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Nielsen Fixed Point Theory

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1 Introduction.

Let X be a topological space and $f : X \to X$ a continuous map. Many interesting problems in mathematics are related to the existence of fixed points of certain functions, or even to the existence of periodic points, that is, fixed points of $f^r = f \circ ... \circ f$, the iteration of f r-times. Another important question is to find the minimal number of fixed points among all maps f', where f' is a deformation of the map f.

We will start by giving examples of problems which can be analyzed using the theory of fixed points.

Problem 1 - Let X be a compact differentiable surface, like for example the sphere S^2 or the torus T in R^3 or in general, let M be a compact differentiable manifold. One would like to know if M admits an everywhere-nonzero vector field.

One particular case of this problem is when the space is the sphere. The solution corresponds to know if one can globally comb in a continuous way a ball which has hair at every point. In general, suppose this problem has a positive answer and denote by $v(x), x \in M$, an everywhere-nonzero vector field. So for each $x \in M$. $v(x)$ is a nonzero vector in the tangent space of M at the point x. Under certain mild conditions on the vector fields, for each point $x_0 \in M$ there is a unique curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = x_0$, $\gamma'(t) = v(\gamma(t))$ for $t \in (-\varepsilon, \varepsilon)$ and γ is injective. In principle, this positive number ε depends on the point x_0. Nevertheless it can be shown (see [11] for more details about this matter) that there is an $\varepsilon > 0$ which is the same for all points $x_0 \in M$, once we have assumed that M is compact. Now we can define a map $f_t : M \to M$ as follows: given $x \in M$ let $f(x) = \gamma_x(\frac{\varepsilon}{2})$, where γ_x is the unique curve which satisfies $\gamma_x(0) = x$ *and* $\gamma_x'(t) = v(\gamma_x(t))$. Because $v(x)$ is everywhere-nonzero, it follows that f has no fixed point. Also, we observe that the map f_1 can be deformed into the identity map. In fact, let $f_t(x) = \gamma_x((1-t)(\frac{\varepsilon}{2}))$ for $t \in [0,1]$. For $t = 0$ we have $f_0(x) = f(x)$ and for $t = 1$ we have $f_1 = id =$ identity map. We conclude that the existence of an everywhere-nonzero vector field implies the existence of a map $f : M \to M$ which can be deformed into the identity and with no fixed point. Therefore, if every map $f : M \to M$ which can be deformed to the identity has a fixed point, then M does not admit an everywhere-nonzero vector field.

Problem 2 - Given a topological compact group G (this includes the compact Lie Group G), $g_0 \in G$ and a positive integer n, we would like to know if there exists $g_n \in G$ such that $g_n^n = g_0$. As a particular cas , given a matrix M and a positive integer n, one might be interested in knowing if there is a $n-th$ root of M. i.e., if there is a matrix N such that $N^m = M$, where M is either a complex matrix of $U(n)$ (the unitary matrices) or a real matrix of $SO(n)$ (the orthogonal matrices). Another example is the case where the group G is the product of n circles S^1, for an arbitrary multiplication on G, not only the one given by the usual multiplication on S^1 on each coordinate.

This problem can be rephrased as follows: let $\gamma_n, c : G \to G$ be the maps defined by $\gamma_n(x) = x^n$ and $c(g) = g_0$ the constant map. We would like to know if this pair (γ_n, c) has a coincidence i.e., if there exist $g_n \in G$ such that $\gamma_n(g_n) = c(g_n)$ or $g_n^n = g_0$. Hence, we are dealing with a coincidence problem, which can also be regarded as a fixed point problem, by taking $f_n(g) = g^{n+1}g_0^{-1}$.

