Cubo Matemática Educacional
Vol. 2. MAYO 2000

Extrapolation Theory and Some Applications

Lahcen Maniar
Department of Mathematics
Faculty of Sciences Semlalia
B.P. 2390, Marrakesh 40000, Morocco

Abstract

In this paper, we deal with the well-posedness, i.e, the existence, uniqueness and the continuous dependance with respect to initial datas, of the following Cauchy problem

$$(ACP) \quad \left\{ \begin{array}{rcl} u'(t) & = & Au(t) + f(t), \; t \geq 0, \\ u(0) & = & u_0. \end{array} \right.$$

where $A : D(A) \subseteq X \longrightarrow X$ is not necessary densely defined operator and $f : R \longrightarrow X$.

1 Introduction

Let X be a Banach space and $(T(t))_{t\geq 0}$ be a family of bounded linear operators. This family is said to be a semigroup if:
(i) $T(0) = I$.
(ii) $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$,

and a strongly continuous semigroup (or C_0-semigroup) if, furthermore
(iii) $t \longmapsto T(t)x$ is continuous on $[0, +\infty)$ for each $x \in X$.

To each C_0-semigroup, we associate a linear operator, called *generator infinitesimal*, defined as

$$Ax := \lim_{t\to 0^+} \frac{T(t)x - x}{t}, \quad \text{for } x \in D(A) := \{x \in X \ : \ \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists in } X\}.$$

It is known (see [8] and [11]) that the classical semigroup theory ensures the well-posedness of problem (ACP) when A is the generator infinitesimal of a C_0- semigroup or, equivalently (by virtue of the Hille-Yosida theorem), when:
(a) $\overline{D(A)} = X$.
(b)There is $M \geq 1$ and $\omega \in \mathbb{R}$: $\|(\lambda-\omega)^n(\lambda - A)^{-n}\| \leq M$ for each $n \geq 0$ and $Re(\lambda) > \omega$. Furthermore, the solution of (ACP) is given by the variation of constants formula

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds, \qquad t \geq 0.$$

In some applications to partial differential equations the operator A is not a generator of C_0-semigroup because only hypothesis **(b)** is satisfied. Such operators are called *Hille-Yosida* operators.

The aim of this paper is to present the extrapolation approach, developed by Nagel [7] and used by Nagel-Sinestrari [9] to solve problem (ACP), where A is only a Hille-Yosida operator. At the end, we apply this abstract results to the following retarded differential equation

$$\begin{cases} x'(t) &= Bx(t) + Lx_t + f(t), \qquad t \geq 0 \\ x(\tau) &= \varphi(\tau), \quad \tau \in [-r, 0]. \end{cases}$$

B is the generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on a Banach space E and $L : C([-r,0], E) \to E$ bounded linear operator and $x_t : [-r, 0] \longrightarrow E$, $x_t(\tau) := x(t+\tau)$.

2 Extrapolation spaces for Hille-Yosida operators

Abstract extrapolation spaces have been introduced by [7], and used by many authors for various purposes (e.g., [1], [2], [4], [6], [10], [12], ...).

In this section, we fix some notations and recall some basic results on extrapolation spaces for Hille-Yosida operators.

Let X be a Banach space and A be a linear operator with domain $D(A)$. From the Hille-Yosida theorem (cf. [3], Thm. 12.2.4), we have the following result.

Proposition 1 *Let A be a Hille-Yosida operator on X. Then, the part A_0 of A in $X_0 := \overline{D(A)}$ given by*

$$D(A_0) := \{x \in D(A) : Ax \in X_0\}, \quad A_0x := Ax \quad \forall x \in D(A_0)$$

generates a C_0-semigroup $(T_0(t))_{t\geq 0}$ on X_0. Moreover, the set resolvent $\rho(A)$ of A is included in $\rho(A_0)$ and for each $\lambda \in \rho(A)$, $R(\lambda, A_0) := (\lambda - A_0)^{-1}$ is the restriction of $R(\lambda, A)$ in X_0.

