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Fixed Point Theorems with Applications to Differential Equations

Juan A. Gatica
Department of Mathematics
University of Iowa
Iowa City IA 52242

1 Introduction.

In this exposition we will center on the following problem: if X is a set, $A \subseteq X$, and if $f : A \to X$ is a function, when can we assert the existence of $x_0 \in A$ such that $f(x_0) = x_0$?

We will also center on some of the main uses of such results, namely in differential equations and we will start with a short discussion of discrete dynamical systems.

The most common special case that appears in the literature is that in which $A = X$, plus some topological considerations. As we progress we will be able to see that topological considerations are at the core of the problem. We will also see that in many useful circumstances it is preferable to consider the case $A \subseteq X$.

Besides the intrinsic interest of the question posed let us start to construct some of the applications that these results may have. This will also give us the advantage of realizing which are some important questions connected to our problem.

This paper is only of an introductory nature: it is meant for those who are curious about the subject but have not studied it in any depth yet. It is not intended in any way to be a review of the area.

We start with a deceivingly simple problem: let $D \subseteq R^n$ and $f : D \to D$ a function. We pose the following problem: given $x_0 \in D$, defining $x_{n+1} = f(x_n)$, and defining the solution of this problem as the sequence $\{f^n(x_0)\}_{n=0}^{\infty}$ (where $f^0 = I$, the identity function, and $f^{n+1} = f \circ f^n$), then there is no question about the existence of a unique solution for any problem of this sort and, if f is continuous, then we have continuous dependance of solutions on initial conditions, the two main requisites usually imposed on ordinary differential equations. This problem is called a discrete dynamical system on D (induced by the function f).

Let us first observe that if $x_0 \in D$ is a fixed point of f, then the constant sequence $\{x_n\}_{n=0}^{\infty}, x_n = x_0, n \geq 1, n$ a positive integer, is a solution of this problem. The real problem is: can we predict anything about the behavior of solutions to this problem?

This is certainly an interesting question, which is very much related to the fixed points of the function f. We will try to get some insight into the problem via two strikingly different examples coming from applications to biology.

1.1. Consider a drug that has the following property: after 24 hours of administration of a dose, a proportion $\alpha \in (0,1)$ remains in the patient. If you start with a dose of m_0 (say milligrams) and prescribe that the patient be administered β units every 24 hours thereafter, and this is a long term treatment (for example a Calcium channel blocker, Prozac, an antibiotic, etc.), can you predict the total drug accumulation in the patient (also called the effective dose)?

To answer this question let us first examine what will happen day by day: if we denote by x_n = amount of drug in the patient exactly after the n^{th} administration of the drug, then we certainly have that $x_{n+1} = \alpha x_n + \beta, x_0 = m_0$.

This is a simple algebraic problem that any student knowing about linear functions should be able to solve, with some calculations along the way. We will look at it in a slightly different manner.

Consider the function $f : [0,\infty) \to [0,\infty), f(x) = \alpha x + \beta$ and the discrete dynamical system it induces. Then we see that if we let $x_0 = m_0$, we must have that $x_{n+1} = f(x_n), x_0 = m_0$. Thus we have the discret dynamical system induced by this function.

Observe that the only fixed point of f is $x = \frac{\beta}{1-\alpha}$ and some elementary calculus shows that, no matter what the value of $m_0, \lim_{n\to\infty} x_n = \frac{\beta}{1-\alpha}$. This shows that a fixed point of f (in this case the only fixed point of the function) serves as a predictor of what the system will ultimately do).

1.2. Consider the discrete logistic equation, which can be posed as follows: let $\alpha \in (0,4]$ and define $f : [0,1] \to [0,1], f(x) = \alpha x(1-x)$. The restriction $\alpha \in (0,4]$ is imposed so that the function is not identically zero and so that it maps $[0,1]$ into itself.

