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COMPUTATIONAL ALGEBRAIC ANALYSIS OF SYSTEMS DIFFERENTIAL EQUATIONS

Daniele C. Struppa
Department of Mathematical Sciences
George Mason University
Fairfax, VA 22030
USA.

1 Introduction

In this paper, I would like to show how some techniques from the theory of Gröbner bases can be used to attack and solve some problems in analysis and in the theory of partial differential equations.
The differential equations I will be dealing with are very simple, since they are both linear and with constant coefficients. On the other hand, they present some interesting problems, because I will be dealing with systems of differential equations and I will be asking some rather deep questions
In this paper, I will show how well known techniques from algebraic analysis (mostly developed in the 60's and 70's by mathematicians such as Ehrenpreis, Malgrange, and Palamodov) can now be made effective and computational.
This will allow us to do some concrete computations and solve problems which appear otherwise intractable. Since research has proceeded very rapidly in this area in the last couple of years, I will only provide what could be considered an introduction, and I will refer the reader to my forthcoming book [6] for more details as well as for the most updated results in the field.

2 Two concrete examples

Let me begin with two concrete examples The first one, well known, is the Cauchy Riemann system which defines holomorphic functions.

If $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $z_j = x_j + iy_j$ $(j = 1, \dots, n)$, a function

$$f : \mathbb{C}^n \longrightarrow \mathbb{C}$$

$$f = u(x_1, \dots, x_n; y_1, \dots, y_n) + iv(x_1, \dots, x_n; y_1, \dots, y_n)$$

is said to be holomorphic if it satisfies the Cauchy-Riemann system

$$\frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right) := \frac{\partial f}{\partial \bar{z}_j} = 0, \quad j = 1, \dots, n$$

or, equivalently

$$\begin{cases} \dfrac{\partial u}{\partial x_j} = \dfrac{\partial v}{\partial y_j} \\ \\ \dfrac{\partial u}{\partial y_j} = -\dfrac{\partial v}{\partial x_j} \end{cases} \quad j = 1, \dots, n$$

which gives a system of $2n$ differential equations in 2 unknown functions (u and v). On the other hand, in its complex form, we have n differential equations for 1 unknown function f (complex valued).

These two systems can be represented, in matrix form, by

$$\begin{bmatrix} \dfrac{\partial}{\partial x_1} & \dfrac{-\partial}{\partial y_1} \\ \dfrac{\partial}{\partial y_1} & \dfrac{\partial}{\partial x_1} \\ \cdots & \\ \vdots & \vdots \\ \cdots & \\ \dfrac{\partial}{\partial x_n} & \dfrac{-\partial}{\partial y_n} \\ \dfrac{\partial}{\partial y_n} & \dfrac{\partial}{\partial x_n} \end{bmatrix}_{2n\times 2} \begin{bmatrix} u \\ v \end{bmatrix}_{2\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{2n\times 1}$$

or, with the use of complex coordinates,

$$\begin{bmatrix} \dfrac{\partial}{\partial x_1} + i\dfrac{\partial}{\partial y_1} \\ \vdots \\ \dfrac{\partial}{\partial x_n} + i\dfrac{\partial}{\partial y_n} \end{bmatrix}_{n\times 1} [f]_{1\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$$

This system is so well known that entire books are devoted to it and, in fact, entire courses are devoted to it. There are a few properties of the solutions of this system, i.e. of holomorphic functions, which I would like to state (though well known) as they offer us an example of what we will be looking for.

Property 1: Removability of compact singularities. The famous Hartogs' theorem (1906) shows that if K is a compact subset of $\mathbb{C}^n$ such that $\mathbb{C}^n \setminus K$ is connected, and if $n \geq 2$, then every holomorphic function in $\mathbb{C}^n \setminus K$ extends uniquely to an entire function (i.e., a function holomorphic on all of $\mathbb{C}^n$).

This means that holomorphic functions in several variables do not allow compact singularities (in striking contrast with the 1-variable case where $f(z) = \frac{1}{z}$ obviously has an isolated compact singularity).