Problem 3 - There is an important concept called *topological entropy* of a map $f : X \to X$ where f is continuous and X is a compact metric space. From [9] we have:

Definition: *Let (X, d) be a compact metric space and $T : X \to X$ continuous. A set $E \subset X$ is (n, ϵ) separated if for any $x, y \in E$ with $x \neq y$ there is a j, $0 \leq j < n$, such that $d(T^j(x), T^j(y)) > \epsilon$. Let $S_n(\epsilon)$ denote the largest cardinality* *of any (n, ϵ) separated set in X, and let*

$$S_\epsilon(T) = lim\ sup\ n^{-1} log S_n(\epsilon).$$

The topological entropy of T, $h(T)$, is given by the formula:

$$h(T) = lim_{\epsilon \to 0} S_\epsilon(T).$$

The problem is to find or estimate $h(T)$.

More details about this number can be found in [9] where one reads "the topological entropy essentially gives the asymptotic exponential growth rate of the number of orbits of T up to any accuracy and arbitrarily high period". We will not solve this problem here but by the end we will be able to understand a Theorem from N. V. Ivanov [3], which describes a lower bound for this number in terms of an invariant which arises from fixed point theory. We also compute the entropy of an explicit example.

In what follows we will present some ideas and results of the theory for fixed points, including the Nielsen Theory, and at the end we will illustrate how we solve or at least approach the problems mentioned above from the point of view of Nielsen Theory. Finally let me point out that for a more complete and advanced expository paper about Fixed Point theory see [2].

2 Examples on D^n and S^1

Given $f : X \to X$ when does it have a fixed point? If every map f has a fixed point, then we say that the space X has the fixed point property (denoted by fpp).

Definition: *The map $g : X \to X$ is a deformation of $f : X \to X$ if there exist $H : X \times I \to X$ such that $H(x, 0) = f$ and $H(x, 1) = g$. In this case we say that f is homotopic to g.*

Let D^n be the closed unit disk in the Euclidean space R^n. If $n = 1$ let us denote D^1 by $I = [-1, 1]$. The following interesting result catches our attention:

Theorem: *Let $f : I \to I$ be a continuous function. Then there exists $x \in I$ such that $f(x) = x$, i.e. f has a fixed point.*

The original proof of this result goes back to Bernard Bolzano 1817. The proof is quite simple nowadays and can be done as follows. Consider the map $g(x) = x - f(x)$. Either $f(1) = 1$, so we have a fixed point or $g(1) = 1 - f(1) > 0$. Similarly, either $f(-1) = -1$ or $g(-1) < 0$. By the intermediate value theorem in calculus we have that g vanishes in some point $x \in I$. So it follows that f has a fixed point.

It was a challenge to study the similar problem for $n \geq 2$.

In the beginning of this century Brouwer proved what is now called:

Brouwer Fixed Point Theorem: *Let D^n be the unit disk in the Euclidean n-space R^n and let $f : D^n \to D^n$ be any map. Then f has a fixed point.*

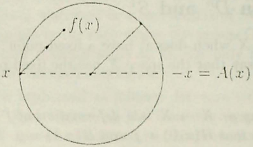
An elementary proof for the case $n = 2$ can be done using the intuitive idea of continuity, without being very rigorous.

Let $f : D^2 \to D^2$ be a continuous map. Suppose it does not have fixed points. Let the boundary of D^2 be denoted by ∂D^2, which is homeomorphic to the circle S^1. Consider a self map of the boundary of D^2, defined as follows:

$$\varphi : S^1 \to S^1, \quad \varphi(x) = \frac{f(x) - x}{||f(x) - x||}$$

Observe that $\varphi(x) \neq x$, for all x.

If $A : S^1 \to S^1$ denotes the antipodal map, i.e., $A(x) = -x$, we can deform φ to A because the angle between $\varphi(x)$ and $-x$ is always less than $\frac{\pi}{2}$. See figure below:



But $A : S^1 \to S^1$ is just rotation of degree π and hence can be deformed to the identity map.

