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Some Mathematical Models for Wave Propagation

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ABSTRACT. This paper describes mathematical models related to wave propagation. The main goal is to motivate the reader towards the mathematical theory related to applications and research in this field. The starting point is very simple differential equations governing wave propagation problems. Different wave applications are described. More complicated and interesting wave models and wave phenomena follow. The presentation is done in a very descriptive form, thus avoiding technical details. Some current themes of research are discussed.

1 Introduction to modelling

It is very difficult to give a single precise definition for a wave (c.f. Whitham [31], pag. 3). Nevertheless one can give examples, explain what is the physics involved, express how the mathematical problem is formulated and possibly show how to solve it. If the analytical solution is hard to find, one can study the problem numerically.

Before we get started with waves, we define three levels of modelling: the conceptual (physical) model, the mathematical model and the numerical model.

These are fundamental topics in Applied Mathematics. In particular in Scientific Computing. In this paper we make a brief description about every one of these levels of modelling, but we will focus on the "middle one", that is, the mathematical model.

We can find several different types of waves in Nature. As a few examples we have surface water waves in the ocean or rivers [18, 31], acoustic waves [3, 8, 9] in the underground or underwater [5, 15], biological waves [19], chemical waves [19, 32], and even waves in football stadiums ("La Ola"). All these waves are, of course, of different nature. In section 3 some specific applications are presented. But note that they all have something in commom: some information is being *propagated* (passed on). For example in "La Ola" the information is "stand up, raise your arms, and sit down". By looking around one knows when this *information* reaches them and therefore we see a very well defined wave propagating at a very well defined speed. This is one the beautiful aspects of "La Ola". The information travels around a stadium in a pratically constant speed and form. As we shall define below, this characterizes a *travelling wave*. This does not happen for all waves. Note also that nothing is carried with the wave but *information*. You do not have to hold on to your distracted friend to prevent the wave from carrying him away!

Every hypothesis regarding the "real world version" of a given type of wave will be refered to as the *conceptual model* for the wave propagation problem. Many times we refer to this as the **physical model** even when we are talking of a biological or chemical problem. Commonly we abuse of the word *physics*.

This paper is structured as follows. We start in section 2 by defining what we mean by a mathematical model. In section 3 we present very simple equations governing wave propagation problems. This is specially motivated for readers without a background in Partial Differential Equations (PDEs). As we move along we try to present more complicated/interesting wave models and wave propagating phenomena. We do this in a very descriptive manner in order to achieve our main goal of motivating the reader towards *mathematical aspects of wave propagation*. In the course of doing so we present some current themes of research. We hope that our goals are achieved and that the references provided fill in the gaps of our presentation.

2 Mathematical modelling

Given a physical model we can make choices regarding the mathematical model to be used. In other words, we select the mathematical object to be analyzed. As an extrememly simple example consider a physical model which tells us that the velocity of an object is given by the function $f(y,t)$ where $y(t)$ expresses position in time. The mathematical model (i.e. mathematical object of interest) can be either the *ordinary differential equation* (ODE), together with its initial condition,

$$\frac{dy}{dt}(t) = f(y(t),t), \quad y(t_0) = y_0,$$

or the *integral equation*

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y(s),s)ds.$$

Note that in the integral equation formulation the initial condition has been automatically imposed, since the integral vanishes at $t = t_0$. This equation is obtained by integrating both sides of the ODE formulation. The same problem has been "mathematically translated" in two different (but equivalent) ways (i.e. two different mathematical objects). It is always very helpful to make the translation back to english. The back–and–forth translation tends to enhance our intuition regarding the model. For example the ODE model can be translated as *the velocity of a given particle (with its position represented by $y(t)$) varies according to the particle's position and according to time.* This means that if the particle reaches the same position at different times, its velocity can be different. This is clearly expressed (mathematically) by $f(y(t),t)$. Note that the translation of the integral equation would be different. Instead of *velocity* we have that the *position, in time, of the particle is updated by adding to its initial position the sum of all instantaneous velocities, up to the time of interest.* This is what the integral term tells us, if we think of integration as summation.

