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Asymptotic Solutions of Linear Differential Equations

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1 Introduction

Linear differential equations are used in a number of areas of science and technology, in order to describe phenomena which are close to an equilibrium state. If the coefficients in the differential equation are constant functions, then the solution of the differential equation can be expressed in terms of elementary functions, for example exponential and trigonometric functions. However for more general variable coefficients, the solution is written in terms of integrals, which may be difficult to compute. For a complete introduction to the subject of ordinary differential equations, we refer to [1].

The methods of *asymptotic analysis* allow one to find elementary formulas which give an excellent approximation to the solution in certain limiting cases. In this paper we consider the behavior of solutions when the independent variable t approaches infinity. For linear differential equations whose coefficients approach a constant, we show that the solution is well approximated by the exact solution of the correponding equation with constant coefficients.

2 Asymptotic Solutions of First Order Linear Equations

We can solve a first-order linear equation in terms of two integrations. If these integrals can be performed by calculus, then we have an explicit solution and no further analysis is necessary. However in many cases these integrals do not result in elementary functions, so that we must resort to other methods to study the solution.

In this section we study the behavior of the solution of the first-order linear equation

$$y' = p(t)y + q(t) \qquad y(t_0) = y_0 \tag{1}$$

when $t \longrightarrow \infty$, where $p(t)$ and $q(t)$ tend to limits when $t \longrightarrow \infty$. We expect that the solution will resemble that of the same equation when $p(t), q(t)$ are constant: $p(t) \equiv p$, $q(t) \equiv q$.

The method of integrating factors tells us that the general solution of (1) can be obtained by first multiplication of both sides by a suitable chosen function. This allows us to solve an equivalent equation in which $p(t)$ is replaced by zero. Then the general solution can be obtained by a single integration. We will illustrate this below for the case $p(t) = p$, $q(t) = q$.

In this case we multiply by e^{-pt} and perform the integration. This leads to the explicit elementary solution formula for $p \neq 0$:

$$y(t) = y_0 e^{p(t-t_0)} + (q/p)[e^{p(t-t_0)} - 1] \qquad p \neq 0,$$

In case $p = 0$ we obtain $y(t) = y_0 + q(t - t_0)$.

The three cases ($p > 0$, $p = 0$, $p < 0$), present different intuitive pictures. In case $p > 0$, the solution is proportional to an exponential function e^{pt}, which tends to infinity, plus an additional constant. We can express the asymptotic behavior by dividing by e^{pt}, to obtain a limit. In case $p = 0$ we have linear growth, whereas in case $p < 0$ we have convergence to a steady state. The mathematics is described as follows:

i) $p > 0: \quad \lim_{t\to\infty} \frac{y(t)}{e^{pt}} = (y_0 + \frac{q}{p})e^{-pt_0}$;

In case $p = 0$ the asymptotic form is even simpler, since the solution is a linear function of t and so

ii) $p = 0: \quad \lim_{t\to\infty} \frac{y(t)}{t} = q$;

Finally, in case $p < 0$, the exponential function tends to zero, so that

iii) $p < 0: \quad \lim_{t\to\infty} y(t) = -\frac{q}{p}$.

In other words, *either we have exponential growth, linear growth or convergence to a constant*. In what follows we will show that the same conclusions apply in the case that $p(t), q(t)$ are non-constant.

We will frequently use the notation $f(t) = O(g(t)), t \to \infty$, whose precise meaning is that there exist constants $T > 0, M > 0$ so that $|f(t)| \leq M g(t)$ for all $t \geq T$.

2.1 First-level asymptotics for constant coefficient

In this subsection we obtain the first approximation to the solution for large time in case $p(t)$ is a constant. The results to be proved are stated as follows.

Theorem 1 *Let $y(t)$ be a solution of the first-order linear differential equation $y' = py + q(t)$ where the function $q(t)$ is bounded: $|q(t)| \le M, t \ge t_0$. Then the solution has the following asymptotic behavior:*

i) $p > 0$: $\lim_{t\to\infty} \frac{y(t)}{e^{pt}} = C$, *a constant.*

ii) $p = 0$: $\lim_{t\to\infty} \frac{y(t)}{t} = q_0$, provided that $\lim_{t\to\infty} q(t) = q_0$.

iii) $p < 0$: $\lim_{t\to\infty} y(t) = -\frac{q_0}{p}$, provided that $\lim_{t\to\infty} q(t) = q_0$.

Before giving the proof, we note that in every case the solution is obtained by means of the integrating factor e^{-pt}. When we multiply the equation by e^{-pt} we obtain an equation which we can solve by one integration and write the solution in the form

$$y(t) = y(t_0)e^{p(t-t_0)} + \int_{t_0}^{t} e^{p(t-s)}q(s)\,ds. \tag{2}$$

Proof. We consider separately the cases $p > 0, p = 0, p < 0$.

Case i: $p > 0$: Referring to the general formula (2), we see that the improper integral $\int_{t_0}^{\infty} e^{-ps}q(s)\,ds$ is convergent, so that we can write

$$\int_{t_0}^{t} e^{-ps}q(s)\,ds = \int_{t_0}^{\infty} e^{-ps}q(s)\,ds - \int_{t}^{\infty} e^{-ps}q(s)\,ds.$$

The second term is $O(e^{-pt}), t \to \infty$ and we have the asymptotic representation

$$\frac{y(t)}{e^{pt}} = C + O(e^{-pt}), \qquad C := y(t_0)e^{-pt_0} + \int_{t_0}^{\infty} e^{-ps}q(s)\,ds.$$

Example 1 *What is the first-level asymptotic approximation of the solution of the equation $y' = 3y + 4 + \sin 2t$ with the initial condition $y(0) = 3$.*

Solution. In this case we have $t_0 = 0, p = 3$ and the improper integral can be computed explicitly as $\int_0^{\infty} e^{-3s}\sin 2s\,ds = 2/13$. The first-level asymptotic approximation is $y(t) = e^{3t}[(3 + 2/13) + O(e^{-3t})], t \to \infty$ •

Case ii: $p = 0$: In this case the solution of (1) is written as the integral

$$y(t) = y(t_0) + \int_{t_0}^{t} q(s)\,ds.$$

If $q(t)$ has a limit q_0 when $t \to \infty$, then $(1/t)\int_{t_0}^t q(s)\,ds$ has the same limit and we can write $\lim_{t\to\infty} t^{-1}\int_{t_0}^t q(s)\,ds = q_0$ and we have the asymptotic formula $y(t_0) = t[q_0 + \epsilon(t)$ where $\epsilon(t) \to 0$ when $t \to \infty$ and where we have incorporated the initial condition into the term $\epsilon(t)$.