On the space X_0, for each $\lambda \in \rho(A)$, we introduce a new norm by

$$\|x\|_{-1}^{\lambda} = \|R(\lambda, A_0)x\|, \quad x \in X_0.$$

The completion of $\left(X_0, \|\cdot\|_{-1}^{\lambda}\right)$ is noted by X_{-1}^{λ}. It is easy to show that the norms $\|\cdot\|_{-1}^{\lambda}$, $\lambda \in \rho(A)$, are equivalent. Hence, the space X_{-1}^{λ} is independent of λ and be called the *extrapolation space* of X_0 associated to A_0 and will be denoted by X_{-1}.

¿From the equality

$$T_0(t)R(\lambda, A_0)x = R(\lambda, A_0)T_0(t)x, \ t \geq 0 \text{ and } x \in X_0,$$

one can show easily that, for each $t \geq 0$, the operator $T_0(t)$ can be extended to a unique bounded operator on X_{-1} denoted by $T_{-1}(t)$.

It is easy to see that the family $(T_{-1}(t))_{t\geq 0}$ is a C_0-semigroup on X_{-1}, called the *extrapolated semigroup* of $(T_0(t))_{t\geq 0}$. We summarize the properties of this new semigroup in the following theorem.

Theorem 2 *The following properties hold*
(i) $\|T_{-1}(t)\|_{\mathcal{L}(X_{-1})} = \|T_0(t)\|_{\mathcal{L}(X_0)}$
(ii) $D(A_{-1}) = X_0$.
(iii) $\lambda - A_{-1}$, $\lambda \in \rho(A)$, *is the unique continuous extension of* $\lambda - A_0 : D(A_0) \subseteq X_0 \to X_{-1}$ *to an isometry from* X_0 *to* X_{-1}.

The relationship between the spaces X_0, X and X_{-1} is given in the following proposition.

Proposition 3 *Let $\lambda \in \rho(A)$. For the norm*

$$\|x\|_{-1} := \|R(\lambda, A)x\|, \quad x \in X,$$

we have that $X_0 := \overline{D(A)}$ is dense in $\left(X, \|\cdot\|_{-1}\right)$. Hence, the extrapolation space X_{-1} is also the completion of $\left(X, \|\cdot\|_{-1}\right)$ and $X \hookrightarrow X_{-1}$. Moreover, the operator, for each $\lambda - A_{-1}$, $\lambda \in \rho(A)$, is an extension of $\lambda - A$ to X_{-1},, $(\lambda - A_{-1})^{-1} X = D(A)$ and $(\lambda - A_{-1})^{-1} X_0 = D(A_0)$.

3 Inhomogeneous Cauchy problems for Hille-Yosida operators

Let $A : D(A) \subset X \longrightarrow X$ be a Hille-Yosida operator on a Banach space X. Let $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$\|T_0(t)\|_{\mathcal{L}(X_0)} \leq Me^{\omega t}, \quad t \geq 0,$$

where $(T_0(t))_{t\geq 0}$ is the semigroup generated by the part A_0 of A on $X_0 := \overline{D(A)}$.

Let A_{-1} the extension of A defined in the previous section. We are interested now in the following Cauchy problem.

$$(ACP)_{-1}\begin{cases} u'(t) &= A_{-1}u(t) + f(t),\ t \geq 0, \\ u(0) &= u_0. \end{cases}$$

We have seen, in the previous section, that A_{-1} generates a semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1}. Then, Phillips theorem ([5], Thm. 1.3) provides conditions on f and u_0 such the differentiable solution of $(ACP)_{-1}$ exists and is given by

$$u(t) = T_{-1}(t)u_0 + \int_0^t T_{-1}(t-s)f(s)ds, \qquad t \geq 0. \tag{1}$$

When f is an X-valued function the term integral in (1) has some important properties given in the following lemma.

Lemma 4 *[9] For $f \in L^1(\mathbb{R}_+, X)$, the following properties hold*

(*i*) $\int_0^t T_{-1}(t-s)f(s)ds \in X_0$, *for all $t \geq 0$.*

(*ii*) $\left\|\int_0^t T_{-1}(t-s)f(s)ds\right\| \leq Ce^{\omega t}\int_0^t e^{-\omega s}\|f(s)\|\,ds$,

where C is independent from t and f.

