that $e = {}_fG + e_0$ is a solution of E_r. In fact, one has $\kappa(e) = \kappa({}_fG) + \kappa(e_0) = \kappa({}_fG) = f$. □

LEMMA 1.2 - DETERMINATION OF GREEN KERNELS OF LINEAR DIFFERENTIAL OPERATORS. 1) *Let* $\mathbb{G} \in (C_0^\infty(E' \boxtimes F))'$ *be a Green kernel of a linear differential operator* $\kappa : C^\infty(E) \to C^\infty(F)$ *of order* r. *Then also* $\widetilde{\mathbb{G}} \equiv \mathbb{G} + \omega_0 \otimes \beta$, $\forall \beta \in (C_0^\infty(F))'$, $\forall \omega_0 \in \mathcal{S}ol(\overline{\overline{E}}_r)$ *is a Green kernel of* κ.

2) *A kernel* $\mathbb{G} \in (C_0^\infty(E' \boxtimes F))' \cong (C_0^\infty(E'))' \bigotimes (C_0^\infty(F))'$ *can be identified with a section* $G : M \times M \to J\mathcal{D}^\infty(E')^* \bigotimes \Lambda_n^0(M) \boxtimes J\mathcal{D}^\infty(F)^* \bigotimes \Lambda_n^0(M)$ *such that* $\mathbb{G}(\alpha \otimes f) = \int_{M \times M} < G, D^\infty \alpha \otimes D^\infty f >$. *One has the following local representation:*

$$\mathbb{G}(\alpha \otimes f) = \sum_U \sum_{|\gamma|,|\beta| \leq \infty} \int_{U \times U} g_{U \times U} G_i^{\beta\gamma j}(x, x')(\partial_\gamma . \alpha_j)(x)(\partial_\beta . f^j)(x') d\mu(x) d\mu(x')$$

where $G_i^{\beta\gamma j}$ *are the local components of* G, *and* $g_{U \times U}$ *is a partition of unity subordined to the covering* $\{U \times U\}$ *of* $M \times M$. *Furthermore, if* $\mathbb{G}$ *is identified with a section of* $(E' \boxtimes F)' \to M \times M$ *then* $\mathbb{G}$ *is identified with local functions* G_i^j *on* $M \times M$ *that are called* **Green functions**. *Furthermore, if* G_i^j *are Green functions of the linear differential operator of order* r, $\kappa. : C^\infty(E) \to C^\infty(F)$, *they must satisfy the following local equations:*

$$(1.1) \qquad \sum_{|\gamma| \leq r} \kappa_i^{\gamma j}(x)(\partial_{x,\gamma} . G_u^i)(x, x') = \delta_u^j \delta(x, x')$$

where $\sum_{|\gamma| \leq r} \kappa_i^{\gamma j}(x) \partial_{x,\gamma}$ *is the local representation of* κ. *Taking into account point (1) we get also that the general solution of* (1.1) *can be written*

$$(1.2) \qquad G_i^j(x, x') \equiv \underline{G}_i^j(x, x') + u^j(x) g_i(x')$$

where $\underline{G}_i^j(x, x')$ *is any particular solution of* (1.1), (u^j) *are the local components of any solution of the linear equation* $\sum_{|\gamma| \leq r} \kappa_i^{*\gamma j}(x)(\partial_{x,\gamma} . u^j)(x) = 0$ *and* g_i *are local arbitrary functions on* M.[5]

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[5] In particular, if κ is an hyperbolic differential operator over a global hyperbolic

PROOF OF LEMMA 1.2 - 1) First let us note that $\omega_0 \otimes \beta$ belongs to the same space to which belongs $\mathbb{G}$. In fact, one has the following commutative diagram:

$$\begin{array}{ccc}
(C_0^\infty(E' \boxed{\times} F))' & \cong & (C_0^\infty(E'))' \bigotimes (C_0^\infty(F))' \\
\uparrow & & \uparrow j\otimes 1 \\
C^\infty(E) \bigotimes (C_0^\infty(F))' & = & C^\infty(E) \bigotimes (C_0^\infty(F))' \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}$$

Furthermore, ${}_f(\omega_0 \otimes \beta)(\kappa^*(\alpha)) = 0$, $\omega_0 \in \mathcal{S}ol(\widetilde{\widetilde{E}}_r)$, $\forall \alpha \in C_0^\infty(F')$, $\forall f \in C_0^\infty(F)$. In fact, one has: ${}_f(\omega_0 \otimes \beta)(\kappa^*(\alpha)) =< \omega_0 \otimes \beta, f \otimes \kappa^*(\alpha) >= \beta(f).\omega_0(\kappa^*(\alpha)) = \beta(f).\widetilde{\kappa}(\omega_0)(\alpha) = 0$. Then, we also have: $(\mathbf{1} \otimes \widetilde{\kappa})(\mathbb{G} + \omega_0 \otimes \beta) = (\mathbf{1} \otimes \widetilde{\kappa})(\mathbb{G}) + (\mathbf{1} \otimes \widetilde{\kappa})(\omega_0 \otimes \beta) = \mathbb{D}$. Thus we have proved the theorem.

2) In local coordinates we must have:

$0=$

$$\sum_U \int_{U\times U} g_{U\times U} \sum_{|\omega|,|\epsilon|\le r} [G_u^{\omega\epsilon i}(x,x')(\partial_\epsilon.\kappa^*(\alpha)_i)(x)(\partial_\omega.f^u)(x') - \alpha_i(x) f^i(x')] d\mu(x) d\mu(x').$$

Now, if G_i^j are Green functions, the above expression can be written:

$$0 = \sum_U \int_{U\times U} g_{U\times U} [G_u^i(x,x')\kappa^*(\alpha)_i(x) f^u(x') - \alpha_j(x) f^u(x')\delta_u^j \delta(x,x')] d\mu(x) d\mu(x')$$

As this expression must hold for any f we can write:

$$G_u^i(x,x')\kappa^*(\alpha)_i(x) = \alpha_j(x)\delta_u^j \delta(x,x')\,.$$

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space-time, there exists a unique Green function $G^+(x,x')$, (resp. $G^-(x,x')$), supported in $\mathcal{E}^+(x')$, i.e. the future of x', (resp. $\mathcal{E}^-(x')$, i.e. the past of x'), $\forall x' \in M$. Then, $\widetilde{G}(x,x') \equiv G^+(x,x') - G^-(x,x')$ is called the **propagator** of κ and satisfies the following Cauchy problem:

$$\left\{\begin{array}{l}
\sum_{|\gamma|} \kappa_i^{\gamma j}(x)(\partial x_\gamma.\widetilde{G}_u^i(x,x')=0 \\
\widetilde{G}_u^i(x,x')|_{x'^0=x^0}=0 \\
(\partial x'_0.\widetilde{G}_u^i(x,x')|_{x'^0=x^0}=-\delta(x,x')|_{x'^0=x^0}
\end{array}\right\}$$

where $(x^\alpha)=(x^0,x^1,x^2,x^3)$ is any adapted coordinate system on M, i.e., ∂x_0 is a time-like vector field.

Taking into account that $\kappa^*(\alpha)_i = \sum_{|\gamma|\le r} \kappa_i^{*\gamma j}(\partial_{x,\gamma}.\alpha_j)$, we can also write

$$\sum_{|\gamma|\le r} G_u^j(x,x')\kappa_j^{*\gamma i}(x)(\partial_{x,\gamma}.\alpha_i)(x) = \alpha_j(x)\delta_u^j\delta(x,x') .$$

From the definition of adjoint of κ and taking into account that the above equation holds for any α we get equations (1.1). □

LEMMA 1.3 - *Let κ_1 be a differential operator on M between the vector fiber bundles $E \to M \leftarrow F$, of order $\le r$ and class $C^h, r \le h \le s-r$. Let κ_2 be another differential operator between F' and E'. Define* **Green operator** *of (κ_1,κ_2) any differential operator G of order $\le r-1$ (and class $C^h, h \ge 1$) between $E \bigotimes F'$ and $\Lambda^0_{n-1}(M)$, such that $< \kappa_1(u), v > - < u, \kappa_2(v) >= dG(u \otimes v)$. If $\kappa_2 = \kappa_1{}^*$, we say that G is a* **Green operator** *for κ_1.*[6]

1) *Let κ be a linear differential operator as given in above lemma. Let $A \subset M$ be a compact oriented domain in M. Then, one has*

(1.3)

$$\int_A < \kappa(e), v > - \int_A < e, \kappa^*(v) >= \int_{\partial A} G(e\otimes v),\ \forall e \in C^\infty(E), v \in C^\infty(F').$$

In particular , if $\partial A = \emptyset$, one has: $\int_A < \kappa(e), v >= \int_A < e, \kappa^*(v) >$. Furthermore, if M is a compact manifold formula (1.3) *can be written in*

[6] For example the Green operator of a second order differential operator κ, locally written as follows $\kappa(s)=[a_j^{i\alpha\beta}(\partial x_\alpha \partial x_\beta.s^j)+b_j^{i\alpha}(\partial x_\alpha.s^j)+c_j^i s^j]e_i$, $\forall s\in C^\infty(E)$, where $\{e_i\}$ is a local basis for $C^\infty(E)$, can be locally written in the following way: $G(s\otimes\gamma)= \sum_\alpha(-1)^{\alpha-1}\{a_i^{j\alpha\beta}[\gamma_j(\partial x_\beta.s^i)-s^i(\partial x_\beta.\gamma_j)]+\bar{b}_i^{j\alpha}\ {}^i\gamma_j\}dx^1\wedge\ldots\wedge\widehat{dx^\alpha}\wedge\ldots\wedge dx^n$, with $\bar{b}_i^{j\alpha}\equiv b_i^{j\alpha}-(\partial x_\beta.a_i^{j\alpha\beta})$, $\forall\gamma\in C^\infty(F')$, $\gamma=\gamma_j\epsilon^j\otimes dx^1\wedge\ldots\wedge dx^n$, where $\{\epsilon^j\}$ is a local basis for $C^\infty(F^*)$. In fact, $\kappa^*(\gamma)=[(\partial x_\alpha\partial x_\beta.(a_i^{j\alpha\beta}\gamma_j)-(\partial x_\alpha.(b_i^{j\alpha}\gamma_j))+c_i^j\gamma_j]\epsilon^i\otimes dx^1\wedge\ldots\wedge dx^n$. Furthermore:

$$\begin{aligned}<\kappa(s),\gamma>-<s,\kappa^*(\gamma)>&=[a_i^{j\alpha\beta}(\partial x_\alpha\partial x_\beta.s^i)\gamma_j-(\partial x_\alpha\partial x_\beta(a_i^{j\alpha\beta}\gamma_j))s^i\\ &\quad+b_i^{j\alpha}(\partial x_\alpha.s^i)\gamma_j-(\partial x_\alpha(b_i^{j\alpha}\gamma_j))s^i]dx^1\wedge\ldots\wedge dx^n\\ &=(\partial x_\alpha.\textstyle\sum_\alpha\{a_i^{j\alpha\beta}[\gamma_j(\partial x_\beta.s^i)-s^i(\partial x_\beta.\gamma_j)]+\bar{b}_i^{j\alpha}s^i\gamma_j\})dx^1\wedge\ldots\wedge dx^n.\end{aligned}$$

the following way:

$$(1.4)\qquad \int_M <\kappa(e),v> - \int_M <e,\kappa^*(v)> = \int_{\partial M} G(e\otimes v).$$

The Green operator of a linear differential operator of order r identifies a section $G\in C^\infty\left(J\mathcal{D}^{r-1}(E\otimes F')^*\bigotimes\Lambda^0_{n-1}M\right)$. In particular, if $r=1$, one has $G\in C^\infty\left(E^*\otimes\Lambda^0_{n-1}\ M\bigotimes(F')^*)\right)$.

2) Let $\kappa.: C^\infty(E)\to C^\infty(F)$ be a linear differential operator of first order between vector fiber bundles $\pi: E\to M$, and $\pi': F\to M$ on a differential manifold M of dimension n. Then, one has the following relation between the symbol $\sigma(\kappa)$ of κ and Green operator G: $<\sigma(\kappa)(\lambda\otimes e),v> = \lambda\wedge G(e\otimes v)$, $\forall\lambda\in\Omega^1(M)\equiv C^\infty(\Lambda^0_1M)$.

3) Let $\kappa.: C^\infty(E)\to C^\infty(F)$ be a linear differential operator of order r over a compact manifold M of dimension n. Then the distributional extension $\widetilde{\kappa}:(C^\infty_0(F'))'\to(C^\infty_0(E'))'$ of κ is related to its formal adjoint κ^* by the following formula:

$$(1.5)\qquad \widetilde{\kappa}(\omega)(\alpha)=\omega(\kappa^*(\alpha))+\widetilde{G}|_{\partial M}(\omega)(\alpha),$$

where $\widetilde{G}|_{\partial M}(\omega)\in(C^\infty_0(F'))'$ is the distribution supported on ∂M identified by the ω and the Green operator of κ. Furthermore, if $r=1$ above equation can be written as follows: $\widetilde{\kappa}(\omega)(\alpha)=\omega(\kappa^*(\alpha))+\widetilde{\sigma}_1(\kappa)|_{\partial M}(\omega)(\alpha)$, where $\widetilde{\sigma}_1(\kappa)|_{\partial M}(\omega)$ is the distribution supported on ∂M identified by the symbol $\sigma_1(\kappa)$ of κ.

4) As particular cases we have:

(a) For the Green kernel $\mathbb{G}$ of κ we have, ${}_f\mathbb{G}\in(C^\infty_0(E'))'$, $\forall f\in C^\infty(F)$. Thus we get

$$(1.6)\qquad \widetilde{\kappa}({}_f\mathbb{G})(\alpha)={}_f\mathbb{G}(\kappa^*(\alpha))+\widetilde{G}|_{\partial M}({}_f\mathbb{G})(\alpha),\quad \alpha\in C^\infty_0(F').$$

If $r=1$ one can also write:

$$(1.7)\qquad \widetilde{\kappa}({}_f\mathbb{G})(\alpha)={}_f\mathbb{G}(\kappa^*(\alpha))+\widetilde{\sigma}_1|_{\partial M}({}_f\mathbb{G})(\alpha),\quad \alpha\in C^\infty_0(F').$$

(b) *If the Green kernel $\mathbb{G}$ is such that it can be identified with a section $G \in C^\infty(E \boxtimes F)$, then we can write:*

$$(1.8) \qquad \int_M < \kappa({}_fG), \alpha > = \int_M < {}_fG, \kappa^*(\alpha) > + \int_{\partial M} G({}_fG \otimes \alpha).$$

If $r = 1$ we can also write:

$$(1.9) \qquad \int_M < \kappa({}_fG), \alpha > = \int_M < {}_fG, \kappa^*(\alpha) > + \int_{\partial M} < \sigma_1(\kappa)({}_fG), \alpha > .$$

Furthermore the Green kernel satisfies the following equivalent equations:

(a) $\widetilde{\kappa}({}_f\mathbb{G}) = {}_f\mathbb{D}, \forall f \in C^\infty(F)$

(b) ${}_f\mathbb{G}(\kappa^*(\alpha)) + \widetilde{G}|_{\partial M}({}_f\mathbb{G})(\alpha) = \int_M < f, \alpha >, \ \forall f \in C^\infty(F), \alpha \in C_0^\infty(F')$

(c) *If $\mathbb{G}$ is identified by a section $G \in C^\infty(E \boxtimes F)$ the above equations become:* $\kappa({}_fG) = {}_f\mathbb{D}, \forall f \in C^\infty(F)$

$$\int_M < {}_fG, \kappa^*(\alpha) > + \int_{\partial M} < \widetilde{G}|_{\partial M}({}_fG), \alpha > = \int_M < f, \alpha > .$$

If $r = 1$ we can also write

(a) $\widetilde{\kappa}({}_fG) = {}_f\mathbb{D}, \forall f \in C^\infty(F)$;

(b) ${}_f\mathbb{G}(\kappa^*(\alpha)) + \widetilde{\sigma}_1(\kappa)|_{\partial M}({}_f\mathbb{G})(\alpha) = \int_M < f, \alpha >, \forall f \in C^\infty(F), \alpha \in C_0^\infty(F')$;

(c) *If $\mathbb{G}$ is identified by a section $G \in C^\infty(E \boxtimes F)$ the above equations become:* $\kappa({}_fG) = {}_f\mathbb{D}, \forall f \in C^\infty(F)$;

$$\int_M < {}_fG, \kappa^*(\alpha) > + \int_{\partial M} < \sigma_1(\kappa)({}_fG), \alpha > = \int_M < f, \alpha > .$$

PROOF OF LEMMA 1.3 - 1) In fact, one has the canonical isomorphism:

$$Hom\left(J\mathcal{D}^{r-1}(E \bigotimes F'); \Lambda^0_{n-1}M\right) \cong J\mathcal{D}^{r-1}(E \bigotimes F')^* \bigotimes \Lambda^0_{n-1}M.$$

So, for $r = 1$ we get

$Hom\,(E \bigotimes F'; \Lambda^0_{n-1}M) \cong E^* \bigotimes (F')^* \bigotimes \Lambda^0_{n-1}M \cong E^* \bigotimes \Lambda^0_{n-1}M \bigotimes (F')^*$.

2) If $\kappa_1. : C^\infty(E) \to C^\infty(F), \kappa_2. : C^\infty(F') \to C^\infty(E')$ are two differential operators of the first order, such that the symbols $\sigma_\lambda(\kappa_1)$ and $\sigma_\lambda(\kappa_2)$, for any differential 1-form $\lambda \in \Omega^1(M)$ are skew-symmetric: $\sigma_\lambda{}^*(\kappa_1) + \sigma_\lambda(\kappa_2) = 0$, then the operator $\gamma(\kappa_1, \kappa_2) : C^\infty(E \bigotimes F') \to \Omega^n(M)$, given by $\gamma(\kappa_1,\kappa_2)(u \otimes v) = < \kappa_1(u), v > - < u, \kappa_2(v) >$ is a differential operator of the first order. The symbol of this operator determines the morphism $\omega : C^\infty(E \bigotimes F') \to \Omega^{n-1}(M)$ such that $\sigma_\lambda(\gamma(\kappa_1,\kappa_2))(u\otimes v) = \lambda \wedge \omega(u \otimes v)$. Thus, the operators $\gamma(\kappa_1,\kappa_2)$ and ω have the same symbol, and, hence, differ by an operator of zero order. By substituting, if necessary, the operator κ_2 with $\kappa_2 + \kappa_2{}'$, where $\kappa_2{}' \in Hom(C^\infty(F'), C^\infty(E'))$, we get for each operator $\kappa_1 \in Diff_1(C^\infty(E), C^\infty(F))$, the operator $\kappa_1{}^* \in Diff_1(C^\infty(F'), C^\infty(E'))$, that is the adjoint of the operator κ_1 for which $\gamma(\kappa_1, \kappa_1{}^*) = d\omega$. Thus $\omega = G$.

3) One has the following Green formula: $< \kappa(e), \alpha > - < e, \kappa^*(\alpha) >= dG(e \otimes \alpha)$, $\forall e \in C^\infty(E)$, $\alpha \in C^\infty_0(F')$. Then, taking into account the canonical immersion $j : C^\infty(E) \to (C^\infty_0(E'))'$, we get:

$$\begin{aligned}\widetilde{\kappa}(j(e),\alpha) &= \int_M < \kappa(e), \alpha >= \int_M < e, \kappa^*(\alpha) > + \int_M dG(e\otimes\alpha)\\ &= j(e)(\kappa^*(\alpha)) + \int_{\partial M} G(e\otimes\alpha) = j(e)(\kappa^*(\alpha)) + \widetilde{G}|_{\partial M}(\omega)(\alpha)\end{aligned}$$

where $\widetilde{G}|_{\partial M}(e) \in (C^\infty_0(F'))'$ is the distribution with support on ∂M, identified by G, i.e., $\widetilde{G}|_{\partial M}(e)(\alpha) = \int_{\partial M} G(e\otimes\alpha)$. Then, we get: $\widetilde{\kappa}(j(e))(\alpha) = j(e)(\kappa^*(\alpha)) + \widetilde{G}|_{\partial M}(e)(\alpha)$. On the other hand, as $C^\infty(E)$ is dense in $(C^\infty_0(E'))'$, for any $\omega \in (C^\infty_0(E'))'$ we can write

$$\begin{aligned}\widetilde{\kappa}(\omega)(\alpha) = \widetilde{\kappa}(\sum_s j(e_s))(\alpha) &= \sum_s \int_M < \kappa(e_s), \alpha >\\ &= \sum_s \int_M < e_s, \kappa^*(\alpha) > + \sum_s \int_M dG(e_s\otimes\alpha)\\ &= \sum_s j(e_s)(\kappa^*(\alpha)) + \sum_s \int_{\partial M} G(e_s\otimes\alpha).\end{aligned}$$

Thus, we have $\omega(\kappa^*(\alpha))+\widetilde{G}|_{\partial M}(\omega)(\alpha) = j(\sum_s e_s)(\kappa^*(\alpha))+\sum_s \widetilde{G}|_{\partial M}(e_s)(\alpha)$.
4) The proof follows from above results. □

Now, from our assumptions on R and using results given in ref.[5], we get that we can associate to R a distribution $F[R] \in (C_0^\infty(E'))'$, that is a distributive solution of E_r iff R is a multivalued solution of $E_r \subset J\mathcal{D}^r(E) \subset J^r_n(E)$. Moreover, from above lemmas, we get that the relation between the Green kernel of κ and $F[R]$ is the following: (♣): $F[R] = {}_f\mathbb{G} + \omega_0$, where $\omega_0 \in Sol(\overline{\widetilde{E}}_r)$, with $\overline{\widetilde{E}}_r$ the linear equation associated to $\widetilde{E}_r$ [7]. Furthermore, if $\partial R = R_1 \bigcup R_2$, with $R_1 \in [R_2]_{E_r} \in \Omega^{E_r}_{n-1}$, then we say that $F[R]$ satisfies the boundary conditions $R_1 \bigcup R_2 \subset E_r$. If $R' \in \Omega_c(R_1, R_2)$ is another solution of E_r that satisfies the same boundary conditions than R, then $F[R'] = {}_f\mathbb{G} + \omega'_0$, where $\omega'_0 \in Sol(\overline{\widetilde{E}}_r)$. Let us denote by $[F[R]]$ the element belonging to the space $[Sol(\widetilde{E}_r)]$ corresponding to $F[R]$ by means of the following commutative diagram:

$$\begin{array}{ccccc}
0 & \rightarrow & Sol(\widetilde{E}_r) & \rightarrow & (C_0^\infty(E'))' \\
 & & \downarrow & & \downarrow \bar{\pi} \\
0 & \rightarrow & [Sol(\widetilde{E}_r)] & \rightarrow & (C_0^\infty(E'))'/Sol(\overline{\widetilde{E}}_r) \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

Then, the equivalence class $[F[R]]$ of $F[R]$ in the space $[Sol(\widetilde{E}_r)]$ is identified by a unique distribution: ${}_f\mathbb{G}$. This proves that ${}_f\mathbb{G}$ is an invariant of the classic-limit statistical set $\Omega_c(R_1, R_2)$. □

COROLLARY 1.1 - GENERALIZED GREEN KERNELS AND SOLUTIONS OF AFFINE PDEs. *Any solution R of an affine PDE $E_r \equiv \ker_f\kappa \subset J\mathcal{D}^r(E) \subset J^r_n(E)$ that satisfies the boundary conditions $\partial R = R_1 \bigcup R_2$, where R_1 and R_2 are such that:* (i) $\pi_{r,0}|_R : R \rightarrow E$ *is a proper application;* (ii) $\omega_1{}^{(r)}(R) = 0$; *identifies a distributive kernel:* (♠): $\mathbb{F}[R] = F[R] \otimes \widetilde{f} \in (C_0^\infty(E' \boxtimes F'))'$, *given by* $\mathbb{F}[R](\alpha \otimes \phi) = F[R](\alpha) \int_M <$

[7] $\widetilde{E}_r$ is the distributional equation associated to E_r.