Let $S^1(r)$ be the circle of radius $0 < r \leq 1$ inside of D^2. By continuity, the map $\varphi_r : S^1(r) \to S^1(r)$, defined in a similar fashion, can also be deformed to the identity for all $0 < r$. This implies that the map φ_r is surjective. The origin **0** is the limit of any sequence $\{x_i\}$, where $\{x_i\} \in S^1(r_i)$ and the values r_i converge to zero. Using the fact that all φ_r are surjective, it is not difficult to find two sequences $\{a_i\}, \{b_i\}$, $a_i, b_i \in S^1(r)$ so that the vectors $\varphi_{r_i}(a_i)$ and also $\varphi_{r_i}(b_i)$ have constant directions, but different ones. Both converge to the direction of $f(\vec{0}) - \vec{0} = f(\vec{0})$. This forces $f(\vec{0}) = \vec{0}$ which contradicts the initial hypothesis.

The arguments given above cannot be used for $n > 2$. Other ingredients should be introduced in order to solve the problem when $n > 2$.

Based in the Brouwer Fixed Point Theorem we have that any map $f : D^n \to D^n$ has a fixed point. This means that D^n has the fixed point property (FPP). Despite the fact that a given function f can have many fixed points, one observe that there is a map f' which is a deformation of f which has exactly one fixed point, namely the constant map at value $\vec{0}$, i.e. $f'(\vec{x}) = \vec{0}$ for all $\vec{x}$. This minimal number of fixed points among all maps f' which are a deformation of a given function f, is a meanfull number. This number will be more interesting in the next example.

The next space which is interesting to be analised is the circle. In fact, this example will bring a new feature that did not appear in the study of maps on the unit ball D^n.

Let $S^1 \subset \mathbb{C}$ be the unit circle considered as a subset of the complex numbers. For each integer $n \in \mathbb{Z}$ we have a map defined as follows

$$f_n(z) = z^n.$$

If $n = 1$ we have that $f_1 = id$, the identity map. So $Fix(id) = S^1$. Nevertheless, we can deform the map such that the deformed map is fixed point free, i.e. it has no fixed point. In order to do this, take the family of maps indexed by $t \in [0, 1]$ such that θ_t is the rotation of degree $t\frac{\pi}{2}$. So $\theta_0 = id$ and θ_i is the rotation of $\frac{\pi}{2}$, i.e. multiplication by the imaginary number i, which is fixed point free.

Now let us consider $n \neq 1$. We have

$$Fix(f_n) = \{z|z^n = z\} = \{z|z^{n-1} = 1\}.$$

So $Fix(f_n)$ is a finite set which contains $|n - 1|$ points. We can make a first attempt to deform f_n to f'_n fixed point free. In case this is not possible we can ask what is the minimal number of fixed points among all maps f'_n which are homotopic to f. In fact we can prove:

Theorem: *If a map $f : S^1 \to S^1$ is homotopic to f_n for some $n \neq 0$, then f has at least $|n - 1|$ points.*

Proof: Consider the diagram:

$$\begin{array}{ccccc} & & & & \mathbb{R} \\ & & & & \downarrow{\scriptstyle exp} \\ [0,1] & \xrightarrow{p} & S^1 & \overset{f_n}{\underset{f'_n}{\rightrightarrows}} & S^1 \end{array}$$

5

where $exp : \mathbb{R} \to S^1$ is given by $exp(t) = e^{2\pi it}$ *and* $p : [0,1] \to S^1$ is the restriction of the map exp.

