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Almost Periodic Functions

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Almost Periodic functions were invented by Harald Bohr. Writing about Harald Bohr, Ray Redheffer, [18], says "Bohr added to the theories of summability, Dirichlet series and did important work on the zeta function; but his outstanding creation was the theory of almost periodic functions. Although his proofs were simplified by Weyl, Wiener and de la Vallee Poussin, practically all the results were first obtained by Bohr himself. Bohl, sometimes names as predecessor did not formulate the main problems; and the theory is that mathematical rarity-a one man job".

Harald Bohr was the younger brother of the physicist Niels Bohr. It was said that you could explain who Niels Bohr was to a Dane by saying he was Harald's brother, and in the rest of the world, one could explain Harald Bohr by saying he was Niels' brother. This was because Harald was a star forward on the 1908 Danish national soccer team which won the silver medal at the Olympic games. (The best Denmark has ever done). When he defended his dissertation, there were over 200 people present, most of them soccer fans. He was such an inspiring lecturer that for his sixtieth birthday, his students composed a cantata in his honor.

So what are these almost periodic functions and why did Bohr invent them? There are two motivations, the first was the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} e^{-x\,\ell n(n)}\, e^{-iy\,\ell n(n)}$$

is a Dirichlet series that for fixed $x > 1$ is a convergent series in y which is a linear combination of exponentials resembling a Fourier series but not with exponents in the usual form. It is an example of an almost periodic function. The hope that the regularity exhibited by almost periodic functions might resolve the Riemann Hypothesis has not materialized.

Another example is given in the Almagest of Ptolemy. It is a theory attributed to Hipparchus and is called the method of epicycles. Let P be a planet or the moon. The model of epicycles is shown in Figure 1.

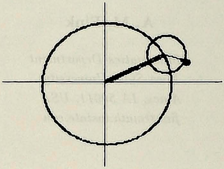

Figure 1

The motion of P may be written as

$$r(t) = r_1 e^{i\lambda t} + r_2 e^{i\mu t}$$

r_j are vectors, where μ and λ are real constants. When applied to the moon, for example, this is not a very good approximation. Copernicus showed that by adding a third circle one could get a better approximation to the observed data. This suggests that if $R(t)$ is the true motion of the moon then there exist vectors r_j and real numbers λ_j such that for all $t \in \mathbb{R}$,

$$R(t) = \sum_{j \in J} r_j e^{i\lambda_j t} \tag{1}$$

The astronomer Bohl, mentioned above, began to investigate finite sums as in (1). Unless the λ_j are rationally related (see the example below), the sum is not a periodic function. It was Bohr who turned the question around. What functions may be approximated in this way? What are the intrinsic properties of such functions? For some history see the introduction to Bohr's book [4].

To begin to see what such a characterization might look like consider the function

$$F(t) = \cos t + \cos \sqrt{2}t.$$

Note that $F(0) = 2$ but $F(t) < 2$ for all other values of t. So it is not a periodic function. However, if $\varepsilon > 0$ is given there are many numbers τ for which $F(\tau) > 2 - \varepsilon$. It is a number theory problem to construct them. It follows from the uniform continuity of F and other considerations that if the graph of F is translated by such a number, then the graphs will be close for all real numbers t. For periodic functions, translating the graph by any multiple of a period results in a match. For the above example this is almost true. This is basically the definition of almost periodic functions. It extends this translation property of periodic functions.

So now we begin with a complex valued function F which is continuous on $\mathbb{R}$. If $\varepsilon > 0$ is given, consider the set of real numbers.

$$T(F,\varepsilon) = \{\tau \ : \ |F(t+\tau) - F(t)| < \varepsilon \text{ for all } t \in \mathbb{R}\}.$$

This is called the ε-translation set of F. We say that F is an almost periodic function if for every $\varepsilon > 0$ this set is *relatively dense* in $\mathbb{R}$, that is, there is an L such that every interval of $\mathbb{R}$ of length at least L has one ε-translation number in it. L is called the inclusion length. For periodic functions $T(F,\varepsilon)$ contains all multiples of the period so any $L >$ period will work. So all periodic functions are also almost periodic.