$\phi, f >$. *Define* $\mathbb{F}[R]$ *the* **generalized Green kernel** *of the singular solution* $R \subset E_r$ *that satisfies the boundary conditions* $\partial R = R_1 \bigcup R_2$. *The relation between* $\mathbb{F}[R]$ *and the Green kernel* $\mathbb{G}$ *of* κ *is given by the following formula:* $\mathbb{F}[R] = {}_f\mathbb{G}[R] \otimes \widetilde{f} \bmod Sol(\widetilde{\overline{E}}_r) \otimes \widetilde{f}$.

PROOF. It is a direct consequence of Theorem 1.1. □

THEOREM 1.2 - GREEN KERNELS AND PROPAGATORS FOR NONLINEAR PDEs. 1) *Let* $E_r = \ker_\chi \kappa \subset J\mathcal{D}^r(W)$ *be a PDE given as a kernel of a differential operator of order* r*:* $\kappa : J\mathcal{D}^r(W) \to K$, *with respect to a section* C^∞*:* $\chi : M \to K$ *of the fiber bundle* $\underline{\pi} : K \to M$. *Then, for any section* $s \in C^\infty(W)$, *solution of* E_r, *and* $j[\chi] = \partial\widetilde{\chi}$, *where* $\widetilde{\chi}$ *is a deformation of* χ, *we can associate to* E_r *an affine equation* $E_r[s] = \ker_{j[\chi]} J[s] \subset J\mathcal{D}^r(s^*vTW)$, *where* $J[s] : C^\infty(s^*vTW) \to C^\infty(\chi^*vTK)$ *is the linearized of* κ *at the section* s. *Define* $E_r[s]$ **Jacobi equation** *of* E_r *at the solution* s. *Furthermore, define* **Green kernels**, *(resp.* **propagator***),* $G[s]$ *of* E_r, *at the section* s, *the Green kernels, (resp. propagator) of* $E_r[s]$.

2) *Assume that* $E_r = \ker_f \kappa \subset J\mathcal{D}^r(W)$ *is a PDE as given in the above point* (1). *Then, the Green kernels (resp. propagator),* $G[s]$ *identifies an integral manifold* (**integral bordism**) $R \in \Omega_c(R_1, R_2)[s]$, *i.e., belonging to the classic limit of a statistical set of* $E_r[s]$, *if* $G[s]$ *satisfies to the boundary condition* $\partial R = R_1 \bigcup R_2 \subset E_r[s]$.

3) *Let* R_1 *and* R_2 *be two admissible integral compact closed manifolds of dimension* $(n-1)$ *contained into* E_r *such that the following conditions are satisfied:*[8] (i) $R_1 \in [R_2]_{E_r} \in \Omega^{E_r}_{n-1}$; (ii) *There exists a vector fiber bundle neighborhood* $\overline{E}_r[s] \subset E_r$ *such that* $R_1, R_2 \subset E_r[s]$. *Then the equivalence class* $[G[s]]$, *identified by the Green kernels, (resp. propagator),* $G[s]$, *is invariant for* $\Omega_c(R_1, R_2)[s]$, *i.e., the set of solutions* V *of* E_r *with* $\partial V = R_1 \bigcup R_2$ *and such that* $V \subset E_r[s]$.

4) *Furthermore, if* E_r *is a formally integrable PDE and* $\Omega_c(R_1, R_2)$ *is restricted to the regular solutions of* $E_r \subset J^r_n(W)$, *then* $G[s]$ *is an invariant of* $\Omega_c(R_1, R_2)$.

[8] The admissibility is in the sense of definition given in ref.[12].

PROOF. For the full proof of 1) and 2) see ref.[1]. Here, let us emphasize only that $E_r[s]$ is an affine equation, that is an affine bundle over M, with associated vector bundle the linearized equation of E_r at s: $(D^r s)^* vTE_r \subset (D^r s)^* vTJ\mathcal{D}^r(W) \cong J\mathcal{D}^r(s^* vTW)$. Therefore, with respect to a solution of $E_r[s]$, one has the identification of $E_r[s]$ with $(D^r s)^* vTE_r \equiv \overline{E}_r[s]$, hence one has the identification of $E_r[s]$ with a submanifold of E_r.

3) It is a direct consequence of above two points and taking into account Lemma 1.3.

4) In fact, one has the following short exact sequence: $0 \to \Omega_c(R_1, R_2)[s] \to \Omega_c(R_1, R_2)$. Moreover, if E_r is formally integrable, taking into account of Theorem 4.5 in ref.[12] that relates bordism to formal integrability, we get also the following short exact sequence: $\Omega_c(R_1, R_2) \to \Omega_c(R_1, R_2)[s] \to 0$. Hence, from the above point, it follows that we can characterize $\Omega_c(R_1, R_2)$ also by means of the Green kernel, (resp. propagator), $G[s]$. □

2 - GENERALIZED KLEIN-GORDON EQUATION

In this and in the following sections we shall apply the results of section 1 to obtain the canonical quantizations of field equations that describe particle physics. In particular, in this section we shall consider scalar fields. In order to include in our formulation also scalar massive neutrinos we will introduce a generalized Klein-Gordon equation on an hyperbolic space-time and, after studied its geometrical structure, we will obtain the canonical quantization of the equation following the general geometric method of quantization of PDEs introduced by us in [5-7]. Finally we prove that our geometric approach to the canonical quantization of such a generalized Klein-Gordon equation preserves the microscopic causality even if particles are massive neutrinos.

Let M be a space-time as considered in the Appendix A1. The **generalized Klein-Gordon** equation is the submanifold $(GKG)_\chi \subset J\mathcal{D}^2(E^c)$, $\pi : E^c \equiv M \times \mathbf{C} \to M$, obtained as the kernel of the following fiber bundle morphism: $\mathcal{K}_\chi \equiv (\Box + \chi) : J\mathcal{D}^2(E^c) \to E^c$, where $\Box$ is the d'Alembertian for scalar fields with respect to the metric g on M, $\chi \equiv \xi R + \bar{\chi}$, with

$\xi \in \mathbf{R}$, R is the Ricci scalar curvature and $\bar{\chi} \in \mathbf{R}$, (square of **mass**).[9] In adapted coordinates $(GKG)_\chi$ can be locally written as follows:

$$(2.1) \qquad \{F \equiv g^{\alpha\beta} z_{\alpha\beta} - \lceil^{\alpha}_{\beta\gamma} g^{\gamma\beta} z_\alpha + \chi z = 0\}$$

where $(x^\alpha, z, z_\alpha, z_{\alpha\beta})$ are coordinates on $J\mathcal{D}^2(E^c)$ induced by fibered coordinates (x^α, z) on $\pi : E^c \to M$. Furthermore, $\lceil^{\alpha}_{\beta\gamma}: U \subset M \to \mathbf{R}$ denote the connection coefficients induced by the metric g. Note that equation (2.1) is equivalent to the product two times of a same equation: $(GKG)_\chi \cong (GKG)^{\mathbf{R}}_\chi \times (GKG)^{\mathbf{R}}_\chi \subset J\mathcal{D}^2(E) \times J\mathcal{D}^2(E) \cong J\mathcal{D}^2(E^c)$, where $E = M \times \mathbf{R} \to M$, ($E^c$ is just the complexification of E: $E^c = \mathbf{C} \otimes_{\mathbf{R}} E$), and $(GKG)^{\mathbf{R}}_\chi = \ker(\mathcal{K}^{\mathbf{R}}_\chi) \subset J\mathcal{D}^2(E)$, with $\mathcal{K}^{\mathbf{R}}_\chi \equiv (\Box + \chi) : J\mathcal{D}^2(E) \to E$. Thus in order to discuss the equation $(GKG)_\chi$ it is enough to consider $(GKG)^{\mathbf{R}}_\chi \subset J\mathcal{D}^2(E)$, that, in real coordinates $(x^\alpha, y, y_\alpha, y_{\alpha\beta})$ on $J\mathcal{D}^2(E)$, looks like $F^{\mathbf{R}} \equiv g^{\alpha\beta} y_{\alpha\beta} - \lceil^{\alpha}_{\beta\gamma} g^{\gamma\beta} y_\alpha + \chi y = 0$.

REMARK 2.1 - As $\mathcal{K}^{\mathbf{R}}_\chi$ is an epimorhism of constant rank 5, it follows that $(GKG)^{\mathbf{R}}_\chi$ is a vector subbundle of $J\mathcal{D}^2(E)$ of dimension $19 - 1 = 18$.

THEOREM 2.1 - *The Cartan distribution* $\mathbb{E}_2$ *of* $J\mathcal{D}^2(E) \cong M \times \mathbf{R} \times \mathbf{R}^4 \times \mathbf{R}^{10} \cong M \times \mathbf{R}^{15}$, *is a distribution of dimension* 14*:* $\mathbb{E}_2 \subset TJ\mathcal{D}^2(E)$*. The Cartan distribution of* $(GKG)^{\mathbf{R}}_\chi$ *is a sub-distribution* $\mathbb{E}_2(GKG)^{\mathbf{R}}_\chi$ *of* $\mathbb{E}_2$ *of dimension* 13 *generated by the vector fields* $\zeta = X^\alpha \left(\partial x_\alpha + y_\alpha \partial y + y_{\alpha\beta} \partial y^\beta\right) + Y_{\alpha\beta} \partial y^{\alpha\beta}$ *on* $J\mathcal{D}^2(E)$ *such that:* $X^\omega \left[g^{\alpha\beta}_{,\omega} y_{\alpha\beta} - (\lceil^{\alpha}_{\beta\gamma,\omega} g^{\gamma\beta} + \lceil^{\alpha}_{\beta\gamma} g^{\gamma\beta}_{,\omega}) y_\alpha\right] + \chi(y_\alpha X^\alpha) - \lceil^{\alpha}_{\beta\gamma} g^{\gamma\beta}(y_{\alpha\delta} X^\delta) + g^{\alpha\beta} Y_{\alpha\beta} = 0$*. Then the solutions of* $(GKG)^{\mathbf{R}}_\chi$ *are* 4*-dimensional integral manifolds of* $\mathbb{E}_2(GKG)^{\mathbf{R}}_\chi$ *that, except for a subset of dimension lower than* 4*, are diffeomorphic projected on* 4*-dimensional*

[9] The numerical factor ξ has two values of particular interest: the so-called **minimally complet case**, ξ=0, and the **conformally complete case**, $\xi=\frac{1}{6}$. In this latter case, if m=0 the field equation $[\Box + \frac{R}{6}]\phi_{\pi_0}=0$ for free pion π_0^+ is conformal invariant. Note that the parameter χ can be, in general, considered a costant-coupling of self-interaction. In the particular case of bradions and luxons, is $\bar{\chi} \geq 0$. In the case of massive neutrinos, instead, one has $\bar{\chi}<0$. We call the numerical factor χ the **geometric mass** of the field. We can have $\chi \gtreqless 0$.

submanifolds of E. *As each* $(GKG)^{\mathbf{R}}_{\chi}$ *is an involutive formally integrable, completely integrable PDE, it follows that in the neighbourhood of each point of* $(GKG)^{\mathbf{R}}_{\chi}$ *we can build a regular solution, i.e., a local 4-dimensional integral submanifold, diffeomorphic to an open set of* M, *by means of the projection* $\pi_2 : J\mathcal{D}^2(E) \to M$.

PROOF. The proof is directly obtained by appling general methods of the geometric theory of PDEs [5,8,12]. □

THEOREM 2.2 - (CAUCHY PROBLEM FOR $(GKG)^{\mathbf{R}}_{\chi}$ WITH THE METHOD OF CHARACTERISTICS). 1) *For the equation* $(GKG)^{\mathbf{R}}_{\chi}$ *we can solve the Cauchy problem by means of characteristics.*

2) *Let* ζ *be a vector field on* $(GKG)^{\mathbf{R}}_{\chi}$ *that represents an infinitesimal symmetry of this equation. If* $N \subset (GKG)^{\mathbf{R}}_{\chi}$ *is a Cauchy hypersurface trasnversal to* ζ, *then* $Y \equiv \bigcup_{t\in J} \phi_t(N)$, $\partial\phi = \zeta$, *is a solution of* $(GKG)^{\mathbf{R}}_{\chi}$, *for a suitable neighborhood* J *of* $0 \in \mathbf{R}$.

PROOF. 1) Let us rewrite the equation $(GKG)^{\mathbf{R}}_{\chi}$ in orthogonal coordinates: $F \equiv u_{00} - u_{xx} - u_{yy} - u_{zz}\chi u = 0$. Therefore, $\dim(GKG)^{\mathbf{R}}_{\chi} = \dim J\mathcal{D}^2(E) - 1 = 19 - 1 = 18$. A characteristic vector field of $(GKG)^{\mathbf{R}}_{\chi}$ is a vector field

$$\zeta = X^{\alpha}\left(\partial x_{\alpha} + u_{\alpha}\partial u + u_{\alpha\beta}\partial u^{\beta} + u_{\alpha\beta\gamma}\partial u^{\beta\gamma}\right)$$

where X^{α} are functions defined by the following equations:

$$\left\{\begin{array}{l} Y_{00}-Y_{11}-Y_{22}-Y_{33}=0 \\ 0=\zeta\rfloor\delta(\theta)\equiv\zeta\rfloor\frac{1}{2}Y_{\alpha\beta}\left(dx^{\alpha}\otimes dx^{\beta}+dx^{\beta}\otimes dx^{\alpha}\right) \end{array}\right\}$$

and the other coordinates functions in the expression of ζ are constrained to satisfy the first prolongation of $(GKG)^{\mathbf{R}}_{\chi}$:

$$((GKG)^{\mathbf{R}}_{\chi})_{+1} : \left\{\begin{array}{l} u_{00\alpha}-u_{11\alpha}-u_{22\alpha}-u_{33\alpha}+\chi u_{\alpha}=0 \\ u_{00}-u_{11}-u_{22}-u_{33}+\chi u=0 \end{array}\right\}.$$

Then we can see that a characteristic strip can be the following: $X^1 = X^2 = X^3 = 0$, $X^0 \neq 0$. In other words:

(2.2)

$$\zeta=X^{0}\left[\partial x_{0}+u_{0}\partial u+(u_{11}+u_{22}+u_{33}-\chi u)\partial u^{0}+u_{0k}\partial u^{k}+(u_{11\gamma}+u_{22\gamma}+u_{33\gamma}-\chi u_{\gamma})\partial u^{0\gamma}+ u_{0\chi\gamma}\partial u^{\chi\gamma}\right].$$

One can also directly see that ζ is tangent to $(GKG)^{\mathbf{R}}_{\chi}$ and it belongs to the Cartan distribution!! Furthermore, we can see also that ζ can be considered the characteristic distribution for the following subequation $(E_2) \subset (GKG)^{\mathbf{R}}_{\chi}$:

$$(E_2) \subset (GKG)^{\mathbf{R}}_{\chi} : \left\{ \begin{array}{l} u_{00}-(u_{11}+u_{22}+u_{33})+\chi u=0 \\ u_{11}+u_{22}+u_{33}=0 \\ u_{\alpha\beta}=0, \quad \alpha\neq\beta. \end{array} \right\}.$$

For any Cauchy data N of (E_2), transversal to ζ, given by (2.2) we can generate a 4-dimensional integral manifold of (E_2) that is contained into $(GKG)^{\mathbf{R}}_{\chi}$, hence it is a solution of $(GKG)^{\mathbf{R}}_{\chi}$. In particular if $N = D^2s(N_0)$, where $N_0 \subset M$ is a space-like submanifold of M, and s is a solution of $(GKG)^{\mathbf{R}}_{\chi}$, then $Y \equiv \bigcup_{t\in\mathbf{R}} \phi_t(N)$ is a regular solution of $(GKG)^{\mathbf{R}}_{\chi} \subset J\mathcal{D}^2(E)$, where ϕ is the flow generated by ζ on $(GKG)^{\mathbf{R}}_{\chi}$.
2) This a direct application of a general property of PDEs. (See e.g. refs.[5,8,12].) Furthermore, by using the same calculations given in ref.[4] to obtain the infinitesimal symmetry algebra for the Klein-Gordon equation, we can see that also for the generalized Klein-Gordon equation the infinitesimal symmetry algebra $\mathfrak{s}((GKG)^{\mathbf{R}}_{\chi})$ of $(GKG)^{\mathbf{R}}_{\chi}$ is generated by the second order holonomic prolongation of the following vector fields $\breve{\zeta} : X^{\alpha}\partial x_{\alpha} + fy\partial y : E|_U \to T(E|_U)$, where X^{α} and f are local numerical functions on M solutions of the following linear PDE:

$$\left\{ \begin{array}{l} \Box f-\frac{\chi}{\mathrm{tr}\,(g)}\mathrm{tr}\,(\mathcal{L}_{\breve{\zeta}}g)=0 \\ -(\partial x_{\alpha}\partial x_{\beta}.X^{\omega})g^{\alpha\beta}-(\partial x_{\alpha}.X^{\omega})h^{\alpha}-X^{\alpha}(\partial x_{\alpha}.h^{\omega})+2fh^{\omega}+2(\partial x_{\sigma}.f)g^{\omega\sigma}+\frac{h^{\omega}\chi}{\mathrm{tr}\,(g)} \\ \mathrm{tr}\,(\mathcal{L}_{\breve{\zeta}}g)=0 \\ \breve{\zeta}\equiv X^{\alpha}\partial x_{\alpha}:U\subset M\to TM, \quad h^{\omega}\equiv\Gamma^{\omega}_{\beta\gamma}g^{\gamma\beta}:U\subset M\to\mathbf{R} \end{array} \right\}.$$

□

By using the geometric approach to quantize PDEs, formulated by A.Prástaro in refs.[5-8], we can prove the following important theorem.

