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Topology and the non-existence of magnetic monopoles

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Abstract

Most of the work being done to unify General Relativity and Quantum Mechanics tries to represent General Relativity in the Quantum Mechanics language. We propose here an approach to represent Quantum Mechanics in the language of Relativity. In order to introduce discretness into the language of Relativity we consider the classical invarients of homotopy theory, in particular the index of a vector field. We insist that these invariants be treated as physical quantities, independent of choices of observers and conventions. Following this prescription we found an argument that pseudo-vectorfields should have zero index. Hence magnetic monopoles should not exist. We give extended philosophical arguments that the index should play an important role in Mathematics, and hence Physics, based on a novel definition of Mathematics and the meaning of the underlying unity of Mathematics.

1. Introduction

There are two successful theories in Physics: Quantum Mechanics and General Relativity. There is considerable work being done to unify them. Mostly, this work tries to represent Relativity as a form of Quantum Mechanics. To a mathematician, however, General Relativity seems to make Physics well-defined and clear, [O], [S–W], [P], whereas Quantum Mechanics is full of tricks and arguments with coordinate systems and formal manipulations, [Su]. So why not try to make Quantum Mechanics look like Relativity ?

To do this, we want some way of introducing discreteness and quantum numbers into smooth and continuous space-time. In Quantum Mechanics this is done by eigenvalues and eigenvectors and symmetries. We believe that in Relativity, one can do it by using the homotopy invariants of Algebraic Topology. A good way to proceed is to insist that any homotopy invariant which arises in a physical way should be treated as a physical object. The easiest invariant and most flexible to use is the index of a vector field. Hence the Index Principle.

Principle of Invariance of Index *The index of any "physical" vector field is invariant under changes of coordinates and orientation of space–time. If the index is undefined, it signals either radiation or unrealistic physical hypotheses.*

Consequence *Every "physical" pseudo–vector field has index zero or the index is undefined.*

Now the magnetic vector field $\overrightarrow{B}$ is a pseudo–vector field. That means if we change the orientation of space, $\overrightarrow{B}$ changes to $-\overrightarrow{B}$. Now $\text{Ind}(-V) = (-1)^n \text{ Ind}(V)$ where n is the dimension of the manifold on which V is defined. Thus $\text{Ind}(-\overrightarrow{B}) = (-1)^3 \text{ Ind}(\overrightarrow{B})$. So either $\text{Ind}(\overrightarrow{B})$ is not defined or $\text{Ind}(\overrightarrow{B}) = 0$. Now a magnetic monopole will give rise to a $\overrightarrow{B}$ with index ± 1. As this is inconsistent with the Invariance of Index Principle, we predict that magnetic monopoles do not exist.

We remark that since the magnetic vector field $\overrightarrow{B}$ changes for each observer, we provide a more sophisticated version of the argument. This involves the "fibre bundle of space-time" and will be discussed in section 6. But the essence of the argument is the one above.

Magnetic monopoles were predicted by Dirac based on an alteration of Maxwell's equations which made them more symmetric, [F]. Despite Dirac's ideas, magnetic monopoles have not been found in nature. More recently magnetic monopoles were predicted using the topological nontriviality of certain principal bundles, [B]. Our non-existence argument is also based on Topology, but our argument is more direct. Note that it does not depend on Maxwell's equations. It is a new kind of argument for Physics and so we devote considerable philosophical discussion in sections 2 and 3 as to why it is reasonable.

In section 2, we discuss why the index of vector fields should play an important role in Mathematics. This involves the question of the underlying unity of Mathematics. In section 3, methods of applying the index in Mathematics are discussed. In section 4, the definition and key properties of the index are listed. In section 5, natural extension of the ideas of index and vector fields to fibre bundles are made. In section 6, we give a more sophisticated argument that magnetic monopoles do not exist using the electromagnetic field tensor F. We introduce a fibre bundle of space–time which clarifies what we mean by the index undefined. Finally in section 7, we discuss how the index of different zeros of a vector field act like electric charge of particles, and following this suggestion we speculate that the transfer theorem for vertical vector bundles combined with the existence of antiparticles gives inductive evidence that particles may be discribed by appropriate vertical vector fields on certain fibre bundles.

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tions.

2. The Unity of Mathematics

We take the following definition of Mathematics:

Definition *Mathematics is the study of well-defined concepts.*

Now well-defined concepts are creations of the human mind. And most of those creations can be quite arbitrary. There is no limit to the well-defined imagination. So if one accepts the definition that Mathematics is the study of the well-defined, then how can Mathematics have an underlying unity? Yet it is a fact that many savants see a underlying unity in Mathematics, so the key question to consider is:

Question *Why does Mathematics appear to have an underlying unity?*

If mathematical unity really exists then it is reasonable to hope that there are a few basic principles which explain the occurrence of those phenomena which persuade us to believe that Mathematics is indeed unified; just as the various phenomena of Physics seem to be explained by a few fundamental laws. If we can discover these principles it would give us great insight into the development of Mathematics and perhaps even insight into Physics.

Now what things produce the appearance of an underlying unity in Mathematics? Mathematics appears to be unified when a concept, such as the Euler characteristic, appears over and over in interesting results; or an idea, such as that of a group, is involved in many different fields and is used in Science to predict or make phenomena precise; or an equation, like De Moivre's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

yields numerous interesting relations among important concepts in several fields in a mechanical way.

Thus the appearance of underlying unity comes from the ubiquity of certain concepts and objects, such as the numbers π and e and concepts such as groups and rings, and invariants such as the Euler characteristic and eigenvalues, which continually appear in striking relationships and in diverse fields of Mathematics and Physics. We use the word *broad* to describe these concepts.