Case iii: $p < 0$: In this case we write the solution formula (2) as

$$y(t) = y(t_0)e^{p(t-t_0)} + \int_{t_0}^t e^{p(t-s)}[q(s) - q_0]\,ds + q_0\int_{t_0}^t e^{p(t-s)}\,ds.$$

The first term tends to zero and the third term can be explicitly evaluated to see that it tends to $-q_0/p$. It remains to show that the second integral tends to zero. Given $\epsilon > 0$, choose T so that $|q(s) - q_0| < \epsilon|p|$ for $s > T$ and write

$$\int_{t_0}^t e^{p(t-s)}[q(s) - q_0]\,ds = \int_{t_0}^T e^{p(t-s)}[q(s) - q_0]\,ds + \int_T^t e^{p(t-s)}[q(s) - q_0]\,ds.$$

The first integral is over a fixed interval and the integrand is less than an constant times e^{pt}, which tends to zero; therefore the integral tends to zero when $t \to \infty$. For the other integral we have

$$\int_T^t e^{p(t-s)}|q(s) - q_0|\,ds \le \epsilon|p|\int_T^t e^{p(t-s)}ds = \epsilon(1 - e^{p(t-T)}) < \epsilon.$$

This completes the proof that $\lim_{t\to\infty} y(t) = -q_0/p$.

Example 2 *Find the first-level asymptotic approximation of the solution of the equation* $y' + 3y = 4 + \frac{6}{1+t^2}$.

Solution. In this case $q(t) = 4 + 1/(1+t^2)$ so that $q_0 = 4$. Therefore the first-level asymptotic approximation is $\lim_{t\to\infty} y(t) = \frac{4}{3}$ •

In this sub-section we have given the first-order asymptotics of the solution when $t \to \infty$. If no further details are known about $q(t)$, there is little that can be said about a more detailed asymptotic formula for the solution.

2.2 Higher-level asymptotic expansion

In case more details are available about the behavior of the function $q(t)$, we can obtain more details about the solution $y(t)$ when $t \longrightarrow \infty$. Again we consider separately the cases $p < 0, p = 0, p > 0$. In every case we asssume that the function $q(t)$ has the form

$$q(t) = q_0 + \frac{q_1}{t} + \cdots + \frac{q_N}{t^N} + O\left(\frac{1}{t^{N+1}}\right) \qquad t \longrightarrow \infty$$

For example, if $q(t) = (4+t)/(2+t)$, then $q(t) = 1 + 2/t + O(1/t^2), t \to \infty$.

Asymptotic expansion in case $p = 0$:

In case $p = 0$ we define $Y(t) = y(t) - q_0 t - q_1 \ln t$. Then Y satisfies the equation $Y' = q(t) - q_0 - (q_1/t) = O(1/t^2), t \to \infty$. Hence the improper integral exists, call it $C = \int_{t_0}^\infty Y'(t)dt$. We have proved that

$$y(t) = q_0 t + q_1 \ln t + C + O(1/t), \qquad t \to \infty.$$

In order to obtain further terms in the asymptotic expansion we write

$$\epsilon(t) = -\int_t^\infty [\frac{q_2}{s^2} + \cdots + \frac{q_N}{s^N} + O(\frac{1}{s^{N+1}})]\,ds.$$

Each of these integrals is elementary and we finally obtain the result.

$$y(t) = q_0 t + q_1 \ln t + C - \frac{q_2}{t} - \frac{q_3}{2t^2} - \cdots - \frac{q_N}{Nt^{N-1}} + O(\frac{1}{t^N}), \qquad t \to \infty.$$

Example 3 *Find the first three terms in an asymptotic expansion for the solution of the equation* $y' = \frac{1}{1+t^2}, t \to \infty$.

Solution. In this case we have $q_0 = 0, q_1 = 0, q_2 = -1, q_3 = 0, q_4 = +1$. Thus $C = \int_0^\infty q(s)ds = \frac{\pi}{2}$ and $y(t) = \frac{\pi}{2} - \frac{1}{t} + \frac{1}{3t^3} + O(\frac{1}{t^5})$ •

Asymptotic expansion in case $p > 0$:

We now discuss the case $p > 0$. Following the analysis of the previous subsection, we must analyze the integral

$$\int_{t_0}^t e^{-ps} q(s)\,ds = \int_{t_0}^\infty e^{-ps} q(s)\,ds - \int_t^\infty e^{-ps} q(s)\,ds.$$

The first integral is a constant, independent of t. To analyze the second integral we use the hypothesis on $q(t)$ and do each term separately. Thus

$$\int_t^\infty \frac{e^{-ps}}{s^k} = -\int_t^\infty \frac{1}{s^k} d(\frac{e^{-ps}}{p}) = \frac{1}{pt^k} e^{-pt} + \int_t^\infty \frac{1}{pk} \frac{e^{-ps}}{s^{k+1}}\,ds.$$

This can be repeated to obtain additional terms containing higher powers. In general, the coefficient of t^{-N} will contain terms involving the coefficients $q_1, \ldots, q_N$. The resulting expansion is written as

$$y(t) = Ae^{pt} + C_0 + \frac{C_1}{t} + \cdots + \frac{C_N}{t^N} + O(\frac{1}{t^{N+1}}),$$

where the constants $A, C_0, \ldots, C_N$ are obtained from the above procedure. We illustrate with an example.

