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UNIQUENESS FOR HIGHER DIMENSIONAL TRIGONOMETRIC SERIES

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Abstract

Five uniqueness questions for multiple trigonometric series are surveyed. If a multiple trigonometric series converges everywhere to zero in the sense of spherical convergence, of unrestricted rectangular convergence, or of iterated convergence, then that series must have every coefficient being zero. But the cases of square convergence and restricted rectangular convergence lead to open questions.[1]

1 Introduction

Let $\mathbb{T}^d = [0,1)^d$ be the d dimensional torus. This means that $\mathbb{T}^d$ is a bounded part of d dimensional Euclidean space, but that addition is modulo 1 in each coordinate. Let $\{\varphi_n(x)\}_{n=1,2,..}$ be a real or complex valued system of functions that are in $L^2(\mathbb{T}^d) = \{f : \mathbb{T}^d \longrightarrow \mathbb{C} : \int_{\mathbb{T}^d} |f|^2 \, dx < \infty\}$. If the

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inner products in L^2, $\langle \varphi_m, \varphi_n \rangle = \int_{\mathbb{T}^d} \varphi_m(x)\, \overline{\varphi_n(x)} dx$, where the bar denotes complex conjugate, satisfy $\langle \varphi_m, \varphi_n \rangle \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$,we call the system orthonormal (ON). Given an ON system and a function f on $\mathbb{T}^d$, it is often possible to represent f as an infinite linear combination of the elements of the system. The word "represent" may be given a variety of meanings, but in this work we will narrow the possibilities down by always demanding that the linear combination, $\sum a_n \varphi_n(x)$ be everywhere pointwise convergent to the value $f(x)$,

$$f(x) = \sum a_n \varphi_n(x) \quad \text{for each } x \in \mathbb{T}^d. \tag{1.1}$$

Even with x fixed, so that the series in definition (1.1) is a series of numbers, the notion of "represents" is still incomplete because it depends on what it means for this series of numbers to converge. As we study different systems here, we will carefully explain what "converges" means for each system. Once we have settled on an ON system and a definition of convergence, two natural questions immediately arise.

I: Existence. Which functions are representable?

II: Uniqueness. Does any function have at least two distinct representations?

To study the first question seriously, we would have to introduce the notion of completeness and then restrict ourselves to complete ON systems. The first question is very interesting and has a vast literature associated with it, but we will not consider it here. In particular , the notion of completeness will play no role whatsoever for us.

Turning to the question of uniqueness, we can immediately reduce the general question II to what seems to be a special case, a case involving the function 0, i.e., the function which has the value 0 at each $x \in \mathbb{T}^d$. Now the function 0 will always have the trivial representation $0 = \sum 0\varphi_n(x)$, and the function 0 may also have a nontrivial representation, $0 = \sum a_n \varphi_n(x)$, with some $a_n \neq 0$.

Definition 1. *If an ON system $\{\varphi_n\}$ and a method of convergence are given, say that uniqueness holds if* 0 *has no nontrivial representation, i. e., if $0 = \sum a_n \varphi_n(x)$ implies that all $a_n = 0$.*

To see that this definition is fully general, fix $\{\varphi_n\}$ and the meaning of convergence. Suppose that there is a function f with two distinct representations so that $f(x) = \sum a'_n \varphi_n(x)$ for each $x \in \mathbb{T}^d$, where $a_n \neq a'_n$ for some n.

Then we will have $0 = f(x) - f(x) = \sum(a_n - a_n')\varphi_n(x)$ which is a nontrivial representation of 0. Thus *any* instance of nonuniqueness immediately leads to the seemingly special case of nonuniqueness given in the definition.

We will begin with an example where uniqueness does not hold.

Example 1 (Haar functions). *Let* $d = 1$, $\{h_n(x)\}$ *be the Haar functions, and let convergence mean* $\sum a_n h_n(x) = \lim_{N\to\infty} \sum_{n=0}^{N} a_n h_n(x)$. *Then uniqueness does not hold.*

Proof. What I shall take for the Haar functions is only a subset of the standard Haar functions. My Haar functions are defined as follows. For all n $h_n(0) = 0$, $h_o(x) = 1$ for $0 < x < 1$.

$$h_1(x) = \begin{cases} 2^{\frac{0}{2}} & 0 < x < \frac{1}{2} \\ -2^{\frac{0}{2}} & \frac{1}{2} < x < 1 \end{cases},$$

$$h_2(x) = \begin{cases} 2^{\frac{1}{2}} & 0 < x < \frac{1}{2^2} \\ -2^{\frac{1}{2}} & \frac{1}{2^2} \leqslant x < \frac{1}{2} \\ 0 & \frac{1}{2} \leqslant x < 1 \end{cases}, \ldots.,$$

$$h_n(x) = \begin{cases} 2^{\frac{n-1}{2}} & 0 < x < \frac{1}{2^n} \\ -2^{\frac{n-1}{2}} & \frac{1}{2^n} \leqslant x < \frac{1}{2^{n-1}} \\ 0 & \frac{1}{2^{n-1}} \leqslant x < 1 \end{cases}, \ldots.$$

Note that

$$\int_0^1 h_n(x)^2 \, dx = \int_0^{2^{-(n-1)}} 2^{n-1} dx = 1$$

and if $n > m$, $\int_0^1 h_m(x)h_n(x)dx = \int_0^{2^{-(n-1)}} h_n(x) \, dx = 0$, so the Haar functions are an ON system. Now

$$h_0(x) + h_1(x) = \begin{cases} 2^1 & 0 < x < \frac{1}{2^1} \\ 0 & \text{otherwise} \end{cases},$$

$$h_0(x) + h_1(x) + 2^{\frac{1}{2}} h_2(x) = \begin{cases} 2^2 & 0 < x < \frac{1}{2^2} \\ 0 & \text{otherwise} \end{cases}, \ldots,$$

$$h_0(x) + h_1(x) + \ldots + 2^{\frac{n-1}{2}} h_n(x) = \begin{cases} 2^n & 0 < x < \frac{1}{2^n} \\ 0 & \text{otherwise} \end{cases}, \ldots$$

Consider the sum $h_0(x) + \sum_{n=1}^{\infty} 2^{\frac{n-1}{2}} h_n(x)$. If $x = 0$ every term is 0, while if $x > 0$, as soon as n is such that $\frac{1}{2^n} \leqslant x$, the sum of the first $n+1$ terms is 0. But for such an x and n, for every $m > n$ we have $h_m(x) = 0$. so that the infinite sum is the same as the sum of the first $n+1$ terms. It follows that

$$h_0(x) + \sum_{n=1}^{\infty} 2^{\frac{n-1}{2}} h_n(x) = 0 \text{ for every } x \in [0,1). \tag{1.2}$$

Since (1.2) is a nontrivial representation for 0, uniqueness fails for the Haar system.

□

For several other examples of ON systems where nonuniqueness occurs, see reference [AWa1].

Example 2 (trigonometric functions). *Let* $d = 1$, $\{t_n(x)\}_{n=0,1,2,\ldots} = \{1, cos2\pi x, sin2\pi x, cos2\pi 2x, \ldots, cos2\pi nx, sin2\pi nx, \ldots\}$ *be the trigonometric functions, and let convergence mean* $\sum a_n t_n(x) = \lim_{N\to\infty} a_0 + \sum_{k=1}^{N} (a_{2k-1}t_{2k-1}(x) + a_{2k}t_{2k}(x))$. *Then uniqueness holds.*

Already from these two one dimensional examples we see that the meaning of convergence can be delicate. For example, the series

$$1 - 1 + 1 - 1 + \ldots$$

is divergent with respect to the definition given in the first example, because the sequence of its partial sums is then $1, 0, 1, 0, \ldots$, but it converges to 1 with respect to the method of convergence given in the second example since there every partial sum is 1.