Given $x_0 \in [0,1]$, then the solution with this initial condition is given by $x_{n+1} = f(x_n)$, and again we want to get some insight of what solutions do. The first observation is that 0 is a fixed point of f and, accordingly, the sequence which is constantly zero is a solution to the problem. Another obvious solution arises by picking $x_0 = 1$, yielding the solution $\{x_n\}_{n=0}^{\infty}, x_0 = 1, x_n = 0, n \geq 1$.

We see immediately that if $0 < \alpha \leq 1$, then 0 is the only fixed point of f and if

$x_0 \in [0,1]$, then the solution $\{x_n\}_{n=0}^{\infty}$ is such that $\lim_{n\to\infty} x_n = 0$ (this is not difficult to show).

If $1 < \alpha \leq 4$, then we have also a positive fixed point of f, namely $x = (\alpha - 1)/\alpha$, and we thus have two constant solutions. With not much work it can be shown that if $1 < \alpha < 3$ then we have that if $x_0 \in (0,1)$, then the solution $\{x_n\}_{n=0}^{\infty}$ is such that $\lim_{n\to\infty} x_n = (\alpha - 1)/\alpha$. Thus the fixed points of f, in this range of values of α, are predictors of the behavior of all solutions to the problem.

If $\alpha = 3$, then we have that the fixed point 2/3 is still approached by solutions but that the convergence is very slow.

If $\alpha > 3$ then many things start to occur, such as the appearance of periodic solutions and, as the value of α gets larger, the more complicated the behavior of the solutions of the problem, starting with the existence of asymptotically stable periodic solutions of period two (here "asymptotically stable" means, intuitively, that if you start near the periodic solution, then you stay near the solution for all values of n and, as n increases to infinity, the difference of the solutions goes to zero), then of period four, then eight and so on up to a point when periodic solutions are no longer asymptotically stable and solutions become extremely dependent on their initial value. The study of this mapping is fascinating but we will leave it here for the purposes of this paper.

The next type of application that we are interested in is the existence of solutions to initial value problem for ordinary differential equations. The problem can be posed as follows: let $D \subseteq R^{n+1}$, $(t_0, x_0) \in D$ (where we take $t_0 \in R, x_0 \in R^n$), and $f : D \to R^n$ a function. $\phi : I \to R^n$ such that: I is an interval of positive length, $t_0 \in I$, $(t, \phi(t)) \in D$ for all $t \in I$, ϕ is differentiable and $\phi'(t) = f(t, \phi(t)), t \in I$, is said to be a solution of the initial value problem

$$\begin{aligned} x' &= f(t,x) \\ x(t_0) &= x_0. \end{aligned}$$

We now have the problems: a) When do initial value problems have solutions?, b) If an initial value problem has at least two solutions that are "intrinsically" different, what can be said of the set of all solutions to the initial value problem?

To get a grasp of the problem, we start by assuming that D is an open subset of R^{n+1} and that f is a continuous function. We will say that a function $\psi : I \to R^n$ is a solution of the integral equation $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$ if I is an interval containing t_0, ψ is continuous, if the graph of ψ is contained in G and:

$$\psi(t) = x_0 + \int_{t_0}^{t} f(s, \psi(s)ds, t \in [a,b].$$

The setting just explained will allow us to transform the initial value problem into a fixed point problem for some operator defined in a subset of a function space.

This illustrates two facts: when dealing with differential equations, the space of "unknowns" is a space of functions and that finding solutions in this type of setting is equivalent to finding fixed points of functions defined in rather "large" spaces.

Finally we explain a problem from differential equations of a different sort, called a boundary value problem for a second order differential equation. We will deal only with an illustrative special case, which includes as a particular case the Emden-Fowler equation; other types of conditions that may be imposed would allow the problem to become "singular" at 0.