Using this definition, it is not particularly easy to show that the class of all almost periodic functions, labeled AP is an algebra, that is it is closed under sums, differences, products, scalar multiples and finally under quotients provided the denominator has a positive lower bound to its modulus. One can consider vector functions also in which case this class would be closed under scalar multiples and sums. In both cases it is also closed under composition with functions that are uniformly continuous on the appropriate closed domain.

The main results for AP are:

1. Every function in AP is bounded and uniformly continuous on R.

2. AP contains all finite sums (labeled as a trigonometric polynomial) of the form

$$p(t) = \sum_{j=1}^{n} a_j e^{i\lambda_j t}$$

3. The Approximation Theorem: If f is in AP and $\varepsilon > 0$ is given, then there is a trigonometric Polynomial p such that

$$|f(t) - p(t)| < \varepsilon \text{ for all } t \in \mathbb{R}.$$

4. For each f in AP there is a uniquely defined Fourier series

$$f \sim \sum_{j=1}^{\infty} a_j e^{i\lambda_j t}.$$

Let us look for a moment at the Fourier series in 4. For ordinary Fourier series

$$f \sim \sum_{j=-\infty}^{\infty} a_j e^{itj}$$

we know that the coefficients a_j are given by the usual formula

$$a_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) e^{-ijt} dt$$

which comes from the orthogonal relations for the exponential functions. In the AP case the orthogonal relation is given by the mean value. This is given in property 5.

5. For each f in AP there exists the mean value,

$$M\{f\} = \lim_{t\to\infty} \frac{1}{T} \int_0^T f(s) ds.$$

As a matter of fact, $M\{f\} = \lim_{T\to\infty} \frac{1}{T} \int_a^{a+T} f(s) ds$ uniformly in a.

Now the formula for the coefficients of the *ap* function f is given by

$a_j = M\{fe^{-i\lambda_j t}\}$ so that the Fourier series for f looks like

$$f \sim \sum_j M\{fe^{-i\lambda_j t}\} e^{i\lambda_j t}.$$

6. The set of λ_j for which the mean value $M\{fe^{-i\lambda_j t}\} \neq 0$ is countable, and we have Parseval's equation

$$M\{|f|^2\} = \sum_j |a_j|^2.$$

The proof of these basic theorems may be found in a variety of places, see Bohr [3], Besicovitch [2], or Fink [11]. The Approximation Theorem and Parseval's Equation are in a sense equivalent and the various approaches to the theory are different. Sometimes the approximation theorem is proved first and some times the Parserval relation is. The approximation theorem gives an easy proof for the sums and products of *ap* functions being *ap*.

Since one of the applications of *ap* functions was to astronomy and generally to differential equations, we might consider the simplest case.

Theorem 1. *Consider the scalar differential equation*

$$y' = My + f(t) \tag{2}$$

where f is an ap function and M is a nonzero constant. If $Re(M) \neq 0$ then there is a bounded solution which is almost periodic.

Proof: It is easy to see that if $Re(M) > 0$, then the unique bounded solution of (2) is given by

$$y(t) = \int_{-\infty}^{t} e^{-M(t-s)} f(s)\, ds \tag{3}$$

and that $|y(t+\tau) - y(t)| \leq [1/Re(M)] \sup |f(t+\tau) - f(t)|$.

Consequently $T(y, \varepsilon) \supset T(f, Re(M)\varepsilon)$. Since the latter is relatively dense, so is $T(y, \varepsilon)$.
Thus y is *ap*. If $Re(M) < 0$, the integral in (3) is from t to infinity.