THEOREM 2.3 - (CANONICAL QUANTIZATION OF $(GKG)_\chi$). *Let us assume that the space-time is a globally hyperbolic 4-dimensional manifold. 1) The PDE $(GKG)^{\mathbf{R}}_\chi$ admits the following canonical quantization for real scalar fields:* (♠♠): $[\hat{s}(x), \hat{s}(x')] = i\hbar\widetilde{G}(x;x'|\chi)\ \mathbf{1}$, *where* $\mathbf{1} = id_{\mathcal{H}}$, *for some suitable locally convex vector space* $\mathcal{H}$. *The propagator* $\widetilde{G}$ *is the Green kernel solution of the following Cauchy problem:*[10]

$$(2.3)\qquad \left\{\begin{array}{l} (\Box_{x'}+\chi)\widetilde{G}(x;x'|\chi)=0 \\ \widetilde{G}(x;x'|\chi)_{x^0=x'^0}=0 \\ \left(\partial x'_0.\widetilde{G}\right)(x;x'|\chi)|_{x^0=x'^0}=-\delta(x;x')|_{x^0=x'^0} \end{array}\right\}.$$

One has the following equal time commutator:

$$(2.4)\qquad \left\{\begin{array}{l} [\hat{s}(x),\hat{s}(x')]|_{x^0=x'^0}=0 \\ [\hat{s}(x),\dot{\hat{s}}(x')]|_{x^0=x'^0}=-i\hbar\delta(x;x')|_{x^0=x'^0}\ \mathbf{1} \end{array}\right\}.$$

2) The PDE $(GKG)_\chi$ admits the following canonical quantization of the complex scalar fields $s=s_1+is_2$:

$$[\hat{s}_i(x),\hat{s}_j(x')]=i\hbar\delta_{ij}\widetilde{G}(x;x'|\chi)\ \mathbf{1},\ i,j=1,2;\ \Rightarrow [\hat{s}_i(x),\dot{\hat{s}}_j(x')]=-i\hbar\delta_{ij}\delta(x;x')|_{s^0=x'^0}\ \mathbf{1},\ i,j=1,2.$$

Furthermore, we get also for the full complex field $s=s_1+is_2$ *and its c.c.* $\bar{s}=s_1-is_2$ *the following commutation relations:*

$$\left\{\begin{array}{l} [\hat{s}(x),\hat{s}(x')]=[\widehat{\bar{s}}(x),\widehat{\bar{s}}(x')]=0 \\ [\hat{s}(x),\widehat{\bar{s}}(x')]=2i\hbar\widetilde{G}(x;x'|\chi)\ \mathbf{1} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} [\hat{s}(x),\dot{\hat{s}}(x')]=[\widehat{\bar{s}}(x),\dot{\widehat{\bar{s}}}(x')]=0 \\ [\hat{s}(x),\dot{\widehat{\bar{s}}}(x')]|_{x^0=x'^0}=-2i\hbar\delta(x;x')|_{x^0=x'^0}\ \mathbf{1} \end{array}\right\}.$$

3) The microscopic causality is conserved also for scalar massive neutrinos (takions) having a geometric mass $\chi \equiv \xi R + \bar{\chi} < 0$.

PROOF. 1) The set of physical observables defined on the classic limit $\Omega((GKG)^{\mathbf{R}}_\chi)_c$ of the quantum situs $\Omega(GKG)^{\mathbf{R}}_\chi$ of $(GKG)^{\mathbf{R}}_\chi$ has a natural

[10] Note that the second condition in (2.3) implies that we exlcude instantaneous propagations, so to conserve the usual microscopic causality. (See also Appendix A1.)

structure of Lie algebra. More precisely a function $f : E \to \mathbf{R}$ identifies a physical observable characterized by a function $A : C^\infty(E) \to C^\infty(M, \mathbf{R}) \to \mathbf{R}$, where the last composition is obtained by means of some measure on M. To A we can associate a current $jA : C^\infty(E) \to C^\infty(E)$, $jA(s) = vd \circ s$, where vd is the vertical differential. If $B : C^\infty(E) \to \mathbf{R}$ is another physical observable, we can consider the corresponding current jB calculated in correpondence of a solution of $(GKG)^{\mathbf{R}}$ for any vector field $\nu \in T_s C^\infty(E)$ belonging to the set of solutions of the Jacobi equation of $(GKG)^{\mathbf{R}}_\chi$: $J[s].\nu = -jA(s)$. Then we define the following bracket

$$(A,B)(s)=\widetilde{G}(jA(s)\otimes jB(s)\otimes\eta)=G^+(jA(s)\otimes jB(s)\otimes\eta)-G^-(jA(s)\otimes jB(s)\otimes\eta) \tag{2.5}$$

where $G^\pm \in C^\infty_0(E\boxtimes E)')'$ are the two Green kernels of $J[s]$ such that ${}_\phi G^+$ has support in the future of $supp(\phi)$ and ${}_\phi G^-$ has support in the past of $supp(\phi)$. So $(J[s] \otimes 1)G^\pm = (1 \otimes J[s])G^\pm = \mathbb{D}$, where $\mathbb{D}$ is the Dirac kernel of $J[s]$. [11] One can see that above bracket (2.5) satisfies the conditions for a Lie algebra, that we denote by $\mathcal{P}(GKG)^{\mathbf{R}}_\chi$. Then, the canonical quantization is obtained by means of the following bracket: $[\hat{A}, \hat{B}](s) = i\hbar\widetilde{G}(jA(s)\otimes jB(s)\otimes\eta)$ $\mathbf{1}$ with η the canonical volume form on M and $\mathbf{1} = id_{\mathcal{H}(s)}$, with $\mathcal{H}(s)$ a suitable locally convex topological vector space associated to the section s. The corresponding expectation value for

[11] The Green kernel $G(x;x'|\chi)$ of the generalized Klein-Gordon equation is a solution of the following equation $(\square_{x'}+\chi)G(x;x'|\chi)=\delta(x,x')$. In a geodesically convex domain, there exist two fundamental solutions G^+ and G^- which vanish outside the future cone $\mathcal{E}^+_x$ and outside the past cone $\mathcal{E}^-_x$ respectively. Then, the propagator is given by $\widetilde{G}(x;x'|\chi)=G^+(x';x|\chi)-G^-(x';x|\chi)$. $\widetilde{G}(x;x'|\chi)$, $G^+(x;x'|\chi)$, $G^-(x;x'|\chi)$ are real and $G^+(x;x'|\chi)=G^-(x';x|\chi)$. It follows that $\widetilde{G}(x;x'|\chi)=-\widetilde{G}(x';x|\chi)$. The solution of (2.3), in the Minkowsky space-time, is the following for $\chi\neq 0$, $\widetilde{G}(x;x'|\chi)=\frac{1}{4\pi r}(\partial r.J_0(\sigma(\chi)\sqrt{|\chi|(t^2-r^2)}))$ $H(t+r)-\frac{1}{4\pi r}(\partial r.J_0(\sigma(\chi)\sqrt{\chi(t^2-r^2)}))H(r-t)$ and for $\chi=0$, $\widetilde{G}(x;x'|\chi)=-\frac{1}{4\pi r}[\delta(r-t)-\delta(r+t)]$, with $r\equiv|\mathbf{x}'-\mathbf{x}|$, $t\equiv x'^0-x^0$, J_0 the regular Bessel function and $H(\xi)$ is the Heaviside function. (Here we use the notation $(x^\alpha)=(x^0,\mathbf{x})=(x^0,x^1,x^2,x^3)$.) Furthermore, $\sigma(\chi)=+1$ if $\chi>0$ and $\sigma(\chi)=-1$ if $\chi<0$.

any $\phi \in \mathcal{H}(s)$, $\phi' \in \mathcal{H}(s)$ is as follows: $< \phi'|[\hat{A}, \hat{B}](s)|\phi >= i\hbar\widetilde{G}(jA(s) \otimes jB(s) \otimes \eta) < \phi'|\phi >$.[12] In particular, for observables identified by means of scalar fields, and the measure on M taken as the Dirac measure at some event $p \in M$, where M is splited in time and space with respect to some relativistic frame, we get a quantum algebra that interpretates the canonical quantization of $(GKG)^{\mathbf{R}}_{\chi}$. The solution of (2.3) can be finded similarly to wath made for the usual Klein-Gordon equation. Furthermore, the derivative with respect to x'^0 of (♠♠), taking $x'^0 = x^0$, and considering equations (2.3), gives us the following commutator $[\hat{s}(x), \dot{\hat{s}}(x)]|_{x^0=x'^0} = i\hbar\left(\partial x'_0.\widetilde{G}(x; x'|\chi)\right)|_{x^0=x'^0}\ \mathbf{1} = -i\hbar\delta(x; x')|_{x^0=x'^0}\ \mathbf{1}$.

2) It is a direct consequence of the fact that $(GKG)_{\chi} \cong (GKG)^{\mathbf{R}}_{\chi} \times (GKG)^{\mathbf{R}}_{\chi}$.

3) Even if the geometric mass χ is negative, the generalized Klein-Gordon operator $\square + \chi$ remains of hyperbolic type, so it admits a propagator $\widetilde{G}(x, x'|\chi)$ with support in $\mathcal{E}^-(x') \cup \mathcal{E}^+(x')$, $\forall x' \in M$. Therefore, the quantum commutator $[\hat{s}(x), \hat{s}(x')] = i\hbar\widetilde{G}(x, x'|\chi)$ respects the microscopic causality. □

THEOREM 2.4 - (PROPAGATOR AND TUNNEL EFFECTS IN $(GKG)^{\mathbf{R}}_{\chi}$). 1) *The integral bordism group* $\Omega_3^{(GKG)^{\mathbf{R}}_{\chi}}$ *of the generalized Klein-Gordon equation is trivial:* $\Omega_3^{(GKG)^{\mathbf{R}}_{\chi}} = 0$. *Therefore are admmissible tunnel effects, i.e., solutions with change of sectional topology.*

2) *If the scalar field is in interaction with some other field such that it produces a current* $f \in C^{\infty}(E)$, *i.e., the PDE considered is the following affine PDE*

$$ {}_f(GKG)^{\mathbf{R}}_{\chi} \subset J\mathcal{D}^2(E): \quad \{g^{\alpha\beta}y_{\alpha\beta} - \Gamma^{\alpha}_{\beta\gamma}g^{\gamma\beta}y_{\alpha} + \chi y = f\}. $$

---

[12] The choice of the quantum commutator allows us also to recognize quantum spectral measures $E{:}(\Omega((GKG)^{\mathbf{R}}_{\chi})_c, \Sigma) \circ\!\!\rightarrow \mathcal{L}(\mathcal{H})$ associated to a random function $f{:}\Omega((GKG)^{\mathbf{R}}_{\chi})_c \rightarrow \mathbf{R}$. More precisely, if a scalar measure μ is recognized on $\Omega((GKG)^{\mathbf{R}}_{\chi})_c$, one has $< \phi'|E(X)|\phi >= \int_X \hat{f}_{\phi,\phi'} d\mu$, for any $X \in \Sigma \equiv$Borel σ-algebra of $\Omega((GKG)^{\mathbf{R}}_{\chi})_c$, where $\hat{f}{:}\Omega((GKG)^{\mathbf{R}}_{\chi})_c \rightarrow \mathcal{L}(\mathcal{H})$ is the map associated to the canonical quantization of f. $\hat{f}$ is determined by means of its spectral measure on $\mathbf{R}$. (See also refs.[5,8].)

We can associate a generalized propagator to any singular solution V of ${}_f(GKG)^{\mathbf{R}}_{\chi}$, (that realizes a tunnel effect, if $\partial V = N_1 \bigcup N_2$, with $\pi_n(N_1) \neq \pi_n(N_2)$, for some $n \geq 0$, $n \in \mathbf{N}$, where $\pi_n(-)$ are the Hurewitz homotopy group-functors).

PROOF. 1) The proof can be directly obtained appling some general theorems given by A.Prástaro in order to calculate integral bordism groups in PDEs [8-12], and taking into account that ${}_f(GKG)^{\mathbf{R}}_{\chi}$ is a conic equation with trivial cohomoly: $H^s((GKG)^{\mathbf{R}}_{\chi}) = 0$, $\forall s \neq 0$, $H^0((GKG^{\mathbf{R}})_{\chi}) = \mathbf{R}$.
2) It is a direct consequence of point (1) and Theorem 1.1. More precisely, if V is the integral manifold (quantum cobord) cobording two 3-dimensional admissible integral manifolds N_0 and N_1 contained into ${}_f(GKG)^{\mathbf{R}}_{\chi}$, such that the mapping $\pi_2|_V : V \to M$ is a proper application and $\omega_1^{(2)}(V) = 0$, where $\omega_1^{(2)}(V)$ is the characteristic of Stiefel-Whitney of V, then the propagator (generalized Green kernel) $G[V]$ between N_0 and N_1 is identified with the following kernel $G[V] \in (C_0^{\infty}(E' \boxtimes E')'$, given by $G[V](\alpha \otimes \phi) = F[V](\alpha) \int_M < \phi, f >$, where $E' \equiv E^* \bigotimes \Lambda_4^0 M$, and $F[V] \in C_0^{\infty}(E')$ is the distribution associated to V. [13] $\square$

3 - GENERALIZED DIRAC EQUATION

Here we want to formulate a generalized Dirac equation in order to include also fermionic massive neutrinos-takions. Let (M, g) be as before a 4-dimensional space-time and let $\pi : E \to M$ be a vector bundle over M identified with the complexificated Clifford bundle over M: $E \equiv \bigcup_{p \in M} E_p$, $E_p \equiv \bigoplus_{s \geq 0} T_0^s(\mathbf{C} \bigotimes_{\mathbf{R}} T_pM) / \mathbb{I}_p$, where $\mathbb{I}_p$ is the subspace of $\bigoplus_{s \geq 0} T_0^s(\mathbf{C} \bigotimes_{\mathbf{R}} T_pM)$ generated by elements of the type $v \otimes v - g^c(v, v)\mathbf{1}$, where $\mathbf{1}$ is the unity of $\mathbf{R}$ and g^c the scalar product on $\mathbf{C} \bigotimes_{\mathbf{R}} T_pM$ defined by $g^c(z_1 \otimes v_1, z_2 \otimes v_2) = z_1 z_2 \otimes g(v_1, v_2)$. Then, each fiber E_p becomes a

---

[13] The physical intepretation for the generalized propagator $G[V]$, $\partial V = N_0 \bigcup N_1$, is that it represents the amplitude probability $<N_0|N_1>$ for the transition from an extendon N_0 to another N_1. (Compare, e.g., with the exposition of extendons given in ref.[15].) More precisely $<N_0|N_1> = G[V](\alpha \otimes \phi)$, where α and ϕ represent the physical states of N_0 and N_1 respectively.

complexificated Clifford algebra, hence it is possible to define a product (Clifford product) on E_p.

THEOREM 3.1 - 1) *The Levi-Civita connection on M lifts on E.*
2) *The Clifford connection is flat iff the space-time manifold M is flat. This is equivalent to say that C_1 is completely integrable iff (M, g) is flat.*

PROOF. 1) Let us recall that the Clifford connection is a first order PDE on the fiber bundle $\pi : E \to M$, $C_1 \subset J\mathcal{D}(E)$, identified by a section $\underset{C}{\rceil} : E \to J\mathcal{D}(E)$ of $\pi_{1,0} : J\mathcal{D}(E) \to E$, such that the following diagram is commutative:

$$\begin{array}{cccccccc} & C_1 & & \cong & & E & & \\ & \| & & & & \| & & \\ & C_1 & \subset & J\mathcal{D}(E) & \overset{\underset{C}{\rceil}}{\longleftarrow} & E & \longrightarrow & M \\ J\mathcal{D}(\gamma) & \uparrow & & \uparrow J\mathcal{D}(\gamma) & & \uparrow \gamma & & \| \\ & (LC)_1 & \subset & J\mathcal{D}(TM) & \underset{\underset{LC}{\rceil}}{\longleftarrow} & TM & \longrightarrow & M \\ & \| & & & & \| & & \\ & (LC)_1 & & \cong & & TM & & \end{array}$$

where $(LC)_1$ is the canonical connection on TM identified by means of the metric g. Furthermore, γ is the canonical monomorphism of vector bundles over M given by composition: $TM \to TM \otimes \mathbf{C} \to E$. If $\{x^\alpha\}$ is a coordinate system on M and ∂x_α is the natural basis induced on the tangent bundle, the monomorphism γ induces a set of sections of E denoted by $\gamma_\alpha \equiv \gamma(\partial x_\alpha)$. Set $\gamma_{\alpha_1 \dots \alpha_p} \equiv \gamma_{\alpha_1} \cdots \gamma_{\alpha_p}$. We get the following relation

$$\gamma_{\alpha_1 \dots \alpha_p} = (-1)^q \gamma_{\alpha_1} \cdots \gamma_{\alpha_{p-q-1}} \gamma_{\alpha_p} \gamma_{\alpha_{p-q}} \cdots \gamma_{\alpha_{p-1}} + \sum_{1 \le r \le q} (-1)^{r+1} 2 g_{\alpha_p \alpha_{p-r}} \gamma_{\alpha_1} \cdots \widehat{\gamma_{\alpha_{p-r}}} \cdots \gamma_{\alpha_{p-1}}.$$

The set $\{1, \gamma_{\alpha_1 \dots \alpha_p}\}_{0 \le \alpha_1 < \dots < \alpha_p \le 3, 1 \le p \le 4}$ is a local basis for sections of $\pi : E \to M$. Hence, if $\psi : M \to E$ is such a (local) section, we get the following local representation in the natural basis induced by the coordinate system $\{x^\alpha\}$ on M:

$$\psi = \phi 1 + \psi^\alpha \gamma_\alpha + \dots + \sum_{0 \le \alpha_1 < \dots < \alpha_p \le 3} \psi^{\alpha_1 \dots \alpha_p} \gamma_{\alpha_1 \dots \alpha_p} + \dots + \psi^{0123} \gamma_{0123}$$
$$\equiv \psi^B \gamma_B$$

where $(\gamma_B) \equiv (1, \gamma_\alpha, \ldots, \gamma_{\alpha_1 \ldots \alpha_p}, \ldots, \gamma_{0123})$, $0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_p \leq 3$. $(\psi^B) \equiv (\phi, \psi^\alpha, \ldots, \psi^{0123})$ are **C**-valued local functions on M. Therefore $\dim_{\mathbf{C}} E_p = 16$, where E_p is the fiber of E over $p \in M$. As a consequence we get the following induced fibered system of coordinates on E: $\{x^\alpha, z, z^B\} \equiv \{x^\alpha, z, z^\alpha, \ldots, z^{\alpha_1 \ldots \alpha_p}, \ldots, z^{0123}\}$, where x^α are **R**-valued and the other ones are **C**-valued. If we denote by $\{x^\alpha, \dot{x}^\alpha\}$ the induced fibered coordinates system on TM, we get that the monomorphism γ can be locally written as follows:

$$\left\{\begin{array}{r} x^\alpha \circ \gamma = x^\alpha \\ z \circ \gamma = 0 \\ z^\alpha \circ \gamma = \dot{x}^\alpha \\ \ldots \\ z^{\alpha_1 \ldots \alpha_p} \circ \gamma = 0 \\ \ldots \\ z^{0123} \circ \gamma = 0 \end{array}\right\}.$$

Then, denoting by $\{x^\alpha, \dot{x}^\alpha, \dot{x}^\alpha_\beta\}$ the induced coordinates on $J\mathcal{D}(TM)$ and by $\{x^\alpha, z, z^B, z_\beta, z^B_\beta\}$ the induced coordinates on $J\mathcal{D}(E)$, we obtain the following local expression of $J\mathcal{D}(\gamma)$.

$$\left\{\begin{array}{r} x^\alpha \circ J\mathcal{D}(\gamma) = x^\alpha \\ z \circ J\mathcal{D}(\gamma) = 0 \\ z^\alpha \circ J\mathcal{D}(\gamma) = \dot{x}^\alpha \\ \ldots \\ z^{\alpha_1 \ldots \alpha_p} \circ J\mathcal{D}(\gamma) = 0 \\ \ldots \\ z^{0123} \circ J\mathcal{D}(\gamma) = 0 \end{array}\right\}.$$

Furthermore, the Levi-Civita connection can be written as follows:

$$(LC)_1 \subset J\mathcal{D}(TM): \quad \{\dot{x}^\alpha_\beta + \dot{x}^\gamma \Gamma^\alpha_{\beta\gamma} = 0\}$$

where $\Gamma^{\alpha}_{\beta\gamma}= g^{\alpha\delta}[\beta\gamma,\delta] = \frac{1}{2}g^{\alpha\delta}[(\partial x_{\gamma}.g_{\beta\delta})+(\partial x_{\beta}.g_{\gamma\delta})-(\partial x_{\delta}.g_{\gamma\beta})]$ are local numerical functions on M representing the usual Levi-Civita connection symbols. Then, the induced Clifford connection $C_1 \subset J\mathcal{D}(E)$, is locally written as follows:

$$C_1 \subset J\mathcal{D}(E) : \left\{ \begin{array}{l} z^B_{\lambda} + \underset{C}{\Gamma}{}^B_{\lambda A} z^A = 0 \\ \underset{C}{\Gamma}{}^B_{\lambda A} \equiv \underset{C}{\Gamma}{}^{\beta_1\dots\beta_p}_{\lambda\alpha_1\dots\alpha_p} \equiv \Gamma^{\beta_1}_{\lambda\alpha_1}\delta^{\beta_2\dots\beta_p}_{\alpha_2\dots\alpha_p} + \dots + \Gamma^{\beta_p}_{\lambda\alpha_p}\delta^{\beta_1\dots\beta_{p-1}}_{\alpha_1\dots\alpha_{p-1}} \end{array} \right\}$$

where $\underset{C}{\Gamma}{}^B_{\lambda A} \equiv -(z^B_{\lambda}\circ \underset{C}{]})\circ\gamma_A \equiv -\underset{C}{]}{}^B_{\lambda}\circ\gamma_A \equiv -\underset{C}{]}{}^B_{\lambda A}$ are the connection coefficients given as $\mathbf{R}$-valued local functions on M. Then, the absolute differential $\underset{C}{\nabla}\psi \equiv \underset{C}{\Gamma}\circ D\psi : M \to J\mathcal{D}(E) \to T^*M\otimes E$ of a section $\psi : M \to E$, locally written $\psi = \phi 1 + \psi^{\alpha}\gamma_{\alpha} + \dots + \psi^{\alpha_1\cdots\alpha_p}\gamma_{\alpha_1\dots\alpha_p} + \dots + \psi^{0123}\gamma_{0123} \equiv \psi^B\gamma_B$, has the following local expression:

$$\underset{C}{\nabla}\psi = (\underset{C}{\nabla}{}_{\lambda}\psi)^B dx^{\lambda}\otimes\gamma_B = [(\partial x_{\lambda}.\psi^B) + \underset{C}{\Gamma}{}^B_{\lambda A}\psi^A]dx^{\lambda}\otimes\gamma_B. \tag{3.1}$$