(*iii*) *The function $[0,+\infty[\ni t \longmapsto \int_0^t T_{-1}(t-s)f(s)ds \in X_0$ is continuous.*

To investigate the inhomogeneous Cauchy problem (ACP), we will substitute it via a homogeneous one on a product space. For this purpose, let us recall that the translation semigroup $(S(t))_{t\geq 0}$ on $L^1(\mathbb{R}_+, X_{-1})$, defined by

$$S(t)f(s) := f(t+s), \qquad s, t \in \mathbb{R}_+$$

is a C_0-semigroup with the generator B, given by

$$D(B) = W^{1,1}(\mathbb{R}_+, X_{-1}), \text{ and } Bf(s) := f'(s),\ s \in \mathbb{R}_+ \text{ a.e.} \quad \text{for} f \in D(B).$$

Using this C_0-semigroup, we can construct a new semigroup, which will provide the solution of the inhomogeneous Cauchy problem $(ACP)_{-1}$.

Theorem 5 *On the Banach space* $Y_{-1} := X_{-1} \times L^1(\mathbb{R}_+, X_{-1})$, *with the norm* $\left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|_{Y_{-1}} := \|x\|_{-1} + \|f\|_{L^1(\mathbb{R}_+, X_{-1})}$, *we define the operators*

$$G_{-1}(t) \begin{pmatrix} x \\ f \end{pmatrix} := \begin{pmatrix} T_{-1}(t)x + \int_0^t T_{-1}(t-s)f(s)ds \\ S(t)f \end{pmatrix}, \quad \begin{pmatrix} x \\ f \end{pmatrix} \in Y_{-1} \text{ and } t \in \mathbb{R}_+.$$

Then, the family $(G_{-1}(t))_{t\geq 0}$ *is a* C_0*-semigroup on* Y_{-1} *and its generator* $\mathcal{A}$ *is given by*

$$\begin{aligned} D(\mathcal{A}) &= X_0 \times W^{1,1}(\mathbb{R}_+, X_{-1}), \\ \mathcal{A} \begin{pmatrix} x \\ f \end{pmatrix} &= \begin{pmatrix} A_{-1}x + f(0) \\ f' \end{pmatrix}. \end{aligned}$$

Proof. By the definition of operators $G_{-1}(t)$, one can see easily that $(G_{-1}(t))_{t\geq 0}$ is a C_0-semigroup on Y_{-1}. Let $\begin{pmatrix} x \\ f \end{pmatrix} \in X_0 \times W^{1,1}(\mathbb{R}_+, X_{-1})$, one has

$$\begin{aligned} \frac{1}{h}\left(G_{-1}(h)\begin{pmatrix} x \\ f \end{pmatrix} - \begin{pmatrix} x \\ f \end{pmatrix}\right) &= \begin{pmatrix} \frac{1}{h}[T_{-1}(h)x - x + \int_0^h T_{-1}(h-s)f(s)ds] \\ \frac{1}{h}(S(h)f - f) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{h}[T_{-1}(h)x - x] + \frac{1}{h}\int_0^h T_{-1}(h-s)f(s)ds \\ \frac{1}{h}(S(h)f - f) \end{pmatrix}. \end{aligned}$$

Since $x \in X_0 = D(A_{-1})$, then $\lim_{h\to} \frac{1}{h}[T_{-1}(h)x - x] = A_{-1}x$. We have also that $\frac{1}{h}\int_0^h T_{-1}(h-s)f(s)ds$ converges to $f(0)$ in X_{-1} and that $\frac{1}{h}(S(h)f - f) - f' \longrightarrow 0$, as $h \to 0$ in $L^1(\mathbb{R}_+, X_{-1})$. Hence, $\begin{pmatrix} x \\ f \end{pmatrix}$ belongs to $D(\mathcal{A}_{-1})$ and

$$\mathcal{A}_{-1}\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} A_{-1}x + f(0) \\ f' \end{pmatrix}.$$

Conversely, we can show easily that $D(\mathcal{A}_{-1}) \subset X_0 \times W^{1,1}(\mathbb{R}_+, X_{-1})$.