Example 4 *Find three non-zero terms in the asymptotic expansion of the solution of the equation* $y' = 3y + \frac{1}{1+t}, t \to \infty$.

Solution. From the above discussion, we must integrate by-parts:

$$\int_t^\infty \frac{e^{-3s}}{1+s}\,ds = -\frac{1}{3}\int_t^\infty \frac{1}{1+s} d(e^{-3s})\,ds$$

$$= \frac{e^{-3t}}{3(1+t)} + \frac{1}{3}\int_t^\infty \frac{e^{-3s}}{(1+s)^2}\,ds.$$

The new integral can again be integrated-by-parts with the result

$$\int_t^\infty \frac{e^{-3s}}{(1+s)^2}\,ds = \frac{e^{-3t}}{3(1+t)^2} - \frac{1}{3}\int_t^\infty \frac{e^{-3s}}{(1+s)^3}\,ds.$$

Combining this and simplifying yields the asymptotic expansion

$$y(t) = e^{3t}[y_0 + \int_0^\infty \frac{e^{-3s}}{1+s}ds] - \frac{1}{3(1+t)} - \frac{1}{9(1+t)^2} + O(\frac{1}{t^3}).$$

The remainder terms can be put in the standard form of ascending powers of $1/t$ by expanding $1/(1+t) = 1/t - 1/t^2 + O(1/t^3), 1/(1+t)^2 = 1/t^2 + O(1/t^3), t \to \infty$ with the final result

$$y(t) = e^{3t}[y_0 + \int_0^\infty \tfrac{e^{-3s}}{1+s}ds] - \tfrac{1}{3t} + \tfrac{2}{9t^2} + O(\tfrac{1}{t^3}) \quad \bullet$$

Asymptotic expansion in case $p < 0$:

It is also possible to obtain an asymptotic expansion in case $p < 0$. In order to determine the form of the expansion, we first note that for any monomial term $1/s^k$, we have

$$\int_{t_0}^t e^{p(t-s)} \tfrac{1}{s^k} ds = -\tfrac{1}{p} \int_{t_0}^t \tfrac{1}{s^k} d(e^{p(t-s)}) = -\tfrac{1}{pt^k} + O(\tfrac{1}{t^{k+1}}).$$

When we apply this to each of the terms in the assumed expansion of $q(s), s \to \infty$, we find that the solution has the asymptotic expansion

$$y(t) = y_0 + \tfrac{y_1}{t} + \cdots + \tfrac{y_n}{t^n} + O(\tfrac{1}{t^{n+1}}).$$

The constants $y_0, y_1, \ldots$ can be found by substituting this into the differential equation and equating coefficients of t^{-1}, t^{-2}, etc. This leads to the equations

$$py_0 = q_0, \quad py_1 = q_1, \quad -y_1 + py_2 = q_2, \quad -2y_2 + py_3 = q_3, \ldots.$$

These equations can be solved for $y_0, y_1, y_2, \ldots$ to obtain an asymptotic expansion of the solution.

Example 5 *Find an asymptotic expansion of the solution of the equation* $y' + 3y = \frac{1}{1+t}, t \to \infty$.

Solution. The right side of the equation has the asymptotic expansion

$$\tfrac{4}{1+t} = \tfrac{4}{t(1+\frac{1}{t})} = \tfrac{4}{t}[1 - \tfrac{1}{t} + \tfrac{1}{t^2} + \cdots + (-1)^n \tfrac{1}{t^n} + \cdots]$$

$$= \tfrac{4}{t} - \tfrac{4}{t^2} + \tfrac{4}{t^3} + \cdots$$

This leads to the equations $3y_0 = 4, 3y_1 = -4, -y_1 + 3y_2 = -4, -y_2 + 3y_3 = 4, \ldots$ and the asymptotic expansion

$$y(t) = \tfrac{4}{3} - \tfrac{4}{3t} + \tfrac{8}{9t^2} + \tfrac{44}{27t^3} + O(\tfrac{1}{t^4}) \quad \bullet$$

2.3 First-level asymptotics for variable coefficient

In the previous sub-sections we have determined the asymptotic behavior of the solution of the first-order linear equation (1) in case the coefficient function $p(t)$ is a constant. In order to formulate the results in the more general case of variable $p(t)$, we first introduce some notions regarding exponential growth.

Definition A function $f(t), t \geq t_0$ is *strongly exponential with exponent* p if $\lim_{t\to\infty} e^{-pt}f(t)$ exists and is non-zero.

Example 6 $f(t) = 4e^{3t} + 7e^{2t}$ is strongly exponential with $p = 3$.

Definition A function $f(t), t \geq t_0$ is *weakly exponential with exponent* p if

$$\lim_{t\to\infty} t^{-1}\log|f(t)| = p.$$

Example 7 $f(t) = t^2e^{5t}$ is weakly exponential with $p = 5$ (but not strongly exponential).

We now formulate and prove a result which is valid for the general first-order linear equation $y' = p(t)y + q(t)$. The proof below can be omitted without loss of continuity.

Theorem 2 *Let $y(t)$ be the solution of the first-order linear differential equation $y' = p(t)y + q(t)$ where the function $q(t)$ is bounded: $|q(t)| \leq M, t \geq t_0$, and the function $p(t)$ has a limit: $\lim_{t\to\infty} p(t) = p$. Then the solution has the following asymptotic behavior.*

i) $p < 0$: $\lim_{t\to\infty} y(t) = -\frac{q_0}{p}$, provided that $\lim_{t\to\infty} q(t) = q_0$.

iia) $p > 0$: *There exists a solution $Y_0(t)$ which is weakly exponential with exponent zero.*

All other solutions are weakly exponential with exponent p: $\lim_{t\to\infty} \frac{\log|y(t)|}{t} = p$.

iib) $p > 0$: *If, in addition the improper integral $\int_{t_0}^{\infty}(p(t) - p)dt < \infty$ is convergent, then all other solutions $y(t)$ are strongly exponential with exponent p.*

iii) $p = 0$: *Every solution $y(t)$ is weakly exponential with exponent zero. If, in addition the improper integral $\int_{t_0}^{\infty} p(t)\,dt$ is convergent, and* $\lim_{t\to\infty} q(t) = q_0$, *then* $\lim_{t\to\infty} \frac{y(t)}{t} = q_0$.