We are given a continuous function $f : (0,1) \times [0,\infty) \to [0,\infty)$, $f(x,y)$ integrable as a function of x for any fixed $y \in [0,\infty), \alpha, \beta, \gamma, \delta$ nonnegative real numbers such that $\rho = \gamma\beta + \alpha\gamma + \alpha\delta > 0$, and we are asked to find, if they exist, all functions $\phi : [0,1] \to [0,\infty)$ which are continuously differentiable on $[0,1]$ ($\phi \in C^1([0,1])$), (taking one sided derivatives at the end points of the interval) and they are twice continuously differentiable on $(0,1)$ ($\phi \in C^2((0,1))$), with the following properties: $\phi''(x) = -f(x,\phi(x)), x \in (0,1)$ and that $\alpha\phi(0) - \beta\phi'(0) = 0, \gamma\phi(1) + \delta\phi'(1) = 0$.

The problem is symbolically written as:

$$\begin{aligned} y'' + f(x,y) &= 0 \\ \alpha y(0) - \beta y'(0) &= 0 \\ \gamma y(1) + \delta y'(1) &= 0. \end{aligned}$$

For example, if $f(x,y) = a(x)y^p, p > 0$, we have an obvious solution, namely the identically zero solution; then the problem of interest is to determine the existence of a non-negative non-constant solution. This problem has been surveyed by Wong [W] and the partial differential equations version (which we can also attack using the methods explained below) has been surveyed by Lions [L].

If we allow for $f : (0,1) \times (0,\infty) \to (0,\infty)$, then we would have as examples $f(x,y) = a(x)y^{-p}, p > 0$, a problem that is singular at 0. There are many existence, multiplicity, approximation results for boundary value problems of this type.

One way to attack this problem is, as in the initial value problem, to transform it into an integral equation on which one can use fixed point theory. To simplify the notation we will assume that $\alpha = 1, \beta = 0, \gamma = 1, \delta = 0$.

Define $G : [0,1] \times [0,1] \to [0,\infty)$ by:

$$\begin{aligned} G(x,t) &= (1-x)t, \ 0 \le t \le x, \\ G(x,t) &= x(1-t), \ x \le t \le 1. \end{aligned}$$

This function is called the Green's function for the problem

$$\begin{aligned} y'' &= 0 \\ y(0) &= y(1) = 0. \end{aligned}$$

Again for simplicity of exposition, we will assume that $f(x, y)$ is increasing as a function of y for each fixed $x \in (0,1)$. Then it follows that if we denote by $K = \{\phi \in C[0,1] : \phi(x) \geq 0, x \in [0,1]\}$, we can define $T : K \to K, T(\phi)(x) = \int_0^1 G(x,t) f(t, \phi(t))dt$, and our assumptions make sure that this map is well defined.

Observe that if $\phi \in K$ then $T(\phi)(x) = \int_0^x (1-x)tf(t, \phi(t))dt + \int_x^1 x(1-t)f(t, \phi(t))dt$, and it is clear that $T(\phi)'(x) = -\int_0^1 tf(t, \phi(t))dt + \int_x^1 f(t, \phi(t))dt$. It follows that $T(\phi)''(x) = -f(x, \phi(x)), x \in (0,1)$, and so if $T(\phi) = \phi$, we have a solution of our boundary value problem. If ϕ is a solution of our boundary value problem then clearly $T(\phi) = \phi$.

2 Classical Fixed Point Theorems and Some Applications.

We will now state some of the most important fixed point theorems, without proof, and we will see the standard ways to generalize them as well as to point out, with some detail, how they are used in applications to differential equations and dynamical systems. These are by no means the only important applications of these results, and we will try to mention, in passing some of the other applications that make these results so useful.

Before we start with the theorems, it is advisable to state some results that will clarify the natural extensions they have. This will be our start point.

In all of our discussion we will restrict ourselves to the following settings: the space we will deal with will always be either complete metric spaces or the closure of bounded open subsets of Banach spaces (vector spaces over the reals with a norm that induces a complete metric).

One fact that we need to keep in mind is that if X is a finite dimensional normed space then all norms are equivalent, meaning that all norms generate the same topology in the space. This is not true in Banach spaces which are not of finite dimension as a vector space.