We should point out that there are many numerical models that use the ODE formulation, by approximating the derivative dy/dt, using Taylor series methods such as Euler's method, Runge–Kutta, ... There are other numerical models that use the integral formulation and perform the numerical integration by approximating $f(y(s),s)$ by a polynomial in s (c.f. Trapezoidal rule, Adams–Bashforth, Adams–Moulton,...). Details can be found in Burden & Faires [7]. The interpretation of the numerical integration method is that, instead of instanteanous velocities, we use average velocities over small time intervals, and add up the average distances travelled over each small time interval.

In the next section we present some mathematical models for wave propagation and briefly describe some of their features and applications. Details of the corresponding physical models will be omitted for brevity, but can be found in the references given.

3 Wave models and their applications

We start with the simplest of all mathematical wave models:

$$\frac{\partial u}{\partial t} + C\frac{\partial u}{\partial x} = 0, \qquad u(x,0) = u_0(x).$$

The initial condition (at time $t = 0$) is the known function u_0. The function $u_0(x)$ represents the initial wave profile. The function $u(x,t)$ represents the wave, that is, the information being propagated. As mentioned earlier it can be water elevation or the concentration of a pollutant etc...

In order to make our presentation simpler and quicker, we will "pull" solutions out of nowhere and check that they indeed satisfy the problem of interest and/or have the properties we are seeking. The techniques for constructing solutions can be found in books containing partial differential equation (PDE) [4, 13, 14].

It is easy to verify that $u(x,t) = u_0(x - Ct)$ is the solution to the problem above. First, at $t = 0$, $u(x,0) = u_0(x - 0)$. Finally by the chain rule, it is easy to check that

$$\frac{\partial u_0}{\partial t} + C\frac{\partial u_0}{\partial x} = -Cu_0' + Cu_0' = 0.$$

The prime indicates the derivative of u_0 with respect to its argument:

$$u_0' = \frac{du_0}{d\xi}, \quad \text{where} \quad \xi(x,t) = x - Ct.$$

Take, for example, the Gaussian initial profile

$$u_0(x) = e^{-x^2},$$

represented in figure 1. The formula for the solution shows that we have a *travelling Gaussian wave* propagating with speed C. Note that at a later time $t = L/C$, the wave has travelled over a distance equal to L without changing its shape. Basically

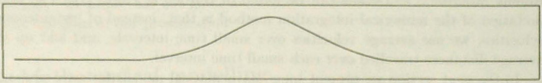

Figure 1. Initial wave profile: $u_0(x) = e^{-x^2}$

the initial profile has been translated by this amount $(u(x, L/C) = u_0(x - L))$.

This is why it is called a *travelling wave*.

We can write another "one–way" wave model, now for waves travelling to the left with speed C. The mathematical model is the wave equation

$$\frac{\partial u}{\partial t} - C\frac{\partial u}{\partial x} = 0, \qquad u(x,0) = u_0(x).$$

The *travelling wave* solution is of the form $u(x,t) = u_0(x + Ct)$.

Can these two "one–way" models be used to construct a "two–way" wave model? By two–way we mean a model that allows both right and leftgoing waves. The answer is yes and is given by the second order PDE

$$\frac{\partial^2 u}{\partial t^2} - C^2\frac{\partial^2 u}{\partial x^2} = 0, \quad \text{with} \quad u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

In this particular case we have that

$$\frac{\partial^2 u}{\partial t^2} - C^2\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} - C\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + C\frac{\partial}{\partial x}\right)u = 0.$$

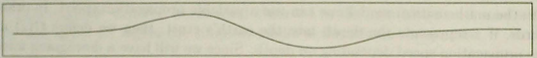

Figure 2. Initial water wave profile: $\eta_0(x) = -2xe^{-x^2}$

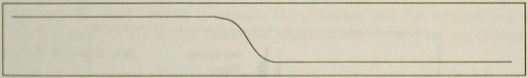

Figura 3. Initial pressure wave profile: $p(x,0) = tanh(-x/0.05)$.