Compare broad concepts with *deep* concepts. The depth of an idea seems to be a function of time. As our understanding of a field increases, deep concepts become elementary concepts, deep theorems are transformed into definitions and so on. But something broad, like the Euler characteristic, remains broad, or becomes broader as time goes on. The relationships a broad concept has with other concepts are forever.

The Function Principle *Any concept which arises from a simple construction of functions will appear over and over again throughout Mathematics.*

We assert the principle that function is one of the broadest of all mathematical concepts, and any concept or theorem derived in a natural way from that of functions must itself be broad. We will use this principle to assert that the underlying unity of Mathematics at least partly stems from the breadth of the concept of function. We will show how the breadth of category and functor and equivalence and e and π and de Moivre's formula and groups and rings and Euler Characteristic all follow from this principle. We will subject this principle to the rigorous test of a scientific theory: It must predict new broad concepts. We make such predictions and report on evidence that the predictions are correct.

The concept of a function as a mapping $f\colon X \to Y$ from a source set X to a target set Y did not develop until the Twentieth Century. The modern concept of a function did not even begin to emerge until the middle ages. The beginnings of Physics should have given a great impetus to the notion of function, since the measurements of the initial conditions of an experiment and the final results gives implicitly a function from the initial states of an experiment to the final outcomes; but historians say that the early physicists and mathematicians never thought this way. Soon thereafter calculus was invented. For many years afterwards functions were thought to be always given by some algebraic expression. Slowly the concept of a mapping grew. Cantor's set theory gave the notion a good impulse but the modern notion was adopted only in the Twentieth Century. See [ML] for a good account of these ideas.

The careful definition of function is necessary so that the definition of the composition of two functions can be defined. Thus $f \circ g$ is only defined when the target of g is the source of f. This composition is associative: $(f \circ g) \circ h = f \circ (g \circ h)$ and f composed with the identity of either the source or the target is f again. We call a set of functions a *category* if it is closed under compositions and contains the identity functions of all the sources and targets.

Category was first defined by S. Eilenberg and S. MacLane and was employed by Eilenberg and N. Steenrod in the 1940's to give homology theory its functorial character. Category theory became a subject in its own right, it's practitioners joyfully noting that almost every branch of Mathematics could be organized as a category. The usual definition of category is merely an abstraction of functions closed under composition. The functions are abstracted into things called morphisms and composition becomes an operation on sets of morphisms satisfying exactly the same properties that functions and composition satisfy. Most mathematicians think of categories as very abstract things and are surprised to find they come from such a homely source as functions closed under composition.

A *functor* is a function whose source and domain are categories and which preserves composition. That is, if F is the functor, then $F(f \circ g) = F(f) \circ F(g)$. This definition also is abstracted and one says category and functor in the same breath.

Now consider the question: What statements can be made about a function f which would make sense in every possible category? There are basically only four statements since the only functions known to exist in every category are the identity functions. We can say that f is an identity, or that f is a *retraction* by which we mean that there is a function g so that $f \circ g$ is an identity, or that f is a *cross-section* by which we mean that there is a function h so that $h \circ f$ is an identity, or finally that f is an *equivalence* by which we mean that f is both a retraction and a cross-section. In the case of equivalence the function h must equal the function g and it is called the *inverse* of f and it is unique.

Retraction and cross-section induce a partial ordering of the sources and targets of a category, hereafter called the *objects* of the category. Equivalences induce an equivalence relation on the objects and give us the means of making precise the notion that two mathematical structures are the same.

Now consider the self equivalences of some object X in a category of functions. Since X is both the source and the target, composition is always defined for any pair of functions, as are inverses. Thus we have a *group*. The definition of a group in general is just an abstraction, where the functions become undefined elements and composition is the undefined operation which satisfies the group laws of associativity and existence of identity and inverse, these laws being the relations that equivalences satisfy. The notion of functor restricted to a group becomes that of *homomorphism*. The equivalences in the category of groups and homomorphisms are called *isomorphisms*.

The concept of groups arose in the solution of polynomial equations, with the first ideas due to Lagrange in the late eighteenth century, continuing through Abel to Galois. Felix Klein proposed that Geometry should be viewed as arising from groups of symmetries in 1875. Poincare proposed that the equations of Physics should be invariant under the correct symmetry groups around 1900. Since then groups have played an increasingly important role in Mathematics and in Physics. The increasing appearance of this broad concept must have fed the feeling of the underlying unity of Mathematics. Now we see how naturally it follows from the Function Principle.

If we consider a set of functions S from a fixed object X into a group G, we can induce a group structure on S by defining the multiplication of two functions f and g to be $f * g$ where $f * g(a) = f(a) \cdot g(a)$. Here a runs through all the elements in X and "$\cdot$" is the group multiplication in G. This multiplication can be easily shown to satisfy the laws of group multiplication. The same idea applied to maps into the Real Numbers or the Complex Numbers gives rise to addition and multiplication on functions. These satisfy properties which are abstracted into the concepts of abelian rings. If we consider the set of self homomorphisms of abelian groups and use composition and addition of functions, we get an important example of a non-commutative ring. The natural functors for rings should be ring homomorphisms. In the case of a ring of functions into the Real or Complex numbers we note that a ring homomorphism h fixes the constant maps. If we consider all functions which fix the constants and preserve the addition, we get a category of functions from rings to rings; that is, these functions are closed under composition. We call these functions *linear transformations*. They contain the ring homomorphisms as a subset. Study the equivalences of this category. We obtain the concepts of *vector spaces* and linear transformations after the usual abstraction.