Call $g_n = p \circ f_n$ and $g'_n = p \circ f'_n$. We call a lifting of a map $g : [0,1] \to S^1$, a map $\tilde{g} : [0,1] \to \mathbb{R}$ such that $g = exp \circ \tilde{g}$. It is well known that for given points $x_0 \in exp^{-1}(f_n(1))$, $x'_0 \in exp^{-1}(f'_n(1))$ there are liftings $\tilde{f_n}, \tilde{f'_n} : [0,1] \to \mathbb{R}$ of g_n, g'_n respectively, such that $\tilde{f_n}(0) = x_0$ and $\tilde{f'_n}(0) = x'_0$ (see [4]). It is simple to see that a point $e^{2\pi it}$ of S^1 is a fixed point of f_n if and only if $\tilde{f_n}(t) - t$ is an integer (the same for f'_n). From [4] we also have either for f_n or for any deformation f'_n of f_n, that $\tilde{f'_n}(1) - \tilde{f'_n}(0) = n$. This implies that the number of solutions of $\tilde{f'_n}(t) - t \in \mathbb{Z}$ is at least $|n-1|$.

Remark: First, let us observe that any map $f : S^1 \to S^1$ is indeed homotopic to some f_n, for exactly one n. Therefore, by the result above, any map f'_n which is deformation of f_n has the property that $\#Fix(f'_n) \geq |n-1|$. Since f_n has exactly $|n-1|$ fixed points, this is the minimal number of fixed points among all maps f'_n which can be deformed to f_n. In the next section we will define a number which will be very useful in computing this minimal number.

3 Nielsen Theory

In the early 1920's, J. Nielsen studied the fixed points of homeomorphisms of compact surfaces. Due to his work a theory was developed to study the fixed points of a finite complex and nowadays we refer to it as Nielsen Fixed Point Theory. We will, for the sake of simplicity, give a description of this theory in the case where the spaces are compact orientable surfaces. For references on the subject we mention [1] and [7, 8].

Let $f : S \to S$ be a continuous map on a surface S and $x \in Fix(f)$ an isolated fixed point. Since S is a surface, there is a small neighborhood around x homeomorphic to the unit closed disk D^2 of $\mathbb{R}^2$. Define $i(x, f)$, the index of x, as follows: by identifying the points of the circle S^1, the boundary of D^2, with the boundary of the neighborhood above, we have a map $\varphi : S^1 \to S$ given by

$\varphi(x) = \frac{f(x)-x}{||f(x)-x||}$. This map is homotopic to the map $z \mapsto z^n$ for some n, so it has degree n. Then we define the index of x, $i(x, f)$, to be n.

If $F \subset Fix(f)$ is an isolated set of fixed points (not just a single point), we can also define the index, $i(F, f) \in \mathbb{Z}$, which in the case of a finite set is just the sum of the indices of its points. See [5] for more details.

The notion of a local index defined above, is the first ingredient to define the notion of Nielsen number. The second one is an equivalence relation on the set of the fixed points.

Let $f : X \to X$ be a map. Two points $x_1, x_2 \in Fix(f)$ are called Nielsen equivalent if there exists a path $\lambda : [0, 1] \to X$ such that $\lambda(0) = x_1$, $\lambda(1) = x_2$ and λ can be deformed to $f(\lambda)$ relative to the end points $\{x_0, x_1\}$.

Definition: *The equivalent classes of $Fix(f)$ given by the above relation are called the Nielsen classes of f.*

Let $\{F_1, ..., F_r\}$ be the Nielsen classes of a function f. Each class $F_i \subset Fix(f)$ has an index $i(F_i, f) \in \mathbb{Z}$.

Definition: *We say that F_i is essential if $i(F_i, f) \neq 0$.*

Finally we can define the Nielsen number as follows:

Definition: *The Nielsen number of f, denoted by $N(f)$, is the number of essential Nielsen classes.*

This number has very nice properties. Let us denote by $MF[f]$ the minimal number of $Fix(f')$ where f' runs over all maps homotopic to f. Of course $MF[f_1] = MF[f_2]$ if f_1 can be deformed to f_2. Nielsen proved:

Theorem: *The Nielsen number $N(f)$ is a homotopy invariant, i.e. $N(f_1) = N(f_2)$ if f_1 is homotopic to f_2. Further, $N(f) \leq \mathrm{MF}[f]$, i.e. the Nielsen number is a lower bound for the minimal number of fixed points in the homotopy class.*

The hard and important problem of computing $MF[f]$ is a great challenge. The above Theorem shows that the Nielsen number is an important tool to estimate $MF[f]$. For maps $f : S \to S$, where S is a compact surface, the problem is still open in most of the cases.