For simplicity we have chosen that the second initial condition is zero. The solution is of the form $u(x,t) = 1/2(u_0(x - Ct) + u_0(x + Ct))$. The initial profile splits into half, with two identical travelling waves, one propagating to the left and another to the right, both without changing their shapes.

The models presented above are called *linear advection equations*. They are also known as *transport models*. The initial *information* u_0 is being transported with velocity C. The mathematical models presented above describe several (very simple) physical models. We mention a few.

Lets change notation so that instead of u we write $\eta(x,t)$ representing the water elevation about the undisturbed level $y = 0$ of the ocean or of a river. In other words, the free surface of the ocean is described, in time and space, by the expression $y = \eta(x,t)$. In its simplest version, the *evolution* of this free surface is governed by the second order advection equation given above. Let the initial condition be the wave profile given by $\eta_0(x)$. Then $y = \eta_0(x - Ct)$ represents a travelling water wave moving to the right with speed C. In the example given in figure 2 the wave has a depression ahead of it (where $\eta < 0$).

In another physical model the variable of interest is pressure and therefore we change notation again and use $p(x,t)$, where $p_0(x)$ decribes the initial pressure profile. Pressure waves are of interest in Meteorology, where we commonly hear about high/low pressure fronts and so on. An example of such a profile is given in figure 3. Of course the true Weather models [12, 30] are far more complex then the advection model presented here. Pressure waves are also of interest in Geophysics, and are called *acoustic waves*. In this kind of application a pressure pulse it sent into the crust of the earth (figure 4). The pressure pulse is similar to the Gaussian presented in Figure 1. Through the properties of the reflected waves geophysicists try to predict the existence of oil reservoirs in the subsurface. The reflected waves, indicated in the figure by $U(z,t)$, are generated because of the changes in the properties of the earth's crust. To incorporate this aspect into the mathematical model one can use a variable propagation speed. In other words, if z represents the depth into the earth's crust, then we define $C(z)$ as the propagation speed depending on depth. Since we will have a downgoing wave (the Gaussian pulse) and upgoing waves (the reflected waves) it is important to use a "two-way" wave equation

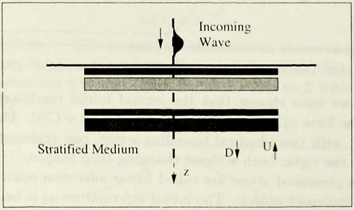

Figure 4. Acoutic pressure wave entering a layered earth crust.

(c.f. figure 4). The mathematical model, used in this kind of analysis, is presented in the next section.

To give another example, we mention one in *environmental modelling*. Suppose that a factory has released a cloud of pollutant in the air. The concentration of the pollutant is given by $c(x,t)$ and the initial configuration of the cloud is $c_0(x)$. The cloud is "mathematically transcribed" as the function presented in figure 5. In this simple model we have a one-dimensional cloud of pollutant and the pollutant concentration is (effectively) constant over the whole cloud. We say *effectively* because

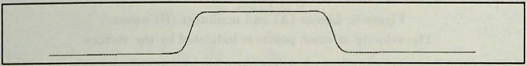

Figure 5. Initial pollutant wave profile
$c_0(x) = tanh((x - 1.0)/0.05) - tanh((x - 2.0)/0.05)$.

the change from zero to one is very fast. The three–dimensional analogue of this cloud would be a rectangular block of pollutant floating in the air. Note that in this one–dimensional model the cloud is at a fixed height. This height does not show up in the model. This is an indication tha we are considering constant winds over the vertical direction.

Now suppose that the worst wind observed in the area blows west–east with constant speed C. Again the simplest model is the advection equation

$$\frac{\partial c}{\partial t} + C\frac{\partial c}{\partial x} = 0, \qquad c(x,0) = c_0(x).$$

In this case the wind literally transports the pollutant and the solution is given by $c(x,t) = c_0(x - Ct)$. In this model the shape of the cloud does not change as it travels towards east.

Now let us contrast these linear models with a nonlinear wave model:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \qquad u(x,0) = u_0(x).$$

This important model is known as Burgers equation. It can be rewritten as

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0, \qquad u(x,0) = u_0(x).$$

The nonlinearity is clearly seen in the x-derivative term. The solution is given implicitly as $u(x,t) = u_0(x - u(x,t)t)$. It is easy to see that the initial condition has been satisfied.