Lemma 1 *Let X be a Banach space and $G \subseteq X$. If Y is a Banach space and H is homeomorphic to G with homeomorphism g (i.e. $g : G \to H$ is one to one, continuous and with continuous inverse) and if $f : G \to G$ is continuous and has a fixed point, then $g \circ f \circ g^{-1} : H \to H$ also has a fixed point.*

Definition 2 *Let X be a Banach space and $K \subseteq X$. We will say that K is a retract of X if there is a function $R : X \to K$, which is continuous and such that $R(x) = x$ whenever $x \in K$.*

Theorem 3 *(Dugundji [D]). Let X be a Banach space and $K \subseteq X$ be a closed convex subset. Then K is a retract of X.*

Theorem 4 *(Brouwer's Fixed point Theorem [B1]). Let $C \subseteq R^n$ be a closed bounded convex set. Then if $f : C \to C$ is continuous, it has at least one fixed point, i.e. there is $x_0 \in C$ such that $f(x_0) = x_0$.*

Corollary 5 *Let $K \subseteq R^n$ be homeomorphic to a closed bounded convex set. Then if $f : K \to K$ is continuous, f has a fixed point in K.*

This result is at the heart of many important applications, ranging from the existence of solutions to a system of n equations (not necessarily linear) in n unknowns to the existence of nonnegative eigenvalues with nonnegative eigenvectors for certain types of linear operators (the Perron-Frobenius theorem which states that if all the entries of a square matrix are nonnegative, then the result is true).

There are elementary proofs of Brouwer's theorem but they require the construction of some elaborate results that would extend the length of this presentation beyond what we have in mind, so we will not present it here.

We will illustrate one of the uses of Brouwer's result which implies the existence of critical points for an autonomous differential equation. For simplicity we will restrict ourselves to dimension two.

Let $f : R^2 \to R^2$ be a continuous function. We will say that f is locally Lipschitzian on R^2 if the following holds: if $C \subseteq R^2$ is a closed bounded set, then there exists a number $L_C \geq 0$ such that for any $x, y \in C$ we have $||f(x) - f(y)|| \leq L_C||x - y||$.

Now consider the differential equation

$$x' = f(x).$$

We will make the following assumption for this differential equation: given any $x_0 \in R^2$ the initial value problem $x' = f(x), x(0) = x_0$ has a solution $\psi_{x_0} : [0, \infty) \to R^2$.

Because of the locally Lipschitzian condition one can show that any initial value problem has a unique solution defined on $[0, \infty)$ and that the following happens: if $\{x_n\}_{n=1}^{\infty}$ is any convergent sequence in R^2, say with $\lim_{n\to\infty} x_n = x_0$, then the sequence of solutions is such that $\lim_{n\to\infty} \psi_{x_n}(t) = \psi_{x_0}(t)$ uniformly on compact subsets of $[0, \infty)$.

If $x_0 \in R^2$, we define the positive semiorbit through this point to be the set $\gamma^+(x_0) = \{\psi_{x_0}(t) : t \geq 0\}$, and we consider it as a curve on which the only allowable parametrizations are given by solutions of our differential equation. Because of the uniqueness of solutions to initial value problems and the fact that the equation is autonomous (meaning that there is no explicit t dependence in the equation itself), it turns out that if $x, y \in R^2$ then either $\gamma^+(x) \cap \gamma^+(y) = \phi$ (the empty set) or $\gamma^+(x) \subseteq \gamma^+(y)$ or $\gamma^+(y) \subseteq \gamma^+(x)$.

Now suppose that $\psi : [0, \infty) \to R^2$ is a solution of our differential equation and suppose that there is $T > 0$ such that $\psi(0) = \psi(T)$. Define $\zeta : [0, \infty) \to R^2$ by: $\zeta(t) = \psi(t + T)$ and observe that $\zeta'(t) = \psi'(t + T) = f(\psi(t + T)) = f(\zeta(t))$ so we have a solution of our differential equation. Furthermore $\zeta(0) = \psi(T) = \psi(0)$ so, by uniqueness of solutions of initial value problems we must have $\psi(t + T) = \psi(t), t \geq 0$.