On the other hand if the surface is the Torus, then the situation is much simpler and we will have a full understanding of this case. For this, we need to introduce another invariant and a celebrated result, the Lefschetz-Hopf fixed point theorem.

4 Lefschetz-Hopf fixed point theorem

Let X be a finite complex (i.e. a space built up from finite unions of points, triangles, thetahedrons,.... e.t.c.) and $f: X \to X$ a continuous map. We have the induced homomorphism in homology, $(f_*)_i : H_i(X) \to H_i(X)$ for each $i \geq 0$. Since X is a finite complex, these abelian groups vanish for $i > n$ for some integer n, and are finitely generated abelian groups. For a finitely generated abelian group A, let $F(A)$ be the quotient of A by it is torsion part, which is a well defined subgroup. $F(A)$ is a finitely generated torsion free abelian group and therefore isomorphic to a sum of $\mathbb{Z}'s$, i.e. $\mathbb{Z} \oplus ... \oplus \mathbb{Z}$. Further if $\varphi : A \to A$ is a homomorphism, then it induces a homomorphism $\overline{\varphi} : F(A) \to F(A)$.

For a given basis $B = \{e_1, ..., e_n\}$ of $F(A) \simeq \mathbb{Z} \oplus ... \oplus \mathbb{Z}$ we have the matrix of $\overline{\varphi}$ with respect to the basis B. This matrix M_B has a trace, which is the sum of the elements of the diagonal. We denote this number by $tr(\overline{\varphi})$ where this number is well defined, i.e., it depends only on the homomorphism of $\overline{u}$ and not on the chosen bases.

Now we can state the famous Lefschetz-Hopf fixed point theorem. First we define the Lefschetz-Hopf trace.

Definition: *The Lefschetz-Hopf trace of a map $f : X \to X$, where X is a finite simplicial complex, is given by $L(f) = \sum_{q=0}^{\infty} (-1)^q tr(\overline{f_{*q}})$, where $(\overline{f_{*q}} : F(H_q(X)) \to F(H_q(X))$ is the homomorphism induced by f.*

Lefschetz-Hopf Fixed Point Theorem: *Let X be a finite simplicial complex and $f : X \to X$ a continuous map. If $L(f) \neq 0$ then f has at least one fixed point.*

This was the first moment that homology theory was used in the study of fixed points. Let us point out that this result has the Brouwer Fixed Point Theorem as a consequence, since we have the following:

Corollary: *Let $f : X \to X$ be a continuous map, where X is a connected finite complex. If X has the property that $H_i(X, Z)$ is a finite torsion group for $i > 0$, then f has at least one fixed point.*

Proof: The homology $H_i(X, Z)$ modulo the torsion is the trivial group for $i > 0$, and $H_0(X, Z) \approx Z$. Therefore any map $f : X \to X$ induces $(f_*)_0 : H_0(X, Z) \to H_0(X, Z)$, the homomorphism $(f_*)_0 : \mathbb{Z} \to \mathbb{Z}$ which is the identity.

Therefore $L(f) = 1 \neq 0$ and by Lefschetz-Hopf theorem it follows that f has a fixed point and we have the result. In particular, the space D^n satisfies the hypothesis of the corollary and the Brouwer fixed point theorem follows.

As we can see the Lefschetz-Hopf Fixed Point Theorem opens many possibilities to explore fixed point problems. Just to give a sample, another space that satisfies the hypothesis of the above corollary is the quotient of an even dimensional sphere by identifying antipodal points, the even dimensional Projective Space.