Proofs: In case i), we can reduce to the case $p(t) = -1$ by a simple change of independent variable. If we define τ so that $\tau'(t) = -p(t)$, and $Y(\tau) = y(t)$, then the new function $Y(\tau)$ satisfies $Y' = -Y + q(t)/p(t)$. Appealing to the result in section 2.3.1, we conclude that $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} q(t)/p(t) = q_0/p$, as required.

In case ii), we appeal to the solution formula by integrating factors:

$$y(t) = \frac{y(t_0) + \int_{t_0}^{t} \pi(s)q(s)\,ds}{\pi(t)}$$

where $\pi(t) = e^{-\int_{t_0}^t p(s)\,ds}$. From the hypothesis, $t^{-1}\log\pi(t) \to -p < 0$, in particular $\pi(t) \to 0$. Let the solution $Y_0(t)$ be defined as

$$Y_0(t) = \frac{-\int_t^\infty \pi(s)q(s)\,ds}{\pi(t)}.$$

Clearly the integrals are convergent and the function $Y_0(t)$ satisfies the first-order linear equation. Given $0 < \epsilon < 2p$, we can find $t_0 > 0$ so that $e^{-(p+\epsilon/2)t} < \pi(t) < e^{-(p-\epsilon/2)t}$ for $t > t_0$. Thus for $t > t_0$,

$$\left|\int_t^\infty \pi(s)q(s)\,ds\right| \le M\int_t^\infty e^{-(p-\epsilon/2)s}\,ds = Q\frac{e^{-(p-\epsilon/2)t}}{p-\epsilon/2}.$$

Combining this with the lower bound for $\pi(t)$, we have for $t > t_0$,

$$|Y_0(t)| \le \frac{Q}{p-\epsilon/2}e^{\epsilon t},$$

which was to be proved. Now the general solution of the equation $y' = p(t)y+q(t)$ is written

$$y(t) = \frac{C}{\pi(t)}+Y_0(t) = \frac{1}{\pi(t)}\left(C+Y_0(t)\pi(t)\right) \tag{3}$$

where

$$C = y(t_0) + \int_{t_0}^\infty \pi(s)q(s)ds.$$

Note that $C \ne 0$ if and only if $y(t) \not\equiv Y_0(t)$. The first term of (3) satisfies

$$\lim_{t\to\infty} t^{-1}\log\frac{1}{\pi(t)} = p.$$

The second factor has a nonzero limit, provided that $C \ne 0$. Therefore we conclude that $\lim_{t\to\infty} t^{-1}\log|y(t)| = p$, as required.

If, in addition the improper integral $\int_{t_0}^\infty (p(t) - p)dt$ is convergent, then we can write $\int_{t_0}^t p(t)dt = p(t - t_0) + L(t)$, where $\lim_{t\to\infty} L(t)$ exists. The solution is written

$$e^{-p(t-t_0)}y(t) = e^{L(t)}\left(y(t_0) + \int_{t_0}^t \pi(s)q(s)ds\right)$$

If $C \ne 0$, then the right side has a non-zero limit, which proves that $y(t)$ is strongly exponential with exponent p.

In case iii) $p = 0$ and we can apply all of the above steps to conclude $t^{-1}\log|y(t)| \to 0$, thus weakly exponential with exponent zero. If, in addition the improper integral $\int_{t_0}^\infty p(t)\,dt$ is convergent, then we have

$$y(t) = e^{L(t)}\left(y(t_0) + \int_{t_0}^t \pi(s)q(s)ds\right)$$

Since $\pi(s) \to 1$ when $t \to \infty$, we see that $t^{-1}\int_{t_0}^\infty \pi(s)q(s)ds \to q_0$ Hence $\lim_{t\to\infty}\frac{y(t)}{t} = q_0$, which was to be proved. •

Remark: In case $p = 0$ we cannot expect in general that $y(t)/t$ will also remain bounded, as we had for the case $p(t) \equiv 0$. Consider, for example the equation $y' = \frac{a}{t}y$ which has the solution $y(t) = t^a$ where a is an arbitrary real parameter. If $a > 1$ this satisfies $\frac{y(t)}{t} \to \infty$.

Summary of Techniques Introduced

The solution of a first order linear equation with constant coefficient is either of exponential growth, of linear growth or convergent to a constant when the right side tends to a constant. If the right side has an asymptotic expansion in powers of t^{-1}, the solution has a corresponding asymptotic expansion. For linear equations with a variable coefficient which tends to a constant, the solution has a corresponding asymptotic behavior, the precise details of which depend upon the speed of convergence to the constant.

Exercises

Find the first-level asymptotic approximation of the solution of the following differential equations.

1. $y' - 3y = \sin(t^2)$
2. $y' + 3y = 5 + \frac{\cos t}{\sqrt{t}}$
3. $y' = 5 + \frac{\cos t}{\sqrt{t}}$
4. $y' = 5y + \cos(4t + 2)$
5. $y' + \frac{1-t^2}{1+t^2}y = \frac{4-t^2}{2+t^2}$
6. $y' - \frac{1-t^2}{1+t^2}y = 5\cos 3t$

For each of the following differential equations, find the first three non-zero terms in an asymptotic expansion when $t \to \infty$.