To check that $u_0(x - ut)$ satisfies Burgers equation, use the chain rule.

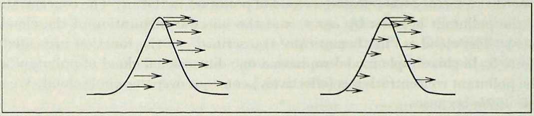

Figure 6. Linear (A) and nonlinear (B) waves.
The velocity at some points is indicated by the vectors.

This case is a little harder than for the linear equation. The nonlinear solution is constructed by the method of characteristics [13] where we follow the orbit $x(t;x_0)$ of a point located at $x(0) = x_0$ at time $t = 0$. It can be shown that indeed one should verify that $u_0(x - u(x(0),0)t)$ (or $u_0(x - u_0(x_0)t)$) satisfies Burgers equation. Observe that the propagation speed is given by the values of the initial condition and therefore it varies from point to point. Lets us use the water wave interpretation again, where the notation would be $\eta(x,t) = \eta_0(x - \eta t)$ representing the water elevation. In the linear case all points along the wave profile move with speed C, and therefore the wave does not change its shape (figure 6(A)). In the nonlinear case of Burgers equation the higher points of the wave profile move faster than the lower points (figure 6(B)). Therefore this nonlinear wave is not a *travelling wave*. Moreover, for the example considered, the wave will eventually break. When it breaks the derivative of our solution, at the breaking point, ceases to make sense in a classical way. A numerical example of a Burgers equation application to water waves is given in figure 7. We see a Gaussian pulse propagating to the right and becoming steeper and steeper. It eventually breaks forming a discontinuity known as a *hydraulic jump (bore)*. In other applications, such as gas dynamics, this discontinuity is called a *shock*. The oscillations along the wavecrest are due to the numerical method [21].

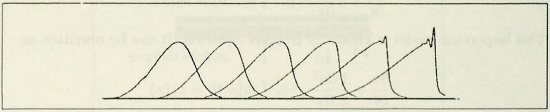

Figure 7. A nonlinear breaking wave propagation to the right.
The initial condition is given in the left, by the dotted-line Gaussian.

This is a typical behaviour of, for example, flood waves [25]. Two MATLAB programs are presented in [25] modelling this situation. The model is such that it rains for 7 days at a point 500km upstream from a city. The numerical experiment shows a flood wave developing and breaking into a bore. The Burgers equation model, and its related wave breaking behaviour, appears in other applications such traffic flow [14], shock tube models [31] in oil recovery and so on.

Some wave models might come in the form of a system of PDEs. For example the shallow water equations are

$$\frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x} = 0$$

$$\frac{\partial \eta}{\partial t} + h_0\frac{\partial u}{\partial x} = 0 \qquad (1)$$

where g stands for the acceleration due to gravity, h_0 is the constant depth of the region studied, $u(x,t)$ is the horizontal velocity of fluid particles and $\eta(x,t)$ is the wave elevation as before. The solution will be a combination of right and leftgoing waves of the form $f(x \pm (gh_0)^{1/2}t)$. The propagation speed depends on the depth. If we differentiate the first equation with respect to t, the second with respect to x, multiply it by g and subtract them, we get a second order wave equation in $u(x,t)$ as before, with $C^2 = gh_0$.

Now we might consider a wave in the presence of a *diffusive mechanism*. Consider the linear, one-dimensional, advection-diffusion equation

$$\frac{\partial c}{\partial t} + C\frac{\partial c}{\partial x} = \kappa\frac{\partial^2 c}{\partial x^2}, \quad c(x,0) = c_0(x).$$

The diffusion coefficient is κ. This type of equation is common in meterological and environmental flows. It is usually a prototypical equation for testing numerical methods. The prototypical three-dimensional model is usually written as

$$\frac{\partial c}{\partial t} + U\frac{\partial c}{\partial x} + V\frac{\partial c}{\partial y} + W\frac{\partial c}{\partial z} = \kappa\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right), \quad c(x,y,z,0) = c_0(x,y,z).$$