5 Maps on the Torus

Now we analyze fixed points of maps $f : T \to T$. More details about this can be found in [6].

I) - The classification of maps $f : T \to T$ up to homotopy

One way to obtain a map $f : T \to T$ is as follows: Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that the matrix of F is an integral matrix M_F. Since the Torus can be regarded as the quotient of the plane $\mathbb{R}^2$ by the subgroup $\mathbb{Z} \times \mathbb{Z}$, the linear transformation F induces a map $\overline{F} : T \to T$. The description of the homotopy classes of maps in the Torus can be done in a simple way based on the fact that given any map $f : T \to T$, there is a linear transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$ as above, such that the induced map $\overline{F} : T \to T$ is homotopic to f. Also if two linear transformations $F_1, F_2 : \mathbb{R}^2 \to \mathbb{R}^2$ induces homotopic maps $\overline{F}_1, \overline{F}_2 : T \to T$ then $F_1 = F_2$. To see this, for each map $f : T \to T$ we have $(f_*)_1 : H_1(T) \to H_1(T)$. The group $H_1(T) \approx \mathbb{Z} \oplus \mathbb{Z}$, so $(f_*)_1$ is a homomorphism of $\mathbb{Z} \oplus \mathbb{Z}$. The set of homomorphisms, $\mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z})$ can be identified with $M_2(\mathbb{Z})$, i.e. the set of integral matrices 2×2. It turns out that the matrix of $(f_*)_1$ is precisely the matrix of the linear transformation F such that $\overline{F}$ belongs to the homotopy class of f. It is well known that if two maps f, g are homotopics, then $(f_*)_1, (g_*)_1$ have the same matrices.

Now we can state the classification of the homotopy classes of maps $f : T \to T$ denoted by $[T, T]$.

The correspondence $f \to (f_*)_1$ induces a bijection $[T, T] \longleftrightarrow M_2(Z)$.

II) - The Lefschetz number of a map $f : T \to T$.

Let $\alpha, \beta \in H^1(T) = \mathbb{Z} \oplus \mathbb{Z}$ be such that $\{\alpha, \beta\}$ is a basis for the first cohomology group. The map f induces a homomorphism $f^* : H^1(T) \to H^1(T)$ where $f^*(\alpha) = a\alpha + b\beta$ $f^*(\beta) = c\alpha + d\beta$.

The matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is, in fact, the one given in the classification above.

A generator of $H^2(T, Z)$ can be obtained as a product of α by β, $\alpha \cup \beta$, the cup product of α and β. It turns out that $f^*(\alpha \cup \beta) = det \begin{pmatrix} a & c \\ b & d \end{pmatrix} \alpha \cup \beta$.

Then we can compute $L(f)$:

$L(f) = tr(f_{*0}) - tr(f_{*1}) + tr(f_{*2}) = 1 + a - d + ad - bc = (a-1)(d-1) - bc = det \begin{pmatrix} a-1 & c \\ b & d-1 \end{pmatrix} = det(M - I)$.

So, the Lefschetz number of f has a very nice formula. In particular, if we consider the iterated of f, i.e. $f^n = f \circ ... \circ f$, then $L(f^n) = det(M^n - I)$.

III) - The minimal problem.

Following the same idea which was used in the case of maps on the circle S^1, we will construct a particular set of functions on the Torus. Give an integral matrix $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ of order 2×2, let $T_M : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which relative to the canonical basis has matrix M. Since the entries are integers, this linear transformation T_M induces a map $f_M : T \to T$.

Let us divide our matrices into two classes:

i) The 2×2 integral matrices M with $det(M - I) = 0$

ii) The 2×2 integral matrices M where $det(M - I) \neq 0$.

We leave to the reader to verify the following facts: In case ii), the number of fixed points are $det(M - I)$, therefore equal to $N(f)$. By the Nielsen theorem it follows that MF$[f] = N(f)$, since we found a map in the homotopy class where $\#Fix(f) = N(f)$.
For the case i), f can be deformed to fixed point free.