Property 2: Compatibility conditions for solutions of the non-homogeneous Cauchy-Riemann system. Let's consider, for simplicity, $n = 2$ and the nonhomogeneous Cauchy-Riemann system

$$\begin{cases} \dfrac{\partial f}{\partial \bar{z}_1} = g_1 \\[2ex] \dfrac{\partial f}{\partial \bar{z}_2} = g_2 \end{cases}$$

with f unknown and g_1, g_2 given on a convex set in $\mathbb{C}^2$ (maybe the entire $\mathbb{C}^2$). It was shown in the late 50's that such a non-homogeneous system has a solution f if, and only if,

$$\frac{\partial g_1}{\partial \bar{z}_2} = \frac{\partial g_2}{\partial \bar{z}_1}.$$

In other words, the obviously necessary compatibility conditions are also sufficient. More generally, if we consider the case of $n \geq 2$ equations

$$\begin{cases} \dfrac{\partial f}{\partial \bar{z}_1} = g_1 \\ \vdots \\ \dfrac{\partial f}{\partial \bar{z}_n} = g_n \end{cases}$$

then it is known that the system has a solution f if, and only if, all the obvious compatibility conditions are satisfied, i.e. if and only if

$$\frac{\partial g_j}{\partial \bar{z}_k} = \frac{\partial g_k}{\partial \bar{z}_j} \quad j, k = 1, \dots, n.$$

Property 3: Vanishing of n–dimensional cohomology. This last property (known as the Malgrange Theorem) is much more abstract and hard to describe than the previous ones. Nevertheless, it seems worth spending a couple of minutes on it. In one complex variable, it is well known that every open set is a domain of holomorphy. This fact can be stated in several, equivalent, ways:
3.1. for any open set Ω, there is a function f, holomorphic on Ω, which cannot be continued outside of Ω.
3.2. the equation $\frac{\partial f}{\partial \bar{z}} = g$ can be solved in any open set Ω, for any C^∞ function g.
3.3. for every open set $\Omega \subseteq \mathbb{C}$, it is $H^1(\Omega, \mathcal{O}) = 0$. This symbolic expression means the following:
let $\{U_i\}$ be a reasonable open covering for Ω , and consider a collection $\{f_{ij}\}$ of functions which are holomorphic on $U_i \cap U_j$ (whenever $U_i \cap U_j \neq \phi$) with $f_{ij} = -f_{ji}$. Then if, for any indexes i, j, k such that $U_i \cap U_j \cap U_k \neq \phi$ we have

$$f_{ij} + f_{jk} + f_{ki} \equiv 0 \text{ on } U_i \cap U_j \cap U_k;$$

then there are holomorphic functions $h_i \in \mathcal{O}(U_i)$ such that

$$f_{ij} = h_i - h_j \text{ on } U_i \cap U_j.$$

As it turns out, properties 3.1, 3.2. and 3.3 are true for any open set in $\mathbb{C}$, but not for any open set in $\mathbb{C}^n$. In fact, the Hartogs' phenomenon shows us that an open set such as $\Omega = \mathbb{C}^n \backslash K$ would not satisfy 3.1. It is nevertheless true that 3.1, 3.2, and 3.3 are equivalent for any open set $\Omega \subseteq \mathbb{C}^n$. Interestingly enough, a very good geometrical characterization exists now for the open sets $\Omega \subseteq \mathbb{C}^n$ for which 3.1 (and therefore 3.2 and 3.3) hold. These sets are called "pseudoconvex." In the sixties, however, Malgrange discovered how to correctly generalize property 3.3 to the case of all open sets in $\mathbb{C}^n$. While it is not true that $H^1(\Omega, \mathbb{C}^n) = 0$ for every open set in $\mathbb{C}^n$, it is true that

$$H^n(\Omega, \mathbb{C}^n) = 0 \text{ for all open sets } \Omega \subseteq \mathbb{C}^n.$$

Rather than explaining the general result, let me quickly say what it means for $n = 2$.
Take an open covering $\{U_j\}$ of $\Omega \subseteq \mathbb{C}^n$ and take any family

$$\{f_{\alpha\beta\gamma} : f_{\alpha\beta\gamma} \in \mathcal{O}(U_\alpha \cap U_\beta \cap U_\gamma), \text{ and for every permutation } \sigma,$$

$$f_{\alpha\beta\gamma} = sgn\,(\sigma)\, f_{\sigma(\alpha\beta\gamma)}\}_{\alpha\beta\gamma}.$$