By using results on the curvature of PDEs, (see ref.[12]), we can state that the curvature $\underset{C}{R}$ of the Clifford connection can be identified with a section $\underset{C}{R} : M \to \Lambda^0_2 M \otimes E$ locally written as follows:

$$\left\{ \begin{array}{l} \underset{C}{R} = \displaystyle\sum_{0\le\alpha,\beta\le 3, A, B} \underset{C}{R}{}_{\alpha\beta}{}^B{}_{.A}\, dx^{\alpha}\wedge dx^{\beta}\otimes\gamma_B\otimes\gamma^A \\ \underset{C}{R}{}_{\alpha\beta}{}^B{}_{.A} = (\partial x_{\alpha}.\underset{C}{\Gamma}{}^B_{\beta A}) - (\partial x_{\beta}.\underset{C}{\Gamma}{}^B_{\alpha A}) + \underset{C}{\Gamma}{}^B_{\alpha D}\underset{C}{\Gamma}{}^D_{\beta A} - \underset{C}{\Gamma}{}^B_{\beta D}\underset{C}{\Gamma}{}^D_{\alpha A}. \end{array} \right\}$$

Here $\underset{C}{R}{}_{\alpha\beta}{}^B{}_{.A}$ are $\mathbf{R}$-valued local functions on M.[14] Now the curvature of a connection is zero iff the equation representing the connection is completely integrable. On the other hand we can easily compute $\underset{C}{R}{}_{\alpha\beta}{}^B{}_{.A}$ and

[14] It is easy to prove the following formula for any section $\psi=\psi^B\gamma_B$: $<\underset{C}{R},\psi>=\underset{C}{R}{}_{\alpha\beta}{}^B{}_{.D}\psi^D dx^{\alpha}\wedge dx^{\beta}\otimes\gamma_B$ with $\underset{C}{R}{}_{\alpha\beta}{}^B{}_{.D}\psi^D\gamma_B=(\underset{C}{\nabla}{}_{\alpha}\underset{C}{\nabla}{}_{\beta}-\underset{C}{\nabla}{}_{\beta}\underset{C}{\nabla}{}_{\alpha})(\psi^B\gamma_B)$ that relates the curvature of the Clifford connection to the Clifford covariant derivative.

we get the following formula:

$$\underset{C_{\alpha\beta}}{R}{}^{B}{}_{.C} \equiv \underset{C}{R}_{\alpha\beta}{}^{\delta_1\ldots\delta_p}{}_{.\gamma_1\ldots\gamma_p} = R^{\delta_1}_{\alpha\beta\cdot\gamma_1}\delta^{\delta_2\ldots\delta_p}_{\gamma_2\ldots\gamma_p} + \ldots + R_{\alpha\beta}{}^{\delta_p}{}_{.\gamma_p}\delta^{\delta_1\ldots\delta_{p-1}}_{\gamma_1\ldots\gamma_{p-1}}.$$

Therefore, $\underset{C}{R}_{\alpha\beta}{}^{B}{}_{.C} = 0$ iff the Levi-Civita curvature $R_{\alpha\beta}{}^{\delta}{}_{.\gamma} = 0$. Thus the theorem is proved. □

THEOREM 3.2 - 1) *Let M be a space-time such that one has the following isomorphism: $E \cong S^* \otimes_{\mathbf{C}} S$, for a suitable vector bundle $S \to M$.*[15] *Then the Clifford connection induces the following linear connection on S:*

$$(3.2)\qquad C_1^S \subset J\mathcal{D}(S): \begin{Bmatrix} y^r_\lambda + \underset{S_\lambda}{\Gamma}{}^r_s y^s = 0 \\ \underset{S}{\Gamma}{}_{\lambda}{}^{r}_{s} = C^{-1}{}^{rb}_{sa} D_{\lambda}{}^{a}_{b} \end{Bmatrix}.$$

Here $\underset{S}{\Gamma}{}_{\lambda}{}^{r}_{s}$ *are* **C***-valued local functions on M and*

(3.3)
$$\begin{Bmatrix} C^{sa}_{rb} \equiv 4\delta^{as}_{rb} - \gamma_{\alpha r}{}^{a}\gamma^{\alpha s}{}_{b} \in L(M(4;\mathbf{C})) \\ D_{\lambda}{}^{a}_{b} \equiv \Gamma^{\beta\delta}_{\lambda}\gamma_{\beta}{}^{a}_{s}\gamma_{\delta}{}^{s}_{b} - (\partial x_\lambda.\gamma_{\alpha}{}^{a}_{r})\gamma^{\alpha r}{}_{b} \in M(4;\mathbf{C}), \quad \forall\lambda = 0,1,2,3. \end{Bmatrix}$$

2) *One has the following consistency condition between Levi-Civita connection and γ_α-matrices:*

$$(3.4)\qquad 2\Gamma^{\beta}_{\lambda\beta}\mathbf{1} = (\partial x_\lambda.\gamma_\alpha)\gamma^\alpha + \gamma^\alpha(\partial x_\lambda.\gamma_\alpha) \in M(4;\mathbf{C}).$$

3) *Moreover by using equation (3.4) we can also write:*

$$(3.5)\qquad \underset{S}{\Gamma}{}_{\lambda} = \frac{1}{4}\Gamma^{\beta\delta}_{\lambda}\gamma_\beta\gamma_\delta \in M(4;\mathbf{C}).$$

---

[15] It is well known [3] that if M is endowed with an almost complex structure J such that g is Hermitian, (this appens e.g. if M is an almost Hermitian manifold or an almost Kählerian manifold), and such that it admits a $Spin^c$-structure (i.e., the second Stiefel-Whitney class $w_2 \in H^2(M;\mathbf{Z}_2)$ is zero), then the bundle E is canonically isomorphic to $Hom_{\mathbf{C}}(S;S) \cong S^* \otimes_{\mathbf{C}} S$, where $S \equiv \Lambda E_1$, with $E_1 = \ker(\omega - id_E)$, where ω is the involution canonically induced by J. One has $\dim_{\mathbf{C}} S = 2^2 = 4$. $S \to M$ is called the **spinors bundle**. The sections of S over M represent spinor fields with spin $s = \frac{1}{2}$.

PROOF. Under our hypotheses there is a unique first order linear connection on S, $C_1^S \subset J\mathcal{D}(S)$, such that the following diagram is commutative:

$$
\begin{array}{ccccccc}
E & \overset{j}{\cong} & S^*\otimes S & \cong & C_1^{S^*}\otimes C_1^S & \cong & C_1 \\
 & & \| & & \cap & & \cap \\
 & & S^*\otimes S & \xrightarrow{\overset{*}{\lceil}_S \otimes \lceil_S} & J\mathcal{D}(S^*\otimes S) & \cong & J\mathcal{D}(E) \\
 & & \uparrow \bar{\pi} & & \uparrow J\mathcal{D}(\bar{\pi}) & & \\
 & & S^*\times S & \xrightarrow{\overset{*}{\lceil}_S \times \lceil_S} & J\mathcal{D}(S^*)\times J\mathcal{D}(S) & & \\
 & & \| & & \cup & & \\
 & & S^*\times S & \cong & C_1{}^*\times C_1^S & &
\end{array}
$$

In order to give a local representation of the isomorphism $C_1 \cong C_1^{S^*}\otimes C_1^S$, let us first locally characterize the isomorphism $j : C \cong L(S) \cong S^*\otimes S$. We get $j(\psi = \phi 1 + \phi^B\gamma_B) = \phi j(1) + \phi^B j(\gamma_B)$, where $j(1) = id_S \in L(S)$, $j(\gamma_B) = j(\gamma_{\alpha_1\dots\alpha_p}) = j(\gamma_{\alpha_1})\dots j(\gamma_{\alpha_p})$, with $j(\gamma_\alpha) \in L(S)$. Hence if $\{y_b \otimes y^a\}_{1\leq a,b\leq 4}$ is a coordinate system on $S^*\otimes S$, we obtain a representation of $j(\gamma_\alpha)$, (and as a consequence of any operator $j(\gamma_B)$), as a 4×4 matrix-function $(y_b \otimes y^a \circ j(\gamma_\alpha)) \equiv (\gamma_\alpha{}^a_b)$ such that $j(\gamma_\alpha)(p) \in M(4;\mathbf{C})$ for any $p \in M$.[16] (For abuse of notation we shall simply write $\gamma_B = j(\gamma_B)$.) The matrices $(\gamma_B{}^a_b)$, $B = \alpha_1\dots\alpha_p$, $0 \leq \alpha_i \leq 3$, $1 \leq a, b \leq 4$, are called **Dirac matrices**. For example, if M is the Minkowsky space-time and x^α is a system of cartesian coordinates, we can identify an isomorphism $j : E \cong L(S)$ such that if $j(\gamma_\alpha) = \gamma_\alpha{}^a_b\theta^b \otimes e_a$, where $\{\theta^b\}$ and $\{e_a\}$ are the corresponding local bases on S^* and S respectively, one has the following constant matrices for $(\gamma_\alpha{}^a_b)$:

$$
\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},
$$

16 We use small greek indexes for space-time indexes, (they run from 0 to 3), and small italic indexes for spinor indexes, (they run from 1 to 4).

$$\gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix};$$

One usually puts $\gamma_5 \equiv \gamma_{0123}$. If one takes anoter coordinate system $\bar{x}^\alpha$ the corresponding matrices $(\bar{\gamma}_{\alpha}{}^{a}_{b})$ are related to $\gamma_{\alpha}{}^{a}_{b}$ by the following transformations:

$$\bar{\gamma}_{\alpha}{}^{m}_{l} = A^{\beta}_{\alpha}\Lambda^{r}_{l}\Lambda^{m}_{s}\gamma_{\beta}{}^{s}_{r}, \quad (A^{\beta}_{\alpha}) = (\partial\bar{x}_{\alpha}.x^{\beta}) \in SO(1,3), (\Lambda^{s}_{p}) = (\partial\bar{y}_{p}.y^{s}) \in$$

$Spin(1,3)$.

Of course $(\bar{\gamma}_{\alpha}{}^{a}_{b})$, are not more, in general, constant matrices. Now, recall that a linear connection on $S \to M$, $C^S_1 \subset J\mathcal{D}(S)$: $\{y^a_\alpha + \underset{S}{\lceil}{}_{\alpha}{}^{a}_{b} y^b = 0\}$, identifies a unique linear connection on the dual bundle $S^* \to M$, such that the following diagram is commutative:

$$\begin{array}{ccc}
C^S_1 \times C^{S^*}_1 & \subset & J\mathcal{D}(S\times S^*) \\
\| \wr & & \| \wr \\
S \times S^* & \xrightarrow{\underset{S}{\rceil} \times \underset{S}{\rceil}^*} & J\mathcal{D}(S)\times J\mathcal{D}(S^*) \\
0 \downarrow & & \downarrow J\mathcal{D}(<,>) \\
J\mathcal{D}(M\times \mathbf{C}) & = & J\mathcal{D}(M\times \mathbf{C})
\end{array}$$

More precisely: $C^{S^*}_1 \subset J\mathcal{D}(S^*)$: $\{y_{k\alpha} - \underset{S}{\lceil}{}_{\alpha}{}^{j}_{k} y_j = 0\}$. Then we have:

$$\begin{aligned}
\underset{C}{\nabla}{}_\lambda \gamma_\alpha &= \lceil^{\beta}_{\lambda\alpha}\gamma_\beta = \lceil^{\beta}_{\lambda\alpha}\gamma_{\beta}{}^{r}_{s}\theta^s \otimes e_r \\
&= \underset{S^*\otimes S}{\nabla}{}_\lambda(\gamma_{\alpha}{}^{r}_{s}\theta^s \otimes e_r) = (\partial x_\lambda.\gamma_{\alpha}{}^{r}_{s})\theta^s \otimes e_r + \gamma_{\alpha}{}^{r}_{s}(\underset{S^*}{\nabla}{}_\lambda \theta^s)\otimes e_r + \\
&\quad \gamma_{\alpha}{}^{r}_{s}\theta^s \otimes (\underset{S}{\nabla}{}_\lambda e_r) \\
&= (\partial x_\lambda.\gamma_{\alpha}{}^{r}_{s})\theta^s \otimes e_r - \gamma_{\alpha}{}^{r}_{s}\underset{S}{\lceil}{}^{s}_{\lambda p}\theta^p \otimes e_r + \gamma_{\alpha}{}^{r}_{s}\theta^s \otimes \underset{S}{\lceil}{}^{q}_{\lambda r} e_q \\
&= \left[(\partial x_\lambda.\gamma_{\alpha}{}^{r}_{s}) - \gamma_{\alpha}{}^{r}_{p}\underset{S}{\lceil}{}^{p}_{\lambda s} + \gamma_{\alpha}{}^{q}_{s}\underset{S}{\lceil}{}^{r}_{\lambda q}\right]\theta^s \otimes e_r.
\end{aligned}$$

Therefore, we must have

$$\Gamma^{\beta}_{\lambda\alpha}\gamma_{\beta}{}^{r}_{s} = (\partial x_{\lambda}.\gamma_{\alpha}{}^{r}_{s}) - \gamma_{\alpha}{}^{r}_{p}\underset{S}{\Gamma}{}^{p}_{\lambda s} + \gamma_{\alpha}{}^{q}_{s}\underset{S}{\Gamma}{}^{r}_{\lambda q}.$$

This relation can be also rewritten in matrix form in the following way:

$$\Gamma^{\beta}_{\lambda\alpha}\gamma_{\beta} = (\partial x_{\lambda}.\gamma_{\alpha}) + \underset{S}{\Gamma}{}_{\lambda}\gamma_{\alpha} - \gamma_{\alpha}\underset{S}{\Gamma}{}_{\lambda}, \tag{3.6}$$

where $\gamma_{\beta}(p), (\partial x_{\lambda}.\gamma_{\alpha})(p), \underset{S}{\Gamma}{}_{\lambda}(p) \in M(4;\mathbf{C})$, $\forall p \in M$. Set $\gamma^{\alpha} = g^{\alpha\beta}\gamma_{\beta}$. Then one has $\gamma^{\alpha}\gamma_{\alpha} = 4\,\mathbf{1}$. (This follows directly by contracting both side of the relation $\gamma_{\alpha}\gamma_{\beta} + \gamma_{\beta}\gamma_{\alpha} = 2g_{\alpha\beta}\,\mathbf{1}$, by $g^{\alpha\beta}$ and taking into account that $g^{\alpha\beta}g_{\alpha\beta} = 4$.) From (3.6) we get also the following equations:

$$\Gamma^{\beta}_{\lambda\alpha}\gamma_{\beta}\gamma^{\alpha} = (\partial x_{\lambda}.\gamma_{\alpha})\gamma^{\alpha} + \underset{S}{\Gamma}{}_{\lambda}\gamma_{\alpha}\gamma^{\alpha} - \gamma_{\alpha}\underset{S}{\Gamma}{}_{\lambda}\gamma^{\alpha}$$

$$\Gamma^{\beta}_{\lambda\alpha}\gamma^{\alpha}\gamma_{\beta} = \gamma^{\alpha}(\partial x_{\lambda}.\gamma_{\alpha}) + \gamma^{\alpha}\underset{S}{\Gamma}{}_{\lambda}\gamma_{\alpha} - \gamma^{\alpha}\gamma_{\alpha}\underset{S}{\Gamma}{}_{\lambda}.$$

Therefore, we get also:

$$\left\{\begin{array}{l} \Gamma^{\beta\delta}_{\lambda}\gamma_{\beta}\gamma_{\delta} = (\partial x_{\lambda}.\gamma_{\alpha})\gamma^{\alpha} + 4\underset{S}{\Gamma}{}_{\lambda} - \gamma_{\alpha}\underset{S}{\Gamma}{}_{\lambda}\gamma^{\alpha} \\ \Gamma^{\beta\delta}_{\lambda}\gamma_{\delta}\gamma_{\beta} = \gamma^{\alpha}(\partial x_{\lambda}.\gamma_{\alpha}) + \gamma^{\alpha}\underset{S}{\Gamma}{}_{\lambda}\gamma_{\alpha} - 4\underset{S}{\Gamma}{}_{\lambda} \end{array}\right\}. \tag{3.7}$$

Of course the second of above equations is equivalent to the first one.[17] By addition of both equations (3.7) we get equation (3.4).[18] In order to obtain

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[17] Note also that the first term on the left in above equation can be written as follows: $\Gamma^{\beta\delta}_{\lambda}\gamma_{\beta}\gamma_{\delta} = \Gamma^{\beta}_{\lambda\beta}1 + \Gamma^{[\beta\delta]}_{\lambda}\gamma_{\beta}\gamma_{\delta}$. In fact we have: $\Gamma^{\beta\delta}_{\lambda}\gamma_{\beta}\gamma_{\delta} = \Gamma^{\beta\delta}_{\lambda}\frac{1}{2}(\gamma_{\beta}\gamma_{\delta} + \gamma_{\delta}\gamma_{\beta}) + \Gamma^{\beta\delta}_{\lambda}\frac{1}{2}(\gamma_{\beta}\gamma_{\delta} - \gamma_{\delta}\gamma_{\beta}) = g_{\beta\delta}\Gamma^{\beta\delta}_{\lambda}1 + \Gamma^{[\beta\delta]}_{\lambda}\gamma_{\beta}\gamma_{\delta}$
$= \Gamma^{\beta}_{\lambda\beta}1 + \Gamma^{[\beta\delta]}_{\lambda}\gamma_{\beta}\gamma_{\delta}$.

[18] Taking into account that $\gamma^{\alpha}\gamma_{\alpha} = \gamma_{\alpha}\gamma^{\alpha} = 4\,\mathbf{1}$, we get that the consistency condition (3.4) can be also written in the following way: $2\Gamma^{\beta}_{\lambda\beta}\mathbf{1} = (\partial x_{\lambda}.\gamma_{\alpha})\gamma^{\alpha} - (\partial x_{\lambda}.\gamma^{\alpha})\gamma_{\alpha} = \gamma^{\alpha}(\partial x_{\lambda}.\gamma_{\alpha}) - \gamma_{\alpha}(\partial x_{\lambda}.\gamma^{\alpha})$.

the explicit expression for $\underset{S}{\Gamma}{}_\lambda$ in the general case, let us rewrite equation (3.6) as an equation in $M(4;\mathbf{C})$ in the following way: $C^{as}_{rb}\underset{S}{\Gamma}{}_{\lambda s}^{r} = D_{\lambda b}^{a}$, $\forall\lambda = 0,1,2,3$, with $D_{\lambda b}^{a} \equiv \Gamma_{\lambda}^{\beta\delta}\gamma_{\beta s}^{a}\gamma_{\delta b}^{s} - (\partial x_\lambda.\gamma_{\alpha r}^{a})\gamma^{\alpha r}{}_{b}$ and $C \equiv 4\mathbf{1}\otimes\mathbf{1} - \gamma_\alpha\otimes\gamma^\alpha$. So, assuming that the matrix $(C^{sa}_{rb}) \in L(M(4;\mathbf{C}))$, that represents a linear application of the 16-dimensional vector space $M(4;\mathbf{C})$, over $\mathbf{C}$, is invertible, we get for $\underset{S}{\Gamma}{}_\lambda$ the following solution:

$$\underset{S}{\Gamma}{}_{\lambda s}^{r} = C^{-1rb}{}_{sa}D_{\lambda b}^{a}. \tag{3.8}$$

Therefore, the general expression of the connection $C_1^S \subset J\mathcal{D}(S)$, induced from the Clifford connection on E, has the local expression given in (3.1). Now let us prove that we can write $\underset{S}{\Gamma}{}_\lambda$ in the Lychnerowicz's form (3.5). In fact, let us substitute (3.5) in the first equation (3.7). We get:

$$\Gamma_{\lambda}^{\beta\delta}\gamma_\beta\gamma_\delta = (\partial x_\lambda.\gamma_\alpha)\gamma^\alpha + \Gamma_{\lambda}^{\beta\delta}\gamma_\beta\gamma_\delta - \frac{1}{4}\gamma_\alpha\Gamma_{\lambda}^{\beta\delta}\gamma_\beta\gamma_\delta\gamma^\alpha. \tag{3.9}$$

On the other hand we can see that

$$\gamma_\alpha\gamma_\beta\gamma_\delta\gamma^\alpha = g_{\beta\delta}. \tag{3.10}$$

Therefore, from (3.9) we get

$$\frac{1}{4}\Gamma_{\lambda\beta}^{\beta}\,\mathbf{1} = (\partial x_\lambda.\gamma_\alpha)\gamma^\alpha. \tag{3.11}$$

Now, let us substitute (3.5) in the second equation of (3.7). We similarly get:

$$\frac{7}{4}\Gamma_{\lambda\beta}^{\beta}\,\mathbf{1} = \gamma^\alpha(\partial x_\lambda.\gamma_\alpha). \tag{3.12}$$

Therefore, we see that (3.5) is a solution of both (3.7) iff (3.11) and (3.12) are both respected. On the other hand by adding these equations we get

(3.4). This proves that conditions (3.11) and (3.12) are not new requirements but are automatically satisfied thanks to the consitency condition (3.4). □

COROLLARY 3.1 - 1) *The absolute differential of a controvariant spinor field* $\psi^a e_a : M \to S$, *is* $\underset{S}{\nabla}\psi = \underset{S}{\Gamma} \circ D\psi = (\underset{S}{\nabla}_\lambda \psi)^r dx^\lambda \otimes e_r$, *where the spinor-covariant derivative* $(\underset{S}{\nabla}_\lambda\psi)^r$ *is given by*

$$(3.13)\qquad (\underset{S}{\nabla}_\lambda\psi)^r = (\partial x_\lambda \psi^r) + \underset{S}{\Gamma}{}_{\lambda}{}^{r}_{s}\psi^s = (\partial x_\lambda \psi^r) + C^{-1}{}^{rb}_{sa} D_{\lambda}{}^{a}_{b}\psi^s.$$