So we can conclude that for a given map $f : T \to T$ we have $|L(f)| = N(f) = MF[f]$.

6 Back to the original problems

We started this exposition with three problems. Now we will show how to analyze them via the Nielsen fixed point theory.

Problem 1 - The existence of an everywhere-nonzero vector field on a compact differentiable manifold implies that the identity can be deformed to a map $f : M \to M$ which is fixed point free. But $L(f) = L(id) = \sum_{q=0}^{\infty}(-1)^q tr(id_*)_q =$

$\sum_{q=0}^{\infty} rk(H_q(X))$. This number is known as the Euler characteristic of X. So, in order to have an everywhere-nonzero vector field it is necessary to have the Euler characteristic of X different from zero.

In particular if S_h is the surface of genus h, we know that $H_0(S_h) = \mathbb{Z}$, $H_1(S_h) = \mathbb{Z}^{2h}$, $H_2(S_h) = \mathbb{Z}$ and it follows that $L(id) = 1 - 2h + 1 = 2 - 2h$.

Hence the only orientable compact surface which might admit an everywhere-nonzero vector field is the Torus. By explicit construction we know that such vector field exists. It suffices to consider at each point $(x_0, y_0) \in S^1 \times S^1$ the unit vector which is tangent to the circle $S^1 \times y_0$.

Problem 2 - The case $G = S^1$ with the product given by multiplication of the complex numbers. Given $g_0 = e^{2\pi i t_0} \in S^1$ and n an integer we have $g'(k) = e^{\frac{2\pi i t_0}{n} + \frac{2k\pi}{n}}$ $k = 0, ..., n-1$ the n-th roots of g_0. So for this particular group structure on S^1, there are n roots. Suppose now $u : S^1 \times S^1 \to S^1$ is another multiplication. Does the n-root still exist? If so, how many are there? If $G = S0(m)$, the $m \times m$ orthogonal matrices, or $G = U(m)$, the $m \times m$ unitary matrices, we have the multiplication given by the usual multiplication of matrices. It is not clear that the n-root of a matrix exists. In [[1], III.F], one find the following result:

Theorem: *Let G be a topological group which is a connected finite complex, then G is divisible, i.e.,given any $k \geq 2$ and $a \in G$, there is a solution of the equation $x^k = a$.*

We sketch here the argument for $G = \underbrace{S^1 \times ... \times S^1}_{m}$ and for $G = SU(m)$.

For $G = \underbrace{S^1 \times ... \times S^1}_{m}$, the map $f : G \to G$ given by $f(x) = g_0.x^{n+1}$ induces in H^1 (independently of the multiplication) the homomorphism given by the matrix

$$M = \begin{pmatrix} n+1 & 0 & \dots & \\ 0 & n+1 & & \\ & & \ddots & \\ & & & n+1 \end{pmatrix}$$

i.e. the diagonal matrix which has $n+1$ in the diagonal. We see that $L(f) = det(M - I) = n^m \neq 0$.

2) At least for $G = U(m)$ with the usual multiplication we have that $H^*(U(m), \mathbb{R}) = \wedge(x_1, x_3, ..., x_{2m-1})$ and $(f_{*1})(x_{2i-1}) = (n+1)x_{2i-1}$. By [10], we have $L(f) = m^m \neq 0$ so it has a root.

A similar proof works for $SO(m)$, where the corresponding calculation is a little more complicated than in the case $U(m)$.

Problem 3 - Here we restrict ourselves to present a result due to Ivanov, see [3], without mentioning any word about its proof. Then we exemplify its use in the case of maps on Torus.