If $M = ia$ for a real, then the change of variable $z = ye^{iat}$ reduces (2) to an equation of the form

$$y' = F(t) \tag{4}$$

with F *ap*. The well know mathematician Kahane [5] gave the opinion that Bohr's most remarkable theorem was the following.

Theorem 2. *If F is ap then any solution of (4) is ap if and only if it is bounded.*

Bohr's proof is lengthy and complicated. I will give a proof below once we have some more general theorems about *ap* functions.

The result of Theorem 1 can be extended to systems of differential equations.

Theorem 3. *(Bohr-Neugebauer) Let M be a constant matrix and f be an almost periodic function. Then all bounded solutions of (2) are almost periodic. If all the* *eigenvalues of M have nonzero real part, then all solutions are almost periodic.*

Sketch of Proof. The matrix M is similar to a triangular matrix T. If $C^{-1}MC = T$, then change variables by $u = Cy$. Then u satisfies a differential equation of the form $u' = Tu + g$ with g almost periodic. Now one of the scalar equations in this system is of the form (2) to which we can apply the above theorems. By induction look at the next component.

There is a wide spread interest in almost periodic solutions to differential equations since they tend to be the ones that are stable. For a long book on this see Yoshizawa [19].

Solomon Bochner was able to find alternate proofs to the approximation theorem. These are used, for example, in the book by Besicovitch. In addition Bochner [6] was the first to notice that there was another characterization of almost periodicity that used concepts more usually associated with Analysis.

We would like to formulate this alternative definition to an *ap* function. Let us introduce the notation f_s for the translated function $f_s(t) = f(t+s)$. So that the number τ is an ε translation number if $\|f - f_s\| < \varepsilon$ where $\| \quad \|$ is the norm defined by $\|f\| = \sup_{t\in\mathbb{R}} |f(t)|$. We want to look at the family of functions $\{f_s | s \in \mathbb{R}\}$. I will prove that it is totally bounded. That is, given ε, there are a finite number of translates so that every translate is within ε of one of them.

Proof If ε is given and s is any real number than there is a number $\ell(s) \in T(f,\varepsilon)$ in the interval $[-s, -s+L]$, where L is the inclusion length of $T(f,\varepsilon)$. That is $|f(t+\ell(s)) - f(t)| < \varepsilon$ for all t. Letting $t = u + s$, we have that

$$|f(u+s+l(s)) - f(u+s)| < \varepsilon \quad \text{for all } u.$$

This is the statement that f_s, is within ε of $f_{s+\ell(s)}$ in the norm $\| \quad \|$. Furthermore, $0 < s + \ell(s) < L$. That is, every translate of f is ε close to some translate from the interval $[0, L]$. Since f is uniformly continuous, then there is a δ such that $|f(t+s) - f(t)| < \varepsilon$ for all t if $|s| < \delta$. So if $x \in [0, L]$ let $n\delta$ be within δ of x and $n\delta \in [0, L]$. Then $\sup |f(t+x) - f(t+n\delta)| = \sup |f(u+x-n\delta) - f(u)| < \varepsilon$. So every translate from $[0, L]$ is within ε from one of the finite number of translates of the form $n\delta$. Now put the above two together and ones gets that every translate is within $2\,\varepsilon$ of one of the translates $n\delta$. So the family $\{f_s | s \in \mathbb{R}\}$ is totally bounded. In the complete metric space of bounded functions on $\mathbb{R}$ with $\| \quad \|$ norm, total boundedness of a set is equivalent to the closure being compact. Define $H(f)$ to be the closure of the family $\{f_s | s \in \mathbb{R}\}$. This is called the *hull of* f. This introduces new functions. Let g be in $H(f)$. Then there is a sequence t_n so that $f(t+t_n)$ converges uniformly to $g(t)$. Since $\sup |f(t+t_n) - g(t)| = \sup |f(t) - g(t-t_n)|$

we see that f is the hull of g. In fact $H(f) = H(g)$.

We have proved half of Bochner's Theorem, Bochner [6].