We conclude from this that $\gamma^+(\psi(0))$ is a closed curve. Let us assume further that ψ is not a constant function. Then there will be a minimum T for which this happens and $\gamma^+(\psi(0))$ is a simple closed curve, which we will denote simply by γ (and is parametrized by ψ).γ is homeomorphic to the unit circle.

Jordan's curve theorem now tell us that $R^2 \setminus \gamma$ consists of exactly two connected components one bounded, the other unbounded. We denote by C the union of γ with the bounded component. C is homeomorphic to the closed unit disk. We will show that there is $x_0 \in C$ such that $f(x_0) = 0$.

For each $\tau > 0$, define the function $\phi_\tau : C \to C, \phi_\tau(x) = \psi_x(\tau)$. By what was explained above this mapping sends C into itself since no solution staring in the interior of C can ever intersect γ, the mapping is continuous and so, by Brouwer's theorem it must have a fixed point x_τ. Then $\psi_{x_\tau}(\tau) = x_\tau = \psi_{x_\tau}(0)$, and it follows that $\psi_{x_\tau}(t + \tau) = \psi_{x_\tau}(t), t \geq 0$.

Pick a sequence $\{\tau_n\}_{n=0}^\infty$ such that $\tau_n > 0, n \in N, \lim_{n\to\infty} \tau_n = 0$. We then have a sequence of fixed points $\{x_{\tau_n}\}_{n=1}^\infty$, which remains in the compact set C. This sequence has a convergent subsequence which we will relabel it with the same indices and we will denote by x_0 its limit. We recall that the sequence of solutions $\{\psi_{x_{\tau_n}}\}_{n=1}^\infty$ converges uniformly on compact sets to ψ_{x_0}.

Let $t > 0$. Then for any n there is an integer $l_n(t)$ such that $l_n(t)\tau_n \leq t < (l_n(t) + 1)\tau_n$, and we have $\psi_n(l_n(t)) = x_{\tau_n}$. We get:

$$||\psi_{x_0}(t) - x_0|| \leq ||\psi_{x_0}(t) - \psi_{x_{\tau_n}}(t)|| + ||\psi_{\tau_n}(t) - x_{\tau_n}|| + ||x_{\tau_n} - x_0||.$$

Now it is clear that, as $n \to \infty$ the right hand side has limit zero and so $\psi_{x_0}(t) = x_0$, concluding that ψ_{x_0} is a constant solution and hence $f(x_0) = 0$.

Our next step will be to consider Banach spaces of infinite dimension. Before getting into this new topic we would like to remind you that a degree theory exists in finite dimensional normed spaces which allows one to conclude, in many situations, the existence of a fixed point for a continuous mapping of the form $f : \overline{G} \to R^n$, where G is a bounded open nonempty subset of R^n, and f has no fixed points in the boundary of G.

The next fixed point result that we will discuss is the natural generalization of Brouwer's Theorem to Banach spaces of infinite dimension.

Theorem 6 *(Schauder's fixed Point Theorem [S]) Let X be a Banach space, $C \subseteq X$ a closed bounded convex and nonempty. If $f : C \to C$ is a continuous map such that $\overline{f(C)}$ is a compact set then f has at least one fixed point in C.*

We will proceed to show one of the applications of this result, in the area of ordinary differential equations.

Since all norms in finite dimensional vector spaces, in all the spaces R^k we will use the norm $\|(x_1, x_2, ..., x_k)\| = \sum_{i=1}^{i=n} |x_i|$, and we will denote, for $x \in R^k, r > 0, B(x,r) = \{z \in R^k : \|z - x\| < r, B[x,r] = \{x \in R^k : \|z - x\| \leq r\}$.

Lemma 7 *Let $G \subseteq R^{n+1}$ be an open set, $(t_0, x_0) \in G, f : G \to R^n$ continuous. Then there exists $\rho > 0$ such that the initial value problem*

$$\begin{aligned} x' &= f(t,x) \\ x(t_0) &= x_0 \end{aligned}$$

has a unique solution defined on the interval $[t_0 - \rho, t_0 + \rho]$.