7. $y' - 3y = \frac{6-t^2}{2+t^2}$
8. $y' + \frac{1}{1+t^2}y = 3t$
9. $y' - 3y = \frac{6-t^2}{2+t^2}$
10. $y' = \frac{e^{-t}}{t}$

(11) Suppose that $p(t) \to 0$ so that $\int_{t_0}^{\infty} p(s)\,ds$ is convergent, and that $q_0 = \lim_{t\to\infty} q(t)$ is supposed to exist. Show that any solution of the equation $y' = p(t)y + q(t)$ satisfies $\lim_{t\to\infty} t^{-1}y(t) = q_0$.

(12) Suppose that $p < 0$ and let $y(t)$ be any solution of the equation $y' = py + a + b\cos ct$, where b, c are nonzero constants. Show that $\lim_{t\to\infty} y(t)$ does not exist, but that $\lim_{t\to\infty} t^{-1}\int_0^t y(s)\,ds = -a/p$.

(13) Suppose that $p < 0$ and that the function $q(t)$ satisfies $|q(t)| \leq M$ and $\lim_{t\to\infty} t^{-1}\int_0^t q(s)ds = q_0$. Prove that any solution $y(t)$ of the equation $y' = py + q(t)$ satisfies the corresponding limiting relation: $\lim_{t\to\infty} t^{-1}\int_0^t y(s)\,ds = -q_0/p$.

(14) Suppose that $p > 0$ and that the function $q(t)$ satisfies the condition that $\int_{t_0}^{\infty} |\, q\ (s)\ | e^{-ps}\,ds < \infty$. Show that the solution $y(t)$ of the equation $y' = py + q(t)$ satisfies $\lim_{t\to\infty} \frac{y(t)}{e^{pt}} = C$, a constant.

(15) Under the conditions of the previous exercise, show that if $C \neq 0$, then the solution satisfies the limiting relation $\lim_{t\to\infty} \frac{\log|y(t)|}{t} = p$. [Hint: Write $y(t) = e^{pt}(C + \epsilon(t))$, where $\epsilon(t) \to 0$. Now take logarithms and divide by t].

Which of the following functions are a) strongly exponential, b) weakly exponential, c) both, d) neither?

16. $f(t) = 5e^t + 2e^{3t}$
17. $f(t) = 5e^t + t^2e^{3t}$
18. $f(t) = t^2e^t + 2e^{3t}$
19. $f(t) = e^{t^2}$
20. $f(t) = \sum_{k=1}^{\infty}(-1)^k e^{-kt}$

3 Asymptotic Solutions of Second-Order Equations

In the previous section we obtained asymptotic solutions for first order linear equations, beginning with the explicit integral formula provided by the method of integrating factors. When we pass to second-order linear equations, it is not in general possible to write the solution in terms of integrals. However we can often convert the differential equation into an integral equation, which can be analyzed by the methods of asymptotic analysis.

As a guiding principle in all problems of asymptotic analysis, we try to express a complicated function in terms of a *simpler function* and a *smaller function*. The detailed expression of this depends on the problem at hand. In the case of differential equations, it is natural to look for the "simpler function" as the solution of a simpler differential equation.

As a practical guide to understanding the proofs below, we note that all of the formulas are derived from the variation-of-parameters representation of the solution of the equation under discussion. In this representation, we will often replace an integral over a finite interval by suitable improper integrals, whenever possible. These are more easily suited to asymptotic analysis, and will lead to interesting asymptotic formulas for the solution.

3.1 Harmonic Oscillator with external force

Many electrical and mechanical systems can be modelled by the second-order equation

$$y'' + \omega^2 y = g(t). \tag{4}$$

For example, $y(t)$ could represent the displacement from equilibrium of a spring, under the influence of external forces. In general we suppose that $g(t)$ is a continuous function which tends to zero when $t \longrightarrow \infty$ so that $\int_{t_0}^{\infty} |g(t)|dt < \infty$ for some t_0.

The general solution of equation (4) can be written as an integral, by the method of variation of parameters; in detail

$$y(t) = y(t_0)\cos\omega(t-t_0) + \frac{y'(t_0)}{\omega}\sin\omega(t-t_0) + \omega^{-1}\int_{t_0}^{t}\sin\omega(t-s)\,g(s)\,ds.$$

In many cases the integral cannot be evaluated explicitly, but the asymptotic form can be obtained. To do this, we write for $t \ge t_0$

$$\int_{t_0}^{t}\sin\omega(t-s)g(s)ds = \int_{t_0}^{\infty}\sin\omega(t-s)g(s)ds - \int_{t}^{\infty}\sin\omega(t-s)g(s)ds.$$

The convergence of the last two improper integrals follows from the hypothesis that $\int_{t_0}^{\infty}|g(t)|dt < \infty$. The sine function is less than or equal to 1 in absolute value, so that the second improper integral satisfies

$$\left|\int_{t}^{\infty}\sin\omega(t-s)g(s)ds\right| \le \int_{t}^{\infty}|g(s)|ds$$

which tends to zero when $t \longrightarrow \infty$, again from the convergence of the improper integral of g. The first improper integral can be expanded from the addition formula for the sine function:

$$\sin\omega(t-s) = \sin\omega(t-t_0)\cos\omega(t_0-s) + \cos\omega(t-t_0)\sin\omega(t_0-s)$$

$$\begin{aligned}\int_{t_0}^{\infty}\sin\omega(t-s)g(s)\,ds &= \sin\omega(t-t_0)\int_{t_0}^{\infty}\cos\omega(t_0-s)\,g(s)\,ds \\ &+ \cos\omega(t-t_0)\int_{t_0}^{\infty}\sin\omega(t_0-s)\,g(s)\,ds.\end{aligned}$$

This is a solution of the homogeneous equation $y'' + \omega^2 y = 0$, while the other improper integral tends to zero when $t \longrightarrow \infty$. The above computations can be summarized in the following proposition.