If we set now

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} := G_{-1}(t)\begin{pmatrix} u_0 \\ f \end{pmatrix},$$

by the definition of $G_{-1}(t), t \geq 0$, it follows that its first component $u(\cdot)$ is the unique continuous (mild) solution of $(ACP)_{-1}$ in the space X_{-1}. ■

To solve the Cauchy problem (ACP) in the space X, we need to construct a new semigroup on product space.

Theorem 6 *The Banach space $Y := X_0 \times L^1(\mathbb{R}_+, X)$ with the sum norm is embedded continuously in the space Y_{-1} and the restrictions $G(t)$ of $G_{-1}(t)$ form a strongly continuous semigroup on Y whose the generator is given by*

$$\begin{aligned} D(\mathcal{A}) &= \{\begin{pmatrix} x \\ f \end{pmatrix} : x \in D(A), f \in W^{1,1}(\mathbb{R}_+, X), Ax + f(0) \in X_0\} \\ \mathcal{A}\begin{pmatrix} x \\ f \end{pmatrix} &= \begin{pmatrix} Ax + f(0) \\ f' \end{pmatrix} \end{aligned} \tag{2}$$

Proof. It is obvious to see that $Y \hookrightarrow Y_{-1}$ and Y is invariant under $G_{-1}(t)$, by Lemma 4 (i). The strong continuity of $(G(t))_{t\geq 0}$ on Y can be deduced easily from Lemma 4 (iii). Hence by Lemma 2.6 in [9], it follows that the generator $\mathcal{A}$ of $(G(t))_{t\geq 0}$ is the part in Y of the operator $\mathcal{A}_{-1}$, given in Theorem 5. That is the set of $\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}_{-1}) \cup Y$ and $\mathcal{A}_{-1}\begin{pmatrix} x \\ f \end{pmatrix} \in Y$, which is exactly the set defined by (2). ■

Using now this new semigroup, one can give conditions on f and u_0 ensuring the existence of differentiable solutions of the Cauchy problem (ACP).

Theorem 7 *Let $A : D(A) \subseteq X \longrightarrow X$ be a Hille-Yosida operator. Let $x \in D(A)$ and $f \in W^{1,1}(\mathbb{R}_+, X)$ such that*

$$Ax + f(0) \in \overline{D(A)}.$$

Then, problem (ACP) has a unique solution $u \in C^1(\mathbb{R}_+, X_0)$ given by

$$u(t) = T_0(t)u_0 + \int_0^t T_{-1}(t-s)f(s)ds, \quad t \geq 0. \tag{3}$$

Proof. It is clear that, each $\begin{pmatrix} x \\ f \end{pmatrix}$ verifying hypotheses of the theorem belongs to $D(\mathcal{A})$. Then, $G(t)\begin{pmatrix} x \\ f \end{pmatrix}$ is differentiable and

$$\frac{d}{dt}G(t)\begin{pmatrix} x \\ f \end{pmatrix} = \mathcal{A}G(t)\begin{pmatrix} x \\ f \end{pmatrix}$$

Hence, the first coordinate u of $G(t)\begin{pmatrix} x \\ f \end{pmatrix}$ is differentiable and, from the definition of $G(t)$ and $\mathcal{A}$, is given by (3) and satisfies the problem (ACP). ■

4 Retarded differential equations

As an application of abstract results of the previous section, we consider the following retarded partial differential equation

$$(RDE)\quad \begin{cases} \frac{d}{dt}x(t) &= Bx(t) + Lx_t + f(t), \ t \geq 0, \\ x(\tau) &= \varphi(\tau), \ \text{ for } \tau \in [-r,0], \end{cases}$$

where B is the generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on a Banach space E, L is a bounded linear operator from $C_E := C([-r,0],E)$ into E and f is an E-valued function defined on $\mathbb{R}_+$.