The remarkable result that forms example 2 was proved by Georg Cantor in 1870. Before stating his result, we will change to a notation that is a little more convenient for discussing trigonometric series.

Notation 1. *From now on, by* $\mathbb{T}^d$ *we will mean* $[0, 2\pi)^d$ *with addition defined modulo* 2π *in each dimension. We will henceforth only consider systems of the form* $\{e^{inx}\}$, *where* $nx = n_1x_1 + \ldots + n_dx_d$. *When* $d = 1$, *convergence will be defined by* $\sum_{n=-\infty}^{\infty} c_n e^{inx} = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{inx}$. *When* $d = 2$, *to avoid subscripts we will write* e^{iMX} *with* $M = (m, n)$, $X = (x, y)$, *and* $MX = mx + ny$.

For each d our system is orthogonal, but it is not exactly ON , since $\int_{\mathbb{T}^d} |e^{inx}|^2 \, dx = (2\pi)^d$, rather than 1. Wen $d = 1$, taking Euler's formula, $e^{ix} = \cos x + i \sin x$ into account, we now have the nth term expressed as $c_{-n}e^{-inx} + c_n e^{inx}$, rather than as $a_{2n-1} \cos(2\pi nx) + a_{2n} \sin(2\pi nx)$. To see the connection, we calculate that when $c_n = \frac{a_n - ib_n}{2}$ and $c_{-n} = \overline{c_n} = \frac{a_n + ib_n}{2}$ there follows

$$c_{-n}e^{-inx} + c_n e^{inx} = a_n \cos nx + b_n \sin nx.$$

Theorem 1 (Cantor). *Let* $d = 1$. *If* $\sum_{n=-\infty}^{\infty} c_n e^{inx} = 0$ *for all* $x \in T$, *then all* $c_n = 0$.

For the details of the proof and a discussion of the history and significance of this celebrated theorem, see reference [A] . We will however here list the major steps of the proof, because they seem to be the starting point for all known generalizations to higher dimensions.

(1) Establish the Cantor-Lebesgue Theorem, that $|c_n| + |c_{-n}| \longrightarrow 0$,

(2) show that the Riemann function $F(x) = c_0 \frac{x^2}{2} + \sum_{n \neq 0} \frac{c_n}{n^2} e^{inx}$ is continuous,

(3) establish the consistency of Riemann summability, that the Schwarz second derivative D^2 defined by

$$D^2 F(x) = \lim_{h \to 0} \frac{F(x+h) - 2F(x) + F(x-h)}{h^2} \tag{1.3}$$

satisfies

$$D^2 F(x) = \lim_{h \to 0} c_0 + \sum_{n \neq 0} c_n e^{inx} \left(\frac{\sin \, n\frac{h}{2}}{n\frac{h}{2}} \right)^2 = 0, \quad \text{and}$$

(4) prove Schwarz's Theorem, that continuous functions with identically zero Schwarz second derivative are of the form $ax + b$.

2 Uniqueness for Multiple Trigonometric Series

The remainder of this paper will discuss the following five questions. All are framed for dimension $d \geqslant 2$.

(1) Does uniqueness hold if convergence means iterated convergence?

(2) Does uniqueness hold if convergence means unrestricted rectangular convergence?

(3) Does uniqueness hold if convergence means spherical convergence?

(4) Does uniqueness hold if convergence means restricted rectangular convergence?

(5) Does uniqueness hold if convergence means square convergence?

Before discussing these questions we will define the five methods of convergence.

We will restrict ourselves henceforth to $d = 2$, since this case is sufficiently general to display the full complexity of the issues.

Fix $X = (x, y)$ and set $a_M = a_M(X) = c_M e^{iMX}$. There are many ways to add up the terms of $\mathrm{T} = \sum a_M$, but we will consider here only the five methods listed above.

Definition 2. *Let* $|M| = \sqrt{m^2 + n^2}$ *and for each real* $r \geqslant 0$, *define*

$$T^r = \sum_{|M| \leqslant r} a_M$$

to be the rth spherical partial sum of T. *We say* T *converges spherically to* t *if*

$$\lim_{r \to \infty} T^r = t.$$

For double indices $M = (m, n)$ and $P = (p, q)$, say that $M \geqslant P$ if $m \geqslant p$ and $n \geqslant q$, and for any real number r, let $\underline{r}$ denote the double index (r, r).

Definition 3. *For* $N \geqslant \underline{0}$, *define*

$$T_N = \sum_{-N \leqslant M \leqslant N} a_M$$

to be the N th rectangular partial sum of T. *Let* r *be a nonnegative integer. We define*

$$T_{\underline{r}} = \sum_{-\underline{r} \leqslant M \leqslant \underline{r}} a_M$$

to be the r th square partial sum of T *and say* T *is square convergent to* t *if*

$$\lim_{r \to \infty} T_{\underline{r}} = t.$$

If we define a different norm $\|\cdot\|$ by $\|M\| = \max\{|m|, |n|\}$, we see that $T_{\underline{r}}$ can also be expressed as

$$T_{\underline{r}} = T_{(r,r)} = \sum_{\|M\| \leqslant r} a_M.$$

The various methods of summing T can be viewed geometrically. For example, the spherical (circular, when $d = 2$) partial sum T^r are so named because the indices associated with the terms of T appearing in the partial sum are exactly the indices contained within the closed origin-centered sphere of radius r. Spherical summation is a plausible way of adding up all the terms of T because any fixed index M is in all spheres of sufficiently large radius; equivalently, a_M is included in all sufficiently late spherical partial sums. Similarly, if $(r, s) \geqslant \underline{0}$, the indices M satisfying $-(r, s) \leqslant M \leqslant (r, s)$ are exactly the indices contained within the closed origin-centered rectangle with lower left corner $(-r, -s)$ and upper right corner (r, s). In particular, if $r = s$, the rectangle is actually a square. (If $\|M\| = r$, then M is on the edge of the square.) Again, as $r \longrightarrow \infty$, any fixed index M is in all sufficiently late squares; equivalently, a_N is included in all sufficiently late square partial sums. In short, any method which eventually captures all points of the index set $\mathbb{Z}^2$ is a plausible summation method. The three methods yet to be described will also eventually capture the entire index set.

Definition 4. *Say that T is iteratively convergent to t if both of the nested limits*

$$\sum_{\mu=-\infty}^{\infty}\left(\sum_{\nu=-\infty}^{\infty} a_{\mu\nu}\right) = \lim_{m\to\infty} \sum_{\mu=-m}^{m}\left(\lim_{n\to\infty} \sum_{\nu=-n}^{n} a_{\mu\nu}\right) \quad \text{and} \quad \sum_{\nu=-\infty}^{\infty}\left(\sum_{\mu=-\infty}^{\infty} a_{\mu\nu}\right)$$

are equal to t.