Now we consider a category of homomorphisms of abelian groups. We ask the same question which gave us equivalence and groups, namely: What statements can be made about a homomorphism f which would make sense in every possible category of abelian groups? Now between every possible abelian group there is the trivial homomorphism $0: A \to B$ which carries all of A onto the identity of B. Also we have for every integer N the homomorphism from A to itself which adds every element to itself N times, that is multiplication by N.

Thus for any homomorphism $h: A \to B$ there are three statements we can make which would always make sense. First $N \circ h$ is the trivial homomorphism 0, second that there is a homomorphism $\tau: B \to A$ so that $h \circ \tau$ is multiplication by N, and third that $\tau \circ h$ is multiplication by N. So we can give to any homomorphism three non-negative integers: The *exponent*, the *cross-section degree*, and the *retraction degree*. The *exponent* is the smallest positive integer such that $N \circ h$ is the trivial homomorphism 0. If there is no such N then the exponent is zero. Similarly the *cross-section degree* is the smallest positive N such that there is a τ, called a *cross-section transfer*, so that $h \circ \tau$ is multiplication by N. Finally the *retraction degree* is the smallest positive N such that there is a τ, called a *retraction transfer*, so that $\tau \circ h$ is multiplication by N.

In accordance with the Function Principle, we predict that these three numbers will be seen to be broad concepts. Their breadth should be less than the breadth of equivalence, retraction and cross-section because the concepts are valid only for categories of abelian groups and homomorphisms. But exponent, cross-section degree and retraction degree can be pulled back to other categories via any functor from that category to the category of abelian groups. So these integers potentially can play a role in many interesting categories. In fact for the category of topological spaces and continuous maps we can say that any continuous map $f: X \to Y$ has exponent N or cross-section degree N or retraction degree N if the induced homomorphism $f_*: H_*(X) \to H_*(Y)$ on integral homology has exponent N or cross-section degree N or retraction degree N respectively.

As evidence of the breadth of these concepts we point out that for integral homology, cross-section transfers already play an important role for fibre bundles. There are natural transfers associated with many of the important classical invariants such as the Euler characteristic and the index of fixed points and the index of vector fields, [B-G], [G1] and the Lefschetz number and coincidence number and most recently the intersection number, [G-O]. And a predicted surprise relationship occurs in the case of cross-section degree for a map between two spaces. In the case that the two spaces are closed oriented manifolds of the same dimension, the cross-section degree is precisely the absolute value of the classical Brouwer degree. The retraction degree also is the Brouwer degree for closed manifolds if we use cohomology as our functor instead of homology, [G1].

The most common activity in Mathematics is solving equations. There is a natural way to frame an equation in terms of functions. In an equation we have an expression on the left set equal to an expression on the right and we want to find the value of the variables for which the two expressions equal. We can think of the expressions as being two function f and g from X to Y and we want to find the elements x of X such that $f(x) = g(x)$. The solutions are called *coincidences*. Coincidence makes sense in any category and so we would expect the elements of any existence or uniqueness theorem about coincidences to be very broad indeed. But we do not predict the existence of such a theorem. Nevertheless in Topology there is such a theorem. It is restricted essentially to maps between closed oriented manifolds of the same dimension. It asserts that locally defined coincidence indices add up to a globally defined coincidence number which is given by the action of f and g on the homology of X. In fact this coincidence number is the alternating sum of traces of the composition of the umkehr map $f_!$, which is defined using Poincare Duality, and g_*, the homomorphism induced by g. We predict, at least in Topology and Geometry, more frequent appearances of both the coincidence number and also the local coincidence index and they should relate with other concepts.

If we consider self maps of objects, a special coincidence is the fixed point $f(x) = x$. From the point of view of equations in some algebraic setting, the coincidence problem can be converted into a fixed point problem, so we do not lose any generality in those settings by considering fixed points. In any event the fixed point problem makes sense for any category. Now the relevant theorem in Topology is the Lefschetz fixed point theorem. In contrast to the coincidence theorem, the Lefschetz theorem holds essentially for the wider class of compact spaces. Similar to the coincidence theorem, the Lefschetz theorem has locally defined fixed point indices which add up to a globally defined Lefschetz number. This Lefschetz number is the alternating sum of traces of f_*, the homomorphism induced by f on homology. This magnificent theorem is easier to apply than the coincidence theorem and so the Lefschetz number and fixed point index are met more frequently in various situations than the coincidence number and coincidence indices.

In other fields fixed points lead to very broad concepts and theorems. A linear operator gives rise to a map on the one dimensional subspaces. The fixed subspaces are generated by *eigenvectors*. Eigenvectors and their associated eigenvalues play an important role in Mathematics and Physics and are to be found in the most surprising places.

Consider the category of C^∞ functions on the Real Line. The derivative is a function from this category to itself taking any function f into f'. The derivative practically defines the subject of calculus. The fixed points of the derivative are multiples of e^x. Thus we would predict that the number e appears very frequently in calculus and any field where calculus can be employed. Likewise consider the set of analytic functions of the Complex Numbers. Again we have the derivative and its fixed point are the multiples of e^z. Now it is possible to relate the function e^z defined on a complex plane with real valued functions by

$$e^{(a+ib)} = e^a(\cos(b) + i\sin(b)).$$

We call this equation de Moivre's formula. This formula contains an unbelievable amount of information. Just as our concept of space–time separation is supposed to break down near a black hole in Physics, so does our definition–theorem view of Mathematics break down when considering this formula. Is it a theorem or a definition? Is it defined by sin and cos or does it define those two functions?

Up to now the function principle predicted only that some concepts and objects will appear frequently in undisclosed relationships with important concepts throughout Mathematics. However the de Moivre equation gives us methods for discovering the precise forms of some of the relationships it predicts. For example, the natural question "When does e^z restrict to real valued functions?" leads to the "discovery" of π. From this we might predict that π will appear throughout calculus type Mathematics, but not with the frequency of e. Using the formula in a mechanical way we can take complex roots, prove trigonometric identities, etc.