Then Malgrange's Theorem says that if

$$f_{\beta\gamma\delta} - f_{\alpha\gamma\delta} + f_{\alpha\beta\delta} - f_{\alpha\beta\gamma} \equiv 0\ \forall\,(\alpha,\beta,\gamma) \text{ s.t. } U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta \neq \phi$$

then we can find a collection

$$\left\{h_{\alpha\beta} : h_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)\ h_{\alpha\beta} = sgn\,(\sigma)\, h_{\sigma(\alpha\beta)}\right\}_{\alpha,\beta}$$

such that

$$f_{\alpha\beta\gamma} = h_{\beta\gamma} - h_{\alpha\gamma} + h_{\alpha\beta} \text{ on } U_\alpha \cap U_\beta \cap U_\gamma \neq \phi.$$

The road to the generalization for $n \geq 2$ is now clear (though not necessarily pleasant to write). Is this an important and interesting result? The answer is yes, as it is the basis for the theory of hyperfunctions, but I will not be able to explore this concept in this paper (and see, e.g. [2] and [11]).
Let me now get to a second concrete example, for which much less is known (or, maybe I should say, was known) and which was one of the motivations for most of my research in these last few years. I want to introduce the Cauchy-Fueter operator. This operator was originally defined as an attempt to extend complex analysis to the case of the skew-field of quaternions defined by $\mathbb{H} = \{q = x_0 + ix_1 + jx_2 + kx_3 : x_0, ..., x_3 \in \mathbb{R}\}$, where i, j, and k are three imaginary units. There are, in complex analysis, two complementary points of view which one can adopt. The Weierstrass approach, which defines holomorphicity in terms of the convergence of Taylor series, and the Cauchy-Riemann approach, which defines holomorphicity in terms of the Cauchy-Riemann system. In seeking a naive theory of quaternion analysis, it was immediately apparent that the Weierstrass route was not available (it would coincide with the theory of real analytic functions). On the other

hand, one could consider a new Cauchy-Riemann like operator which could act on functions

$$f : \mathbb{H} \longrightarrow \mathbb{H}_{\mathbb{C}} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = \{q = z_0 + iz_1 + jz_2 + kz_3 : z_0, \ldots, z_3 \in \mathbb{C}\}.$$

If $q = (q_1, \ldots, q_n) \in \mathbb{H}^n$ and $q_t = x_{0t} + ix_{1t} + jx_{2t} + kx_{3t}$ $(t = 1, \ldots, n)$ a function

$$f : \mathbb{H}^n \longrightarrow \mathbb{H}_{\mathbb{C}}$$

$$f = f_0(q_1, \ldots, q_n) + if_1(q_1, \ldots, q_n) + jf_2(q_1, \ldots, q_n) + kf_3(q_1, \ldots, q_n)$$

is said to be regular (or quaternionic-holomorphic) if

$$\frac{\partial f}{\partial \bar{q}_t} := \frac{1}{4}\left(\frac{\partial f}{\partial x_{0t}} + i\frac{\partial f}{\partial x_{1t}} + j\frac{\partial f}{\partial x_{2t}} + k\frac{\partial f}{\partial x_{3t}}\right) = 0, \quad t = 1, \ldots, n.$$

Given the quaternionic decomposition of f, this last equation can be re-written in matrix form as:

$$\begin{bmatrix} \frac{\partial}{\partial x_{01}} & \frac{-\partial}{\partial x_{11}} & \frac{-\partial}{\partial x_{21}} & \frac{-\partial}{\partial x_{31}} \\ \frac{\partial}{\partial x_{11}} & \frac{\partial}{\partial x_{01}} & \frac{-\partial}{\partial x_{31}} & \frac{\partial}{\partial x_{21}} \\ \frac{\partial}{\partial x_{21}} & \frac{\partial}{\partial x_{31}} & \frac{\partial}{\partial x_{01}} & \frac{-\partial}{\partial x_{11}} \\ \frac{\partial}{\partial x_{31}} & \frac{-\partial}{\partial x_{21}} & \frac{\partial}{\partial x_{11}} & \frac{\partial}{\partial x_{01}} \\ \cdots & \cdots & \cdots & \cdots \\ \vdots & & & \vdots \\ \frac{\partial}{\partial x_{0n}} & \cdots & \cdots & \cdots \\ \vdots & & & \\ \frac{\partial}{\partial x_{3n}} & \frac{-\partial}{\partial x_{2n}} & \frac{\partial}{\partial x_{1n}} & \frac{\partial}{\partial x_{0n}} \end{bmatrix}_{4n\times 4} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}_{4\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{4n\times 1}$$

Let me point out that the solutions of this system are actually physically significant as they represent massless, right handed, neutrino fields.
So one might want to ask for these object the same questions we just saw are well known for the case of the Cauchy-Riemann system. In the next sections, we will explore exactly these questions.