Proof. To see this, let $r > 0$ be such that there $B((t_0, x_0), r) \subseteq G$. By the continuity of f there is $M > 1$ such that $\|f(t,x)\| \leq M, (t,x) \in B[(t_0, x_0), r/2]$. Choose $\rho = r/4$.

Now let $X = C([t_0 - \rho, t_0 + \rho], R^n)$ with the supremum norm, define $\phi_0 : [t_0 - \rho, t_0 + \rho] \to R^n$ by $\phi_0(t) = x_0,\ t \in [t_0 - \rho, t_0 + \rho]$ and consider $B[\phi_0, \rho]$ in our space X. It is a closed, bounded, convex subset of it.

Define $T : B[\phi_0, \rho] \to X$ by: $T(\psi)(t) = x_0 + \int_{t_0}^{t} f(s, \psi(s))ds$. Notice that if $\psi \in B[\phi_0, \rho]$ then

$$\begin{aligned} \|T(\psi)t - \phi_0(t)\| &= \left\| \int_{t_0}^{t} f(s, \psi(s))ds \right\| \\ &\leq \left| \int_{t_0}^{t} \|f(s, \psi(s)\|ds \right| \\ &\leq |t - t_0| M \\ &\leq \rho M \\ &\leq (r/2M)M \\ &= r/2 \\ &\leq \rho,\ t \in [t_0 - \rho, t_0 + \rho]. \end{aligned}$$

It follows that $\|T(\psi)\| \leq \rho$, so $T(B[\phi_0, \rho]) \subseteq B[\phi_0, \rho]$ and so we can consider this as a mapping from the closed ball into itself. T is continuous since we can use the standard theorems concerning the interchange of limit of a sequence of functions and the integral.

Also, if $\psi \in B[\phi_0, \rho]$ then the graph of $T(\psi)$ is contained in $B((t_0, x_0), r)$, since:

$$\begin{aligned} \|(t, T(\psi)(t)) - (t_0, x_0)\| &\leq |t - t_0| + \|T(\psi)(t) - x_0\| \\ &\leq r/4 + r/4 \\ &= r/2 \\ &< r. \end{aligned}$$

Now we need to show that $\overline{T(B[\phi_0,\rho])}$ is compact.

Let $\{\psi_k\}_{k=1}^{\infty}$ be a sequence in $B[\phi_0,\rho]$ and observe that:

$$\begin{aligned} ||T(\psi_k)'(t)|| &= ||f(t,\psi(t))|| \\ &\leq M. \end{aligned}$$

It follows that $\{T(\psi_k)\}_{k=1}^{\infty}$ is an equicontinuous uniformly bounded sequence of functions and, by the Arzela-Ascoli Theorem, it must have a convergent subsequence, implying that $\overline{T(B[\phi_0,\rho])}$ is compact.

By Schauder's Theorem T has a fixed point in $B[\phi_0,\rho]$, implying that the integral equation has a solution in $[t_0-\rho, t_0+\rho]$, yielding the desired result.

The last result we address is Banach's Contraction Principle, which is used to prove existence and uniqueness of solutions of initial value problems, among other applications.

Theorem 8 *(Banach's Contraction Principle [B]). Let (X,d) be a complete metric space and let $f : X \to X$ a mapping such that there exists $\alpha \in (0,1)$ such that: $x,y \in X \Rightarrow d(f(x),f(y)) \leq \alpha d(x,y)$. Then f has a unique fixed point in X.*

We will not prove this result although the proof is remarkably simple (pick $x_0 \in X$ and prove that the sequence of iterates of f at x_0 is a Cauchy sequence, hence convergent and then use the continuity of f to show that the limit must be a fixed point of f). Uniqueness follows by contradiction.

Banach's Theorem is one of the standard tools to prove uniqueness of solutions to initial value problems. It is also valid in spaces that are not Banach spaces, but the existence of $\alpha \in (0,1)$ restricts the usefulness of the result.