2) *The absolute differential for the covariant spinor field* $\varphi = \varphi_a\theta^a : M \to S^*$ *is given by:* $\underset{S^*}{\nabla}\varphi = \overset{*}{\underset{S}{\Gamma}} \circ D\varphi = (\underset{S^*}{\nabla}_\rho\varphi)_s dx^\rho \otimes \theta^s$, *where the spinor-covariant derivative* $(\underset{S^*}{\nabla}_\rho\varphi)_s$ *is given by:*

$$(3.14)\qquad (\underset{S^*}{\nabla}_\rho\varphi)_s = (\partial x_\rho . \varphi_s) - \underset{S}{\Gamma}{}_{\rho}{}^{r}_{s}\varphi_r = (\partial x_\rho . \varphi_s) - C^{-1}{}^{rb}_{sa} D_{\rho}{}^{a}_{b}\varphi_r.$$

3) *The curvature of the spinor connection* $C_1^S \subset J\mathcal{D}(S)$ *on* $\pi : S \to M$, *is given by a morphism of vector fiber bundles over* M, $\underset{S}{R} : C_1^S \to \Lambda_2^0 M \otimes S$ *locally written, as a section over* M, *as follows:*

$$(3.15)\qquad \left\{ \begin{array}{l} \underset{S}{R} = \displaystyle\sum_{0\le\alpha,\beta\le 3, 1\le a,b\le 4} \underset{S}{R}{}_{\alpha\beta}{}^{a}{}_{.b} dx^\alpha \wedge dx^\beta \otimes e_a \otimes \theta^b \\ \underset{S}{R}{}_{\alpha\beta}{}^{a}{}_{.b} = (\partial x_\alpha . \underset{S}{\Gamma}{}_{\beta}{}^{a}_{b}) - (\partial x_\beta . \underset{S}{\Gamma}{}_{\alpha}{}^{a}_{b}) + \underset{S}{\Gamma}{}_{\alpha}{}^{a}_{c}\underset{S}{\Gamma}{}_{\beta}{}^{c}_{b} - \underset{S}{\Gamma}{}_{\beta}{}^{a}_{s}\underset{S}{\Gamma}{}_{\alpha}{}^{c}_{b} \end{array} \right\}$$

where $\underset{S}{R}{}_{\alpha\beta}{}^{a}{}_{.b}$ *are* **C**-*valued local functions on* M. *If we use the Lichnerowicz's formula we get that* (3.15) *gives the following expression for the spinor curvature:*

$$(3.16)\qquad \underset{S}{R}{}_{\alpha\beta}{}^{a}{}_{.b} = \frac{1}{4} R_{\alpha\beta}{}^{\mu}{}_{.\lambda}(\gamma_\mu\gamma^\lambda)^a_b$$

where $R_{\alpha\beta}{}^{\mu}{}_{.\lambda}$ *is the Levi-Civita curvature.*

4) *Similarly to the case of the Clifford connection, we have the following formula:*

$$(3.17)\quad \left\{\begin{array}{l} <\underset{S}{R}, \psi>=\underset{S}{R}{}_{\alpha\beta}{}^{b}{}_{.c}\psi^{c}dx^{\alpha}\wedge dx^{\beta}\otimes e_{b} \\ \underset{S}{R}{}_{\alpha\beta}{}^{b}{}_{.c}\psi^{c}e_{b}=\left(\underset{S}{\nabla}_{\alpha}\underset{S}{\nabla}_{\beta}-\underset{S}{\nabla}_{\beta}\underset{S}{\nabla}_{\alpha}\right)(\psi^{b}e_{b}) \end{array}\right\}$$

that relates the spinor-curvature to the spinor-covariant derivative.

DEFINITION 3.1 - 1) *The* **Dirac operator** *on the Clifford bundle* $\pi : E \to M$ *is a first order linear differential operator* $\underset{C}{P} : C^{\infty}(E) \to C^{\infty}(E)$, *given by* $\underset{C}{P} = i\underset{C}{P}_0$, *where* i *is the imaginary unity and* $\underset{C}{P}_0$ *is defined by means of the following homomorphism of vector bundles over* M*:*[19]

$$\begin{array}{ccccc} JD(E) & \xrightarrow{\underset{C}{\Gamma}} & T^{*}M\otimes E & \overset{g'\otimes 1}{\cong} & TM\otimes E \\ \underset{C}{P}_0\downarrow & & & & \downarrow\gamma\otimes 1 \\ E & & \xleftarrow[a]{} & & E\otimes E \end{array}$$

where a *is the morphism induced by the Clifford multiplication. Therefore, if* $\psi = \psi^{B}\gamma_{B} : M \to E$ *is a section of* $\pi : E \to M$, *we get the following local expression of* $\underset{C}{P}.\psi$*:*

$$\begin{aligned}\underset{C}{P}.\psi =& ig^{\alpha\epsilon}\left[(\partial x_{\alpha}.\psi^{B})+\underset{C}{\Gamma}{}^{B}_{\alpha D}\psi^{D}\right]\gamma_{\epsilon}\gamma_{B}=i\gamma^{\alpha}\left[(\partial x_{\alpha}.\psi^{B})+\underset{C}{\Gamma}{}^{B}_{\alpha D}\psi^{D}\right]\gamma_{B}\\ =& i\gamma^{\alpha}(\underset{C}{\nabla}_{\alpha}\psi)^{B}\gamma_{B}.\end{aligned}$$

2) *If there is an isomorphism* $j : E \cong S^{*}\otimes S$, *where* $S \to M$ *is a vector bundle over* M, *then we can define also a Dirac operator on* S, *(***spinor Dirac operator***), as the first order linear differential operator* $\underset{S}{P} : C^{\infty}(S) \to C^{\infty}(S)$, *given by* $\underset{S}{P} = i\underset{S}{P}_0$, *where* i *is the imaginary unity*

[19] The imaginary unity is really pleonastic. Here it is used in order to solder our formulas with usually ones used in theoretical physics.

and $\underset{S}{P}_0$ is defined by means of the following homomorphism of vector bundles over M:

$$\begin{array}{ccccc} J\mathcal{D}(S) & \xrightarrow{\underset{S}{\Gamma}} & T^*M\otimes S & \overset{g'\otimes 1}{\cong} & TM\otimes S \\ \underset{S}{P}_0\downarrow & & & & \downarrow\gamma\otimes 1 \\ S & \leftarrow & S^*\otimes S\otimes S & \cong & E\otimes S \end{array}$$

Of course there is also a Dirac operator for covariant spinors, i.e., defined on S^*, $\underset{S^*}{P} : C^\infty(S^*) \to C^\infty(S^*)$, with $\underset{S^*}{P} = i\underset{S^*}{P}_0$, where $\underset{S^*}{P}_0$ is defined by means of the following homomorphism of vector bundles over M:

$$\begin{array}{ccccc} J\mathcal{D}(S^*) & \xrightarrow{\underset{S^*}{\Gamma}} & T^*M\otimes S^* & \overset{g'\otimes 1}{\cong} & TM\otimes S^* \\ \underset{S^*}{P}_0\downarrow & & & & \downarrow\gamma\otimes 1 \\ S^* & \leftarrow & S^*\otimes S\otimes S^* & \cong & E\otimes S^* \end{array}$$

The covariant local expression for controvariant spinor field $\psi^a e_a : M \to S$ and covariant spinor field $\varphi = \varphi_a\theta^a : M \to S^*$ are respectively:

$$\begin{aligned} \underset{S}{P}.\psi &= ig^{\lambda\epsilon}\left[(\partial x_\lambda.\psi^r) + \underset{S}{\Gamma}{}^r_{\lambda s}\psi^s\right]\gamma_{\epsilon r}{}^a e_a, \quad \underset{S^*}{P}.\varphi \\ &= ig^{\lambda\epsilon}\left[(\partial x_\lambda.\varphi_s) - \underset{S}{\Gamma}{}^r_{\lambda s}\varphi_r\right]\gamma_{\epsilon b}{}^s\theta^b. \end{aligned}$$

In matrix form we can write:

$$\underset{S}{P}.\psi = i\gamma^\lambda(\underset{S}{\nabla}_\lambda.\psi), \quad \underset{S^*}{P}.\varphi = i\gamma^\lambda(\underset{S^*}{\nabla}_\lambda.\varphi).$$

3) The **Clifford Laplace operator** is the second order linear differential operator on the Clifford bundle $\underset{C}{\Delta} : C^\infty(E) \to C^\infty(E)$ defined as the square of the Dirac operator: $\underset{C}{\Delta} = \underset{C}{P}^2$. So if $\psi = \psi^B\gamma_B$ is the local expression of a section of the fiber bundle $\pi : E \to M$, then the local expression of $\underset{C}{\Delta}.\psi$ is as follows:

(3.18)
$$\begin{aligned} \underset{C}{\Delta}.\psi &= \underset{C}{P}i((\underset{C}{\nabla}_\alpha.\psi)^B\gamma^\alpha\gamma_B) = -(\underset{C}{\nabla}_\beta(\underset{C}{\nabla}_\alpha.\psi))^B\gamma^\beta\gamma^\alpha\gamma_B = -\gamma^\beta\gamma^\alpha(\underset{C}{\nabla}_\beta \\ &\quad (\underset{C}{\nabla}_\alpha.\psi))^B\gamma_B. \end{aligned}$$

THEOREM 3.3 - 1) *The Clifford Laplace operator is related to the curvature $\underset{C}{R}$ of the Clifford connection, (hence also to the Levi-Civita curvature). In fact one has*

$$\underset{C}{\triangle}.\psi = -\frac{1}{2}(\gamma^\beta\gamma^\alpha + \gamma^\alpha\gamma^\beta)(\underset{C}{\nabla}_\beta \underset{C}{\nabla}_\alpha.\psi)^B\gamma_B - \frac{1}{2}(\gamma^\beta\gamma^\alpha - \gamma^\alpha\gamma^\beta)(\underset{C}{\nabla}_\beta \underset{C}{\nabla}_\alpha.\psi)^B\gamma_B$$

$$= g^{\beta\alpha}(\underset{C}{\nabla}_\beta \underset{C}{\nabla}_\alpha.\psi)^B\gamma_B - \frac{1}{2}\gamma^\beta\gamma^\alpha\left[(\underset{C}{\nabla}_\beta \underset{C}{\nabla}_\alpha - (\underset{C}{\nabla}_\alpha \underset{C}{\nabla}_\beta).\psi)^B\right]\gamma_B.$$

On the other hand we have

$$\left[\left(\underset{C}{\triangle}_\beta \underset{C}{\triangle}_\alpha - \underset{C}{\triangle}_\alpha \underset{C}{\triangle}_\beta\right).\psi)^B\right]\gamma_B = \underset{C}{R}_{\beta\alpha}{}^B{}_{.D}\psi^D\gamma_B.$$

Therefore we can also write:

$$\underset{C}{\triangle}.\psi = -\left[g^{\beta\alpha}(\underset{C}{\nabla}_\beta \underset{C}{\nabla}_\alpha.\psi)^B + \underset{C}{R}_{\beta\alpha}{}^B{}_{.D}\psi^D\gamma^\beta\gamma^\alpha\right]\gamma_B. \tag{3.19}$$

2) *Similarly we can define the* **spinor Laplace operator** *as follows:* $\underset{S}{\triangle} \equiv \underset{S}{P}^2 : C^\infty(S) \to C^\infty(S)$. *Then for any spinor field ψ, locally written as $\psi = \psi^a e_a$, we get:*

$$\underset{S}{\triangle}.\psi = -\gamma^\beta\gamma^\alpha(\underset{S}{\nabla}_\beta(\underset{S}{\nabla}_\alpha.\psi))^a e_a. \tag{3.20}$$

Similarly we can find the expression of $\underset{S}{\triangle}.\psi$ in terms of curvature of the spinor connection on $\pi : S \to M$. We have:

$$\underset{S}{\triangle}.\psi = -\left[g^{\beta\alpha}(\underset{S}{\nabla}_\beta \underset{S}{\nabla}_\alpha.\psi)^a + \frac{1}{2}\underset{S}{R}_{\beta\alpha}{}^a{}_{.b}\gamma^\beta\gamma^\alpha\psi^b\right]e_a. \tag{3.21}$$

Moreover if we use for the spinor connection the Lichnerowicz's formula, we can write (3.21) in the following form:

$$\underset{S}{\triangle}.\psi = -\left[g^{\beta\alpha}(\underset{S}{\nabla}_\beta \underset{S}{\nabla}_\alpha.\psi)^a + \frac{1}{8}R_{\beta\alpha\mu\lambda}(\gamma^\beta\gamma^\alpha\gamma^\mu\gamma^\lambda)^a_b\psi^b\right]e_a.$$

Then by using the relation $R_{\beta\alpha\mu\lambda}(\gamma^\beta\gamma^\alpha\gamma^\mu\gamma^\lambda)^a_b = 2\delta^a_b R$, [20] *where* R *is the scalar curvature of the Levi-Civita connection, we get:*[21]

$$(3.22)\qquad \underset{S}{\Delta}.\psi = -\left[g^{\beta\alpha}(\underset{S}{\nabla}_\beta \underset{S}{\nabla}_\alpha.\psi)^a + \frac{1}{4}R\psi^a\right] e_a.$$

Furthermore, for covariant spinor field $\varphi = \varphi_a\theta^a$ *we similarly get:*

$$(3.23)\qquad \left\{\begin{aligned} \underset{S^*}{\Delta}.\varphi &= -\gamma^\beta\gamma^\alpha(\underset{S^*}{\nabla}_\beta(\underset{S^*}{\nabla}_\alpha.\varphi))_a\theta^a \\ &= -\left[g^{\beta\alpha}(\underset{S^*}{\nabla}_\beta \underset{S^*}{\nabla}_\alpha.\varphi)_a + \frac{1}{2}\underset{S}{R}_{\beta\alpha}{}^b{}_{.a}\gamma^\beta\gamma^\alpha\varphi_b\right]\theta^a \\ &= \left[g^{\beta\alpha}(\underset{S^*}{\Delta}_\beta \underset{S^*}{\Delta}_\alpha.\varphi)_a + \frac{1}{4}R\varphi_a\right]\theta^a. \end{aligned}\right\}$$

DEFINITION 3.2 - 1) (**Generalized Dirac-Clifford equations**). *One has two generalized Dirac-Clifford equations:*

(3.24)

$$(DC)_\mp \equiv \ker(\underset{C}{P} \mp m) \subset J\mathcal{D}(E):\quad \left\{\left[i\gamma^\alpha(\underset{C}{\nabla}.\psi)^B \mp m\psi^B\right]\gamma_B = 0\right\}$$

where $m \in \mathbf{R}\bigcup i\mathbf{R}^+$. *An equivalent expression of* (3.24) *is the following:*

$$(3.25)\qquad i\gamma^\alpha\left[(\partial x_\alpha.\psi^B) + \underset{C}{\Gamma}{}^B_{\alpha C}\psi^C\right] \mp m\psi^B = 0.$$

2) (**Generalized Dirac-spinor equations**). *Similarly one has two generalized Dirac-spinor equations:*[22]

$$(3.26)\qquad \left\{\begin{aligned} &(DS)_\mp \equiv \ker(\underset{S}{P} \mp m) \subset J\mathcal{D}(S) \\ &(\underset{S}{P} \mp m)\psi = 0 \Leftrightarrow \left\{i\gamma^{\lambda}{}^a_r\left[(\partial x_\lambda.\psi^r) + \underset{S}{\Gamma}{}^r_{\lambda s}\psi^s\right] \mp m\psi^a = 0\right\} \end{aligned}\right\}.$$

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[20] See Appendix A2 for a proof of this formula.

[21] This expression coincides with the well known formula given by A.Lichnerowicz [1].

[22] We set $(GM)_m \equiv (DC)_-$ or $(GM)_m \equiv (DS)_-$.

The corresponding dual Dirac adjoint equation results:

$$(3.27)\quad \left\{ \begin{array}{l} (DS^*)_{\mp} \equiv \ker(\underset{S^*}{P} \mp m) \subset J\mathcal{D}(S^*) \\ (\underset{S^*}{P} \mp m)\varphi = 0 \Leftrightarrow \left\{ i\gamma^{\lambda}{}_{b}^{s}\left[(\partial x_{\lambda}.\varphi_s) - \underset{S}{\Gamma}{}^{r}_{\lambda s}\varphi_r\right] \mp m\varphi_b = 0 \right\} \end{array} \right\}.$$

3) **(Generalized Klein-Gordon Clifford Equation)**. *This is the following equation:*

(3.28)

$$\left\{ \begin{array}{l} (KGC) \subset J\mathcal{D}^2(E) \\ (\underset{C}{P}^2 - m^2)\psi = 0 \Leftrightarrow (\underset{C}{P} - m)(\underset{C}{P} + m)\psi = 0 \Leftrightarrow (\underset{C}{\Delta} - m^2)\psi = 0 \end{array} \right\}.$$

Tle local expression of (KGC) is the following:

$$(3.29)\quad g^{\beta\alpha}(\underset{C}{\nabla}_{\beta}\underset{C}{\nabla}_{\alpha}\psi)^B + \left(\underset{C}{R}_{\beta\alpha}{}^{B}{}_{.D}\gamma^{\beta}\gamma^{\alpha} + \delta^B_D m^2\right)\psi^D = 0.$$

4) **(Generalized Klein-Gordon Spinor Equation)**. *Similarly we get the following equation:*

$$(3.30)\quad \left\{ \begin{array}{l} (KGS) \subset J\mathcal{D}^2(S) \\ (\underset{S}{P}^2 - m^2)\psi = 0 \Leftrightarrow (\underset{S}{P} - m)(\underset{S}{P} + m)\psi = 0 \Leftrightarrow (\underset{S}{\Delta} - m^2)\psi = 0 \end{array} \right\}$$

Tle local expression of (KGS) is the following:

$$(3.31)\quad g^{\beta\alpha}(\underset{S}{\nabla}_{\beta}\underset{S}{\nabla}_{\alpha}\psi)^a + \left(\underset{S}{R}_{\beta\alpha}{}^{a}{}_{.b}\gamma^{\beta}\gamma^{\alpha} + \delta^a_b m^2\right)\psi^b = 0.$$

In particular by using the Lichnerowicz's formula for the spinor connection we get the following local expression for (KGS):[23]

$$(3.32)\quad g^{\beta\alpha}(\underset{S}{\nabla}_{\beta}\underset{S}{\nabla}_{\alpha}\psi)^a + \left(\frac{1}{4}R + m^2\right)\psi^a = 0.$$

[23] Note that

$$g^{\beta\alpha}(\underset{S}{\nabla}_{\beta}\underset{S}{\nabla}_{\alpha}\psi)^a = g^{\beta\alpha}\{(\partial x_{\beta}\partial x_{\alpha}.\psi^a) + (\partial x_{\alpha}.\psi^b)\underset{S}{\Gamma}{}^{a}_{\beta b} + (\partial x_{\beta}.\psi^b)\underset{S}{\Gamma}{}^{a}_{\alpha b} + \psi^c[(\partial x_{\beta}.\underset{S}{\Gamma}{}^{a}_{\alpha c}) + \underset{S}{\Gamma}{}^{b}_{\alpha c}\underset{S}{\Gamma}{}^{a}_{\beta b}]\}.$$

The corresponding expression for covariant spinor field is the following:

$$(3.33) \qquad g^{\beta\alpha}(\underset{S^*}{\nabla}{}_{\beta}\underset{S^*}{\nabla}{}_{\alpha}\varphi)_a + \left(\frac{1}{4}R + m^2\right)\varphi_a = 0.$$

REMARK 3.1 - The **Dirac conjugated**, (or h.c.), of a spinor field $\psi = \psi^a e_a : M \to S$ is the covariant field $\bar{\psi} = \varphi\gamma^0 : M \to S^*$, where $\varphi = \varphi_a\theta^a$ with $\varphi_a = (\psi^a)^*$, ($* =$ complex conjugated). Then the generalized Dirac equation for $\bar{\psi}$, (**generalized Dirac h.c.**), is the following (written in matrix form): $i\underset{S^*}{\nabla}{}_{\mu}\bar{\psi}\gamma^{\mu} + m\bar{\psi} = 0$. It is important to note that the Dirac equation and its conjugated, must be considered together as they together are the Euler-Lagrange equation of a first order Lagrangian $\mathcal{L} : J\mathcal{D}(S^* \otimes S) \to \mathbf{C}$, defined on $S^* \otimes S$, i.e., defined on the Clifford bundle. More precisely the Lagrangian $\mathcal{L}$, in matrix form, can be written in the following way:

$$(3.34) \qquad \mathcal{L} = \frac{1}{2}i\left[\bar{\psi}\gamma^{\mu}\underset{S}{\nabla}\psi - (\underset{S^*}{\nabla}\bar{\psi})\gamma^{\mu}\psi\right] - m\bar{\psi}\psi.$$

The variation of the action integral with respect to $\bar{\psi}$, (resp. ψ), yields the generalized Dirac equation for ψ, (resp. $\bar{\psi}$). So the Euler-Lagrange equation, $E[\mathcal{L}] \subset J\mathcal{D}^2(S^* \otimes S)$, of $\mathcal{L}$ is the following:

$$E[\mathcal{L}] \subset J\mathcal{D}^2(S^*\otimes S) : \left\{\begin{array}{ll} i\gamma^{\mu}\underset{S}{\nabla}{}_{\mu}\psi - m\psi = 0 & \textbf{(Generalized Dirac-equation)} \\ i(\underset{S^*}{\nabla}{}_{\mu}\bar{\psi})\gamma^{\mu} - m\bar{\psi} = 0 & \textbf{(Generalized Dirac-h.c. equation)} \end{array}\right\}.$$