3 A primer of algebraic analysis

Let us now proceed to talk about the algebraic treatment of differential equations.

As we saw in the previous two examples, our differential operators (constant coefficients and linear) appear as follows. let $P = [P_{ij}]$ be an $r_1 \times r_0$ matrix of polynomials in, n complex variables, and let $D = \left(-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_n}\right)$. Then

$$P(D) := [P_{ij}(D)]$$

is an $r_1 \times r_0$ matrix of differential operators ($2n \times 2$ or $n \times 2$ in the case of the Cauchy-Riemann operator in $\mathbb{C}^n$, $4n \times 4$ in the case of the Cauchy-Fueter operator in $\mathbb{H}^n$).

If we apply the matrix $P(D)$ to a "set" S of functions, we can define a sequence of maps as follows:

$$\begin{array}{l} 0 \longrightarrow S^P \longrightarrow S^{r_0} \xrightarrow{P(D)} S^{r_1} \\ \qquad\;\; \| \\ \qquad \ker\{P(D) : S^{r_0} \longrightarrow S^{r_1}\}. \end{array}$$

For our purposes, S really needs to be a sheaf of functions, but we do not need to be concerned about this detail at this stage. Just think of S as, for example, the space of C^∞ functions.

If we want to study the analytical properties of S^P through algebraic methods, we need to introduce a new polynomial matrix $P^t := [P_{ji}(z)]$ and look at the dual of the previous sequence: set R $= \mathbb{C}\ [z_1, \ldots, z_n]$ and consider

$$0 \longleftarrow M = \text{Coker } P^t = \frac{R^{r_0}}{P^t R^{r_1}} \longleftarrow R^{r_0} \xleftarrow{P^t} R^{r_1}.$$

A famous, fundamental, result of Hilbert, known as the Hilbert's syzygy theorem, states that it is possible to continue this last sequence to a finite resolution

$$0 \longleftarrow M \longleftarrow R^{r_0} \xleftarrow{P_1^t} R^{r_1} \xleftarrow{P_2^t} R^{r_2} \xleftarrow{P_3^t} R^{r_3} \longleftarrow \cdots \xleftarrow{P_m^t} R^{r_m} \longleftarrow 0$$

with $m \leq n$ and $P_1 = P$. Note that Hilbert 's theorem ensures us of the existence of the sequence (which is not unique), and of a bound on its length, but does not say anything about its concrete construction.
If we now take the transpose of this sequence, we obtain a very important complex, namely

$$0 \longrightarrow R^{r_0} \xrightarrow{P_1} R^{r_1} \xrightarrow{P_2} R^{r_2} \longrightarrow \cdots \xrightarrow{P_m} R^{r_m} \longrightarrow 0.$$

Once again, the complex is not unique, but its cohomology is, and we define

$$\mathrm{Ext}^j(M, R) := \frac{\mathrm{Ker} P_{j+1}}{\mathrm{Im} P_j}.$$

Before I explain the relevance of these groups for the analytical questions we discussed before, let me give a simple example of how to construct such a resolution. Let $Q = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$ be a 3×1 matrix of polynomials, and suppose that they have no common factors. Then M $= \frac{R}{Q^t R^3} = \frac{R}{I(Q_1, Q_2, Q_3)}$ where $I(Q_1, Q_2, Q_3)$ is the ideal generated by Q_1, Q_2, Q_3 in R. To search for P_2^t means to search for the kernel of P^t in R^3. It is obvious that elements such as $(-Q_2, Q_1, 0)$, $(-Q_3, 0, Q_1)$, and $(0, -Q_3, Q_2)$ belong to this kernel, but, in fact, under the previous assumptions they generate all of it. So our resolution looks like

$$0 \longleftarrow M \longleftarrow R \xleftarrow{(Q_1, Q_2, Q_3)} R^3 \xleftarrow{\begin{bmatrix} -Q_2 & Q_1 & 0 \\ -Q_3 & 0 & Q_1 \\ 0 & -Q_3 & Q_2 \end{bmatrix}^t} R^3.$$

We now need to build the kernel of this last map, and arguments as before show that such a kernel is generated by the map

$$\begin{aligned} R^3 &\longleftarrow R \\ h(Q_3, -Q_2, Q_1)^t &\longleftarrow\!\shortmid\, h \end{aligned}$$