Theorem 3 *Suppose that* $\omega > 0$ *and the function* $g(t)$ *is continuous with* $\int_{t_0}^{\infty}|g(t)|\,dt < \infty$ *for some* t_0. *Then the solution of the initial-value problem*

$$y'' + \omega^2 y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_1$$

can be written

$$y(t) = C_1\cos\omega(t-t_0) + C_2\omega^{-1}\sin\omega(t-t_0) + \epsilon(t)$$

where the constants C_1, C_2 *are given by*

$$C_1 = y_0 + \omega^{-1}\int_{t_0}^{\infty}\sin\omega(t_0-s)g(s)\,ds,$$
$$C_2 = y_1 + \int_{t_0}^{\infty}\cos\omega(t_0-s)g(s)\,ds$$

and where the function $\epsilon(t) \longrightarrow 0$ *when* $t \longrightarrow \infty$ *so that* $\epsilon(t) \le \omega^{-1}\int_{t}^{\infty}|g(s)|ds$.

Proof. It only remains to identify $\epsilon(t) = -\omega^{-1}\int_{t}^{\infty}\sin\omega(t-s)g(s)ds \le \omega^{-1}\int_{t}^{\infty}|g(s)|ds$.

Example 8 Find the asymptotic form of the solution of $y'' + y = e^{-t}$ when $t \longrightarrow \infty$.

Solution. In this case we can take $t_0 = 0$: the integrals can be computed explicitly with the results

$$\int_0^{\infty}e^{-t}\cos t\,dt = \tfrac{1}{2}, \quad \int_0^{\infty}e^{-t}\sin t\,dt = \tfrac{1}{2}$$

so that the asymptotic form of the solution is

$$\left(y(0) + \tfrac{1}{2}\right)\cos t + \left(y'(0) + \tfrac{1}{2}\right)\sin t \quad \bullet$$

Exercises

Find a suitable asymptotic formula, containing two arbitrary constants, for the solution of the following second-order equations when $t \to \infty$

1. $y'' + 4y = \frac{1}{1+t^2}$
2. $y'' + 4y = 1 + \frac{1}{1+t^2}$
3. $y'' + 25y = 100 + \frac{1}{1+t^2}$
4. $y'' + 4y = \cos t + \frac{1}{1+t^2}$

5. Suppose that we can choose the initial conditions at will. For what choice of the constants y_0, y_1 is it true that the solution of the initial-value problem $y'' + 4y = e^{-t}, y(0) = y_0, y'(0) = y_1$ has the asymptotic form $y(t) = 5\cos 2t + O(1/t), t \to \infty$?

6. Suppose that we can choose the initial conditions at will. For what choice of the constants y_0, y_1 is it true that the solution of the initial-value problem $y'' + 4y = e^{-t}, y(0) = y_0, y'(0) = y_1$ has the asymptotic form $y(t) = O(1/t), t \to \infty$?

7. Suppose that $g(t)$ is a continuous function for which $g(t) \to 0$ when $t \to \infty$ and with a continuous derivative for which $\int_{t_0}^{\infty} |g'(t)|dt < \infty$ for some t_0. Find an asymptotic formula for the solution of the equation $y'' + \omega^2 y = g(t)$ [Hint: Integrate-by-parts in the variation-of-parameters formula for the solution]

8. Use the previous exercise to find an asymptotic formula for the solution of the equation $y'' + 4y = \frac{1}{1+t}, t \to \infty$

3.2 Oscillator with almost-constant frequency

We now consider the homogeneous equation

$$y'' + (\omega^2 + h(t))y = 0$$

where $h(t) \longrightarrow 0$ when $t \longrightarrow \infty$ so that $\int_{t_0}^{\infty} |h(t)|dt < \infty$. This can be written in the form of the previous section as $y'' + \omega^2 y = g(t)$, where $g(t) = -h(t)y(t)$. To apply the results developed there, we must show that $g(t)$ satisfies the hypothesis that $\int_{t_0}^{\infty} |g(t)|dt < \infty$.

Lemma 1 *Any solution of the homogeneous equation satisfies*

$$|y(t)| \le K_1, \quad |y'(t)| \le K_2$$

for suitable constants K_1, K_2.

Proof. If $y(t)$ is identically zero, then the result is automatic. Otherwise, we consider the expression $E(t) = y'(t)^2 + \omega^2 y(t)^2$ which is everywhere positive. Computing the derivative by calculus, we find that

$$\begin{aligned} E'(t) &= 2y'(t)y''(t) + 2\omega^2 y(t)y'(t) \\ &= 2y'(t)(-\omega^2 y(t) - h(t)y(t)) + 2\omega^2 y(t)y'(t) \\ &= -2h(t)y(t)y'(t). \end{aligned}$$

If we now use the inequality $2ab \le (a^2 + b^2)$ with the values $a = y'(t)/\omega, b = \omega y(t)$ we find that

$$E'(t) \le \frac{|h(t)|}{\omega^2} E(t).$$

Forming the logarithmic derivative, we have

$$\begin{aligned} \frac{d}{dt}\log E(t) &= \frac{E'(t)}{E(t)} \\ &\le \frac{|h(t)|}{\omega^2}, \\ \log E(t) &\le \log E(t_0) + \int_{t_0}^{t} \frac{|h(s)|}{\omega^2}\,ds \\ &\le \log E(t_0) + \int_{t_0}^{\infty} \frac{|h(s)|}{\omega^2}\,ds < \infty. \end{aligned}$$

This proves that $E(t) \le K$ for the constant $K = E(t_0)\exp \int_{t_0}^{t} \omega^{-2}|h(s)|\,ds$. But $E(t) \ge \omega^2 y(t)^2$, so that the corresponding inequality is inherited by the function $y(t)$: $y(t)^2 \le \frac{K}{\omega^2}$. Using $E(t) \ge y'(t)^2$, we see that we have the corresponding estimate for $y'(t)$ •

Now we can state the corresponding theorem for the homogeneous equation.