Let us define the asymptotic Nielsen number $N_\infty(f)$ to be $\lim_n sup \frac{1}{n} log(N(f^n))$. From Ivanov [3] we have:

Theorem: *Let f be a continuous mapping of a compact polyhedron into itself. Then $N_\infty(f) \leq h(f)$, i.e. the asymptotic Nielsen number is a lower bound for the entropy.*

Ex. Let $f : T \to T$ be a map whose matrix in homology is $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$.

(For those who have some familiarity, this is an Anosov map on the Torus). We would like to compute $L(f)$ and $L(f^n)$ for all integer $n > 1$. The best way to find the power of a matrix is to see if it is similar to a diagonal matrix. The matrix above has eigenvalues $2 - \sqrt{3}$, $2 + \sqrt{3}$ and

$$L(f) = (2 - \sqrt{3} - 1)(2 + \sqrt{3} - 1) = (1 - \sqrt{3})(1 + \sqrt{3})$$

call $\lambda_1 = 2 - \sqrt{3}$, $\lambda_2 = 2 + \sqrt{3}$. $L(f^n) = (\lambda_1^n - 1)(\lambda_2^n - 1)$. Since $\lambda_1^n \to 0$ we have $lim \frac{1}{n} log|(\lambda_1^n - 1)|| |\lambda_2^n - 1| = lim \frac{1}{n} log(\lambda_2^n) = lim \frac{1}{n} log(\lambda_2^n) = log\lambda_2$. Therefore $h(t) \geq log(2 + \sqrt{3})$.

Finally, it is important to mention that there is another branch of fixed point theory, which deals with spaces which are not finite polyhedron. It has many relations and applications in problems in Functional Analysis. A good reference for this matter is the article by Felix Browder, "Fixed point Theory and nonlinear problems" Bull. Amer. Math. Soc. 9(1983), 1-39. We finish this article with some of his words: "Among the most original and far-reaching of the contributions made by Henri Poincaré to mathematics was his introduction of the use of topological or 'qualitative' methods in the study of nonlinear problems in analysis... The ideas introduced by Poincaré include the use of fixed point theorems, the continuation method, and the general concept of global analysis. Fixed point theory was an integral part of topology at the very birth of the subject in the work of Poincaré in the 1880s. He showed that the solutions to certain important analytic problems could be studied by defining a set X and a function $f: X \to X$ in such a way that the solutions correspond to the *fixed points* of the function f, that is, to the points $x \in X$ such that $f(x) = x$".

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Referencias

[1] **Brown R.F.**, *The Lefschetz fixed point theorem*, Scott Foresmann, Illinois, (1971).

[2] **Brown R.F.**, *Topological Fixed Point Theory, Chapter of the book "History of Topology"*, I. James Ed., to be published by Elsevier.

[3] **Ivanov, N.V.**, *Entropy and Nielsen numbers*, Soviet Math.Dokl. **26**, 63–66 (1982).

[4] **Croom, F.H.**, *Basic Concepts of Algebraic Topology*, Springer-Verlag, (1978).

[5] **Dold, A.**, *Lectures on Algebraic Topology*, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, **200**, Berlin-Heidelberg-New York Springer-Verlag (1972).

[6] **Gonçalves, D. L. and Kiihl, J. C. S.**, *Teoria do Índice*, Publicaçoĉoes do 14 Colóquio Brasileiro de Matemática, (1983).

[7] **Katok, A.B.**, *Smooth Dynamical Systems*,(in Russian) "Mir" Moscow, (1977).

[8] **Kiang, T.H.**, *The theory of fixed point classes*, Berlin-Heidelberg-New York Springer-Verlag (1989).

[9] **Shub, M.**, *Dynamical Systems, filtrations and entropy*, Bull. Amer. Math. Soc. **80** 27–41 (1974).

[10] **Oliveira, E.**, *Some Applications of Coincidence Theory*, Manuscripta Mathematica **86** 159–167 (1995).

[11] **Pontrjagin, L. S.** *Ordinary Differential Equations*, Reading, Addison-Wesley (1962).