Definition 5. *Say that T is unrestrictedly rectangularly convergent to t if*

$$\lim_{min\{|m|,|n|\}\to\infty} T_{(m,n)} = t.$$

We will often abbreviate unrestrictedly rectangularly convergent to just *rectangularly convergent.* Finally,

Definition 6. T *is restrictedly rectangularly convergent to* t *if for any* $e \geqslant 1$, *no matter how large,*

$$\lim_{r\to\infty} \sup \left\{ |T_M - t| : \min\{|m|, |n|\} \geqslant r, \text{ and } \frac{1}{e} \leqslant \left|\frac{m}{n}\right| \leqslant e \right\} = 0.$$

All five methods are symmetric, i.e., if a_M is included in a partial sum and if M' differs from M only by a coordinate signs so that $|m'| = |m|$ and $|n'| = |n|$, then $a_{M'}$ will also be included in that partial sum. There are only two obvious connections between these methods: unrestricted rectangular convergence implies restricted rectangular convergence and restricted rectangular convergence implies square convergence. To better understand these methods of convergence, it may be useful to construct examples which show that there are no other connections between the various methods. For example, if $a_{0n} = (-1)^n$, $a_{1n} = (-1)^{n+1}$, and $a_{mn} = 0$ otherwise, then T is unrestrictedly rectangularly convergent to 0, but T is not iteratively convergent. See [AWe] for other examples. The definition of restricted rectangular convergence is particularly tricky, since the limit must be t for *every* choice of the eccentricity it almost seems that this method is not very different from the unrestricted method. We will see below that the two methods are quite different.

The answers to the five questions are yes, yes, yes, don't know, and don't know. Only the first question is easy. It is a routine induction argument.

Proposition 1. *Uniqueness hold for iterated convergence. If* $\sum c_M e^{iMX}$ *is iteratedly convergent to 0 everywhere, then all* $c_M = 0$.

Proof If $d = 1$, this is Cantor's Theorem. If $d = 2$, our hypothesis asserts that for each fixed y,

$$\lim_{m\to\infty} \sum_{\mu=-m}^{m} C_\mu(y)\, e^{i\mu x} = 0 \text{ for every } x, \tag{2.1}$$

where

$$C_\mu(y) = \sum_{\nu=-\infty}^{\infty} c_{\mu\nu} e^{i\nu y}. \tag{2.2}$$

Cantor's Theorem allows us to conclude from equation (2.1) that $C_\mu(y) = 0$ for each y. Then fixing μ and applying Cantor's Theorem again to equation (2.2) shows that $c_{\mu\nu} = 0$ for all ν. Since μ was arbitrary, the proposition is verified. □

3 Unestricted Rectangular Convergence

The historical path to the two nontrivial positive results did not follow the straightforward path we present here. What actually happened was that first a flawed proof for unrestricted rectangular uniqueness was published in 1919. Since it appeared that the unrestricted rectangular case had been resolved, it was natural for attention to turn to spherical uniqueness, where a proof for d dimensional uniqueness involving extra assumptions on the coefficient size was achieved by Victor Shapiro in 1957.[S] A corollary of one of Shapiro's results was this.

Corollary 1. *If* $\sum c_n e^{inx} = 0$ *spherically for all* $x \in T^d$, *and if*

$$\lim_{r\to\infty} \frac{1}{r} \sum_{r-1<|m|\leqslant r} |c_m| = 0, \tag{3.1}$$

then all $c_n = 0$.

Then in 1971 Roger Cooke found this generalization to the Cantor-Lebesgue Theorem for dimension $d = 2$.[Coo]

Theorem 2 (Cooke) . *Let* $d = 2$. If $\{c_m\}$ *is a doubly indexed set of complex numbers such that*

$$\sum_{|m|=r} c_m e^{imx}$$

tends to zero for almost all x, *then*

$$\sqrt{\sum_{|m|=r} |c_m|^2} \text{ tends to } 0 \text{ as } r \to \infty. \tag{3.2}$$

It is clear from the definition of spherical convergence that spherical convergence at x to 0 (or to any other finite value for that matter) implies that the hypothesis of Cooke's theorem holds at x. Now it turns out that when $d = 2$, the conclusion of Cooke's Theorem implies the validity of condition (3.1) and thus the unconditional spherical uniqueness theorem in dimension $d = 2$. [AWa] , page 42

The pendulum then swung back to the unrestricted rectangular convergence uniqueness question. Just at the time of Cooke's work, Grant Welland and I looked at an argument that Hilda Gehringer had given in 1919 in support of uniqueness for unrestrictedly rectangularly convergent series. We found a gap in the proof that we could not fill. We were able to prove only this.

Theorem 3. *Uniqueness for unrestricted rectangular convergence holds in two dimensions.*

If $S_{mn} = \sum_{\mu,\nu=0,0}^{m,n} A_{\mu\nu}$, we have the simple "Mondrian" identity

$$A_{mn} = S_{mn} - S_{m,n-1} - S_{m-1,n} + S_{m-1,n-1},$$

which leads to a fairly strong (and best possible) Cantor-Lebesgue type theorem. To see why I named this identity after the artist Mondrian, see Figure 1 on page 411 of reference [AWe].

Theorem 4. *If a series is unrestrictedly rectangular convergent everywhere, then the coefficients satisfy*

$$c_m \to 0 \quad \textit{as} \quad \min_i |m_i| \to \infty \quad \textit{and all} \quad c_m \quad \textit{are bounded.} \tag{3.3}$$

The proof of Theorem 3 depended on two "lucky" facts. Lucky fact number one is that in dimension two, this condition implies the Shapiro condition (3.1).

Now as one would suspect, at a single fixed point (x_o, y_o), unrestricted rectangular convergence does not imply circular convergence. Furthermore, it is even possible for a double trigonometric series to be unrestricted rectangular convergent almost everywhere while being circular convergent on at most a set of measure zero.[AWe], p. 418 However, unrestricted rectangular convergence everywhere *does* imply spherical Abel summability everywhere. (The multiple series $\sum a_m$ is spherically Abel summable to s if $\lim_{r\to 0+} \sum a_m e^{-|m|r} = s$.) This was a second stroke of good luck because the corollary mentioned above was to a theorem of Shapiro which postulated that a multiple trigonometric series satisfied Shapiro's condition (3.1) and was everywhere spherical Abel summable to 0 and concluded that all the coefficients were zero.

So dimension two was done, but in retrospect, the proof really was as lucky as it seemed and shed no light at all on the higher dimensional unrestricted rectangular uniqueness question. Twenty years went by without any further progress. Then in the early 1990s came the complete solution, with two independent and completely different proofs.

Theorem 5. *Unrestricted rectangular uniqueness holds in all dimensions.* [Tet], [AFR]

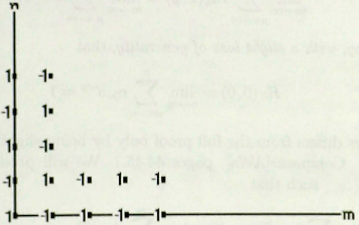
We will discuss both proofs only in dimension two, since all the ideas already come into view there.

Tetunashvili's proof is based on a simple, but powerful idea. He noticed that it is easy to prove uniqueness for iterated convergence. In fact, we have presented the routine induction argument which accomplishes this above. (See the proof of Proposition 1 above.) So he would have an immediate proof of Theorem 5 if he could prove that unrestricted rectangular convergence implies iterated convergence.