Theorem 4 *Suppose that $\omega > 0$ and the function $h(t)$ is continuous with $\int_{t_0}^{\infty} |h(t)|dt < \infty$. Then the solution of the initial-value problem*

$$y'' + (\omega^2 + h(t))y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1$$

can be written

$$y(t) = C_1 \cos\omega(t - t_0) + C_2\omega^{-1} \sin\omega(t - t_0) + \epsilon(t),$$

where the function $\epsilon(t) \longrightarrow 0$ when $t \longrightarrow \infty$, so that $\epsilon(t) \le \omega^{-1}K_1 \int_t^{\infty} h(s)\,ds$ and where the constants C_1, C_2 satisfy

$$C_1 = y_0 + \omega^{-1}\int_{t_0}^{\infty} \sin\omega(t_0 - s)\,h(s)y(s)\,ds \le |y_0| + K_1\omega^{-1}\int_{t_0}^{\infty} |h(s)|ds,$$

$$C_2 = y_1 + \int_{t_0}^{\infty} \cos\omega(t_0 - s)\,h(s)y(s)\,ds \le |y_1| + K_1\int_{t_0}^{\infty} |h(s)|ds.$$

In contrast to the previous section, the constants C_1, C_2 which appear in the asymptotic solution cannot be explicitly identified in terms of the initial data of the problem. However their existence is without question.

Proof of the theorem. It is immediate from lemma 2 that the function $g(t) = h(t)y(t)$ satisfies the condition that $\int_{t_0}^{\infty}|g(t)|\,dt < \infty$. Therefore we can apply the result of theorem 2 to conclude the asserted asymptotic formula. •

The above result will now be extended to certain equations for which the function $h(t) \to 0$ with $\int_{t_0}^{\infty}|h(t)|\,dt = \infty$, for example $h(t) = 1/t$. In order to deal with problems of this type, we develop the method of *phase plane analysis*, whereby we introduce a suitable system of polar coordinates (r,θ) in the plane of (y,y') and re-write the second-order linear differential equation as a system of two non-linear equations for r',θ'. The appropriate polar coordinate formulas are

$$\omega y(t) = r(t)\sin\theta(t), \qquad y'(t) = r(t)\cos\theta(t).$$

We perform the necessary derivatives and use the equation $y'' + (\omega^2 + h(t))y = 0$, resulting in the system of equations

$$\theta'(t) = \omega + \tfrac{h(t)}{\omega}\sin^2\theta(t), \qquad r'(t) = -\tfrac{r(t)}{\omega}h(t)\sin\theta(t)\cos\theta(t).$$

From the hypothesis $h(t) \to 0$, we immediately conclude that $\theta'(t) \to \omega$ and hence $\theta(t)/t \to \omega$ when $t \to \infty$; in particular the solution $y(t)$ is zero infinitely often when $t \to \infty$. In order to estimate the amplitude $r(t)$, we note that if the solution is not identically zero, then $r(t) > 0$ for all t and we can write

$$\begin{aligned}
\log r(t) - \log r(t_0) &= -\frac{1}{\omega}\int_{t_0}^{t} h(s)\sin\theta(s)\cos\theta(s)\,ds \\
&= -\frac{1}{2\omega}\int_{t_0}^{t}\frac{h(s)}{\theta'(s)}\,d(\sin^2\theta(s)) \\
&= -\frac{1}{2\omega}\frac{h}{\sin^2\theta}\theta'\Big|_{t_0}^{t} + \frac{1}{2\omega}\int_{t_0}^{t}\sin^2\theta(s)\,d\Big(\frac{h(s)}{\theta'(s)}\Big).
\end{aligned}$$

In the final integral we write $(h/\theta')' = (\theta'h' - h\theta'')/\theta'^2$ and use the differential equation for θ' to write

$$\theta'' = \tfrac{h'}{\omega}\sin^2\theta + \tfrac{2h\theta'}{\omega}\sin\theta\cos\theta.$$

From the differential equation for θ' we see that $\omega/2 < \theta' < 2\omega$ for large t and so we can estimate the above second derivative by

$$|\theta''| \le \tfrac{|h'|}{\omega} + 2|h|,$$

with the conclusion that

$$\Big|\Big(\tfrac{h(t)}{\theta'(t)}\Big)'\Big| \le 2\tfrac{|h'(t)|}{\omega} + \tfrac{4|h(t)|}{\omega^2}\Big(\tfrac{|h'(t)|}{\omega} + 2|h(t)|\Big).$$

If both h' and h^2 are absolutely integrable, then the final integral is convergent, leading us to the following result.

Theorem 5 *Suppose that $\omega > 0$ and the function $h(t)$ is continuous with a continuous derivative so that $h(t) \to 0$ and for some t_0*

$$\int_{t_0}^{\infty} (h(t)^2 + |h'(t)|)\, dt < \infty.$$

Then the solution of the initial-value problem

$$y'' + (\omega^2 + h(t))y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1$$

can be written $y(t) = \omega^{-1} r(t) \sin\theta(t)$, $y'(t) = r(t)\cos\theta(t)$ where the function $\theta(t) \longrightarrow \infty$ when $t \longrightarrow \infty$, and the function $r(t) \longrightarrow r_\infty$, a finite constant, when $t \longrightarrow \infty$. In particular the solution remains bounded and passes through zero infinitely often for large t.

Exercises

Find a suitable asymptotic formula for the solution of the following second-order equations

1. $y'' + \frac{4t^2}{1+t^2} y = 0$, 2. $y'' + \frac{4t}{1+t} y = 0$.

3.3 Equations with a first-order term

We now extend the discussion to second order equations of the form

$$y'' + p(t)y' + (\omega^2 + h(t))y = 0$$

where $\omega > 0$, the function $h(t)$ satisfies $\int_{t_0}^{\infty} |h(t)|\, dt < \infty$ for some t_0 and $p(t)$ satisfies some corresponding conditions.