To solve the equation (RDE), we use here the technique developed in [6],which consists in formulating this equation as an abstract Cauchy problem in the Banach space $X := E \times C_E$

$$(CP)\quad \begin{cases} \frac{d}{dt}\begin{pmatrix} 0 \\ u(t) \end{pmatrix} &= \mathcal{A}\begin{pmatrix} 0 \\ u(t) \end{pmatrix} + \begin{pmatrix} f(t) \\ 0 \end{pmatrix}, \quad t \geq 0, \\ \begin{pmatrix} 0 \\ u(0) \end{pmatrix} &= \begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \end{cases}$$

where

$$\mathcal{A} := \begin{pmatrix} 0 & B\delta_0 + L - \delta_0' \\ 0 & \frac{d}{d\tau} \end{pmatrix}, \ D(\mathcal{A}) = \{0\} \times \left\{\varphi \in C_E^1 : \varphi(0) \in D(B)\right\},$$

$$\delta_0'\varphi := \varphi'(0), \quad \text{for all } \varphi \in C_E^1 := C^1([-r,0],E).$$

We remark immediately that $D(\mathcal{A})$ is not dense in X. Exactly, one has $X_0 := \overline{D(\mathcal{A})} = \{0\} \times C_E$. But one can show that the operator $\mathcal{A}$ is of Hille-Yosida (see [6] and [12]). Hence, by Hille-Phillips theorem, the part $\mathcal{A}_0$ of $\mathcal{A}$ generates a C_0-semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ on the space $\{0\} \times C_E$. It is easy to see that the operator $\mathcal{A}_0$ is given by

$$\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix},$$

where A is the operator defined on C_E by

$$A\varphi := \varphi', \text{ for } \varphi \in D(A) := \{\phi \in C_E^1 : \phi(0) \in D(B); \phi'(0) = B\phi(0) + L\phi\}.$$

Hence, the operator A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on C_E. This semigroup satisfies

$$T(t)\varphi(\theta) = \begin{cases} \varphi(t+\theta), & t+\theta \leq 0 \\ S(t+\theta)\varphi(0) + \int_0^{t+\theta} S(t+\theta-s)LT(s)ds, & t+\theta \geq 0, \end{cases}$$

and is called the semigroup solution of the homogeneous retarded differential (i.e., $f = 0$).

Since $\mathcal{A}_0$ is a diagonal operator, $(\mathcal{T}_0(t))_{t\geq 0}$ is under the form

$$\mathcal{T}_0(t) = \begin{pmatrix} 0 & 0 \\ 0 & T(t) \end{pmatrix}.$$

Let $(\mathcal{T}_{-1}(t))_{t\geq 0}$ denote the extrapolated semigroup of $(\mathcal{T}_0(t))_{t\geq 0}$. Then, we have the following result.

Theorem 8 *Let $\varphi \in C_E^1$ and $f \in W^{1,1}(\mathbb{R}_+, E)$ such that*

$$\varphi'(0) = B\varphi(0) + L\varphi + f(0)$$

Then, there is a unique solution $x \in C^1([-r, +\infty), E)$ of (RDE) and is given by

$$x(t) := \begin{cases} \varphi(t), & t \in [-r, 0] \\ u(t)(0), & t \geq 0 \end{cases} \tag{4}$$

where u is the function given by (3).
Conversely, if $x \in C^1([-r, +\infty), E)$ is the solution of (RDE) then the function $\begin{pmatrix} 0 \\ x_t \end{pmatrix}$ is the unique solution of (CP).

Proof. If φ and f satisfy hypotheses of the theorem then $\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in D(\mathcal{A})$ and

$$\mathcal{A}\begin{pmatrix} 0 \\ \varphi \end{pmatrix} + \begin{pmatrix} f(0) \\ 0 \end{pmatrix} \in \overline{D(\mathcal{A}_0)} = \{0\} \times C_E.$$

Theorem 7 implies that Cauchy problem (CP) admits a unique differentiable solution $\begin{pmatrix} 0 \\ u \end{pmatrix}$, given by

$$\begin{pmatrix} 0 \\ u(t) \end{pmatrix} = \begin{pmatrix} 0 \\ T(t)\varphi \end{pmatrix} + \int_0^t \mathcal{T}_{-1}(t-s)\begin{pmatrix} f(s) \\ 0 \end{pmatrix} ds, \quad t \geq 0. \tag{5}$$

Hence the function x given by (4) is differentiable and satisfies (RDE). For more details see [6]. ∎

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