But consider the numerical double series given by

$$a_{mn} = \begin{cases} (-1)^{m+n} & \text{if } m \in \{0,1\} \text{ or } n \in \{0,1\} \\ 0 & \text{otherwise} \end{cases}$$

Here is a representation of this series where the value of a_{mn} is attached to the point (m,n).



As soon as both m and n exceed 1, the (m,n) partial sum S_{mn} is 0,

$$\begin{aligned} S_{mn} = \sum_{\mu=-m}^{m} \sum_{\nu=-n}^{n} a_{\mu\nu} &= \sum_{\mu=0}^{m} (-1)^{\mu} \left(\sum_{\nu=0}^{1} (-1)^{\nu} \right) + \sum_{\nu=2}^{n} (-1)^{\nu} \left(\sum_{\mu=0}^{1} (-1)^{\mu} \right) \\ &= 0 + 0 \\ &= 0, \end{aligned}$$

so that $\lim_{min(m,n)\to\infty} S_{mn} = 0$. On the other hand, $S_{0n} = (1+(-1)^n)/2$ and $S_{m0} = (1+(-1)^m)/2$, so that neither $\lim_{n\to\infty} S_{0n}$ nor $\lim_{m\to\infty} S_{m0}$ exist. In other words, this double series in unrestrictedly rectangularly convergent, but not iteratively convergent. Nevertheless, Tetunashvili was able to prove that if a multiple trigonometric series is everywhere unrestrictedly rectangularly convergent to 0, then it is also everywhere iteratively convergent to 0. We will now explain the idea behind Tetunashvili's proof. Assume that a double trigonometric series converges unrestrictedly rectangularly to 0 everywhere, so that for all $(x, y) \in T^2$

$$(3.4)\ T(x,y) = \lim_{min\{m,n\}\to\infty} T_{mn}(x,y) = \lim_{min\{m,n\}\to\infty} \sum_{\mu=-m}^{m} \sum_{\nu=-n}^{n} c_{\mu\nu} e^{i(\mu x+\nu y)} = 0.$$

It is enough to show that this implies that every row sum

$$R_m(x,y) = \lim_{n\to\infty} \sum_{\nu=-n}^{n} c_{m\nu} e^{i(mx+\nu y)} = e^{imx} \lim_{n\to\infty} \sum_{\nu=-n}^{n} c_{m\nu} e^{i\nu y}$$

is identically zero for then the iterated sum will be

$$\lim_{m\to\infty} \sum_{\mu=-m}^{m} R_m(x,y) = \lim_{m\to\infty} \sum_{\mu=-m}^{m} 0 = 0.$$

Let us assume, *with a slight loss of generality*, that

$$R_0(0,0) = \lim_{n\to\infty} \sum_{\nu=-n}^{n} c_{0\nu} e^{i\nu y} = 1.$$

(What follows differs from the full proof only by being slightly less painful notationally. Compare [AWa], pages 44.45.) We will produce constants $\{\lambda_m\}_{m=1,-1,2,-2,\ldots}$ such that

$$(3.5) \qquad 1 + \sum_{m\neq 0} \lambda_m e^{imx} = \lim_{m\to\infty} 1 + \sum_{\mu=-m}^{m} \lambda_\mu e^{i\mu x} = 0 \ \text{ at every } x,$$

contradicting Theorem 1, Cantor's original one dimensional uniqueness theorem. Once and for all, fix $y = 0$ and set

$$A_{mn} = \sum_{\nu=-n}^{n} c_{m\nu},\ m = 0, 1, -1, 2, -2, \ldots;\ n = 0, 1, 2.....$$

We will now need a theorem that Grant Welland and I proved as a lemma for

Theorem 4 which says that all the rectangular partial sums are bounded, $|T_{mn}(x,0)| \leqslant B(x)$. In one dimension, such a theorem amounts to the trivial remark that the partial sums of a convergent series are bounded, while here there is definitely something to prove, even though the constant B is allowed to depend on x.(See [AWe], pages 406-407.) Next, if $E_n = \{x \in \mathbb{T} : B(x) \leqslant n\}$, $\cup E_n = \mathbb{T}$ so that some E_n has positive Lebesgue measure. In other words, there is a set E of positive measure and an absolute constant $\overline{B}$ so that

$$|T_{mn}(0,x)| \leqslant \overline{B} \text{ for all } x \in E, \text{ all } m \geqslant 0, \text{ and all } n \geqslant 0.$$

An important theorem of Paul Cohen asserts that we can expand the above inequality to hold for every $x \in \mathbb{T}$. The reason for this is that we may think of T_{mn} as a trigonometric polynomial of degree m, since

$$T_{mn}(x,0) = \sum_{\mu=-m}^{m} \left(\sum_{\nu=-n}^{n} c_{\mu\nu} \right) e^{i\mu x} = \sum_{\mu=-m}^{m} A_{\mu n} e^{i\mu x},$$

and Paul Cohen's lemma says that for each nonnegative m, there is a bound $B(m) = B(m, \overline{B}, |E|)$ so that

$$|T_{mn}(x,0)| \leqslant B(m)$$

holds for *every* x and *every* n. Even though our constant is no longer absolute, but now depends on m, this is quite a strong fact, since there are infinitely many values of n. This powerful lemma has the further consequence that whenever $|\mu| \leqslant m$,

$$|A|_{\mu n} = \left| \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{\mu=-m}^{m} A_{\mu n} e^{i\mu x} \right) dx \right| \leqslant \left| \frac{1}{2\pi} \int_0^{2\pi} B(m)\, dx \right| = B(m)$$

Recall that $A_{0,n} \longrightarrow 1$ as $n \longrightarrow \infty$. Because $\{A_{1,n}\}$ is bounded by $B(1)$, we may find a first subsequence $\{n_j\}$ and a number $\lambda_1 \in [-B(1), B(1)]$ so that $A_{1,n_j} \longrightarrow \lambda_1$. Because $\{A_{-1,n}\}$ is bounded by $B(1)$, we may find a second subsequence which we will still denote $\{n_j\}$, namely a subsequence of the first one and a number $\lambda_{-1} \in [-B(1), B(1)]$ so that $A_{1,n_j} \longrightarrow \lambda_{-1}$. Because $\{A_{2,n}\}$ is bounded by $B(2)$, we may find a third subsequence which we will still denote $\{n_j\}$, namely a subsequence of the second one and a number $\lambda_2 \in [-B(2), B(2)]$ so that $A_{2,n_j} \longrightarrow \lambda_2$. Continuing this process recursively, after $2m$ repetitions we come to a $2m$th subsequence which we will still denote $\{n_j\}$ and a number λ_{-m} such that $A_{-m,n_j} \longrightarrow \lambda_{-m}$. Furthermore, since each subsequence was extracted from the previous one, at this stage we know that

$$\text{for each } \mu \in [-m, m],\ A_{\mu,n_j} \longrightarrow \lambda_\mu \text{ as } j \longrightarrow \infty.$$

This process generates a number λ_μ for every nonzero integer μ. To see that condition (3.5) holds, it suffices to show that if $x \in T$ and $\varepsilon > 0$ are given, then there is an M so that whenever $m \geqslant M$,

$$\left|1 + \sum_{\mu=-m}^{m} \lambda_\mu e^{i\mu x}\right| \leqslant \varepsilon. \tag{3.6}$$

Let $x \in T$ and $\varepsilon > 0$ be given. Because the original series is unrestrictedly rectangularly convergent at $(x, 0)$, there is an M so that whenever $\min\{m, n\} \geqslant M$, $|T_{mn}(x, 0)| < \varepsilon$. Fix any $m \geqslant M$ and pick n_j beyond M and from the $2m$th subsequence. Then

$$\left|\sum_{\mu=-m}^{m} A_{\mu n_j} e^{i\mu x}\right| = \left|T_{mn_j}(x, 0)\right| < \varepsilon,$$

so letting $j \longrightarrow \infty$, gives inequality (3.6), and consequently the required contradiction.