The wind velocity is given by the constant vector $[U,V,W]^T$, with $U^2+V^2+W^2 = C^2$, and the transported scalar quantity $c(x,y,z,t)$ can be, for example, the concentration of a pollutant, or humidity. In Nachbin & Tabak [25] we have used MATLAB to numerically solve the one-dimensional version of this equation. We consider a cloud of pollutant with concentration $c(x,t)$. The initial cloud shape is given by $c_0(x)$. The wind blows the cloud to the right with speed C and the concentration of the pollutant decays in time due to diffusion. The diffusion mechanism is similar to what happens when a drop of ink falls into the water. The ink "cloud" starts spreading while its concentration decays. In the model above the wind transports the pollutant cloud while it diffuses. Hence this *physical model* has two ingredients (mechanisms): transport and diffusion. The result of a computational experiment is presented in figure 8. Note that the pollutant concentration, in the cloud, decays as it is transported to the right by the wind. Note also that the cloud is spreading due to the diffusion mechanism. This is analogous to dropping some pollutant (e.g. oil) in a river, which flows downstream with speed C.

As the square oil patch flows downstream, it slowly spreads while its concentration goes down.

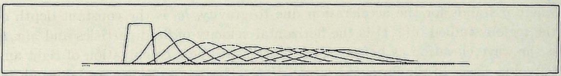

Figure 8. A rightgoing wave in the presence of diffusion.
The initial condition is given by the "square pulse".

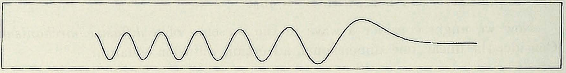

Figure 9. A linear dispersive wave given by the Airy function.

Another interesting mechanism related to wave propagation is called *dispersion*. This terminology is unfortunate since this type of dispersion has nothing to do with the term *dispersion of a pollutant*, which is connected to diffusion. This can be confusing at first. Hence it is very important to be aware in which context *dispersion* is being used. An example of a dispersive equation is the linear Korteweg–de Vries (KdV) equation

$$\frac{\partial \eta}{\partial t} + C\frac{\partial \eta}{\partial x} + \frac{\partial^3 \eta}{\partial x^3} = 0.$$

As the wave propagates to the right it starts to develop an oscillatory tail behind it. The solution resembles (and actually depends on) the Airy function shown in figure 9. A more detailed description is given in Drazin & Johnson [11] (c.f. exercise Q1.10 page 18) where the equation is rewritten with respect to a moving frame of speed C and turns out to be

$$\frac{\partial\eta}{\partial\tau} + \frac{\partial^3\eta}{\partial\chi^3} = 0.$$

We have substituted in the original PDE the composite function $u = u(\chi, \tau)$ with $\chi = x - Ct$ (the moving reference frame) and $\tau = t$. For initial profiles given by $\eta(\chi, 0) = f(\chi)$ the solution is of the form

$$\eta(\chi, \tau) = (3\tau)^{-1/3} \int_{-\infty}^{\infty} f(y) Ai\left(\frac{\chi - y}{(3\tau)^{1/3}}\right) dy.$$

The dispersive wave equation (linear KdV) and the diffusive wave equation, presented earlier, are not that different. Nevertheless the behaviour of their solutions is quite different. One simple way to understand the difference between the *dispersive wave model* and the *diffusive wave model* is by using *Fourier modes*, that is, sinusoidal waves of the form

$$u(x, t) = e^{i(kx - \omega t)} = e^{ik(x - c(k)t)}.$$

The time frequency is given by $\omega = 2\pi/T$, where T is the wave period. The spatial frequency is given by the wavenumber k. The *wavenumber* is defined as $k = 2\pi/\lambda$, where λ is the wavelength. Hence k tell us how many waves fit in an interval of length 2π. The wave's propagation speed is given by $c(k) = \omega/k$. If we substitute this Fourier mode in the "one-way" wave equation we get

$$-i\omega(k) + Cik = 0.$$

We easily conclude that the propagation speed is independent of the wavenumber:

$$c(k) = \frac{\omega(k)}{k} = C = \text{ constant}.$$

If we perform the same substitution for the dispersive wave equation we get

$$\omega(k) = Ck - k^3$$

and

$$c(k) = C - k^2.$$

The propagation speed depends on the spatial frequency of the wave. This is why an oscillatory tail develops behind the propagating wave. Loosely speaking it is like having slower components falling behind because they can not keep up with the leading wavefront (c.f. Figure 9). Note that each individual Fourier mode is a travelling wave. But any combination of 2 or more Fourier modes will *not* generate a travelling wave since each mode propagates at a different speed. They will be out of phase.