Theorem 4. *A continuous function f is in AP if and only if $H(f)$ is compact in the topology of uniform convergence on $\mathbb{R}$.*

Proof. We need to argue that a function with $H(f)$ totally bounded satisfies Bohr's condition. Let ε be given and pick $a_i, i = 1, \dots, n$ for which f_{a_i} is an ε net for $H(f)$. If τ is a given real number, then for some $i(\tau)$ we have $|f(t + a_{i(\tau)}) - f(t + \tau)| < \varepsilon$ for all t. This is equivalent to $|f(u + \tau - a_{i(\tau)}) - f(u)| < \varepsilon$ for all u. So $\tau - a_{i(\tau)} \in T(f, \varepsilon)$. If $L = \max|a_i|$ then $\tau - L \leq \tau - a_{i(\tau)} \leq \tau + L$ so $2L$ is inclusion length for $T(f, \varepsilon)$.

Corollary. *AP is closed under uniform limits.*

Proof We apply Bochner's condition. If f_n is in AP and converges uniformly to g, then let $\|f_N - g\| < \varepsilon$. Invoke an ε net in $H(f_N)$. By the triangle inequality, the same translates of g are a 3 ε net in $H(g)$.

One might ask if AP is closed under differentiation and integration. As above we postpone the integration question but we can answer the differentiation question now.

Theorem 5. *The derivative of an ap function is ap if and only if it is uniformly continuous.*

Proof. If f' is in AP then it is uniformly continuous. For the converse, first consider real f. Then $f_n(t) = n\left\{f\left(t + \frac{1}{n}\right) - f(t)\right\} = f'(t+\delta(t))$ where $|\delta(t)| < \frac{1}{n}$. ($\Delta$ depends on n and t). By uniform continuity, this last function converges uniformly to $f'(t)$ as n becomes infinite. But $f_n(t)$ is *ap* for each n so by the previous corollary, f' is *ap*. If f is complex, apply the real result to the real and imaginary parts separately.

Let us return to Bochner's condition of the previous theorem and formulate it this way. The continuous function f is in AP if from every sequence t_n we can extract a subsequence t'_n, such that $f(t + t'_n)$ converges uniformly on $\mathbb{R}$. For purposes of typography for any sequence $t = \{t_n : n = 1, 2, \dots\}$ we write $T_t f = g$ to mean that $f(t + t'_n)$ converges to g. In each case we will specify the type of convergence. So Bochner's criteria means that from every sequence t, we can extract a subsequence t' so that $T_{t'} f$ exists uniformly.

As explained above, this compactness criteria is equivalent to the total boundedness of the set of translates of f. Bochner and von Neumann [8], realized that this criteria would make sense for complex valued functions defined on arbitrary groups. So there is a theory of almost periodic functions on groups where the mean value can sometimes be given by a Haar integral. The almost periodic functions on groups become important in representation theory. The elements of a finite dimensional bounded representation become almost periodic functions. See Maak [16] for details and a precise statement of the theorem. On a compact group, the almost periodic functions are precisely the continuous functions. The relationship between almost periodic functions and Fourier analysis on groups may be found in Loomis [15].

Let us for the moment return to the notion of the Fourier series of an almost periodic function as in 4. above. The coefficients in the fourier series are given by the mean value formula and come from the othogonality of the exponentials with respect to the mean value. The set $\{\lambda | M\{fe^{-i\lambda t}\} \neq 0\}$ is called the exponents of f. For periodic functions, this set is the additive group of numbers generated by the exponential with smallest positive period. For general almost periodic functions we consider the smallest additive group which contains all the exponents. This object is called the module of f and is denoted by Mod(f). This is a natural object. Thus, if we consider the product of two almost periodic functions we might expect that the fourier series is the formal product of the two fourier series. The exponents in the product become linear combinations of the exponents of the individual functions. For example, in taking the powers of an almost periodic function the exponents will be contained in the Mod(f) so we expect that if we take a composition of f with a uniformly continuous function, the smallest set where we will be sure that the exponents lie is Mod(f). The following theorem is an important one in the theory of almost periodic solutions to differential equations. (See [11, page 62].)