The other proof of Theorem 5 is completely different. Again let the dimension be two for simplicity and again assume that a double trigonometric series converges unrestrictedly rectangularly to 0 everywhere, so that for all $(x, y) \in T^2$, relation (3.4) holds. Following Cantor's program listed under the statement of Cantor's Theorem 1, we start by trying to get some control of the coefficient's size. There is indeed a Cantor-Lebesgue result available here which asserts that the coefficients are uniformly bounded and tend to 0 "in the corners" , that is there holds the relation

$$\lim_{\min\{|m|,|n|\} \to \infty} c_{mn} = 0.$$

Continuing Cantor's program, we next form an analogue of the Riemann function, namely

$$F(x,y) = c_0\frac{x^2y^2}{4} - \frac{x^2}{2}\sum_{n\neq 0}\frac{c_{0n}}{n^2}e^{iny} - \frac{y^2}{2}\sum_{m\neq 0}\frac{c_{m0}}{m^2}e^{iny} + \sum_{m,n\neq 0}\frac{c_{mn}}{m^2n^2}e^{i(mx+ny)}.$$

The Weierstrass M-test shows that F is continuous, being the uniform limit of its partial sums. If we were allowed to differentiate term by term, we would get

$$\frac{\partial^4 F}{\partial^2 x\partial^2 y}(x,y) = \sum_{m,n} c_{mn}e^{i(mx+ny)} = 0.$$

If a twice continuously differentiable (C^2) function F satisfies this differential equation, integration shows that it must have the form

$$F(x,y) = a(y)x + b(y) + c(x)y + d(x). \tag{3.7}$$

It turns out that if the heart of the matter is to show that F has this form. Since termwise differentiation is not justified, we may try to mimic what Cantor did when he used the second Schwarz derivative in place of the ordinary second derivative.

So we introduce the generalized Schwarz derivative $D_{2,2}F(x,y)$ which is defined to be

$$\lim_{\substack{h,k\to 0\\ hk\neq 0}} \frac{\begin{array}{ccc} +1F(x-h,y+k) & -2F(x,y+k) & +1F(x+h,y+k) \\ -2F(x-h,y) & +4F(x,y) & -2F(x+h,y) \\ +1F(x-h,y-k) & -2F(x,y-k) & +1F(x+h,y-k) \end{array}}{h^2k^2}.$$

From assumption (3.4) it readily follows that $D_{2,2}F(x,y)$ is identically zero, so that if F were C^2, the theorem would follow easily. However, F is only known to be continuous. Well, in the one dimensional situation, continuity was sufficient, so it seems natural to conjecture that a continuous function with identically zero $D_{2,2}$ generalized fourth derivative must have the form (3.7) . But the function $E(x,y) = (x+y)|x+y|$ satisfies $D_{2,2}E(x,y)$ identically zero and does not have the proper form.[AFR], p.148 So something more is needed. The example causes mischief because of its symmetry, so an additional desymmetrizing property is required. we introduce the three "connectors,"

$$\lim_{\substack{h,k\to 0\\ hk\neq 0}} \frac{\begin{matrix} +1F(x-h,y+k) & -2F(x,y+2k) & +1F(x+h,y+2k) \\ -2F(x-h,y) & +4F(x,y+k) & -2F(x+h,y+k) \\ +1F(x-h,y-k) & -2F(x,y) & +1F(x+h,y) \end{matrix}}{h^2},$$

$$\lim_{\substack{h,k\to 0\\ hk\neq 0}} \frac{\begin{matrix} +1F(x,y+k) & -2F(x+h,y+k) & +1F(x+2h,y+k) \\ -2F(x,y) & +4F(x+h,y) & -2F(x+2h,y) \\ +1F(x,y-k) & -2F(x+h,y-k) & +1F(x+2h,y-k) \end{matrix}}{k^2}, \text{ and}$$

$$\lim_{\substack{h,k\to 0\\ hk\neq 0}} \frac{\begin{matrix} +1F(x,y+2k) & -2F(x+h,y+2k) & +1F(x+2h,y+2k) \\ -2F(x,y+k) & +4F(x+h,y+k) & -2F(x+2h,y+k) \\ +1F(x,y) & -2F(x+h,y) & +1F(x+2h,y) \end{matrix}}{1}.$$

Notice that as the numerators become less symmetric, the denominators exert less of an impediment. The original hypothesis (3.4) also implies that all three of these connectors are identically zero. Even better, we were able to prove the real variable theorem that a continuous function with identically zero connectors and identically zero Schwarz derivative $D_{2,2}$ must be of the form (3.7).

So the hard part of the proof is the real variable theorem. In order to prove this theorem, we had to come up with an entirely new technique. Given the simplicity of Tetunashvili's proof above, probably the main remaining value in the proof is this technique. In the paper [A1], the technique itself is illustrated by applying it to give a (harder) proof of Schwarz's original theorem, that continuous functions with identically zero Schwarz second derivative (see definition (1.3) above) are of the form $ax + b$. The basic idea is to find for the second difference, an analog of the following "additive interval function" property of the first difference:

$$(3.8) \qquad f(1) - f(0) = \sum_{i=1}^{n} \{f(x_i) - f(x_{i-1})\},$$

where the interval $[0,1] = \cup_{i=1}^{n}[x_{i-1}, x_i]$, $0 = x_0 < x_1 < ... < x_n = 1$. Given a function $F(x)$ defined on $[0,1]$ associate to it a function $f(x,y)$ of two variables defined on the square $S = [0,1] \times [0,1]$ by

$$(3.9) \qquad f(x,y) = F\left(\frac{x+y}{2}\right).$$

Then the function $F(x)$ can be identified with the function f restricted to the diagonal $\{(x,x) : 0 < x < 1\}$, since $F(x) = f(x,x)$. Note that f has the constant value $F(a)$ on the entire line segment passing through (a,a) and having slope -1. Let $\cup_{i=1}^{n} P_i$ be a partition of S into nonoverlapping squares. Then associate to each square P with lower left corner (a,c), upper left corner $(a, c+h)$, upper right corner $(a+h, c+h)$, and lower right corner $(a+h, c)$, the value

$$f(a+h, c+h) - f(a+h, c) - f(a, c+h) + f(a.c).$$

The result is additive and produces this analogue of the decomposition (3.8):

$$f(1,1) - f(1,0) - f(0,1) + f(0,0) =$$
$$\sum_{i=1}^{n} \{f(a_i + h_i, c_i + h_i) - f(a_i + h_i, c_i) - f(a_i, c_i + h_i) + f(a_i, c_i)\},$$

where P_i is the square which has lower left corner (a_i, c_i) and upper right corner $(a_i + h_i, c_i + h_i)$. Now translate this back into a statement about F by means of equation (3.9). We have

$$F(1) - 2F\left(\frac{1}{2}\right) + F(0) = \sum_{i=1}^{n} \{F(t_i) - 2F(m_i) + F(b_i)\},$$

where $t_i = \frac{1}{2}(a_i + c_i) + h_i$, $b_i = \frac{1}{2}(a_i + c_i)$, and m_i is the midpoint of t_i and b_i. To see this idea lead to a proof of Schwarz's Theorem, see [Al]; and to see how it is used in the proof of the higher dimensional analogue involving $D_{2,2}$ and the connectors, see [AFR].