For a diffusive model, the (mathematical) story is completely different. In substituting a Fourier mode into the advection–diffusion equation we get that

$$\omega(k) = Ck - i\kappa k^2.$$

Substituting back for the Fourier mode we have

$$c(x,t) = e^{i(kx-\omega t)} = e^{-\kappa k^2 t}\, e^{ik(x-Ct)}.$$

Now we clearly see the difference between a *dispersive wave model* and *diffusive wave model.* For the diffusive model the propagation speed remains constant and independent of wavenumber. But the wave–amplitude decays exponentially in time. Again we do not have a travelling wave solution, but now for a different reason. The rate of decay of the amplitude is given by the diffusion κ. As we saw before, the interpretation is that as a pollution wave (i.e. cloud) is transported by the wind with speed C, its concentration decays exponentially. One hopes that the decay of the pollutant's concentration is fast enough so that when the cloud reaches an urban area the level of pollution is not dangerous to public health.

We now make comments on a very famous wave model, namely the nonlinear dispersive KdV equation:

$$\frac{\partial \eta}{\partial t} + 6\eta\frac{\partial \eta}{\partial x} + \frac{\partial^3 \eta}{\partial x^3} = 0.$$

This equation is a combination, to some extent, of Burgers equation and the dispersive wave model given above. To make this obvious it is sometimes convenient to rewrite the KdV equation in the form

$$\frac{\partial \eta}{\partial t} + \alpha\eta\frac{\partial \eta}{\partial x} + \beta\frac{\partial^3 \eta}{\partial x^3} = 0.$$

When α is zero we have only dispersion. When $\beta = 0$ we recover Burgers equation. In water waves the nonlinearity parameter α expresses the wave amplitude to depth ratio, while the dispersion parameter β the depth to wavelength ratio. Small values of β indicate that we either are in the regime of long waves or (equivalently) of shallow channels. Thus we have seen that the KdV equation can be written in different ways, depending on the choice of certain scales and frame of reference. The first version above (Drazin & Johnson [11], page 21, with $u = -\eta$) gives place to a simpler integration process. The solution is of the form

$$\eta(x,t) = \frac{C}{2}\,\mathrm{sech}^2\left\{\frac{\sqrt{C}}{2}\,(x - Ct)\right\}$$

and is depicted in figure 10. In this nonlinear dispersive model the *travelling wave*

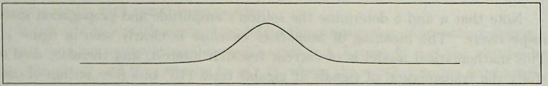

Figure 10: A solitary wave $\eta(x,0) = 1/2\ \mathrm{sech}^2\,(x/2)$.

has a propagation speed that depends on its amplitude. Note that nonlinearity is perfectly balanced by dispersion to give rise to a wave that travels without changing its shape. Nonlinearity alone, as seen earlier, would lead to wavebreaking. Dispersion alone would generate an oscillatory tail behind the wave. At the end, the "dispute" between these two mechanisms (one "pushing" the wave forward and the other "pulling" it from behind) leads to a coherent wave that retains its initial shape.

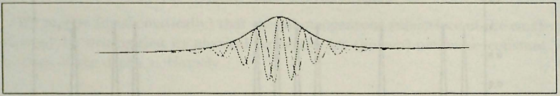

Figure 11: An optical soliton $\psi(x,0) = \mathrm{sech}(x)\,e^{i(2\pi x)}$. The real part is present in a dashed line and the imaginary part in a dotted line.The solid line presents the amplitude envelope $|\psi|$.