Theorem 6. *The following statements are equivalent for f and g in AP:*

1. *Mod(f) $\supset$ Mod(g).*

2. *For every $\varepsilon > 0$, there is a $\delta > 0$ such that $T(f, \delta) \subset T(g, \varepsilon)$.*

3. *$T_t f$ exists implies $T_t g$ exists (in any sense, pointwise or uniform on compact sets or uniform on $\mathbb{R}$).*

For example it follows that for the differential equation (2), the statement after (3) shows that the solution of the differential equation y has Mod(y) $\subset$ Mod(f). In particular, if f is periodic, then y is periodic of the same period.

Both Bohr's and Bochner's criteria for almost periodicity are difficult to work with. For example there are no theorems in Analysis which give the uniform convergence of functions on $\mathbb{R}$. Bochner [7] in a brilliant paper addressed this difficulty and showed that the following is an equivalent definition of almost periodicity.

Bochner's second criteria. *A bounded uniformly continuous function on R is almost periodic if and only if from every two sequences α and β we may extact common subsequence α' and β' such that*

$$T_{\alpha+\beta}f = T_\alpha T_\beta f \tag{5}$$

pointwise. Common sequences means the same choice function so that $\alpha' + \beta'$ is a subsequence of $\alpha + \beta$.

Loosely speaking, sequences of real numbers act on f by the T operation in a group manner provided one takes the appropriate subsequences. In (5) the left hand side is a single limit and the right hand side is an iterated limit. The importance of the criteria is that it is pointwise. In fact, for almost periodic functions the convergence is uniform on $\mathbb{R}$, but as a sufficient condition it only has to hold pointwise. For differential equations, the importance can be seen in viewing the differential equation (2) again. Any bounded solution y (if any) would automatically have a bounded derivative. That means that if t is a sequence, for any interval $[-n, n]$ one can extract a subsequence t' of t so that $T_{t'}y$ exists uniformly there (Arzela-Ascoli). By a diagonalization argument, one can obtain a subsequence s of t for which $T_s y$ converges uniformly on any compact subset of $\mathbb{R}$. From this it would follow from the differential equation that the limit of this function is a solution of the equation (2) with f replaced by $T_s f$.

With the above criteria and the observations of the previous paragraph we are ready to prove the theorem on integration. A proof that some call a swindle.

Theorem 7. *An indefinite integral of an almost periodic function is almost periodic if and only if it is bounded.*

Proof. Recall we are talking about a bounded solution of the differential equation (6).

$$y' = f \tag{6}$$

Fix one such solution calling it F. The general solution of the differential equation is given by $F + c$, c a constant. We are assuming that F is bounded. We choose c so that $F + c$ has infimum $-a$ and supremum a for some positive a. If this solution is ap then so is F. For reasons of typography we may assume that $c = 0$ and assume that F has this property. This means that among all solutions of (6), F has the least norm. Furthermore, no other solution has the same norm, i.e.

F is the unique norm minimizer of the solutions of (4). Let this norm be $a(f)$. Now consider any equation in the null. Let $g \in H(f)$ by $T_t f = g$ (here this can be uniformly on $\mathbb{R}$). By the argument above there is a subsequence of t called s by which $z = T_s F$ exists and z is a solution to $z' = g$.

So first of all, each differential equation in the hull has a bounded solution. Apply the above argument to get a solution $z(g)$ with minimal norm as arranged above for the original equation $y' = f$. Let the norm of this solution be $a(g)$. Now comes the fun part. Since each value of z is the limit of values of F, the norm of z, $\|z\| \leq a(f)$. By minimality, $\|z\| \geq a(g)$, so $a(f) \geq a(g)$. Since the argument is symmetric and f is in $H(g)$ we can argue the reverse inequality and arrive at $a(f) = a(g)$. So all the minimal norms of equations in the hull are the same. Since $\|z\| = a(g)$ it must be that z is the minimizer of the norms of the equation $z' = g$. Minimal solutions go to minimal solutions.