Above remark shows that a theory for spinor fields could be written on the Clifford bundle instead then on spinor bundle only. Furthermore, if we consider also that the isomorphism $E \cong S^* \otimes S$ is conditioned to some very particular categories of space-times, (e.g., almost Hermitian or almost Kählerian space-times), it follows that it should be more convenient to implement any physical theory for "spinor fields" on Clifford bundles. Therefore the local expression of $(GD)_m$ with respect to local coordinates $\{x^\alpha, z^A, z^A_\alpha\}$ on $J\mathcal{D}(E)$ is the following:[24]

$$(GM)_m \subset J\mathcal{D}(E): \quad \left\{ F^A \equiv i\gamma^\alpha \left[z^B_\alpha + z^A \underset{C}{\Gamma}{}^B_{\alpha A} \right] - m\delta^B_A z^A = 0 \right\} \tag{3.35}$$

■

THEOREM 3.4 - (STRUCTURE OF THE GENERALIZED DIRAC EQUATION, TUNNEL EFFECTS AND CANONICAL QUANTIZATION). 1) *For the generalized Dirac equation $(GD)_m$ we have a structure similar to $(GKG)_\chi$, even if it is of the first order. In fact, $(GD)_m$ is an involutive formally integrable, as well completely integrable PDE, hence it is equivalent to its first prolongation $((GD)_m)_{+1} \subset J\mathcal{D}^2(E)$.*
2) *If there is the isomorphism $E \cong S^* \otimes S$, then $(GD)_m$ is isomorphic to the tensor product of two equations: $(GD)_m \cong \underline{(GD)_m} \otimes \overline{(GD)_m}$, where $\overline{(GD)_m}$ is the Dirac hermitiam conjugated of $\underline{(GD)_m}$ such that the following diagram is commutative:*

$$\begin{array}{ccccc} & & ((\underline{(GD)_m} \times \overline{(GD)_m} & \subset & J\mathcal{D}(S^* \times S) \cong J\mathcal{D}(S^*) \times J\mathcal{D}(S) \\ & & \downarrow & & \downarrow \\ (GD)_m & \cong & \underline{(GD)_m} \otimes \overline{(GD)_m} & \subset & J\mathcal{D}(S^* \otimes S) \end{array}$$

Then similarly to what made for the equation $(GKG)_\chi$, we can prove that Cauchy problem can be solved for the equation $(GD)_m$ by the method of characteristics. Furthermore we can also prove that the integral bordism group $\Omega_3^{(GD)_m}$ of $(GD)_m$ is trivial: $\Omega_3^{(GD)_m} = 0$. Therefore, tunnel effects can be observed and propagators associated to these calculated.
3) *In the following we shall specialize on the Dirac equation for spin 1/2, on globally hyperbolic 4-dimensional space-time. Thus we shall consider the following equations:*

$$\left\{ \begin{array}{l} (i\gamma^\mu \nabla_\mu - m)\psi = 0 \\ i\nabla_\mu \bar{\psi}\gamma^\mu - m\bar{\psi} = 0 \end{array} \right\}. \tag{3.36}$$

[24] Note that in the particular case of fermionic bradions and luxons the parameter m is a real number, instead for massive neutrinos it is an imaginary number.

The canonical quantization of the generalized Dirac equation $(GD)_m \subset J\mathcal{D}^2(E)$ is given by the following anticommutation relations:

$$(3.37)\qquad \left\{\begin{array}{l} [\widehat{s_A}(x),\widehat{s_B}(x')]_+=[\widehat{\bar{s}_A}(x),\widehat{\bar{s}_B}(x')]_+=0 \\ [\widehat{s_A}(x),\widehat{\bar{s}_B}(x')]_+=i\hbar\widetilde{G}_{AB}(x;x'|m) \end{array}\right\}$$

where $\widetilde{G}_{AB}(x;x'|m) = \left(\blacksquare_{x'} + m\right)_{AB} G(x;\, \widetilde{x}'|m)$, $(\blacksquare_{x'})_{AB} \equiv (i\gamma^\mu\nabla_\mu)_{AB}$, where $\widetilde{G}(x;x'|m)$ is the propagator of the generalized Klein-Gordon spinor equation. Then, the microscopic causality is conserved also for fermionic massive neutrinos having a **geometric mass** $\chi \equiv \frac{1}{4}R + m^2 < 0$.

PROOF. 1) This can be easily seen by rewriting equation (3.37) in the following splited form:

$$(3.38)\qquad \left\{\begin{array}{l} i\gamma^\alpha\left[\xi^B_\alpha + \underset{C}{\left[{}^{B}_{\alpha A}\right.}\xi^A\right] - m\delta^B_A\xi^A = 0 \\ i\gamma^\alpha\left[\eta^B_\alpha + \underset{C}{\left[{}^{B}_{\alpha A}\right.}\eta^A\right] - m\delta^B_A\eta^A = 0 \end{array}\right.$$

where $z^A \equiv \xi^A + i\eta^A$ and $z^A_\alpha \equiv \xi^A_\alpha + i\eta^A_\alpha$.

2) If there is an isomorphism $E \cong S^* \otimes S$ we can represent $(GM)_m$ in the form $(GM)_m \cong \underline{((GM)_m} \otimes \overline{(GD)_m}$.

3) The propagator $\widetilde{G}$ is the Green kernel solution of the following Cauchy problem:

$$(3.39)\qquad \left\{\begin{array}{l} ((\blacksquare_{x'})^C_A - m\delta^C_A)\widetilde{G}^A_B(x;x'|m)=0 \\ \widetilde{G}^A_B(x;x'|m)|_{x^0=x'^0}=0 \\ \left(\partial x'_0.\widetilde{G}^A_B\right)(x;x')|_{x^0=x'^0}=-\delta^A_B\delta(x;x')|_{x^0=x'^0} \end{array}\right\}+\text{h.c.}.$$

Let us find solutions of the type $\widetilde{G}(x;x'|m) = (\blacksquare_{x'} + m)\,\widetilde{G}(x;x'|m)$. Then we find that this is a solution iff $\widetilde{G}^A_B(x;x'|m)$ satisfies the equation $\left(\underset{S}{\Delta} - m^2\right)\widetilde{G}(x;x'|m) = 0$. This means that $\widetilde{G}(x;x'|m)$ is the propagator of the generalized Klein-Gordon spinor equation. Then we get the usual anticommutation relations:

$$[\widehat{\psi_A}(x),\widehat{\psi_B}(x')]_+=[\widehat{\bar{\psi}_A}(x),\widehat{\bar{\psi}_B}(x')]_+=0$$
$$[\widehat{\psi_A}(x),\widehat{\bar{\psi}_B}(x')]_+=i\hbar\widetilde{G}_{AB}(x;x'|m)$$

On the other hand, as the operator $\underset{S}{\Delta}-m^2$ is hyperbolic its propagator $\widetilde{G}_{AB}(x;x'|m)$ exists and it is unique and with support in $\mathcal{E}^-(x')\cup\mathcal{E}^+(x')$, $\forall x'\in M$. So the quantum commutator $[\hat{\psi}^A(x),\hat{\bar{\psi}}^B(x')]_+=i\hbar\widetilde{G}_{AB}(x;x'|m)$ respects the microscopic causality even if the geometric mass of the field is negative. □

4 - FIELDS IN INTERACTION

Above field equations are, of course, linear PDEs, and represent free fields on a general curved space-time background identified with a globally hyperbolic 4-dimensional manifold. For interacting fields the corresponding PDEs are not more linear. However the general geometric method developed by us to quantize PDEs [5-8] works well also for interacting fields. In fact that method has been formulated to work for nonlinear PDEs also. As an example let us consider the following.

□ (DECAY OF SCALAR PION). We shall study the following decay: $\pi_0^+\to\mu^++\nu_\mu$. Of course the neutrino path is not directly observed. We will assume that the neutrino is massive. In this case we have scalar particles (π_0^+) and Dirac particles (μ^+, ν_μ). So the process is described by a second order PDE $(E_2)\subset J\mathcal{D}^2(E)$, where the fiber bundle E is defined by $E\equiv E_{(\pi_0)}\times_M E_{(\mu)}\times_M E_{(\nu)}$, given by:[25]

(4.1)
$$\left\{\begin{array}{l}(\Box_{x'}+\chi_{(\pi_0)})\varphi=j_{(\pi_0)}\equiv\lambda\bar{\psi}_{(\nu)}\psi_{(\mu)}\\ (i\gamma^\alpha\nabla_\alpha-m_{(\mu)})\psi_{(\mu)}=j_{(\mu)}\equiv\lambda\psi_{(\nu)}\varphi\\ (i\gamma^\alpha\nabla_\alpha-m_{(\nu)})\psi_{(\nu)}=j_{(\nu)}\equiv\lambda\bar{\varphi}\psi_{(\mu)}\end{array}\right\}\left\{\begin{array}{l}i\nabla_\alpha\bar{\psi}_{(\mu)}\gamma^\alpha-m_{(\mu)}\bar{\psi}_{(\mu)}=\bar{j}_{(\mu)}\\ i\nabla_\alpha\bar{\psi}_{(\nu)}\gamma^\alpha-m_{(\nu)}\bar{\psi}_{(\nu)}=\bar{j}_{(\nu)}\end{array}\right\}\{\chi_{(\pi_0)}\equiv\xi_{(\pi_0)}R+m^2_{(\pi_0)}\}$$

[25] Here we use the following notation: $\bar{\varphi}$ is the c.c. of φ and $\bar{\psi}=\psi^*\gamma^0$, with ψ^* the c.c. of ψ.

The local expression of (4.1) is the following:

$$\left\{\begin{array}{r}
g^{\alpha\beta}\varphi_{\alpha\beta}-\Gamma^{\alpha}_{\beta\gamma}g^{\gamma\beta}\varphi_{\alpha}+\chi_{(\pi_0)}\varphi=\lambda\bar{\psi}_{(\nu)j}\psi^j_{(\mu)}\\
(i\gamma^{\alpha}{}^i_j\nabla_\alpha-m_{(\mu)}\delta^i_j)\psi^j_{(\mu)}\equiv i\gamma^{\alpha}{}^i_j\left[(\partial x_\alpha.\psi^j_{(\mu)})+\underset{S}{\Gamma^j_{\alpha s}}\psi^s\right]-m_{(\mu)}\delta^i_j\psi^j_{(\mu)}=\lambda\varphi\psi^i_{(\nu)}\\
(i\gamma^{\alpha}{}^i_j\nabla_\alpha-m_{(\nu)}\delta^i_j)\psi^j_{(\nu)}\equiv i\gamma^{\alpha}{}^i_j\left[(\partial x_\alpha.\psi^j_{(\nu)})+\underset{S}{\Gamma^j_{\alpha s}}\psi^s\right]-m_{(\nu)}\delta^i_j\psi^j_{(\nu)}=\lambda\bar{\varphi}\psi^i_{(\mu)}
\end{array}\right\}+\text{h.c.}$$

Then, the corresponding Jacobi operator has the following components:[26]

$$\left\{\begin{array}{ll}
(J[s].\nu)_{(\pi_0)}\equiv & [\chi_{(\pi_0)}]\nu_{(\pi_0)}-[\Gamma^{\alpha}_{\beta\gamma}g^{\gamma\beta}]_\bullet\nu_{(\pi_0),\alpha}+[g^{\alpha\beta}]_\bullet\nu_{(\pi_0),\alpha\beta}-\delta_{ij}[\lambda\bar{\psi}^i_{(\nu)}]\nu^j_{(\mu)}-\\
& \delta_{ij}[\lambda\psi^j_{(\mu)}]_\bullet\bar{\nu}^i_{(\nu)}\\
(J[s].\nu)^i_{(\mu)}\equiv & [i\gamma^i_{\beta l}g^{\alpha\beta}\Gamma^l_{\alpha j}-m_{(\mu)}\delta^i_j]_\bullet\nu^j_{(\mu)}+i[\gamma^i_{\alpha j}g^{\alpha\beta}]_\bullet\nu^j_{(\mu),\beta}-[\lambda\psi^i_{(\nu)}]_\bullet\nu_{(\pi_0)}-\\
& [\lambda\varphi\delta^i_j]_\bullet\nu^j_{(\nu)}\\
(J[s].\nu)^i_{(\nu)}\equiv & [i\gamma^i_{\beta l}g^{\alpha\beta}\Gamma^l_{\alpha j}-m_{(\nu)}\delta^i_j]_\bullet\nu^j_{(\nu)}+i[\gamma^i_{\alpha j}g^{\alpha\beta}]_\bullet\nu^j_{(\nu),\beta}-[\lambda\bar{\varphi}]_\bullet\nu^i_{(\mu)}-\\
& [\lambda\psi^i_{(\mu)}]_\bullet\bar{\nu}_{(\pi_0)}.
\end{array}\right\}+\text{h.c.}$$

The equations for the Green kernel are given by the following Cauchy problem:

(4.2)

$$\left\{\begin{array}{r}
\chi_{(\pi_0)}G_{(\pi_0)}(x;x')-[\Gamma^{\alpha}_{\beta\gamma}g^{\gamma\beta}]_\bullet(\partial x_\alpha.G_{(\pi_0)})(x;x')+[g^{\alpha\beta}]_\bullet(\partial x_\alpha\partial x_\beta.G_{(\pi_0)})(x;x')-\\
\delta_{ij}[\lambda\bar{\psi}^i_{(\nu)}]_\bullet G^j_{(\mu)}(x;x')-\delta_{ij}[\lambda\psi^i_{(\mu)}]_\bullet\bar{G}^j_{(\nu)}(x;x')=0\\
[i\gamma^i_{\beta l}g^{\alpha\beta}\Gamma^l_{\alpha j}-m_{(\mu)}\delta^i_j]_\bullet G^j_{(\mu)}(x;x')+[i\gamma^i_{\alpha j}g^{\alpha\beta}]_\bullet(\partial x_\beta.G^j_{(\mu)})(x;x')-[\lambda\psi^i_{(\nu)}]_\bullet G_{(\pi_0)}(x;x')-\\
[\lambda\varphi\delta^i_j]_\bullet G^j_{(\nu)}(x;x')=0\\
[i\gamma^i_{\beta l}g^{\alpha\beta}\Gamma^l_{\alpha j}-m_{(\nu)}\delta^i_j]_\bullet G^j_{(\nu)}(x;x')+[i\gamma^i_{\alpha j}g^{\alpha\beta}]_\bullet(\partial x_\beta.G^j_{(\nu)})(x;x')-[\lambda\bar{\varphi}]_\bullet G^i_{(\mu)}(x;x')-\\
[\lambda\psi_{(\mu)}]_\bullet\bar{G}_{(\pi_0)}(x;x')=0\\
\left\{\begin{array}{l}
G(x^0,\mathbf{x};x^0,\mathbf{x}')=0\\
(\partial x'_0.G^{IJ})(x;x')|_{x^0=x'^0}=\delta^{IJ}\delta(x;x')|_{x^0=x'^0}
\end{array}\right\}
\end{array}\right\}+\text{h.c.}$$

Set

$$G(x;x')=\begin{pmatrix}X\\Y^j\\Z^j\end{pmatrix};\quad \bar{G}(x;x')=\begin{pmatrix}\bar{X}\\\bar{Y}^j\\\bar{Z}^j\end{pmatrix}.$$

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[26] Here and in the following the bracket $[-]_\bullet$ denotes "evaluation at a solution of the equation (4.1)".

Then, we get the following expression for the system (4.2):

$$(4.3)\qquad \left\{\begin{array}{l} \left[\square_{x'}+\chi_{(\pi_0)}\right]_{\bullet}X-[\lambda\bar{\psi}_{(\nu)}]_{\bullet}Y-[\lambda\psi_{(\mu)}]_{\bullet}\bar{Z}=0\\ \left[\blacksquare_{x'}-m_{(\mu)}\right]_{\bullet}Y-[\lambda\psi_{(\nu)}]_{\bullet}X-[\lambda\varphi]_{\bullet}Z=0\\ \left[\blacksquare_{x'}-m_{(\nu)}\right]_{\bullet}Z-[\lambda\bar{\varphi}]_{\bullet}Y-[\lambda\psi_{(\mu)}]_{\bullet}\bar{X}=0\\ +\text{Cauchy conditions:}\left\{\begin{array}{l} X|_{x^0=x'^0}=Y|_{x^0=x'^0}=Z|_{x^0=x'^0}=0\\ (\partial x'_0,X)|_{x^0=x'^0}=\delta(x;x')|_{x^0=x'^0}\\ (\partial x'_0,Y)|_{x^0=x'^0}=\delta(x;x')|_{x^0=x'^0}\\ (\partial x'_0,Z)|_{x^0=x'^0}=\delta(x;x')|_{x^0=x'^0} \end{array}\right\} \end{array}\right\}+\text{h.c.}$$

Then the propagator $\widetilde{G}=G^{+}-G^{-}$ is given by means of the retarded and advanced solutions

$$G^{\pm}(x;x')=\begin{pmatrix} X^{\pm}\\ Y^{\pm,j}\\ Z^{\pm,j}\end{pmatrix},\quad \bar{G}^{\pm}(x;x')=\begin{pmatrix} \bar{X}^{\pm}\\ \bar{Y}^{\pm,j}\\ \bar{Z}^{\pm,j}\end{pmatrix},\quad \widetilde{G}(x;x')=\begin{pmatrix} X^{+}(x;x')-X^{-}(x;x')\\ Y^{+}(x;x')-Y^{-}(x;x')\\ Z^{+}(x;x')-Z^{-}(x;x')\end{pmatrix}.$$

As a consequence the quantization of the dynamic equation (4.1) is obtained by considering the following quantum bracket:

$$\left[\widehat{\phi^I}(x),\widehat{\phi^J}(x')\right]_{\pm}=i\hbar\widetilde{G}^{IJ}(x;x')\mathbf{1}$$

where $(\phi^I)=\begin{pmatrix}\varphi\\ \psi^i_{(\mu)}\\ \psi^i_{(\nu)}\end{pmatrix}$.[27] By derivation of above commutator with respect to x'^0, and taking $x^0=x'^0$, and by considering also equation (4.2), we get:

$$\left[\widehat{\phi^I}(x),\dot{\widehat{\phi^J}}(x')\right]_{\pm}|_{x^0=x'^0}=i\hbar\delta^{IJ}(x;x')|_{x^0=x'^0}\mathbf{1}.$$

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[27] $[A,B]_-\equiv AB-BA$ is the commutator or simply $[A,B]$, instead $[A,B]_+\equiv AB+BA$ is the anticommutator. If both A and B are fermionic one has the anticommutator otherwise one uses the commutator.

This completes the procedure to obtain the canonical quantization of our dynamical equation for the decay $\pi_0^+ \to \mu^+ + \nu_\mu$. Note that equation (4.1) admits the zero section $(\varphi, \psi_{(\mu)}, \psi_{(\nu)}) = (0,0,0)$ as a solution. The canonical quantization of (4.1) at the zero section, reduces to the following Jacobi equation:

$$\left\{ \left\{ \begin{array}{l} (\square_{x'} + \chi_{(\pi)})X = 0 \\ (\blacksquare_{x'} - m_{(\mu)})Y = 0 \\ (\blacksquare_{x'} - m_{(\nu)})Z = 0 \end{array} \right\} + \begin{array}{c} \text{Cauchy} \\ \text{conditions} \end{array} : \left\{ \begin{array}{l} X|_{x^0=x'^0} = Y|_{x^0=x'^0} = Z|_{x^0=x'^0} = 0 \\ (\partial x'_0 . X)|_{x^0=x'^0} = \delta(x;x')|_{x^0=x'^0} \\ (\partial x'_0 . Y)|_{x^0=x'^0} = \delta(x;x')|_{x^0=x'^0} \\ (\partial x'_0 . Z)|_{x^0=x'^0} = \delta(x;x')|_{x^0=x'^0} \end{array} \right\} \right\} + \text{h.c.}$$

Then the propagator of (4.1) at the zero section is directly identified by the propagator of the independent fields. So we get the following (anti)commutation relations:

$$[\hat{\varphi}(x), \hat{\varphi}(x')] = [\widehat{\bar{\varphi}}(x), \widehat{\bar{\varphi}}(x')] = 0$$

$$[\hat{\varphi}(x), \widehat{\bar{\varphi}}(x')] = i\hbar \widetilde{G}_{(\pi_0)}(x;x'|\chi_{(\pi_0)}) \ \mathbf{1}$$

$$[\hat{\varphi}(x), \dot{\hat{\varphi}}(x')] = [\widehat{\bar{\varphi}}(x), \dot{\widehat{\bar{\varphi}}}(x')] = 0$$

$$[\hat{\varphi}(x), \dot{\widehat{\bar{\varphi}}}(x')]|_{x^0=x'^0} = -i\hbar \delta(x;x')|_{x^0=x'^0} \ \mathbf{1}$$

$$[\widehat{\psi_{(\mu)A}}(x), \widehat{\psi_{(\mu)B}}(x')]_+ = [\widehat{\bar{\psi}_{(\mu)A}}(x), \widehat{\bar{\psi}_{(\mu)B}}(x')]_+ = 0$$

$$[\widehat{\psi_{(\mu)A}}(x), \widehat{\bar{\psi}_{(\mu)B}}(x')]_+ = i\hbar \widetilde{G}_{(\mu)AB}(x;x'|m_{(\mu)})$$

$$[\widehat{\psi_{(\mu)}}(x), \dot{\widehat{\psi_{(\mu)}}}(x')]_+ = [\widehat{\bar{\psi}_{(\mu)}}(x), \dot{\widehat{\bar{\psi}_{(\mu)}}}(x')]_+ = 0$$

$$[\widehat{\psi_{(\mu)A}}(x), \dot{\widehat{\bar{\psi}_{(\mu)B}}}(x')]_+|_{x^0=x'^0} = i\hbar \delta_{AB}(x;x')|_{x^0=x'^0}$$

$$[\widehat{\psi_{(\nu)A}}(x), \widehat{\psi_{(\nu)B}}(x')]_+ = [\widehat{\bar{\psi}_{(\nu)A}}(x), \widehat{\bar{\psi}_{(\nu)B}}(x')]_+ = 0$$

$$[\widehat{\psi_{(\nu)A}}(x), \widehat{\bar{\psi}_{(\nu)B}}(x')]_+ = i\hbar \widetilde{G}_{(\nu)AB}(x;x'|m_{(\nu)})$$

$$[\widehat{\psi_{(\nu)}}(x), \dot{\widehat{\psi_{(\nu)}}}(x')]_+ = [\widehat{\bar{\psi}_{(\nu)}}(x), \dot{\widehat{\bar{\psi}_{(\nu)}}}(x')]_+ = 0$$

$$[\widehat{\psi_{(\nu)A}}(x), \dot{\widehat{\bar{\psi}_{(\nu)B}}}(x')]_+|_{x^0=x'^0} = i\hbar \delta_{AB}(x;x')|_{x^0=x'^0}$$

where $\widetilde{G}_{(\pi_0)}$, (resp. $\widetilde{G}_{(\mu)}$, $\widetilde{G}_{(\nu)}$), is the propagator for the Klein-Gordon, (resp. Dirac μ-particle, resp. Dirac ν-particle), equation. As an application of Theorem 1.2 we get that $\widetilde{G}[s]$ is an invariant of $\Omega_c(R_1, R_2)$.