This map is obviously injective and so the resolution has its final form

$$0 \leftarrow M \leftarrow R \xleftarrow{(Q_1, Q_2, Q_3)} R^3 \xleftarrow{\begin{bmatrix} -Q_2 & Q_1 & 0 \\ -Q_3 & 0 & Q_1 \\ 0 & -Q_3 & Q_2 \end{bmatrix}^t} R^3 \xleftarrow{\begin{bmatrix} Q_3 \\ -Q_2 \\ Q_1 \end{bmatrix}} R \leftarrow 0.$$

This construction is known as the Koszul complex, and if we look at the Cauchy-Riemann system in three variables z_1, z_2, z_3, we can apply it to

$$Q_1 = i\xi_1 - \eta_1 \quad Q_2 = i\xi_2 - \eta_2 \quad Q_3 = i\xi_3 - \eta_3$$

so that

$$Q_1(D) = \frac{\partial}{\partial \overline{z}_1} \qquad Q_2(D) = \frac{\partial}{\partial \overline{z}_2} \qquad Q_3(D) = \frac{\partial}{\partial \overline{z}_3}$$

But let's now go back to the analytical meaning of the objects we have just constructed. To begin with, a rather deep result of Ehrenpreis, Hörmander et al. shows that P_{j+1} is a compatibility system of P_j. In particular P_2 is a compatibility system for P, so that:

Theorem: The non-homogeneous system

$$P(D)f = g$$

has a solution f if and only if $P_2(D)g = 0$.

The simple Koszul complex we have just constructed shows that

$$\frac{\partial f}{\partial \overline{z}_1} = g_1 \qquad \frac{\partial f}{\partial \overline{z}_2} = g_2 \qquad \frac{\partial f}{\partial \overline{z}_3} = g_3$$

has a solution if and only if

$$\frac{\partial g_i}{\partial \overline{z}_j} = \frac{\partial g_j}{\partial \overline{z}_i} \qquad i, j = 1, 2, 3,$$

exactly as we mentioned in section 2.

A second, important, fact is that the removability of compact singularities can also be read through the resolution of M, and precisely Ehrenpreis and Palamodov showed that.

Theorem: The solutions of $P(D)f = 0$ cannot have compact singularities if and only if

$$\mathrm{Ext}^1(M, R) = 0,$$

i.e., if and only if

$$\mathrm{Im}P_1 = \ker P_2.$$

Again, our simple Koszul computation shows that this is the case for the Cauchy-Riemann system (due to the symmetric nature of the Koszul complex) as well as for other large families of systems.
Finally, I mentioned earlier the vanishing of higher cohomology. I won't spend too many words now, but such vanishing is a consequence of the vanishing of the first few $\mathrm{Ext}^j(M, R)$.
Now that we see the importance of computing these resolutions, we may want to go back to the Cauchy-Fueter system and see what can be said. The main difficulty lies in the fact that this is a genuine rectangular system, and the simple Koszul construction will not work here. There exists a similar, more elaborate, construction, known as the generalized Koszul complex (introduced by Buchsbaum back in the 60's), but if we try to use it here it does not seem to be very effective (see [18] for a precise description of this complex).

4 Algebraic Analysis of the Cauchy-Fueter system

For the sake of simplicity, let's start by looking at the case $n = 2$ (i.e., 8 real variables). The system we are studying acts from S to S^2 or, in matrix form, from S^4 to S^8.
Note that the "usual" constuction of syzygies won't quite work here because

$$\frac{\partial}{\partial \overline{q}_2}\frac{\partial}{\partial \overline{q}_1} \neq \frac{\partial}{\partial \overline{q}_1}\frac{\partial}{\partial \overline{q}_2}.$$

In fact, it turns out that a resolution (a minimal one, in fact) can be constructed as follows:

$$0 \longleftarrow M \longleftarrow R^4 \xleftarrow{P^t} R^8 \xleftarrow{P_2^t} R^8 \xleftarrow{P_3^t} R^4 \longleftarrow 0$$

where P_2 is a (real) 8×8 matrix whose "quaternionic" interpretation is

$$\begin{bmatrix} \bar{q}_1 q_2 & -\bar{q}_1 q_1 \\ \bar{q}_2 q_2 & -\bar{q}_2 q_1 \end{bmatrix} = \begin{bmatrix} \bar{q}_1 q_2 & -\Delta_1 \\ \Delta_2 & -\bar{q}_2 q_1 \end{bmatrix}$$

and P_3 is nothing but a variation of the Cauchy-Fueter system (just as it happened for the Cauchy-Riemann case), with the same sheaf of solutions.