4 Spherical Convergence

As was mentioned in the last section, it took two strokes of good fortune to make the leap from the conditional theorem of Victor Shapiro to the final two dimensional spherical uniqueness theorem. These were Cooke's Theorem and the fact that Cooke's condition (3.2) implies Shapiro's condition (3.1).

The next development took place in 1976 when Bernard Connes found a beautiful extension of Cooke's Theorem to higher dimensions.[Con]

Theorem 6 (Connes) . *If* $\{c_m\}$ *is a multi-indexed set of complex numbers such that*

$$\sum_{|m|=r} c_m e^{imx}$$

tends to zero for almost all x, *then*

(4.1) $$\sqrt{\sum_{|m|=r} |c_m|^2} \text{ tends to } 0 \text{ as } \quad r \to \infty.$$

Unfortunately, when $d \geqslant 3$, condition (4.1) does *not* imply condition (3.1), so this did not immediately lead to an unconditional spherical theorem in higher dimensions. What it did do, however, was set the stage for what is probably the deepest theorem in the entire subject. This was done 1995 by Jean Bourgain, who proved a spherical uniqueness theorem for all dimensions with only condition (4.1) assumed. [B] Just as Cooke's Theorem had removed the side condition (3.1) and thereby converted Shapiro's Theorem into a full strength spherical uniqueness theorem when $d = 2$, so Connes' Theorem means that in reality Bourgain's Theorem has no side condition. Thus we finally had the full spherical uniqueness theorem.

Theorem 7. *If* $\sum c_n e^{inx} = 0$ *spherically for all* $x \in T^d$, *then all*, $c_n = 0$.

The proof of this theorem is very difficult, requiring numerous ideas as well as strong technique. It does, however, follow the steps of Cantor's original proof listed above. Suppose that $c_o = 0$ and that $\sum' c_n e^{inx} = 0$, where the prime denotes the absence of an $n = 0$ term. (This involves no loss of generality.[AWa], page 49) The higher dimensional Riemann function introduced by Shapiro is $F(x) = -\sum' \frac{c_n}{|n|^2} e^{inx}$. It is easy to calculate that the Laplacian of e^{inx} is $-|n|^2$, so, formally, F has zero Laplacian and hence is harmonic. By "formally" I mean that if you were allowed to interchange the summation sign and the Laplacian operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_d^2}$, then you would get $\Delta F(x) = \sum' c_n e^{inx} = 0$. From this it would follow that F were harmonic and hence infinitely differentiable, whence integration by parts would show the coefficients of F and hence the original coefficients to be decreasing faster than any fixed negative power of $|n|$. Then the original series would be absolutely convergent ($\sum |c_n| < x$), so that the theorem would have this one line proof: for any m,

$$(4.2)\ (2\pi)^d c_m = \sum_n{}' c_n \int_{T^d} e^{i(n-m)x} dx = \int_{T^d} \Big(\sum_n{}' c_n e^{inx} \Big) e^{-imx} dx = \int_{T^d} 0 dx = 0.$$

The fallacy in all this is that there is no justification available for the interchange of Δ and $\sum$. Even though ΔF doesn't make sense due to F not being smooth enough, there is a generalized Laplacian $\tilde{\Delta}$ which can be applied to F. The definition is

$$\tilde{\Delta} F(x) = \lim_{\rho \to 0} \frac{c_d}{\rho^2} \left(\frac{1}{m(B(x,\rho))} \int_{B(x,\rho)} F(t)\, dt - F(x) \right),$$

where $B(x, \rho)$ is an x centered solid d dimensional ball of radius ρ,and m denotes Lebesgue measure. Taylor expanding a C^2 function about x shows that the generalized Laplacian agrees with the ordinary one if the constant c_d is chosen appropriately. A straightforward calculation involving Bessel functions shows that the generalized Laplacian of the Riemann function F agrees with the original series and hence is everywhere 0. There is a classical theorem of Rado that if the generalized Laplacian of a continuous function is everywhere 0, then that function must be harmonic. So if F were shown to be continuous, spherical uniqueness would be established.

Showing F to be continuous seems to be very difficult. Here is the logical flow of Bourgain's argument. Suppose that F is not continuous and let Z be the nonempty set of discontinuities of F. A Baire category argument on Z produces a point $p \in Z$ and a solid ball B about p such that if $Z \cap B$ is "thin" (is of measure 0 with respect to a certain harmonic measure) , then F is harmonic and hence continuous on B, contrary to the definition of Z; while if $Z \cap B$ is "thick" , then F must be continuous at p, contrary to $p \in Z$. Bourgain achieves the first contradiction using a Balayage argument. His other contradiction is reached by hard analysis, harmonic measure, and capacity theory. The proof appears in at least three places. There is Bourgain 's original 15 page article[B], a somewhat expanded 22 page version appears in [AWa], and a 42 page version of the proof specialized down to dimension 2 only appears in [A2]. This last version should be the most accessible to the novice.

5 Square Nonuniqueness?

This section will be speculation. There are several obstacles to proving a uniqueness theorem for the remaining two methods, square convergence and restricted rectangular convergence. First of all, there is no Cantor-Lebesgue theorem, at least in the usual sense. What I mean by this is that the usual Cantor-Lebesgue theorem associated with one parameter methods of convergence assumes that the difference of successive partial sums tends to zero everywhere (or at least at every point of a "substantial" subset of $\mathbb{T}^d$), and concludes that the coefficients themselves must be small. But there is a double trigonometric series which is square convergent to a finite value at every point, so that the differences of its partial sums tend to zero everywhere, whose coefficients grow faster than any polynomial. The series is

$$(5.1) \qquad T(x) = 2\sqrt{\pi}\sum_{n=2}^{\infty} n^{3/2}e^{n/\ln n}\cos^2\left(\frac{x}{2}\right)\sin^{2n-2}\left(\frac{x}{2}\right)\cos ny.$$

The partial sums of the series as written coincide with the square partial sums of a double trigonometric series as can be seen from expanding

$$(5.2) \qquad \cos^2\left(\frac{x}{2}\right)\sin^{2n-2}\left(\frac{x}{2}\right) = \left(\frac{e^{ix/2}+e^{-ix/2}}{2}\right)^2\left(\frac{e^{ix/2}-e^{-ix/2}}{2i}\right)^{2n-2}.$$

If $x=\pi$ or $-\pi$, $\cos\left(\frac{x}{2}\right)=0$ so$T(-\pi)=T(\pi)=0$, while if $x \in (-\pi,\pi)$, then $a=\sin^2\left(\frac{x}{2}\right)<1$, so that the series defining T converges absolutely by comparison with $2\sqrt{\pi}/a\sum_{n=2}^{\infty} n^{3/2}e^{n/\ln n}a^n$. To see that the $(0,n)$ coefficients of T grow like $e^{n/\ln n}$ involves multiplying out the right hand side of equation (5.2). It is also true, but somewhat more technical to verify, that this series is also restrictedly rectangularly convergent to a finite value at every point. See [AWa2] for details.