Furthermore, the integral bordism groups of the dynamic equation (4.3) E_2 are given by $\Omega_p^{E_2} \cong \mathbf{Z}_2$, $p = 0, 2$, and $\Omega_p^{E_2} \cong 0$, $p = 1, 3$. In particular we get existence of tunnel effects for global solutions of (4.3).

□ (QUANTUM PARTICLES IN INTERACTION WITH THE GRAVITATIONAL FIELD). Emphasize that we can consider also the interaction of above particles with the gravitational field. In fact the general method pointed out by A.Prástaro to describe the canonical quantization of PDEs can be applied also to this case. In particular we shall reconsider the decay $\pi_0^+ \to \mu^+ + \nu_\mu$ by inserting the interaction with gravitons, i.e., the interaction with the gravitational field as a quantum field. So we will consider the following fiber bundle $\pi : E \equiv E_{(\pi_0)} \times_M E_{(\mu)} \times_M E_{(\nu)} \times_M S_2^0 M \to M$. A section $s = (\varphi, \psi_{(\mu)}, \psi_{(\nu)}, g)$ of π must be a solution of the following second order PDE on E:

(4.4)
$$\left\{\begin{array}{l}(\square+\chi_{(\pi_0)})\varphi=j_{(\pi_0)}\\ (\blacksquare-m_{(\mu)})\psi_{(\mu)}=j_{(\mu)}\\ (\blacksquare-m_{(\nu)})\psi_{(\nu)}=j_{(\nu)}\\ G=-kT, \quad \mathrm{div}\,(T)=0\end{array}\right\}+\text{h.c.}, \left\{\begin{array}{l}G_{\alpha\beta}=R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R\\ R_{\alpha\beta}=R^{\lambda}_{\beta\alpha\lambda}=(\partial x_\alpha \Gamma^{\lambda}_{\beta\lambda})-(\partial x_\lambda \Gamma^{\lambda}_{\alpha\beta})+\Gamma^{\lambda}_{\alpha\sigma}\Gamma^{\sigma}_{\beta\lambda}-\Gamma^{\lambda}_{\sigma\lambda}\Gamma^{\sigma}_{\alpha\beta}\\ R=R^{\alpha}_{\alpha}\end{array}\right\}.$$

where the currents are as before and the stress-tensor T is given by $T = T_{(\pi_0)} + T_{(\mu)} + T_{(\nu)}$ with

$$\left\{\begin{array}{l}T_{(\pi_0)\alpha\beta}=(1-2\xi_{(\pi_0)}\varphi_{/\alpha}\varphi_{/\beta}+(2\xi_{(\pi_0)}-\frac{1}{2})g_{\alpha\beta}g^{\rho\sigma}\varphi_{/\rho}\varphi_{/\sigma}-2\xi_{(\pi_0)}\varphi_{/\alpha\beta}\varphi\\ \qquad +\frac{1}{2}\xi_{(\pi_0)}g_{\alpha\beta}\varphi\square\varphi-\xi_{(\pi_0)}[R_{\alpha\beta}-\frac{1}{2}Rg_{\alpha\beta}+\frac{3}{2}\xi_{(\pi_0)}Rg_{\alpha\beta}]\varphi^2\\ \qquad +\frac{1}{2}[1-3\xi_{(\pi_0)}]m^2 g_{\alpha\beta}\varphi^2\\ T_{(\mu)\alpha\beta}=\frac{i}{2}[\bar{\psi}_{(\mu)}\gamma_{(\alpha}\Delta_{\beta)}\psi_{(\mu)}-(\Delta_{(\alpha}\bar{\psi}_{(\mu)})\gamma_{\beta)}\psi_{(\mu)}]\\ T_{(\nu)\alpha\beta}=\frac{i}{2}[\bar{\psi}_{(\nu)}\gamma_{(\alpha}\Delta_{\beta)}\psi_{(\nu)}-(\Delta_{(\alpha}\bar{\psi}_{(\nu)})\gamma_{\beta)}\psi_{(\nu)}]\end{array}\right\}.$$

The canonical quantization of the dynamic equation (4.4) gives

(4.5) $$[\widehat{s^I}(x), \widehat{s^J}(x')]_{\pm} = i\hbar\widetilde{G}^{IJ}(x;x'|s)\,\mathbf{1}$$

where $\widetilde{G}^{IJ}$ is the propagator, solution of the following Cauchy problem:

$$(4.6)\quad \left\{ \begin{array}{l} (J[s].\widetilde{G})^{IJ} = 0 \\ \widetilde{G}^{IJ}(x^0, \mathbf{x}; x'^0, \mathbf{x}'|s) = 0 \\ (\partial x'_0.\widetilde{G}^{IJ})(x; x'|s)|_{x^0=x'^0} = \delta^{IJ}\delta(x; x')|_{x^0=x'^0} \end{array} \right\}$$

where $J[s]$. is the Jacobi operator of equation (4.4) at the solution s. Then, by derivation of (4.5) with respect to x'^0, and taking $x^0 = x'^0$, and by considering also equation (4.6), we get:

$$(4.7)\quad [\widehat{s^I}(x), \widehat{s^J}(x')]_{\pm}|_{x^0=x'^0} = i\hbar\delta^{IJ}(x; x')|_{x^0=x'^0}\,\mathbf{1}.$$

Then, similarly to previous examples, as the Jacobi operator is hyperbolic we get that the microscopic causality is conserved even if the geometric mass of the neutrino is negative. Furthermore, as the integral bordism groups $\Omega_3^{E_2} = 0$, we get also the existence of global solutions of the dynamic equation with tunnel effects.

APPENDIX : A1 - SPACE-TIME GEOMETRY AND CAUSALITY

In this paper we shall assume that space-time is a connected 4-dimensional non compact manifold M with a local hyperbolic metric, g, signature $(+---)$. So we can locally orient M by means of the volume form $\eta{=}\sqrt{|g|}dx^0{\wedge}dx^1{\wedge}dx^2{\wedge}dx^3$ canonically associated to g [28]. Here we are interested to emphasize the following consequences of such a structure of space-time: 1) The light-cone $C_p{\subset}T_pM$, $g(p)(v,v){=}0$, at $p{\in}M$, has an induced

[28] Recall here the homological meaning of local orientation. Let M be a n-dimensional manifold. A **local orientation** μ_x for M at x is a choice of one of the two possible generators $H_n(M,M\backslash x;\mathbf{Z}){\cong}H_n(\mathbf{R}^n,\mathbf{R}^n\backslash\{0\};\mathbf{Z}){=}$ infinite cyclic. Such a μ_x determines local orientations μ_y for all points y in a small neighborhood of x. In fact, if B is a ball about x, then for each $y{\in}B$ the isomorphisms $H_\bullet(M,M\backslash x;\mathbf{Z}) \overset{\rho_x}{\leftarrow} H_\bullet(M,M\backslash B;\mathbf{Z}) \overset{\rho_y}{\rightarrow} H_\bullet(M,M\backslash y;\mathbf{Z})$ determine a local orientation μ_y. Furthermore, an **orientation** for M is a function which assigns to each $x{\in}M$ a local orientation μ_x such that should exist a compact neighborhood N and a class $\mu_N{\in}H_n(M,M\backslash N;\mathbf{Z})$ such that $\rho_y(\mu_N){=}\mu_y$ for each $y{\in}N$. An **oriented manifold** is a manifold with an orientation. For any oriented manifold

orientation, i.e., we have fixed the positive half cone C_p^+, such that $C_p\setminus\{p\}=C_p^+\cup C_p^-$. 2) Each continuous path-line $\gamma\subset M$ has an induced orientation, by means of the following induced metric: $(\partial t.i^\alpha)(\partial t.i^\beta)g_{\alpha\beta}dt\otimes dt=i^*g\equiv S_2^0(i)\circ g\circ i:\gamma\to M\to S_2^0(M)\to S_2^0(\gamma)$. Hence the induced volume form η_γ on γ is $\eta_\gamma=\sqrt{|\dot{x}^\alpha\dot{x}^\beta g_{\alpha\beta}|}dt$. This implies that if $p,p'\in\gamma$, (γ is an open path-line contained in M), we can answer to the question: does p belong to the future of p'? In fact, if $p\in\gamma_{p'}^+$, we say that p is in the future of p', **with respect to** γ. Here $\gamma_{p'}^+$ is the positive part of $\gamma_{p'}\setminus\{p'\}$. 3) Let p and p' belong to M. We say that p is in the future of p' if there exists a time-like (or light-like) curve γ (i.e., $g(\dot\gamma,\dot\gamma)\ge 0$, $\dot\gamma$=velocity of γ), such that $p,p'\in\gamma$ and $p'\in\gamma_p^+$. We say that two ordered events $p_1,p_2\in\gamma\subset M$ respect the **standard-causality** if p_2 is in the future of p_1 in the above sense, and we write $p_1\ll p_2$. We say that two ordered events $p_1,p_2\in M$ respect the **γ-path-causality**, γ open path-line in M, if p_2 is in the future of p_1 with respect to the fixed γ, and we write $p_1<_\gamma p_2$. Of course the path-causality is a property strictly related to the curve γ considered, and for the same two events can be $p_2<_{\gamma'}p_1$, for $\gamma'\neq\gamma$. Instead the standard-causality is a local property of the space-time. 4) Let us introduce a **relativistic frame** in M, i.e., a couple $\psi\equiv(\phi,\tau)$, where ϕ is a flow $\phi:\mathbf{R}\times M\to M$, such that its velocity $\dot\phi=\partial\phi:M\to TM$ is a time-like vector field, and $\tau:M\to\mathbf{R}$ is a function of constant rank 1 (**proper time**), such that $<d\tau,\dot\phi>=1$. Then, we can represent M in the split form $M\cong\mathbf{R}\times S_\phi$, where S_ϕ is the set of flow-lines of the flow ϕ.[29] Then, at each point $p\in M$, a frame $\psi\equiv(\phi,\tau)$ identifies a 3-dimensional space-like submanifold $M_{\tau(p)}\equiv\{p'\in M|\tau(p')=\tau(p)\}\subset M$. $M_{\tau(p)}$ is, of course, transversal to the flow-lines of the

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M and any compact $K\subset M$, there is one and only one class $\mu_K\in H_n(M,M\setminus K;\mathbf{Z})$ which satisfies $\rho_x(\mu_K)=\mu_x$ for each $x\in K$. In particular, if M itself is compact, then there is one and only one $\mu_M\in H_n(M;\mathbf{Z})$ with the required property. This class μ_M is called the **fundamental homology class** of M.

[29] Here flow is synonymous of 1-parameter group of diffeomorphisms of M, i.e., the set of diffeomorphisms $\{\phi_\lambda\}_{\lambda\in\mathbf{R}}$ has the structure of group induced by the additive structure of group of $\mathbf{R}$. Note also that S_ϕ does not necessitate be a manifold for a generic flow. However, by assuming some conditions of regularity S_ϕ becomes a 3-dimensional manifold. We shall assume that such conditions of regularity will be respected in our category of frames.

frame, and the orientation of M induces an orientation of $M_{\tau(p)}$.[30] As a consequence, we can define a **frame-causality**. More precisely, we say that for two ordered events $p_1,p_2 \in M$, p_2 is in the future of p_1, with respect to the frame ψ if there exists a path connecting p_1 and p_2 that belong to the positive part of M with respect to the spacelike hypersurface $M_{\tau(p_1)}$. We write $p_1 <_\psi p_2$. So we have in M three different order relations and we can consider the following "weakness relations": $p_1 \ll p_2 \Rightarrow p_1 <_\psi p_2$, for any frame ψ, $\Rightarrow p_1 <_\gamma p_2$, for any $\gamma \pitchfork \psi$ [31] . The **principle of general covariance** in Physics requires that all the physical entities should be represented by means of geometrical objects that have a natural way of transformation under local diffeomorphisms of M.[32] Furthermore, by studing the symmetry properties of M, we can disnguish a very important category of frames. In fact, the properties of symmetry of the structure (M,g,η) are described by the pseudogroup $\mathcal{P} \subset aut(M)$ of local diffeomorphisms of M that preserve the metric g and the volume form η: $\mathcal{P} \equiv \{f \in aut(M) | f^* g = g, f^* \eta = \eta\}$. So the jacobian $J(f)(p) = (\partial x_\alpha . f^\beta)(p)$ of f at $p \in M$, must belong to the Lorentz group $SO(1,3)$. Then, we call **rigid frame** one $\psi \equiv (\phi,\tau)$ such that its flow $\{\phi_\lambda\}_{\lambda \in \mathbf{R}}$ is a 1-dimensional subgroup of $\mathcal{P}$. A particulary important subclass of rigid frames is that of the **inertial frames**. These are defined as rigid frames such that their acceleration is zero: $\ddot{\phi} \equiv \nabla_{\dot{\phi}} \dot{\phi} = 0$, hence the corresponding flow-lines are geodesics of M.[33]

PROPOSITION A1.1 - *Rigid frames preserve all the three types of causality.*

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[30] With respect to coordinates $\{x^\alpha\}$ on M adapted to ψ, one has that $M_{\tau(p)}$ is locally characterized by the equation $x^0 - \tau(p) = 0$, and the coordinate lines $x_{k,p}$, $1 \leq k \leq 3$, passing for p are all contained into $M_{\tau(p)}$.

[31] $\gamma \pitchfork \psi$ denotes a curve γ transversal to all the space-like 3-dimensional manifolds transversal to the flow-lines of the frame $\psi \equiv (\phi,\tau)$. Note also that all the above mentioned three types of causality are all directly induced by the structure of locally oriented space-time.

[32] From the mathematical point of view this requires to represent physical entities as sections of particular fiber bundles. (See e.g., refs.[3,4], the book [8] and references quoted there.)

[33] Care of the name "rigid frame". In fact, it is well known that in general relativity does not exist the rigid body. So, with the term rigidity for a frame we refer to its rigidity with respect to the structure of space-time.

PROOF. In fact, the flow associated to a rigid frame is a subgroup of $\mathcal{P}$, hence it preserves the metric and the orientation. □

Of course, there are, also, non-rigid frames that non necessarily preserve causality. Such frames are, for example, ones related to flows of continuum media.

Finally at quantum level we must also consider the **microscopic causality**, i.e., the assumption that the quantum commutator $[\hat{s}(x),\hat{s}(x')]$, for a quantum field $\hat{s}$ must be a distributive kernel with support in $\mathcal{E}^+(x')\cup\mathcal{E}^-(x')$, for any $x'\in M$. This interpreats the bealiving that no propagation of interaction can exist between two events x and x' that do not respect the standard causality. As a by product we get, for example, that between two points x, x' of the space-time M, belonging to a same 3-dimensional space-like submanifold $M_t\subset M$ of equal-time events, with respect to a relativistic frame ψ, no propagation or interaction can exist, hence we should have $[\hat{s}(x),\hat{s}(x')]_{x_0=x'_0=t}=0$. Of course, the microscopic causality is conserved by the frame ψ, as it preserves space-like submanifolds and acts in a natural way on the quantum fields according to the principle of general covariance. Furthermore, the microscopic causality is conserved also by passing from a frame to another frame, as it is frame-independent. These considerations on the concept of causality and its covariance properties are suitable as we want to include, in our geometrical mathematical model of quantized particles-fields with negative square mass. In fact such particles should appear describe, in semiclassical approximation, space-like path-lines in the space-time. In the following, we report curve-types in M by using their possible interpretations as paths for bradions, luxons and takions. Note, however, that the introduction of massive neutrinos in physics is compatible with the principle of the microscopic causality, but has not a natural counterpart at the semiclassical level, where massive neutrinos should be identified with takions, hence with no physical particles. (For details see Example 1.1 and Example 1.2. and Theorem A1.4)[34]

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[34] The existence of particles with imaginary rest mass should solve also the problem of quantum numbers of the universe. In fact, admitting that the actual universe has been generated from the vacuum, by means of a quantum tunnel effect, it is clear that all the quantum numbers of the universe must be zero. In particular the global Casimir invariant P^2 should be zero. This is only possible if also particles with imaginary part exist. On the other hand, recent experimental results on the square mass of neutrinos

TAB.A1.1 - Classification of curves in a locally oriented hyperbolic space-time

Path-line Name	Particle Name	Casimir inv.P^2 *	Physical mass
time-like	bradions	$P^2>0$	yes
light-like	luxons	$P^2=0$	yes
space-like	takions	$P^2<0$	no

(*) $P^2=\mu_0^2c^2$, $\mu_0=$ rest-mass, $c=$ light-velocity.

All above considerations can be putted in a natural covariant way by using the language of categories. (See e.g. refs.[3,4,18].) So we give the following definition.

DEFINITION A1.1 - *The* **category of relativistic frames** *on M is a category $\mathcal{F}(M)$, such that $\psi\equiv(\phi,\tau:M\to\mathbf{R})\in Ob(\mathcal{F}(M))$; $f\in Hom_{\mathcal{F}(M)}(\psi;\psi')$, iff f is a local diffeomorphism of M such that induces a local diffeomorphism $f_{\mathbf{R}}$ on $\mathbf{R}$ such that the following diagram*

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \phi_\lambda \downarrow & & \downarrow \phi'_\lambda \\ M & \xrightarrow{f} & M \\ \tau \downarrow & & \downarrow \tau' \\ \mathbf{R} & \xrightarrow[f_{\mathbf{R}}]{} & \mathbf{R} \end{array}$$

is commutative for each $\lambda\in\mathbf{R}$. In other words, a morphism in $\mathcal{F}(M)$ is a fiber bundle morphism with respect to the structure $\tau:M\to\mathbf{R}$, that commutes with all the diffeomorphisms ϕ_λ. This is equivalent to say that f sends space-like hypersurfaces with respect to ψ into space-like hypersurfaces with respect to ψ' and flow-lines of ψ into flow-lines of ψ'. (Of course for two generic frames, it could happen that the corresponding morphism set should be empty.)

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should confirm this ansatz. (See e.g. ref.[17].) Recall that neutrinos are leptons, i.e., particles with spin and with zero electric charge that do not strongly interact. These are of three types: ν_e=*electronic neutrino*, ν_μ=*muonic neutrino* and ν_τ=*tauonic neutrino*. They are characterized by three different decays: $\nu_e\to e^-+\bar{\nu}_e+p^+$, $\nu_\mu\to\mu^-+\bar{\nu}_\mu+p^+$, $\nu_\tau\to\tau^-+\bar{\nu}_\tau+p^+$.

THEOREM A1.1 - *Let* $f\in Hom_{\mathcal{F}(M)}(\psi;\psi')\neq\emptyset$. *Then f preserves the microscopic causality.*

PROOF. In fact f transforms space-like 3-dimensional submanifolds $N\subset M$, with respect to ψ, into space-like 3-dimensional submanifolds $N'\subset M$, with respect to ψ'. Furthermore, the principle of general covariance assures that if $[\hat{s}(x),\hat{s}(x')]_{x_0=x'_0}=0$, then also $[\widehat{f^*s}(x),\widehat{f^*s}(x')]_{x_0=x'_0}=0$. □

THEOREM A1.2 - *Let us consider the* **subcategory** $\mathcal{F}_R(M)\subset\mathcal{F}(M)$ *of* **rigid frames** *on M. Then any* $f\in aut(M)$ *that is also a fiber bundle homomorphism*

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \tau\downarrow & & \downarrow\tau' \\ \mathbf{R} & \xrightarrow[f_{\mathbf{R}}]{} & \mathbf{R} \end{array} \qquad (A1.1)$$

and that belongs to the stabilizer $S\mathcal{P}$ *of* $\mathcal{P}$ *in* $aut(M)$, *transforms a rigid frame* $\psi\equiv(\phi,\tau)$ *into a rigid frame* $\psi'\equiv(\phi',\tau')$ *with* $\phi'_\lambda=f^{-1}\phi_\lambda f$. *Therefore,* $Hom_{\mathcal{F}_R(M)}(\psi;\psi')\neq\emptyset$ *is contained into the stabilizer* $S\mathcal{P}$ *of* $\mathcal{P}$ *into* $aut(M)$.