There are several interesting comments one could make here. Maybe the most important is the surprising fact that the first syzygies are quadratic. When we first looked at this problem, we were actually looking for linear syzygies as the Cauchy-Riemann system has taught us, while the use of the generalized Koszul complex lead us to cubic syzygies. However, the case of $n = 2$ is way too simple (and it can be "guessed" or obtained by trial and error) to illuminate us on the entire story, nor does it allow for a direct generalization.

Already too complex for hand-computations is the case of $n = 3$ (i.e., 12 real variables). If U_1, U_2, U_3 denote the 4×4 polynomial matrices which we obtain from $\frac{\partial}{\partial \bar{q}_1}, \frac{\partial}{\partial \bar{q}_2}, \frac{\partial}{\partial \bar{q}_3}$ with the usual substitutions, then we used CoCoA (a software developed at the University of Genova, and freely available at http://cocoa.dime.unige.it) to compute the first syzygies for $n = 3$ and we obtained that a generating set for such syzygies was given by

$$\begin{gathered}(U_r U_s^t)\, U_s - (U_s^t U_s)\, U_r = 0 \\ U_r U_s^t U_l + U_r U_l^t U_s - (U_s^t U_l + U_l^t U_s)\, U_r = 0 \\ [U_r, U_s]\, J\, [U_l, I] + [U_s, U_l]\, J\, [U_r, I] + [U_l, U_r]\, J\, [U_s, I] = 0\end{gathered}$$

where $I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$ and $J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

Once again, we get quadratic syzygies, and the resolution looks like

$$0 \longleftarrow M \longleftarrow R^4 \longleftarrow R^{12} \longleftarrow R^{40} \longleftarrow R^{60} \longleftarrow R^{36} \longleftarrow R^8 \longleftarrow 0.$$

So, here, a new (surprising) phenomenon appears, namely the last operator of the resolution is not a transpose of the first one, and we come to understand the uniqueness of the duality which occurs for the Cauchy-Riemann system.

We are now in the position of actually discussing what happens for higher values of n. The key point here is that we need a tool to study the syzygies of the Cauchy-Fueter system, regardless of the number of variables. In addition, the two results we just discussed seem to indicate (but this is fortunately not true) that as the dimension grows, new unexpected syzygies

might appear. Our approach, as described in detail in [2] is based on the study of the Gröbner bases for the module M. One can show, in fact, that such bases can be explicitely computed, and this computation is all we need to establish our fundamental results.
The first, important, result is really an algebraic lemma (we refer the reader to [1] for the necessary definitions:
Lemma: The reduced Gröbner bases for M are given by the columns of the matrix P^t, together with the columns of the matrices $U_rU_s - U_sU_r$.
Using this lemma one can show that for any value of n the following results hold:
Theorem 1: All first syzygies are generated by the formulas given before, and therefore quadratic.
Theorem 2: All other syzygies are linear.
As a corollary of the previous results, one can actually compute the Hilbert-Poincare series for M , and the Betti numbers of the resolutions of M.
One shows then that the length of the resolution for the Cauchy-Fueter system in n quaternionic variables is in fact $2n - 1$. The analytic consequences of this result are quite significant and, as shown in [2], allow us to reconstruct a pretty complete function theory for the sheaf of regular functions.
For the sake of completeness, I should add that there is an important recent paper of Baston [3] in which a different, geometrical approach is used. While Baston's construction is not explicit, he does prove that the first syzygies are quadratic. The recent important work of Soucek and his colleagues [17] show in detail the way in which Baston's approach is related to our own work. In particular, Baston's result is quite important as it uses the specific quaternionic structure of the system, in contrast to our more algebraic approach, in which quaternions disappear. So, in a way, Baston's result more clearly ties quaternions to the resolution of the Cauchy-Fueter system. On the other hand, our approach shows that both Cauchy-Riemann and Cauchy-Fueter are nothing but specific examples of how modern computational algebra can be used to deal with systems of partial differential equations. Our results open a new chapter in the study of such systems and we have shown in a series of articles (for example [2], [5], [7], [14], [16], and [19]) how important systems of physical interest can be dealt with (the examples include Maxwell, Dirac, Proca, Moisil-Theodorescu, and other systems in the context of quaternions, Clifford Algebras, and Octonions).

Our most recent work [15] also begins a series of applications to the variable coefficients case. We refer the reader to our forthcoming book [6].

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