It is logically possible that everywhere square convergence *to zero* is different than everywhere convergence to finite values, but this would involve a completely new and much more delicate type of Cantor-Lebesgue type theorem. So the method of proof used by Cantor, Shapiro, and Bourgain, which involves forming a Riemann function or second integral of the original series by dividing by $|n|^2$ is not very likely to have an analogue here.

Second, I have devoted some effort to trying to prove a conditional square uniqueness theorem in the spirit of Shapiro's work. In other words, I simply add some reasonably mild condition on the coefficients such as $c_{mn} \to 0$ as $max\{|m|, |n|\} \to \infty$. Even such a conditional uniqueness theorem for square convergence seems difficult to achieve at this time.

With all efforts to prove something positive at a standstill, it seems natural to wonder if there might be a counterexample. Since at a fixed point restricted rectangular convergence implies square convergence, proving a uniqueness theorem should be easier for restricted rectangular convergence, while finding a counterexample to uniqueness should be easier for square convergence. So we will move in the direction of trying to find a counterexample to square uniqueness; that is, of trying to construct a double trigonometric series that is square convergent to 0 everyvhere.

We start with an example of a one dimensional trigonometric series that has a *subsequence* of partial sums that converges to zero everywhere. Such a series was discovered by Kozlov.[K] See example 2.2 on pages 187-190 of[AWa1] for two different ways to construct such a sequence. The basic fact used amounts to this.

FACT.Given any trigonometric polynomial $f = \sum_{n=1}^{N} a_n \sin nx$, any $\varepsilon > 0$, any $\eta > 0$, and any integer $M > N$, there can be found a trigonometric polynomial $p = \sum_{n=M}^{M+R} a_n \sin nx$ such that for all $x \in \mathbb{T} \setminus (-\varepsilon, \varepsilon)$, $|f(x) + p(x)| < \eta$.

We will construct two one dimensional trigonometric series, $P(x) = p_1(x) + p_2(x) + ...$, and $Q(y) = q_1(y) + q_2(y) + ...$where every p and q is a linear combination of sine functions, the lowest frequency of each p_{n+1} is greater than the highest frequency of p_n, and the q's have the same property. We will then consider the resulting double trigonometric series

$$T(x,y) = P(x)Q(y).$$

The nth square partial sum of T is exactly the product of the nth partial sum of P and the nth partial sum of Q ,

$$T_{nn}(x,y) = P_n(x)Q_n(y).$$

The plan is to design P and Q is such a way that for (x, y) fixed, for a certain subsequence of n, P_n tends to zero and Q_n is not too big, while for the subsequence consisting of the remaining n, P_n is not too big and Q_n tends to zero.

Let $\{\varepsilon_i\}$ be a sequence of positive nurnbers tending monotonically to 0. Then the intervals $\{\ (-\varepsilon_i, \varepsilon_i)\ \}$ shrink to 0, so the complementary subintervals of $\mathbb{T}$, $I_i = [-\pi, -\varepsilon_i] \cup [\varepsilon_i, \pi]$ increase monotonically to $\mathbb{T} \setminus \{0\}$. Also let $\{\eta_n\}$ be another sequence of positive numbers tending monotonically to 0. Start with $p_1(x) = \eta_1 \sin x$ so that

$$\sup_{x \in I_1} |p_1(x)| \leqslant \eta_1$$

and let m_1 =deg $p_1 = 1$. Then use the FACT to pick p_2 of degree m_2 with frequencies starting at $m_1 + 1 = 2$ so that p_2 satisfies

$$\sup_{x \in I_2} |p_1(x) + p_2(x)| \leqslant \eta_2.$$

This creates a first "bad x zone," $[2, m_2 - 1]$, bad in the sense that for n in this interval, the nth partial sum of P may not be small. So, let q_1 be a nontrivial trigonometric polynomial in y of degree $n_1 = 1$ satisfying

$$\sup_{y \in I_1} |q_1(y)| \leqslant \eta_1,$$

in particular $q_1(y) = \eta_1 \sin y$. Next use the FACT to pick $q_2(y)$ to have frequencies starting with m_2, to be of degree n_2, and to satisfy

$$\sup_{y \in I_2} |q_1(y) + q_2(y)| \leqslant \eta_2.$$

We have

$$T_{11}(x,y) = p_1(x)q_1(y) = \eta_1^2 \sin x \sin y$$

and if n is in the first bad x zone,

$$T_{nn}(x,y) = (p_1(x) + p_2^*(x))(q_1(y)),$$

where p_2^* is a partial sum of p_2 . This creates a first bad y zone; for $n \in [m_2, n_2 - 1]$, the nth partial sum of Q may not be small. For n in the first bad y zone,

$$T_{nn}(x,y) = (p_1(x) + p_2(x))(q_1(y) + q_2^*(y)),$$

where p_2^* is a partial sum of p_2.

Now use the FACT to pick p_3 to have frequencies starting with n_2, to be of degree m_3, and to satisfy

$$\sup_{x \in I_3} |p_1(x) + p_2(x) + p_3(x)| \leqslant \eta_3.$$

The second bad x zone is $[n_2, m_3 - 1]$ and for n in this zone, we have

$$T_{nn}(x,y) = (p_1(x) + p_2(x) + p_3^*(x))(q_1(y) + q_2(y)),$$

where p_3^* is a partial sum of p_3. So use the FACT to pick q_3 to have frequencies starting with m_3, to be of degree n_3, and to satisfy

$$\sup_{y \in I_3} |q_1(y) + q_2(y) + q_3(y)| \leqslant \eta_3.$$

This creates a second bad y zone, $[m_3, n_3 - 1]$ on which the nth partial sum of Q may not be small.

We continue inductively. Having chosen p_{k-1} with frequencies belonging to $[n_{k-2}, m_{k-1}]$ and satisfying

$$\sup_{x \in I_{k-1}} |p_1(x) + \ldots + p_{k-1}(x)| \leqslant \eta_{k-1}$$

and also q_{k-1} with frequencies belonging to $[m_{k-1}, n_{k-1}]$ and satisfying

$$\sup_{x \in I_{k-1}} |q_1(y) + \ldots + q_{k-1}(y)| \leqslant \eta_{k-1},$$

use the FACT to choose first p_k with frequencies belonging to $[n_{k-1}, m_k]$ and satisfying

$$\sup_{x \epsilon I_k} |p_1(x) + \ldots + p_k(x)| \leqslant \eta_k$$

and then use the FACT to choose q_k with frequencies belonging to $[m_k, n_k]$ and satisfying

$$\sup_{x \epsilon I_k} |q_1(y) + \ldots + q_k(y)| \leqslant \eta_k.$$

Notice that if n is in the $(k-1)$th bad x zone $[n_{k-1}, m_{k-1}]$, then

$$T_{nn}(x,y) = (p_1(x) + \ldots + p_k^*(x))(q_1(y) + \ldots + q_{k-1}(y)), \tag{5.3}$$

where $p_k^*(x)$ is a partial sum of $p_k(x)$, while if n is in the $(k-1)$th bad y zone $[m_k, n_k - 1]$, then

$$T_{nn}(x,y) = (p_1(x) + \ldots + p_k(x))(q_1(y) + \ldots, q_k^*(y)), \tag{5.4}$$

where $p_k^*(x)$ is a partial sum of $p_k(x)$.