There is yet another fixed point question to consider: What are the fixed points of the identity map? This question not only makes sense in every category; it is solved in every category! The invariants arising from this question should be even broader than those from the fixed point question. But they are very uninteresting. However, if we consider the fixed point question for functions which are equivalent to the identity under some suitable equivalence relation in a suitable category we may find very broad interesting things. A suitable situation involves the fixed points of maps homotopic to the identity in the topological category. For essentially compact spaces the *Euler characteristic* (also called the Euler–Poincare number) is an invariant of a space whose nonvanishing results in the existence of a fixed point. This Euler characteristic is the most remarkable of all mathematical invariants. It can be defined in terms simple enough to be understood by a school boy, and yet it appears in many of the star theorems of Topology and Geometry. A restriction of the concept of the Lefschetz number, its occurrence far exceeds that of its "parent" concept. First mentioned by Descartes, then used by Euler to study regular polyhedra, the Euler characteristic slowly proved its importance. Bonnet showed in the 1840's that the total curvature of a closed surface equaled a constant times the Euler characteristic of the two dimensional sphere. Poincare gave it its topological invariance by showing it was the alternating sum of Betti numbers. In the 1920's Lefschetz showed that it determined the existence of fixed points of maps homotopic to the identity, thus explaining, according to the Function Principle, its remarkable history up to then and predicting the astounding frequency of its subsequent appearances in Mathematics.

The Euler characteristic is equal to the sum of the local fixed point indices of the map homotopic to the identity. We would predict frequent appearances of the local index. Now on a smooth manifold we consider vector fields and regard them as representing infinitesimally close maps to the identity. Then the local fixed point index is the local index of the vector field.

These considerations lead us to the prediction that a certain equation due to Marston Morse, [M], will play a very active role in Mathematics, and by extension Physics. This equation, which we call the Law of Vector Fields was discovered in 1929 and has not played a role at all commensurate with our prediction up until now.

We describe the Law of Vector fields. Let M be a compact manifold with boundary. Let V be a vector field on M with no zeros on the boundary. Then consider the open set of the boundary of M where V is pointing inward. Let $\partial_- V$ denote the vector field defined on this open set on the boundary which is given by projecting V tangent to the boundary. The Euler characteristic of M is denoted by $\chi(M)$, and $Ind(V)$ denotes the index of the vector field. Then the Law of Vector Fields is

$$Ind(V) + Ind(\partial_- V) = \chi(M)$$

We propose two methods of applying the law of vector fields to get new results and we report on their successes. These successes and the close bond between Physics and Mathematics encourage us to predict that the Law of Vector Fields and its attendant concepts must play a vital role in Physics.

3. The Law of Vector Fields

Just as de Moivre's formula gives us mechanical methods which yields precise relationships among broad concepts, we predict that the Law of Vector Fields will give mechanical methods which will yield precise relationships among broad concepts.

One method is:

1. Choose an interesting vector field V and manifold M.
2. Adjust the vector field if need be to eliminate zeros on the boundary.
3. Identify the global and local Ind V.
4. Identify the global and local index Ind($\partial_- V$).
5. Substitute 3 and 4 into the Law of Vector Fields.

We predict that this method will succeed because the Law of Vector Fields is morally the definition of index, so all features of the index must be derivable from that single equation. We measure success in the following descending order: 1. An important famous theorem generalized; 2. A new proof of an important famous theorem; 3. A new, interesting result. We put new proofs before new results because it may not be apparent at this time that the new result will famous or important.

In category 1 we already have the extrinsic Gauss-Bonnet theorem of differential Geometry [G3], the Brouwer fixed point theorem of Topology [G3], and Hadwiger's formulas of Integral Geometry, [G3], [Had], [Sa]. In category 2 we have the Jordan separation theorem, The Borsuk-Ulam theorem, the Poincare-Hopf index theorem of Topology; Rouche's theorem and the Gauss-Lucas theorem in complex variables; the fundamental theorem of algebra and the intermediate value theorem of elementary Mathematics; and the not so famous Gottlieb's theorem of group homology, [G2]. Of course we have more results in category 3, but it is not so easy to describe them with a few words. One snappy new result is the following: Consider any straight line and smooth surface of genus greater than 1 in three dimensional Euclidean space. Then the line must be contained in a plane which is tangent to the surface, ([G3], theorem 15).

We will discuss the Gauss-Bonnet theorem since that yields results in all three categories as well as having the longest history of all the results mentioned. One of the most well-known theorems from ancient times is the theorem that the sum of the angles of a triangle equals 180 degrees. Gauss showed for a triangle whose sides are geodesics on a surface M in three-space that the sum of the angles equals $\pi + \int_M KdM$, where K is the Gaussian curvature of the surface. Bonnet and Van Dyck pieced these triangles together to prove that for a closed surface M the total curvature $\int_M KdM$ equals $2\pi\chi(M)$. Hopf proved that $\int_M KdM$, where M is a closed hypersurface in odd dimensional Euclidean space and K is the product of the principal curvatures must equal the degree of the Gauss map $\hat{N} : M^{2n} \to S^{2n}$ times the volume of the unit sphere. Then he proved $2deg(\hat{N}) = \chi(M^{2n})$. (Morris Hirsch in [Hi] gives credit to Kronicker and Van Dyck for Hopf's result). For a history of the Gauss-Bonnet theorem see [Gr], pp. 89-72 or [Sp], p. 385. Or see [G4] for a history close to the point of view given here.