A branch of fixed point theory has been developed to try to include the case $\alpha = 1$, when the mapping is called nonexpansive and the results are centered on the finer aspects of the geometrical structure of Banach spaces (see [KG] and [K]).

3 Results in Cones in Banach Spaces.

In many situations we are confronted with either the existence of a "trivial" fixed point (in which case a result asserting only the existence of at least one fixed point is rendered useless) or we need to find fixed points with certain properties (for example, the existence of a nonnegative solution to a differential equation).

There are many results that do assert the existence of two or more fixed points for certain mappings. We will restrict our attention to an approach that can be used for both of the purposes stated above; we will present a few results in this subject, which we will apply to boundary value problems in differential equations.

The standard references to this area are the works of M. A. Krasnoselskii [K1] and H. Amann [A].

Definition 9 *Let X be a Banach space and $K \subseteq X$ be a nonempty closed set. We will say that K is a cone in X if the following conditions hold:*

a) $x, y \in K \Rightarrow x + y \in K$.
b) $x \in K, \alpha \geq 0 \Rightarrow \alpha x \in K$.
c) $K \cap (-K) = \{0\}$

As a first example of the types of results available, we will present a result which can be viewed as an extension of Krasnoselskii's "cone compression" results.

Theorem 10 *([GS1],[T]). Let X be a Banach space, $K \subseteq X$ a cone, $0 < r < R$, denote by D the set $\{x \in K : r \leq ||x|| \leq R\}$ and let $T : D \to K$ be a continuous map such that $\overline{T(D)}$ is a compact set and such that:*

a) $x \in D, ||x|| = R, T(x) = \lambda x \Rightarrow \lambda \leq 1$.
b) $x \in D, ||x|| = r, T(x) = \lambda x \Rightarrow \lambda \geq 1$.
c) $\inf\{||T(x)|| : x \in D, ||x|| = r\} > 0$.
Then T ha a fixed point in D.

Corollary 11 *([GS]). Let X, K, r, R, D, T be as in Theorem 3.2 but now assume that:*

a) $x \in D, ||x|| = R, T(x) = \lambda x \Rightarrow \lambda \geq 1$.
b) $x \in D, ||x|| = r, T(x) = \lambda x \Rightarrow \lambda \leq 1$.
c) $\inf\{||T(x)|| : x \in D, ||x|| = R\} > 0$.
Then T has a fixed point in D.

We will present typical applications of these results to the existence of positive solutions to boundary value problem on ordinary differential equations (they are also applied to semilinear elliptic partial differential equations).

Let $f : (0,1) \times [0,\infty) \to [0,\infty)$ be a continuous function such that $f(x,\cdot)$ is increasing as a function of the second variable for all $x \in (0,1)$, $f(\cdot, y)$ is integrable as a function of the first variable for all $y \in [0,\infty)$, and consider the boundary value problem:

$$\begin{aligned} y'' + f(x,y) &= 0 \\ y(0) &= y(1) = 0. \end{aligned}$$

The prototype of function we are thinking about is $f(x,y) = a(x)y^p, p > 0$.

Theorem 12 *([GOW]). If $\lim_{y\to 0+} \frac{f(x,y)}{y} = \infty, \lim_{y\to\infty} \frac{f(x,y)}{y} = 0$, both limits being uniform on compact subsets of $(0,1)$, the boundary value problem has a non-constant nonnegative solution.*

Proof. We Consider the Banach space of all continuous real valued functions defined on $[0,1]$ with the supremum norm and we let $K = \{\phi \in X : \phi(x) \geq 0$ and the graph of ϕ is concave down$\}$.