We remark that this construction has been carried out in such a way that the partial sums of

$$P(x) = p_1(x) + \ldots + p_k(x) + \ldots$$

have the constant value

$$P_m(x) = p_1(x) + \ldots + p_k(x)$$

for $m = m_k, m_{k+1}, \ldots, n_k - 1$, and the partial sums of $Q(y)$ have the constant value

$$Q_n(y) = q_1(y) + \ldots + q_k(y)$$

for $n = n_k, n_{k+1}, \ldots m_{k+1} - 1$.

Let (x,y) be any point of $\mathbb{T}^2$. If $x = 0$, then $P_n(x) = 0$ for all n, so that

$$\lim_{n\to\infty} T_{nn}(0,y) = \lim_{n\to\infty} 0 \cdot Q_n(y) = 0.$$

Similarly,

$$\lim_{n\to\infty} T_{nn}(x,0) = \lim_{n\to\infty} P_n(x)\cdot 0 = 0.$$

The question is whether the polynomials $\{p_m\}$ and $\{q_n\}$ can be chosen in such a way that for every other pair $(x,y)\in\mathbb{T}^2$,

$$\lim_{n\to\infty} T_{nn}(x,y) = 0.$$

The basic idea of the construction is that every square partial sum of the double series T is a product of two terms and one of these two terms is always very small. The hope for constructing a counterexample to square uniqueness lies in trying to control the other term.

Conjecture 1. *In the above construction it is possible to pick the sequence* $\{\eta_n\}\searrow 0$ *and the trigonometric polynomials* $\{p_m\}$ *and* $\{q_n\}$ *in such a way that for fixed nonzero* x *and* y,

$$\lim_{k\to\infty}\left(\sup_{\ell}|p_{k,\ell}(x)|\right)\eta_{k-1} = 0$$

and also

$$\lim_{k\to\infty}\left(\sup_{\ell}|q_{k,\ell}(y)|\right)\eta_{k} = 0,$$

where $p_{k,\ell}$ *and* $q_{k,\ell}$ *denote the* ℓth *partial sums of* p_k *and* q_k.

Notice that the process of picking the m_i 's and n_i 's is such that

$$1 = m_1 = n_1 < m_2 < n_2 < m_3 < n_3 < \ldots < n_{k-1} < m_k < n_k < \ldots$$

so that every index $n \geqslant 2$ is either in a bad x zone (when $n_{k-1}\leqslant n\leqslant m_k - 1$ for some k) or a bad y zone (when $m_k\leqslant n\leqslant n_k - 1$ for some k). Now fix (x,y) with both x and y not zero. If n is sufficiently large, either it is in $[n_{k-1}, m_k - 1]$ or $[m_k, n_k - 1]$ for a k with the property that both x and y are in I_k. In the former case, from equation (5.3) we have the estimate

$$\begin{aligned}|T_{nn}(x,y)| &= |p_1(x)+\ldots+p_k^*(x)|\,|q_1(y)+\ldots+q_{k-1}(y)|\\ &\leqslant \left(\eta_{k-1}+\sup_{\ell}|p_{k,\ell}(x)|\right)\eta_{k-1}\\ &= o(1)+\left(\sup_{\ell}|p_{k,\ell}(x)|\right)\eta_{k-1},\end{aligned}$$

while in the latter case from equation (5.4) we have the similar estimate

$$\begin{aligned} |T_{nn}(x,y)| &= |p_1(x) + \dots + p_k(x)|\, |q_1(y) + \dots + q_k^*(y)| \\ &\leqslant \eta_k \left(\eta_{k-1} + \sup_\ell |q_{k,\ell}(x)| \right) \\ &= o(1) + \left(\sup_\ell |q_{k,\ell}(y)| \right) \eta_k . \end{aligned}$$

From these two estimates it is immediate that if the conjecture can be satisfied, then the double trigonometric series

$$T(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_m c_n \sin mx \sin ny$$

is everywhere on $\mathbb{T}^2$ square convergent to zero. This would violate uniqueness for square convergence. Unfortunately, both proofs of the FACT, while potentially constructive, have so far only been carried out in a nonconstructive way, so that while it is clear what has to be done to test the conjecture, I have not had the courage to try it.

One thing that can be stated in favor of this program is that it is not ruled out by virtue of producing a counterexample to either the unrestricted rectangular uniqueness Theorem 4 nor the iteratedly uniqueness Proposition 1 discussed above.

Indeed, we can show the following proposition.

Proposition 2. *A double trigonometric series of the form* $T(x,y) = P(x)Q(y)$ *where* P *and* Q *are nontrivial trigonometric series is neither iteratively nor unrestrictedly rectangularly convergent to zero everywhere.*

Proof. Since P is nontrivial,

$$B = \{x \in I : \{P_m(x)\} \text{ does not tend to 0 at } x\}$$

is nonempty, for otherwise Cantor's one dimensional nonuniqueness theorem would be violated. Fix any $x \in B$. There is an extended real number $s \neq 0$ (s may be $+\infty$ or $-\infty$) and a sequence $\{\mu_k\}$ such that

$$\lim_{j\to\infty} P_{\mu_j}(x) = s. \tag{5.5}$$

Similarly, let

$$C = \{y \in \mathbb{T} : \{Q_n(y)\} \text{ does not tend to 0 at } y\},$$

fix a $y \in C$, and find an extended real number $t \neq 0$ and a sequence $\{\nu_k\}$ such that

$$\lim_{k\to\infty} Q_{\nu_k}(y) = t. \tag{5.6}$$

Then

$$\lim_{j,k\to\infty} P_{\mu_j}(x)Q_{\nu_k}(y) = st. \tag{5.7}$$

For $T(x,y)$ to be iteratively convergent to zero at the point (x,y), we must have both

$$\lim_{m\to\infty} (\lim_{n\to\infty} P_m(x)Q_n(y)) = 0,$$

and

$$\lim_{n\to\infty} (\lim_{m\to\infty} P_m(x)Q_n(y)) = 0.$$

Actually, at the point (x,y) neither limit is 0. By symmetry, it is enough to see that the first limit is not zero. Suppose it were zero. Then we would have

$$\begin{aligned} 0 &= \lim_{m\to\infty} (\lim_{n\to\infty} P_m(x)Q_n(y)) \\ &= \lim_{m\to\infty} (P_m(x) \lim_{n\to\infty} Q_n(y)) \\ &= \left(\lim_{m\to\infty} P_m(x)\right)\left(\lim_{n\to\infty} Q_n(y)\right). \end{aligned}$$

Since subsequential limits must agree with limits, it would follow that

$$0 = \left(\lim_{j\to\infty} P_{\mu_j}(x)\right)\left(\lim_{k\to\infty} Q_{\nu_k}(y)\right). \tag{5.8}$$

But, by relation (5.6),

$$t = \lim_{k\to\infty} Q_{\nu_k}(y),$$

and by relation (5.5),

$$s = \lim_{j\to\infty} P_{\mu_j}(x),$$

which contradicts equation (5.8), since $st \neq 0$. Similarly, if T were unrestrictedly rectangularly convergent to zero at (x, y), in particular we would have

$$0 = \lim_{k\to\infty} T_{\mu_k,\nu_k}(x,y) = \lim_{k\to\infty} P_{\mu_k}(x)Q_{\nu_k}(y),$$

which is contrary to relations (5.,5) and (5.6). □

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