Let $f : M \to R^n$ be a smooth map from a compact Riemannian manifold of dimension n to n-dimensional Euclidean space so that f near the boundary ∂M is an immersion. The index of the gradient of $x \circ f : M \to R$, where x is the projection of R^n onto the x-axis, is equal to the difference between the Euler Characteristic and the degree of the Gauss map. Thus

$$Ind(grad(x \circ f)) = \chi(M) - deg\hat{N}.$$

This equation leads to an immediate proof of the Gauss-Bonnet Theorem, since for odd dimensional M and any vector field W, the index satisfies $Ind(-W) = -Ind(W)$. Thus the left side of the equation reverses sign while the right side of the equation remains the same. Thus $\chi(M)$ equals the degree of the Gauss-map, which is the total curvature over the volume of the standard $n-1$ sphere. Now $2\chi(M) = \chi(\partial M)$, so we get Hopf's version of the Gauss-Bonnet theorem.

Note as a by-product we also get $Ind(grad(x \circ f)) = 0$ which is a new result thus falling into category 3. Another consequence of the generalized Gauss-Bonnet theorem follows when we assume the map f is an immersion. In this case the gradient of $x \circ f$ has no zeros, so its index is zero so the right hand side in zero and so again $\chi(M) = deg\hat{N}$. This is Haefliger's theorem [Hae], a category 2 result. Please note in addition that the Law of Vector Fields applied to odd dimensional closed manifolds, combined with the category 2 result $Ind(-W) = -Ind(W)$, implies that the Euler characteristic of such manifolds is zero, (category 2). So the Gauss-Bonnet theorem and this result have the same proof in some strong sense. Also our prediction of the non-existence of magnetic monopoles follows from the same result!

Just as the Gauss-Bonnet theorem followed from considering pullback vector fields, the Brouwer fixed point theorem is generalized by considering the following vector field. Suppose M is an n-dimensional body in R^n and suppose that $f : M \to R^n$ is a continuous map. Then let the vector field V_f on M be defined by drawing a vector from m to the point $f(m)$ in R^n. If the map f satisfies the transversal property, that is the line between m on the boundary of M and $f(m)$ is never tangent to ∂M, than f has a fixed point if $\chi(M)$ is odd (category 1). This last sentence is an enormous generalization of the Brouwer fixed point theorem, yet it remains a small example of what can be proved from applying the Law of Vector Fields to V_f. In fact the Law of Vector Fields applied to V_f is the proper generalization of the Brouwer fixed point theorem.

A second method of producing Mathematics from the Law of Vector Fields involves making precise the statement that the Law defines the index of vector fields, [G–S]. In this method we learn from the Law. The Law teaches us that there is a generalization of homotopy which is very useful. This generalization, which we call *otopy*, not only allows the vector field to change under time, but also its domain of definition changes under time. An otopy is what $\partial_- V$ undergoes when V is undergoing a homotopy. A *proper otopy* is an otopy which has a compact set of zeros. The proper otopy classes of vector fields on a connected manifold is in one to one correspondence with the integers via the map which takes a vector field to its index. This leads to the fact that homotopy classes of vector fields on a manifold with a connected boundary where no zeros appear on the boundary are in one to one correspondence with the integers. This is not true if the boundary is disconnected.

We find that we do not need to assume that vector fields are continuous. We can define the index for vector fields which have discontinuities and which are not defined everywhere. We need only assume that the set of "defects" is compact and never appears on the boundary or frontier of the sets for which the vector fields are defined. We then can define an index for any compact connected component of defects (subject only to the mild condition that the component is open in the subspace of defects). Thus under an otopy, it is as if the defects change shape with time and collide with other defects, and all the while each defect has an integer associated with it. This integer is preserved under collisions. That is, the sum of the indices going into a collision equals the sum of the indices coming out of a collision, provided no component "radiates out to infinity", i.e. loses its compactness.

This picture is very suggestive of the way charged particles are supposed to interact. Using the Law of Vector Fields as a guide we have defined an index which satisfies a conservation law under collisions. The main ideas behind the construction involve dimension, continuity, and the concept of pointing inside. We suggest that those ideas might lie behind all the conservation laws of collisions in Physics.

4. Properties of Index

The Law of Vector Fields is the following: Let M be a compact smooth manifold and let V be a vector field on M so that $V(m) \neq \vec{0}$ for all m on the boundary ∂M of M. Then ∂M contains an open set $\partial_- M$ which consists of all $m \in \partial M$ so that $V(m)$ points inside. We define a vector field, denoted $\partial_- V$ on $\partial_- M$, so that for every $m \in \partial_- M$ we have $\partial_- V(m) =$ Projection of $V(m)$ tangent to $\partial_- M$. Under these conditions we have

$$Ind\ V + Ind\ \partial_- V = \chi(M) \tag{1}$$

where $Ind\ V$ is the index of the vector field and $\chi(M)$ is the Euler characteristic of M. ([M], [G–O], [P]).

The Law of Vector Fields can be used to define the index of vector fields, so the whole of index theory follows from (1). The definition of index is not difficult, but proving it is well-defined is a little involved, [G–S]. The definition proceeds as follows:

a) The index of an empty vector field is zero.
b) If M is a finite set of points and V is defined on all of the M (the vectors are necessarily zero), then $Ind(V) =$ number of points in M.
c) If V is a *proper* vector field on a compact M, by which we mean V has no defects on ∂M, then we set

$$Ind\ V = \chi(M) - Ind(\partial_- V).$$

d) If V is defined on the closure of an open subset U of a smooth manifold M so that the set of defects D is compact and $D \subset U$, then we say V is a *proper vector* field. The index $Ind\ V$ is defined to be $Ind(V|M)$ where M is any compact manifold such that $D \subset M \subset U$.
e) If C is a connected component of D and C is compact and open in D define $Ind_C(V)$ to be the index of V restricted to an open set containing C and no other defects of V.