PROOF. If we restrict to the sub-category $\mathcal{F}_R(M)$ of the rigid frames, then $f\in Hom_{\mathcal{F}_R(M)}(\psi;\psi')\neq\emptyset$ is an element of $aut(M)$ such that $f^{-1}\phi_\lambda f\equiv\phi'_\lambda\in\mathcal{P}$, $\forall\lambda\in\mathbf{R}$. Of course, any fiber bundle morphism $(f,f_{\mathbf{R}})$ like in above commutative diagram $(A1.1)$, with $f\in\mathcal{P}$, transforms a rigid frame $\psi\equiv(\phi,\tau)$ into a rigid frame $\psi'\equiv(\phi',\tau')$, with $\phi'_\lambda=f^{-1}\phi_\lambda f$. However, if $S\mathcal{P}$ is the stabilizer of $\mathcal{P}$ in $aut(M)$, i.e., $f\mathcal{P}f^{-1}\subset\mathcal{P}$, $\forall f\in S\mathcal{P}\subset aut(M)$, then any $f\in S\mathcal{P}\subset aut(M)$, that is also a fiber bundle homomorphism like in $(A1.1)$, transforms a rigid frame $\psi\equiv(\phi,\tau)$ into a rigid frame $\psi'\equiv(\phi',\tau')$, with $\phi'_\lambda=f^{-1}\phi_\lambda f$.[35] Therefore the theorem is proved. □

COROLLARY A1.1 - *If* $f\in Hom_{\mathcal{F}_R(M)}(\psi;\psi')\neq\emptyset$, *then the standard-causality, frame-causality and path-causality, (shortly causality), is conserved, iff* $f\in\mathcal{P}$.

THEOREM A1.3 - *Let* $\psi,\psi'\in Ob(\mathcal{F}(M))$. *If* $f\in Hom_{\mathcal{F}(M)}(\psi;\psi')\neq\emptyset$, *then f preserves the causality iff* $f\in\mathcal{P}$.

PROOF. In fact $\phi'_\lambda=f\phi_\lambda f^{-1}$ has the same time orientation than ϕ_λ as it represents a frame, hence its velocity is timelike and oriented in the future. □

[35] Of course the stabilizer $S\mathcal{P}$ properly contains $\mathcal{P}$. In fact, if $f\in aut(M)$ is such that $f^*g=g$, $f^*\eta=-\eta$, then $(f^{-1}\phi_\lambda f)^*\eta=f^*(\phi^*_\lambda((f^{-1})^*\eta))=f^*(\phi^*_\lambda(-\eta))=-f^*\eta=\eta$. So $f\in S\mathcal{P}$, but $f\notin\mathcal{P}$.

So we can give the following definition.

DEFINITION A1.2 - *We call* **Lorentz category** *on M the little subcategory $\mathcal{F}_L(M)\subset \mathcal{F}(M)$ such that $Hom_{\mathcal{F}_L(M)}(\psi;\psi')\subset\mathcal{P}$. An object of $\mathcal{F}_L(M)$ is called a* **Lorentz frame**.

PROPOSITION A1.2 - *One has the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{F}_R(M) & \subset & \mathcal{F}(M) \\ \cup & & \cup \\ \mathcal{F}_I(M) & \subset & \mathcal{F}_L(M) \end{array}$$

where $\mathcal{F}_I(M)$ is the category of inertial frames.

PROPOSITION A1.3 - *If $f\in Hom_{\mathcal{F}_L(M)}(\psi;\psi')$, then f preserves causality.*[36]

EXAMPLE A1.1 - In the particular case that M is the flat Minkowski space-time, then $\mathcal{P}$ is exactly the Poincaré group P: $\mathcal{P}=P$. Then, Lorentz frames are related by rigid transformations, i.e., elements of the Poincaré group. ■

In the following propositions we relate covariance with differential equations for paths in M, as seen by two different Lorentz frames.

PROPOSITION A1.4 - *Let M be the Minkowsky space-time. Let $E_2\subset J\mathcal{D}^2(\tau:M\to\mathbf{R})$, $F^k(x^\alpha,\dot{x}^\alpha,\ddot{x}^\alpha)=0$, be an ordinary differential equation for paths in M, with respect to coordinates (x^α) adapted to a rigid frame $\psi\equiv(\phi,\tau)$ and let $\bar{E}_2\subset J\mathcal{D}^2(\bar{\tau}:M\to\mathbf{R})$, $\bar{F}^k(\bar{x}^\alpha,\dot{\bar{x}}^\alpha,\ddot{\bar{x}}^\alpha)= 0$, be the corresponding differential equation for paths in M, with respect to another rigid frame $\bar{\psi}\equiv(\bar{\phi},\bar{\tau})$ related to the previous one by means of transformations, $\bar{x}^\alpha=\bar{x}^\alpha(x^\beta)$, such that $(\partial x_\beta.\bar{x}^\alpha)=(A^\alpha_\beta)\in SO(1,3)$. Then we get that the equations $F^k=0$ are obtained from $\bar{F}^k=0$ by means of the following transformations between the second jet-derivative spaces $J\mathcal{D}^2(\tau:M\to\mathbf{R})$ and $J\mathcal{D}^2(\bar{\tau}:M\to\mathbf{R})$:*[37]

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[36] Let us emphasize that the causality is always conserved in the category of Lorentz frames. This last properly contains that of inertial frames.

[37] Here, in order to emphasize different fiber bundle structures that are considered between M and $\mathbf{R}$, we denote the second jet-derivative space for sections of $\tau:M\to\mathbf{R}$ by $J\mathcal{D}^2(\tau:M\to\mathbf{R})$, instead than simply $J\mathcal{D}^2(M)$.

$$\left\{\begin{array}{l}\left\{\begin{array}{l}\bar{x}^{\alpha}=A^{\alpha}_{\beta}x^{\beta}+y^{\alpha}\\ \dot{\bar{x}}^{\alpha}=\dot{A}^{\alpha}_{\beta}x^{\beta}+A^{\alpha}_{\beta}\dot{x}^{\beta}+\dot{y}^{\alpha}\\ \ddot{\bar{x}}^{\alpha}=\ddot{A}^{\alpha}_{\beta}x^{\beta}+2\dot{A}^{\alpha}_{\beta}\dot{x}^{\beta}+A^{\alpha}_{\beta}\ddot{x}^{\beta}+\ddot{y}^{\alpha}\end{array}\right\}\begin{array}{c}\text{with the}\\ \text{conditions}\end{array}:(\dot{A}^{0}_{0}x^{0}+\dot{y}^{0}=0;\quad \ddot{A}^{0}_{0}x^{0}+\dot{A}^{0}_{0}\dot{x}^{0}+\ddot{y}^{0}=0)\\ +\text{constraint equations}\left\{\begin{array}{l}\text{for non-takions:}\{\dot{x}^{0}=1;\quad \ddot{x}^{0}=0\}\\ \text{for takions:}\{\dot{x}^{0}=\ddot{x}^{0}=0\}\end{array}\right\}\end{array}\right\}.$$

The first rigid frame has velocity $\dot{\phi}=\partial x_0$ and the second one $\dot{\bar{\phi}}=\partial \bar{x}_0$. Then the relation between the two is given by the following transformation: $\partial\bar{x}_0=\phi^*\partial x_0=T(\phi^{-1})\circ\partial x_0\circ\phi=A^0_0\partial x_0\circ\phi$. Therefore, the flow-orientation is preserved iff $A^0_0>0$. Now, let $Y_\tau\subset J\mathcal{D}^2(\tau:M\to\mathbf{R})$ be the submanifold defined by $(\dot{x}^0=1,\ddot{x}^0=0)$. Then Y_τ is transformed by means of the above transformation in the submanifold $Y_{\bar{\tau}}\subset J\mathcal{D}^2(\bar{\tau}:M\to\mathbf{R})$, defined by $(\dot{x}^0=1,\ddot{x}^0=0)$, iff the following equation is satisfied: $A^0_0=1$. Therefore, Y_τ is transformed in $Y_{\bar{\tau}}$ iff we restrict to the Poincaré group, i.e., iff the rigid frames considered are in the category $\mathcal{F}_L(M)$. Let, now, $X_\tau\subset J\mathcal{D}^2(\tau:M\to\mathbf{R})$ be the submanifold defined by $(\dot{x}^0=\ddot{x}^0=0)$, then X_τ is transformed in the submanifold $X_{\bar{\tau}}\subset J\mathcal{D}^2(\bar{\tau}:M\to\mathbf{R})$.[38]

THEOREM A1.4 - *The interactions between non-takions and takions cannot be verified (at the semiclassical level) without to introduce some ghost forces whose quanta are not bradions neither takions.*

PROOF. In fact, let $\tau:M\times_{\mathbf{R}}M\to\mathbf{R}$ be the configuration bundle with respect to a rigid frame. On $J\mathcal{D}^2(\tau:M\times_{\mathbf{R}}M\to\mathbf{R})$ we have the following adapted coordinates $(x^0,x^k_1,x^k_2,\dot{x}^0\dot{x}^k_1,\dot{x}^k_2,\ddot{x}^0,\ddot{x}^k_1,\ddot{x}^k_2)$. Then the submanifold for takions is defined by $X_{\bar{\tau}}\equiv\{q\in J\mathcal{D}^2(\tau:M\times_{\mathbf{R}}M)\to\mathbf{R}|\dot{x}^0=0,\ddot{x}^0=0\}\subset J\mathcal{D}^2(\tau:M\times_{\mathbf{R}}M\to\mathbf{R})$ and the submanifold for bradions is defined by $Y_{\bar{\tau}}\equiv\{q\in J\mathcal{D}^2(\tau:M\times_{\mathbf{R}}M\to\mathbf{R})|\dot{x}^0=1,\ddot{x}^0=0\}\subset J\mathcal{D}^2(\tau:M\times_{\mathbf{R}}M\to\mathbf{R})$. Then we see that $Y_{\bar{\tau}}\cap X_{\bar{\tau}}=\emptyset$. Therefore, we cannot have interaction between takions and bradions without to introduce some gosth forces with quanta describing path-lines that satisfies the following condition: $0\leq\dot{x}^0\leq 1$. □

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38 Y_τ, (resp. $Y_{\bar{\tau}}$), represents the submanifold of $J\mathcal{D}^2(\tau:M\to\mathbf{R})$, (resp. $J\mathcal{D}^2(\bar{\tau}:M\to\mathbf{R})$), that is the constraint for bradions-dynamics with respect to the frame ψ, (resp. $\bar{\psi}$). Instead X_τ, and $X_{\bar{\tau}}$, are the corresponding submanifolds for takions.

APPENDIX : A2 - CURVATURE OF THE LEVI-CIVITA CONNECTION AND THE LICHNEROWICZ FORMULA RELATING SUCH CURVATURE WITH THE SCALAR CURVATURE VIA DIRAC-MATRICES

We have used the following relation:

$$(A2.1)\qquad R_{\beta\alpha\mu\lambda}\gamma^{\beta}\gamma^{\alpha}\gamma^{\mu}\gamma^{\lambda}=2R\,1$$

where $R^{\gamma}{}_{.\delta\alpha\beta}=(\partial x_{\alpha}\cdot\Gamma^{\gamma}_{\delta\beta})-(\partial x_{\beta}\cdot\Gamma^{\gamma}_{\delta\alpha})+\Gamma^{\gamma}_{\alpha\omega}\Gamma^{\omega}_{\delta\beta}-\Gamma^{\gamma}_{\beta\omega}\Gamma^{\omega}_{\delta\alpha}$. Now, in ref.[1] A.Lichnerowicz first proved the same relation but with the sign minus in the second term. The motivation of this difference is that A.Lichnerowicz used a definition of Clifford algebra that differes from our just for a sign minus. However, a simple proof of $(A2.1)$ can be obtained just following the line given in [1] and taking into account some symmetry properties of the indexes in the curvature that here we recall:

$$(A2.2)\qquad \left\{\begin{array}{l} R^{\gamma}{}_{.\delta\alpha\beta}=-R^{\gamma}{}_{.\delta\beta\alpha}\\ R^{\gamma}{}_{.\delta\alpha\beta}+R^{\gamma}{}_{.\alpha\beta\delta}+R^{\gamma}{}_{.\beta\delta\alpha}=0\\ R^{\gamma}{}_{.\gamma\alpha\beta}=0\\ R_{\omega\delta\alpha\beta}=R^{\gamma}{}_{.\delta\alpha\beta}g_{\gamma\omega}\end{array}\right\}.$$

Then we get also:

$$(A2.3)\qquad \left\{\begin{array}{l} R_{\omega\delta\alpha\beta}=-R_{\delta\omega\alpha\beta}\\ R_{\omega\delta\alpha\beta}=-R_{\omega\delta\beta\alpha}\\ R_{\omega\delta\alpha\beta}=R_{\alpha\beta\omega\delta}\\ R_{\omega\delta\alpha\beta}+R_{\omega\alpha\beta\delta}+R_{\omega\beta\delta\alpha}=0\\ R^{\delta}{}_{.\delta\alpha\beta}=g^{\omega\delta}R_{\omega\delta\alpha\beta}=0\\ R_{\omega\delta\alpha\beta}g^{\alpha\beta}=0\\ R_{\delta\alpha}=R^{\gamma}{}_{.\delta\alpha\gamma}=g^{\omega\beta}R_{\omega\delta\alpha\beta}=-g^{\omega\alpha}R_{\omega\delta\alpha\beta}\\ R_{\delta\alpha}=R_{\alpha\delta}\\ R=g^{\delta\alpha}R_{\delta\alpha}=g^{\omega\beta}g^{\delta\alpha}R_{\omega\delta\alpha\beta}\end{array}\right\}.$$

Staring from the following geometric object: $T^{\alpha}=R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\beta}\gamma^{\lambda}\gamma^{\mu}$, it is possible to prove above formula $(A2.1)$ by following a road similar to one given in ref.[1]. In fact, from the second equation in $(A2.3)$ we have

$$\begin{aligned}0&=(R^{\alpha}{}_{.\beta\lambda\mu}+R^{\alpha}{}_{.\lambda\mu\beta}+R^{\alpha}{}_{.\mu\beta\lambda})\gamma^{\beta}\gamma^{\lambda}\gamma^{\mu}\\&=R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\beta}\gamma^{\lambda}\gamma^{\mu}+R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\mu}\gamma^{\beta}\gamma^{\lambda}+R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}\gamma^{\mu}\gamma^{\beta}.\end{aligned}$$

Hence we get:

$$(A2.4)\qquad R^{\alpha}{}_{.\beta\lambda\mu}\big(\gamma^{\beta}\gamma^{\lambda}\gamma^{\mu}+\gamma^{\mu}\gamma^{\beta}\gamma^{\lambda}+\gamma^{\lambda}\gamma^{\mu}\gamma^{\beta}\big\}=0.$$

Therefore we have also:

$$\begin{aligned}0&=T^{\alpha}+R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}(-\gamma^{\beta}\gamma^{\mu}+2g^{\beta\mu})+R^{\alpha}{}_{.\beta\lambda\mu}(-\gamma^{\beta}\gamma^{\mu}+2g^{\mu\beta})\gamma^{\lambda}\\&=T^{\alpha}-R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}\gamma^{\beta}\gamma^{\mu}+2g^{\beta\mu}R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}-R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\beta}\gamma^{\mu}\gamma^{\lambda}+2g^{\mu\beta}R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}\\&=T^{\alpha}-R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}\gamma^{\beta}\gamma^{\mu}+4g^{\beta\mu}R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}-R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\beta}(-\gamma^{\lambda}\gamma^{\mu}+2g^{\lambda\mu})\\&=T^{\alpha}-R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}\gamma^{\beta}\gamma^{\mu}+4g^{\beta\mu}R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}+R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\beta}\gamma^{\lambda}\gamma^{\mu}-2R^{\alpha}{}_{.\beta\lambda\mu}g^{\lambda\mu}\gamma^{\beta}\\&=T^{\alpha}-R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}\gamma^{\beta}\gamma^{\mu}+4g^{\beta\mu}R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}+R^{\alpha}{}_{.\beta\lambda\mu}(-\gamma^{\lambda}\gamma^{\beta}+2g^{\beta\lambda})\gamma^{\mu}\\&=T^{\alpha}-R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}\gamma^{\beta}\gamma^{\mu}+4g^{\beta\mu}R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}-R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}\gamma^{\beta}\gamma^{\mu}-2g^{\beta\mu}R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}\\&=T^{\alpha}+2R^{\alpha}{}_{.\beta\lambda\mu}g^{\beta\mu}\gamma^{\lambda}-2R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}\gamma^{\beta}\gamma^{\mu}\\&=T^{\alpha}-2R^{\alpha}{}_{.\beta\mu\lambda}g^{\beta\mu}\gamma^{\lambda}-2R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}\gamma^{\beta}\gamma^{\mu}.\end{aligned}$$

Hence we get:

$$(A2.5)\qquad T^{\alpha}-2R^{\alpha}{}_{\lambda}\gamma^{\lambda}-2R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}\gamma^{\beta}\gamma^{\mu}=0.$$

From $(A2.5)$ we have also:

$$\begin{aligned}0&=T^{\alpha}-2R^{\alpha}{}_{\lambda}\gamma^{\lambda}-2R^{\alpha}{}_{.\beta\lambda\mu}(-\gamma^{\beta}\gamma^{\lambda}+2g^{\beta\lambda})\gamma^{\mu}\\&=T^{\alpha}-2R^{\alpha}{}_{\lambda}\gamma^{\lambda}+2R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\beta}\gamma^{\lambda}\gamma^{\mu}-4R^{\alpha}{}_{.\beta\lambda\mu}g^{\beta\lambda}\gamma^{\mu}\\&=3T^{\alpha}-2R^{\alpha}{}_{\lambda}\gamma^{\lambda}-4R^{\alpha}{}_{\mu}\gamma^{\mu}.\end{aligned}$$

Therefore we get:

$$(A2.6)\qquad T^{\alpha}=2R^{\alpha}{}_{\lambda}\gamma^{\lambda}.$$

Now, by substituting $(A2.6)$ in $(A2.5)$ we get:

$$2R^{\alpha}{}_{\lambda}\gamma^{\lambda}-2R^{\alpha}{}_{\lambda}\gamma^{\lambda}-2R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}\gamma^{\beta}\gamma^{\mu}=0.$$

Hence:

$$(A2.7)\qquad \left\{\begin{array}{l} R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\lambda}\gamma^{\beta}\gamma^{\mu}=0 \\ R^{\alpha}{}_{.\beta\mu\lambda}\gamma^{\mu}\gamma^{\beta}\gamma^{\lambda}=0 \\ -R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\mu}\gamma^{\beta}\gamma^{\lambda}=0. \end{array}\right\}$$

By utilizing equation $(A2.7)$ in $(A2.4)$, or taking into account the definition of T^{α} and equation $(A2.6)$, we get:

$$(A2.8)\qquad R^{\alpha}{}_{.\beta\lambda\mu}\gamma^{\beta}\gamma^{\lambda}\gamma^{\mu}=2R^{\alpha}{}_{\lambda}\gamma^{\lambda}.$$

Then, by contraction on the left of $(A2.8)$ by $\gamma_{\alpha}=g_{\omega\alpha}\gamma^{\omega}$ and taking into account the simmetry property of $R_{\omega\lambda}$, (see equation $(A2.3)$), we get

$$R_{\omega\beta\lambda\mu}\gamma^{\omega}\gamma^{\beta}\gamma^{\lambda}\gamma^{\mu}=R_{\omega\lambda}(\gamma^{\omega}\gamma^{\lambda}+\gamma^{\lambda}\gamma^{\omega})=2R_{\omega\lambda}g^{\omega\lambda}=2R\,\mathbf{1}.$$

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[15] For modern physical interpretations of extended particles (extendons) see e.g. the following book: **NE'EMAN, Y.** & **EIZENBERG, E.**, *Membranes & other extendons (p-branes)*, World Scientific, Singapore 1995.

[16] Interesting physical considerations on quantum fields in curved space-times can be also found in the following two books:
(a) **BIRRELL, N.D. & DAVIES, P.C.**, *Quantum Fields in Curved Space*, Cambridge Univ. Press, Cambridge, 1982.
(b)**NAKANISKHI, N. & OJIMA, I.**, *Covariant Operator Formalism of Gauge Theories and Quantum Gravity*, World Scientific, Singapore, 1990.

[17] In the following we quote some works reporting on experimental results showing that neutrinos necessitate to have a negative square mass:
(a) **CABAM,C., REMBIELINSKI,J., & SMOLÍNSKI,K.**, *Decays of spacelike neutrinos*, (to appear).
(b) **CHODOS,A., HAUSER, A.I., & KOSTELECKY, V.A.**, Phys. Lett. **B150**(1985), 431.
(c) **BOROWSKI,J.G., & REMBIELINSKI, J.**, *Experimental results and the hypothesis of tackions neutrinos*, Proc.Int.Conf. High Energy Phys, Warsaw, July 1996-(hep-ph/9607477).

[18] The formulation in categorial language of the concept of frame-covariance, considered in this paper, has been first introduced by A.Prástaro in some previous papers, (see e.g. refs.[3,4,8] and his works on continuum physics quoted there). Recently **P.CABAN** and **J.REMBIELINSKI** also applied these language in quantum mechanincs in the following paper: *Lorentz-covariant quantum mechanics and preferred frame*, Phys. Rev. A **59**(6)(1999), 4187–4196.