As in the introduction, define $G : [0,1] \times [0,1] \to [0,\infty)$ by:

$$\begin{aligned} G(x,t) &= (1-x)t, 0 \leq t \leq x \\ G(x,t) &= x(1-t), x \leq t \leq 1. \end{aligned}$$

Now we define $T : K \to K$ by:

$$T(\phi)(x) = \int_0^1 G(x,t) f(t,\phi(t))dt.$$

As discussed in the introduction, if $\phi \in K$ then $T(\phi)$ is differentiable on [0,1], so it is certainly continuous on $[0,1]$, it is clear that $T(\phi)(x) \geq 0$, $x \in [0,1]$, and we have shown that $T(\phi)$ is twice differentiable on $(0,1)$ with $(T(\phi))''(x) = -f(x,\phi(x))$ and since f is nonnegative it follows that the graph of $T(\phi)$ is concave down. Thus we may consider $T : K \to K$.

If $D \subseteq K$ is any bounded subset of K, then using our computation of the derivative of $T(\phi)$, we conclude, using the Arzela-Ascoli Theorem, that $\overline{T(D)}$ is a compact set.

We will need one basic fact concerning functions in K : if $\phi \in K$,then we have that $\phi(t) \geq \frac{\|\phi\|}{4}$, $t \in [1/4, 3/4]$ ([GOW]).

Since $\lim_{y \to 0+} \frac{f(x,y)}{y} = \infty$, uniformly on compact sets, we can find $r_0 > 0$ such that $\int_0^1 G(1/2,t)\frac{f(t,r_0)}{r_0} > 1$. We pick $r = 4r_0$ and observe that if $\phi \in K, \|\phi\| = r$ and if $T(\phi) = \lambda\phi$, then:

$$\begin{aligned} T(\phi)(x) &= \lambda\phi(x), x \in [0,1] \\ \int_0^1 G(1/2,t)f(t,\phi(t))dt &= \lambda\phi(1/2) \\ \int_0^1 G(1/2,t)f(t,\phi(t))dt &\leq \lambda r/4 \\ \int_0^1 G(1/2,t)f(t,r/4)dt &\leq \lambda r/4 \\ \int_0^1 G(x,t)\frac{f(t,r_0)}{r_0}dt &\leq \lambda \\ 1 &\leq \lambda. \end{aligned}$$

In particular $\lambda > 1$.

Also we have that $\phi \in K, \|\phi\| = r \Rightarrow \phi(1/2) = \int_0^1 G(1/2,t)f(1/2,\phi(t))dt$. Thus: $\phi(1/2) \geq \int_{1/4}^{3/4} G(1/2,t)f(1/2,r_0)dt = M$, an hence $\inf\{\|T(\phi) : \phi \in K, \|\phi\| = r\} \geq M > 0$.

Finally, we can pick $R > r$ such that $\int_0^1 G(1/2,t)\frac{f(t,R)}{R}dt < 1/4$. If $\phi \in K, \|\phi\| = R$ and if $T(\phi) = \lambda\phi$, then : .

$$\int_0^1 G(1/2,t)f(t,\phi(t))dt = \lambda\phi(1/2)$$

$$\int_0^1 G(1/2,t)f(t,R) \geq \lambda R/4$$

$$4\int_0^1 G(t_0,t)\frac{f(t_0,R)}{R}dt \geq \lambda$$

$$1 \geq \lambda.$$

Now we have satisfied all the conditions of Theorem 3.2 on the set

$$D = \{\phi \in K : r \leq \|\phi\| \leq R\}.$$

The existence of a non-constant nonnegative solution to the boundary value problem follows. ■

We have just proved a result which corresponds to what is frequently called the "sublinear case" and, in the particular example in mind, corresponds to $0 < p < 1$. There is ,of course, the "superlinear case" corresponding to $p > 1$, and it is treated much in the same way, now using Corollary 3.3. The proof is so similar that we will only state the result here.

Theorem 13 *([GOW]). Suppose that, under the general hypotheses on f used before now we require that:* $\lim_{y\to 0+} \frac{f(x,y)}{y} = 0, \lim_{y\to\infty} \frac{f(x,y)}{y} = \infty$, *both limits being uniform on compact subsets of* $(0,1)$. *Then the boundary value problem has at least one nonnegative, non-constant solution.*

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