A key idea in proving this definition is well-defined is a generalization of the concept of homotopy which we call otopy. An *otopy* is what $\partial_- V$ undergoes when V

undergoes a homotopy. The formal definition is as follows: An *otopy* is a vector field V defined on the closure of an open set $T \subset M \times I$ so that $V(m,t)$ is tangent to the slice $M \times t$. The otopy is *proper* if the set of defects D of V is compact and contained in T. The restriction of V to $M \times 0$ and $M \times 1$ are said to be properly otopic vector fields. Proper otopy is an equivalence relation.

The following properties hold for the index:

(2) Let M be a connected manifold. The proper otopy classes of proper vector fields on M are in one to one correspondence via the index to the integers. If M is a compact manifold with a connected boundary, then a vector field V is properly homotopic to W if and only if $Ind\ V = Ind\ W$.

(3) $Ind(V|A \cup B) = Ind(V|A) + Ind(V|B) - Ind(V|A \cap B)$

(4) $Ind(V \times W) = Ind(V) \cdot Ind(W)$

(5) $Ind(-V) = (-1)^{\dim M} Ind(V)$

(6) If V has no zeros, then $Ind(V) = 0$

(7) $Ind(V) = \sum_C Ind_C(V)$ for all compact connected components C, assuming D is the union of a finite number of compact connected components.

For certain vector fields the index is equal to classical invariants. Suppose $f : R^n \to R^n$. Let M be a compact n submanifold. Define V^f by $V^f(m) = f(m)$. If $f : \partial M \to R^n - \vec{0}$, then

(8) $Ind\ V^f = \deg f$.

Suppose $f : U \to R^n$ where U is an open set of R^n. Let $V_f(m) = \overrightarrow{\vec{m} - f(m)}$. Then

(9) $Ind\ V_f =$ fixed point index of f on U.

Suppose $f : M \to N$ is a smooth map between two Riemannian manifolds. Let V be a vector field on N. Let f^*V be the pullback of V on M. We define the pullback by

$$\langle f^*V(m), \vec{v}_m \rangle = \langle V(f(m)), f_*(\vec{v}_m) \rangle.$$

Note that for $f : M \to R$ and $V = \frac{d}{dt}$, we have $f^*V =$ gradient f.

Now suppose that $f : M^n \to R^n$ where M^n is compact and V is a vector field on R^n so that f has no singular points near ∂M and V has no zeros on $f(\partial M)$. Then if $n > 1$.

(10) $Ind\ f^{*}V = \sum w_i v_i + (\chi(M) - \deg \hat{N})$

where $\hat{N} : \partial M \to S^{n-1}$ is the Gauss map defined by the immersion of ∂M if R^n under f, and $v_i = Ind_{c_i}(V)$ for the i^{th} zero of V and w_i is the winding number of the i^{th} zero with respect to $f : \partial M \to R^n$. The winding number is calculated by sending a ray out from the i^{th} zero and noting where it hits the immersed $n-1$ manifold ∂M. At each point of intersection the ray is either passing inward or outward relative to the outward pointing normal N. Add up these point assigning $+1$ if the ray is going from inside to outside, and -1 if the ray goes from outside to inside. This is the generalized Gauss-Bonnet theorem.

Using the property that the Euler–Poincare number is an invariant of homotopy type and the above properties of the index, the following useful properties of the Euler–Poincare number can easily be proved.

1. $\chi(M) = 1$ where M is contractible.
2. $\chi(M_1 \cup M_2) = \chi(M_1) + \chi(M_2) - \chi(M_1 \cap M_2)$ where M_1 and M_2 and $M_1 \cap M_2$ are submanifolds of $M_1 \cup M_2$.
3. $\chi(M_1 \times M_2) = \chi(M_1) \times \chi(M_2)$.
4. $\chi(M) = 0$ when M is a closed odd dimensional manifold.
5. $\chi(\partial M) = 2\chi(M)$ if M is an even dimensional manifold.

5. Vertical Vector Fields

Let $F \to E \xrightarrow{p} B$ be a fibre bundle whose fibre F is a smooth manifold and whose structure group is the group of diffeomorphisms of F. Then we have a vector bundle α over B of vectors tangent to the fibres. That is $\alpha|F$ = tangent bundle of F. A vertical vector field V on E is an assignment to each point e of E a vector in α at e. V might be empty or it might be defined on part of E. A more precise way to express this is that $V : S \to \alpha$ is a cross-section from some subset S of B into α.

If $B = I$, the unit interval, then V is called an otopy. If V is an otopy which is continuous and defined over all of E, then V is called a homotopy.

A vertical vector field V is *proper* if $D \cap p^{-1}(C)$ is compact for all compact subsets C of B where D is the set of defects of V.

A vertical vector field V is *proper with respect to an open set* $U \subset E$ if $(D \cap U) \cap p^{-1}(C)$ is compact for all compact C in B and if V can be extended continuously over the frontier $\overline{U} - U$ so that there are no zeros on $\overline{U} - U$.

If $F \to E \xrightarrow{p} B$ is a fibre bundle so that F is compact and has boundary $\dot{F}$, we say a vertical vector field V is *proper with respect to the boundary* if $D \cap \dot{E} = \emptyset$ where $\dot{E} \subset E$ is the set of points in E on the boundary of some fibre.

Note that V proper with respect to the boundary implies that V is proper with respect to the open set $E - \dot{E}$.