This travelling wave solution was first observed by J. Scott Russel, while riding his horse along the Edinburgh–Glasgow canal in 1834. He was impressed how a "large solitary elevation" travelled forward with great velocity and "preserving its original figure" (c.f. Drazin & Johnson [11] page 8). The Korteweg–de Vries equation was then derived in 1895. This solitary wave is also known as a *soliton*. Solitons are nonlinear travelling waves with special properties. They appear in different wave models. For example in nonlinear optics, modeled through the nonlinear Schrödinger (NLS) equation. Let the amplitude envelope of an optical wave be denoted by $\psi(x,t)$. The nonlinear Schrödinger equation is given by

$$i\,\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} + 2|\psi|^2\psi = 0,$$

and the soliton solution by ([16, 27])

$$\psi(x,t) = a \ \text{sech}(a(x-2bt)) \ \ e^{i(bx+(a^2-b^2)t)}.$$

Note that a and b determine the soliton's amplitude and propagation speed respectively. The meaning of *amplitude envelope* is clearly seen in figure 11. This mathematical model is of current research interest, and therefore used to study the transmission of signals at gigabit rates (10^9 bits (i.e. strings of ones and zeros) per second) in fiber optics communications [6, 16]. The transmission speed is determined by how closely the solitons can be spaced without risk of overlaping at the end of the optical fiber.

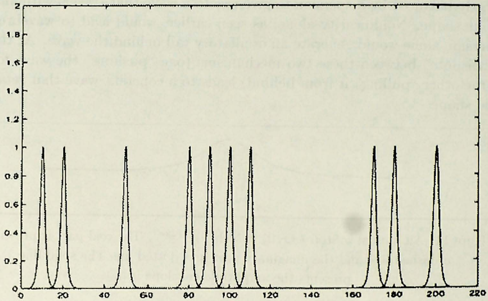

Figure 12: An optical signal propagating to the left.
The information is sent in binary form: 11001001111000001101, left to right.

Overlap implies that the information in the received signal has deteriorated. When this happens one can not distinguish between the *zeros and ones* represented by the solitons. An example of a short signal is given in Figure 12. The fact that nonlinearity and dispersion are perfectly balanced allows the solitons to travel for large distances without much of a deterioration of the signal. As pointed out by Bronski & Kutz [6] experts in the field have been able to achieve a remarkable 100-Gbit/s transmission over 6300km using 20 channels at 5Gbit/s per channel.

4 Wave models in inhomogeneous media

In the previous section we studied waves in homogeneous media. The medium's properties did not change. As a result we had a constant coefficient differential equation and also a constant propagation speed. In this section we consider media with variable properties.

We start with a mathematical model for the propagation of acoustic waves in a layered medium representing the earth's crust (c.f. Figure 4). Let the velocity be given by $u(z,t)$ and the pressure by $p(z,t)$, where the depth z points downwards. The acoustic problem is governed by the system

$$\rho(z)\frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0$$

$$\frac{1}{K(z)}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial z} = 0,$$

where $\rho(z)$ is the density and $K(z)$ the bulk modulus. The initial conditions are

$$u(z,0) = u_0(z) \quad \text{and} \quad p(z,0) = p_0(z).$$

We express (mathematically) that the inhomogeneous subsurface of the earth is layered, by writing that its physical properties are (known) piecewise-constant functions of the depth z. Namely

$$\rho(z) = \rho_n, \quad \text{for} \quad z_{n-1} < z < z_n,$$

$$K(z) = K_n, \quad \text{for} \quad z_{n-1} < z < z_n,$$

where $z_0 = 0$ and the n-th layer is of thickness $z_n - z_{n-1}$. The acoustic wave's propagation speed is given by $c(z) \equiv (K(z)/\rho(z))^{1/2}$. In the case of a homogeneous medium $\rho(z,t) \equiv \rho_0$ and $K(z,t) \equiv K_0$, both constants. It is easy to check that the *velocity and pressure travelling waves* $u(z,t) = f(z - (K_0/\rho_0)^{1/2}t) = p(z,t)/(\rho_0 K_0)^{1/2}$ are solutions in this particular case. Note that this system of differential equations is very similar to the one used for shallow water waves.