Let α and β be arbitrary sequences, we take common subsequences α' and β' to get $T_{\beta'} f = g$, and $T_{\beta'} y = z$ to exist, then further common sequences α'' and β'' of α' and β' to get $T_{\alpha''} g = h$ and $T_{\alpha''} z = w$ to exist, all of these uniformly on compact intervals so that z is a solution of $z' = g$ and $w' = h$. We finally take another common sequence α^* and β^* of α'' and β'' so that $T_{\alpha^*+\beta^*} f$ and $T_{\alpha^*+\beta^*} y = x$ exist. But since f is almost periodic, we have by Bochner's criteria that $T_{\alpha^*+\beta^*} f = h = T_{\alpha^*} T_{\beta^*} f$. Then it follows that $w = T_{\alpha^*} T_{\beta^*} y$ and x are both solutions of the same equation $y' = h$ and they were arrived at by translation. Since minimal solutions go to minimal solutions, both w and x are minimal solutions. By uniqueness they are the same, that is y satisfies Bochner's criterion and is almost periodic. This completes the proof of Bohr's theorem.

There are theorems which make the relationship between the *ap* function and its Fourier series more precise. We give several examples.

Theorem 8. *If the Fourier exponents of an almost periodic function are linearly independent, then it is absolutely summable, that is, if* $f(t) \sim \sum_j a_j e^{it\lambda_j}$ *then* $\sum_j |a_j| < \infty$.

Theorem 9. *If the Fourier exponents of* f *all lie in the interval* $[-a, a]$, *then* f *may be extended into the complex plane as an entire function. Thus* f' *is an almost periodic function and furthermore we have the estimate* $\|f'\| \leq a\|f\|$.

Theorem 10. *If no Fourier exponents of* f *lie in the interval* $[-a, a]$ *then* $\int_0^t f(s)ds$ *is in* AP. *There is an absolute constant* D *so that if* g *is the particular integral of* f *with mean value 0, then* $\|g\| \leq (D/a)\|f\|$.

The proof of Theorem 10 may be found in Bohr's book [4] and Theorems 8 and 9 appear in Fink [11] and elsewhere.

There are other classes of almost periodic functions that are of interest. If the Fourier exponents of f have a finite basis over the integers, then f is called quasi-periodic. That is if there are numbers $a_i, i = 1, 2, \ldots, n$ so that every exponent λ is a linear combination

$$\lambda = \sum_{1}^{n} \lambda_j a_j, \text{ integer } \mathrm{a_j}$$

then f is quasi-periodic. Quasi-periodic functions are important in the applications to mechanics. Loosely speaking, the dimension of the graph of a solution of a system of autonomous differential equations is equal to the number n of number of basis elements. See Cartwright [9] for this result. Formal integration of the Fourier series of a quasi-periodic function leads to the "small divisor problem". This is because when one tries to solve an equation with the Fourier series one must (as explained above) look at the entire group generated by the λ_j. But these numbers are dense in the real numbers except for the periodic case. These are the numbers that appear in the denominators of the formal integral of the Fourier series, so convergence of the resulting series is in doubt. A nice introduction is given by Moser [17].

There is a corresponding concept of almost periodicity in topological dynamics and symbolic systems, but we will not pursue that here.

For other expositions of Almost Periodic Functions, see Cordenanu [10] or Zaidman [20] for functions in abstract spaces. Amerio and Prouse [1] deal with almost periodicity in spaces used to solve partial differential equations. For another book on almost periodic differential equations see Levitan and Zhikov [14]. For a paper on the various definitions of almost periodicity see Fink [12]. For more historical perspective see the introduction to Bohr's book [4] or the Collected Works [5].

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