The above definitions restrict to the concepts of proper homotopy and proper otopy when $B = I$.

If W is a vertical vector field, and if V is the restriction of W to a fixed fibre, then we say the defects of V *interact* via W if they are contained in a connected set of defects of W. An important class of questions is the following. If $F \to E \xrightarrow{p} B$ is a fibre bundle and V is a vector field on a fibre F, is it possible to extend V to a vertical vector field W so that certain defects of V do not interact, or so that the defects of W satisfy some condition such as they are compact?

The extension of vector fields with nonzero indices puts strong conditions on the homology of the fibre bundle as the following results from [B–G] show:

Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a smooth fibre bundle with F a compact manifold with boundary ∂F and B a finite complex. Let V be a proper vertical vector field defined on an open set of E. We assume that V has no zeros on $\dot{E}$. We will call such vector fields *vertical* vector fields for short in the next theorems.

In [B–G], we defined an S-map $\tau_V : B^+ \to E^+$ associated with V. This transfer τ_V has the usual properties:

a) If V is homotopic to a vertical vector field V' by a homotopy of vertical vector fields, so in particular no zeros appear on $\dot{E}$, then $\tau_{V'}$ is homotopic to τ_V.

b) $\tau_V^*(p^*\alpha \cup \beta) = \alpha \cup \tau_V^*(\beta)$ for cohomology theories h^* with cup products.

c) For ordinary homology or cohomology, $p_* \circ \tau_{V_*}$ and $\tau_V^* \circ p^*$ is multiplication by the index of V restricted to a fibre F, denoted $Ind(V|F)$.

$$p_* \circ \tau_{V_*} = Ind(V|F). \tag{11}$$

Also in [B–G] the following theorem is shown. Given fibre bundle $F \xrightarrow{i} E \xrightarrow{p} B$ where V is a vertical vector field, that

$$0 = Ind(V|F)\omega_* : \{X, \Omega B\} \to \{X, F\}$$

is trivial. Here we assume that X is a finite complex, $\Omega B \xrightarrow{\omega} F$ is the transgression map induced by the fibre bundle, $\{X, Y\}$ denotes the group of stable homotopy classes from X to Y.

It follows that

$$0 = Ind(V|F)\omega_* : H_*(\Omega B) \to H_*(F). \tag{12}$$

6. The Nonexistence of Magnetic Monopoles

We have the following picture immerging out of the previous sections. A vector field has a set of connected components of defects. Now under a homotopy these components move around and collide with one another. There is a conservation law which says that the sum of the indices of the components going into a collision is equal to the sum of the indices of the components at the collision is equal to the sum of the indices after the collision, if during the homotopy all the components remain compact and there are only a finite number of them. Thus the index remains conserved unless some component "radiates out to infinity." This suggests particles bearing charges could be modeled as defects of vector fields.

The fact that charge–like conservation follows from a simple topological construct, which depends only on continuity and dimension and pointing inside, suggests that the topological concept of index has physical content.

There is another compelling reason to consider the index as a physical quantity. It is an invariant of General Relativity. This is made precise in the following theorem.

Theorem *Suppose V is a space–like vector field in a space–time S. Suppose M and N are two time–like slices of S which can be smoothly deformed into each other. Suppose D, the set of defects of V, is compact in the region of S where the deformation takes place. Then the index of V projected onto M is equal to the index of V projected onto N.*

This theorem is true since we can set up a proper otopy between the two projected vector fields given the hypotheses of the theorem. This means for any space–like vector field, the index is invariant under any choice of space–like slices. Thus it is mathematically true that the index is an invariant of general relativity, just like proper time, unless there is topological radiation or there is a singularity or strange topology between the two slices so that there is no deformation possible.

Now we will give an argument that Magnetic Monopoles do not exist using Classical Field Theory. See [M–T–W], [P], [T], [F] for the relevant formalism.

Let F be an electromagnetic 2-form on space-time S. Let $\hat{F}$ denote the associated linear transformation on the tangent space of S. Let u denote a time-like unit vector field. Then the electric vector field associated to u is a space-like vector field given by $\overrightarrow{E} = \hat{F}u$. Now consider the 2-form $*F$. Here the $*$ denotes the Hodge dual which depends on the choice of orientation made on S. Now the magnetic field relative to u is given by $-\overrightarrow{B} = (\hat{F^*})u$. Note that $\overrightarrow{B}$ reverses direction if the orientation is changed. The Index Principle asserts that either the index of $-\overrightarrow{B} = (\hat{F^*})u$ is zero, or the vector field is not defined. Now a magnetic monopole will have either index $+1$ or -1. Hence the monopole cannot exist.

We make three remarks on the above argument. The first deals with the question: Why doesn't $\overrightarrow{E}$ reverse sign if we change our conventions of positive charge to negative charge? We say that the vector field u represents a swarm of test charges. If the sign of the test charge changes we still expect that the test charge will accelerate in the same direction because the signs of every charge is reversed. If we insisted on keeping the sign of the test charge the same and changed the signs on all the real charges it would be as if we changed the experiment, so we should not expect that the index of $\vec{E}$ would remain the same.

The second remark deals with the orientation of space-time. We are assuming that changing the orientation of space-time does not intail a change in some experiment used to define the electric and magnetic vectors. If it did, if for example some particle interacts with F by means of an "intrinsic choice of orientation", then the Index Principle would no longer apply. So it is worthwhile to think a bit about changes in orientation.