A very interesting phenomenon happens when the acoustic wave is pulse shaped and broad, compared to the length of the layers. Most of the waves presented in this paper are pulse shaped. For broad pulses different parts of the wave are feeling different layers of the subsurface and therefore travelling at different speeds. This problem is linear but very difficult to analyse mathematically. Several research papers have been published in recent years, addressing this problem

[1, 2, 10, 17, 29]. In some cases the theory involves the asymptotic analysis of stochastic differential equations. This analysis also has been carried out for water waves [20, 22, 23, 24]. Without getting into details, we will describe one interesting result.

Consider that, if needed, a layer can be subdivided into more layers in such a way that the wave's travel (or transit) time is the same over any layer. This means that the time spent by any wave-segment over any layer is constant. This is called a *Goupillaud medium*. Now consider that, due to uncertainties in the subsurface's properties, we model $\rho(z)$ and $K^{-1}(z)$ as a disordered function. This means that their values are random. For example, we can take them to be uniformly distributed in some interval. Hence for each layer we sample the value of ρ_n in the (pre-defined) interval $[\rho_0 - \delta_\rho, \rho_0 + \delta_\rho]$ and the value of K_n^{-1} in $[K_0^{-1} - \delta_K, K_0^{-1} + \delta_K]$. The study of the effect of fine scale layering on a propagating pulse was initiated in 1971 by O'Doherty and Anstey [29]. They gave a quantitative explanation of the *pulse shaping* in terms of the statistics of the reflection coefficients for a Goupillaud medium. *Pulse shaping* takes place as the wave interacts with the inhomogeneous medium and generates reflected waves. These reflected waves also get reflected. This process is called *multiple scattering*. The wave energy starts being spread over larger and larger regions due to the multiple reflections. The mathematical theory shows that the multiple scattering associated with the disordered microstruture, leads to the *apparent attenuation (or diffusion)* of the propagating pulse. Hence the leading pulse is "shaped", by the microstructure, in a diffusive-like manner. The terminology *apparent diffusion* is due to the fact that the mathematical model has no diffusion term.

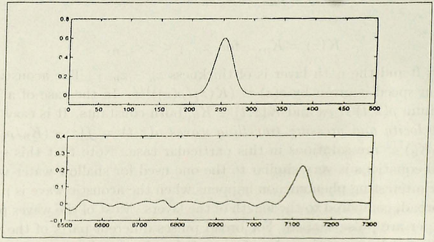

Figure 13: Top figure: initial Gaussian pulse. Bottom figure: pulse under apparent diffusion. Multiply scattered reflection is seen behind the wavefront. The pulse has

travelled approximately (7100-250) * $\Delta x = 6850 * 0.01 = 68.5$ units of length.

The acoustic wave model represents a conservative system of partial differential equations. One way of seeing that there is no diffusive term in the equations is by eliminating the velocity from the system of differential equations. This gives place to a variable coefficient wave equation for the pressure:

$$\frac{\partial^2 p}{\partial t^2} - \left[c^2(z)\rho(z)\right] \frac{\partial}{\partial z}\left(\frac{1}{\rho(z)} \frac{\partial p}{\partial z}\right) = 0.$$

Notice that for a homogeneous medium (ρ and K constant) we recover the second order wave equation as we know it. The surprising fact in the O'Doherty–Anstey problem is that the pulse diffuses about its moving center due to the *disordered multiple scattering* of the wave energy [1, 2, 10, 17, 26, 28]. O'Doherty and Anstey's motivation for studying pulse spreading was to explore whether the scattering associated with fine scale layering in the earth could explain the observed damping of seismic (acoustic) waves used in the oil–exploration industry. A similar phenomenon occurs for water waves, when the topography is disordered [26]. An example of this phenomenon is presented in Figure 13. Note, in the bottom figure, that the pulse's amplitude has decayed and that the pluse is much broader than the original one (top figure). This is the *apparent attenuation* geophysicists observed, connected to the *pulse shaping* described by the mathematical theory.

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