The violation of parity by the weak force is illustrated by describing an experiment which cannot take place if reflected in a mirror. Sudbury does it in describing the beta decay of Cobalt 60 on page 273 of [Su]. and Feynman does it on page 17–11 in volume III of [F–L–S] where he describes the Λ^0 decay. In both cases the experiment would be possible if one changed the handedness conventions in the mirror. That is, assume the reflection carries the change from the right hand rule to the left hand rule. In that case, the axial vectors spin and angular momentum would be reflected the correct way. Feynman describes the mirror reflection very carefully in volume I , chapter 35, section 5 of [F–L–S]. There is no overriding reason why he chooses not to change the handedness in the reflected world.

In Quantum Mechanics the transformation P is given by reflecting the coordinate system. Since Quantum Mechanics concerns itself with formal manipulation of coordinate systems, there is no question about reversing the handedness. It does not seem that changing orientation fits into Quantum Mechanics. But Classical Electrodynamics is unequivocal that the handedness be changed. The change of coordinates of Quantum Mechanics is replaced by the more intrinsic reflection diffeomorphism R. Now R is an isometry of the Lorentzian metric which reverses orientation. Since every measurement must be the same, if we reverse the orientation we can imagine R as an orientation preserving isometry onto the target space–time and every statement involving electromagnetism on the source space–time is in one to one correspondence with those of the target space–time. Then if we change the orientation back so that R is not orientation preserving, the possibility exists that there is some statement no longer true. Thus the arguments of Sudbury and of Feynman are not so convincing since they arbitrarily choose the orientation in the mirror's world.

The third remark deals with the fact that there may be local zeros of $\vec{B}$ with nonzero indices. This seems to violate the Index Principle. Also, $\vec{B}$ changes for changing observers. So we need to modify the index principle by making precise the concept of undefined index. We shall do that below.

The correct generalization of proper otopy, as we mentioned in section 5, is that of a *proper vertical vector field along the fibres of a fibre bundle*

$$M \to E \to B$$

where M is a smooth manifold and V is a vector field along the fibres and proper means that the defects of V are compact over any compact subset of B. Then V restricted to any fibre has the same index.

Now there is a very natural fibre bundle whose fibres are diffeomorphic to a standard space–like slice M of a space-time S. Let B denote the space of smooth space–like imbeddings of M in S. Let G denote the group of self diffeomorphisms M. Then G acts as a transformation group on B by composition on the right. Also G acts on $B \times M$ diagonally. The projection from $B \times M$ to B commutes with these actions and hence induces a map on the quotient spaces which gives rise to the fibre bundle

$$M \to (B \times M)/G \to B/G$$

We will call this fibre bundle the *space-time fibre bundle.*

Now there is a natural way to put vertical vector fields on the space-time fibre bundle given an antisymmetric 2-tensor F^* on S. Now for each M in S there is a normal unit future pointing time-like vector field u. Then $\hat{F}^* u$ is tangent to that slice since $\hat{F}^* u$ is orthogonal to u. See [P], [T]. We now think of $\hat{F}^* u$ as being tangent to the fibre M in the fibre bundle. Thus we have our vertical vector field.

Now $-\overrightarrow{B} = \hat{F}^* u$ where $*F$ is the Hodge dual of the electromagnetic tensor F. Here the $*$ denotes the Hodge dual which depends on the choice of orientation made on S. Now the vector field along the fibre on the space-like fibre bundle arising from $*F$ is the magnetic vector field $\overrightarrow{B}$ on each fibre, and it reverses direction if the orientation is changed. The Index Principle asserts that either the index on each fibre is zero, or the vector field is not proper.

Principle of Invariance of Index *The index of any "physical" proper vertical vector on the space-time fibre bundle is invariant under changes of orientation of space-time. Otherwise the vector field is not proper and it signals either radiation or unrealistic physical hypotheses.*

A careful discription of the electromagnetic forms and their associated operators can be found in [G5] and [G6]. A main result in [G5] is stated in terms of the degree of map, in the generality mentioned previously. Thus it is a new result predicted by the function principle.

7. The index as a quantum number

As we mentioned in section 5, a vector field changing under an otopy has zeros and defects moving under time and colliding and interacting with each other. Each defect has an index, and the sum of the indices coming into a collision is equal to the sum of the indices of the outgoing indices. If we could identify what particles correspond to which defects of the appropriate vector fields, we could derive as a consequence the conservation of the particle's indices.

Now vector fields along the fibre of a fibre bundle are a generalization of the concept of otopy. We can think of these vertical vector fields as representing a collection of possible otopies under certain circumstances. We say that a set of defects *potentially interact* on a fibre if they are contained in a connected component of defects in the total space.

Suppose we have a fibre bundle and a proper vector field defined on the total space so that every vector is tangent along the fibre. What kind of vector fields on a given fibre could be the restriction of the global vector field along the fibre? Equation (11) of section 5 states that there is a transfer on homology whose trace is the index of the vector field restricted to the fibre. Thus the homology of the fibre bundle restricts the possible indices of vector fields on the fibre. For example, consider the principal $SU(2)$ - bundle whose total space is the 7 dimensional sphere and whose base space is the 4 dimensional sphere. This is the Hopf fibration. The homology only permits transfers of trace 0. Hence the index of the restriction of any vector field along the fibre restricted to a fibre must be zero.

So if we could find a physical meaning for the Hopf bundle and if there is a vertical vector field on it which represents all the possible processes, then any particle with index not zero must be annihilated under some process. As an example, consider a classical electron in three space made compact, hence the three sphere. The defects of the electric field are the electron and and the point at infinity. The electron has index -1 and the zero at infinity has index $+1$. If there is a universal vertical vector field for the Hopf bundle which restricts to this Coulomb field in one fibre, then the electron must interact with the zero at infinity. Since all particles are said to have antiparticles which annihilate them, we have inductive evidence that just maybe, principle bundles over the four dimensional sphere with vertical vectorfields may describe particles.

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