In order to perfom the asymptotic analysis, we reduce to the previously studied case by means of the substitution $z(t) = \mu(t)y(t)$ where $\mu(t)$ satisfies the equation $2\mu'(t) = p(t)\mu(t)$, hence $\mu(t) = \exp(\frac{1}{2}\int p(t)dt)$. Applying the second derivative, we find $z'' = \mu y'' + 2\mu' y' + \mu'' y$ and the new differential equation

$$z''(t) + [\omega^2 + h(t) + \tfrac{\mu''}{\mu}]z(t) = 0$$

which is of the form studied in the previous subsection, provided that $\frac{\mu''}{\mu}$ has a finite improper integral. This can be expressed directly in terms of the function $p(t)$ by writing

$$\mu' = \tfrac{1}{2}\mu p, \qquad \mu'' = \tfrac{1}{2}(\mu' p + \mu p') = \mu(\tfrac{p^2}{4} + \tfrac{p'}{2}).$$

If these terms involving $p(t)$ have a finite improper integral, then we can apply the previous results to conclude

$$z(t) = C_1 \cos\omega(t - t_0) + C_2\omega^{-1} \sin\omega(t - t_0) + \epsilon(t)$$

for suitable constants C_1, C_2 and where the function $\epsilon(t) \to 0$ when $t \to \infty$. Returning to the original solution $y(t)$, we can summarize the above discussion as follows:

Theorem 6 *Suppose that $\omega > 0$, that $y(t)$ is a solution of*

$$y'' + p(t)y' + (\omega^2 + h(t))y = 0$$

and that the continuous functions $h(t), p(t)$ satisfy

$$\int_{t_0}^{\infty} |h(t)|\,dt < \infty, \quad \int_{t_0}^{\infty} |p'(t)|\,dt < \infty, \quad \int_{t_0}^{\infty} |p(t)|^2\,dt < \infty,$$

for some t_0. Then we have the asymptotic formula

$$y(t) = \frac{1}{\mu(t)}\left(C_1 \cos\omega(t - t_0) + C_2\omega^{-1}\sin\omega(t - t_0) + \epsilon(t)\right), \quad \mu(t) = \exp\frac{1}{2}\int_{t_0}^{t} p(s)\,ds$$

for suitable constants C_1, C_2 and where the function $\epsilon(t) \to 0$ with

$$\epsilon(t) \le \omega^{-1}\int_t^{\infty}\left(|h(s)| + \tfrac{1}{2}|p'(s)| + \tfrac{1}{4}p(s)^2\right)ds.$$

Example 9 *Find an asymptotic formula for the solution of Bessel's equation* $y'' + \frac{1}{t}y' + y = 0$.

Solution. In this case we have $p(t) = 1/t, \omega = 1, h(t) = 0$. Taking $t_0 = 1$, we have $\mu(t) = \sqrt{t}$. The above discussion provides the asymptotic solution $y(t) = t^{-1/2}[C_1\cos t + C_2 \sin t + O(1/t)]$ for suitable constants C_1, C_2.

It should be noted that the asymptotic methods discussed here do not provide the explicit values of the constants which occur in the asymptotic formulas. These must be obtained by other methods; for example they can often be estimated numerically from computer-assisted graphics.

Exercises

Find a suitable asymptotic formula for the solution of the following second-order equations

1. $y'' + \frac{2}{t}y' + 4y = 0$
2. $y'' + \frac{1}{t}y' + (1 - \frac{m^2}{t^2})y = 0$, where m is a constant.
3. $y'' + (\frac{1}{t} + \frac{3}{t^2})y' + 4y = 0$ [Hint: Apply the standard substitution to remove the y' term].

3.4 Oscillator with variable frequency

We now consider the homogeneous equation

$$y'' + \omega(t)^2\, y = 0$$

where $\omega(t) > 0$ for $t \ge t_0$. For this equation, it is not immediately clear what "simpler equation" can be used to make the asymptotic analysis. We will do this by a two-step reduction to the previous cases, where we introduce a new independent variable and then a suitable integrating factor.

For the first step of the reduction we consider a new independent variable s which defines a new function f so that $y(t) = f(s)$. Computing the successive derivatives, we have by the chain rule

$$y'(t) = f'(s)\frac{ds}{dt}, \qquad y''(t) = f''(s)\left(\frac{ds}{dt}\right)^2 + f'(s)\frac{d^2s}{dt^2}.$$

This leads to the choice

$$\frac{ds}{dt} = \omega(t), \qquad s(t) = \int_{t_0}^{t}\omega(\tau)\,d\tau$$

and to the differential equation in the form

$$f''(s) + \frac{\omega'(t)}{\omega(t)^2} f'(s) + f(s) = 0.$$

The middle term can be expressed directly in terms of the new variable s where we write $\omega(t) = W(s)$, so that $\omega'(t) = W'(s)ds/dt = W'(s)\omega(t)$ which cancels one factor of ω and the yields the differential equation in the form

$$f''(s) + \frac{W'(s)}{W(s)} f'(s) + f(s) = 0.$$

This completes the first step of the reduction.

Example 10 *Find the first step of the reduction of Airy's equation $y'' + ty = 0$ with $t_0 = 0$.*

Solution. In this case we have $\omega(t) = \sqrt{t}$ and $s(t) = (2/3)t^{3/2}$, so that $t = (3s/2)^{2/3}$ and $\omega(t) = W(s) = (3s/2)^{1/3}$, $dW/ds = 1/2(3s/2)^{-2/3}$, $\frac{1}{W} dW/ds = 1/3s$ and the new equation

$$f''(s) + \tfrac{1}{3s} f'(s) + f(s) = 0 \quad \bullet$$

Returning to the theory, the second step of the reduction consists in removing the first derivative term, as we have done in the previous section. This is accomplished by means of the function

$$\mu(s) = \exp \int \tfrac{W'(s)}{2W(s)}\, ds = \sqrt{W(s)}$$

and writing the solution in terms of $F(s) := \mu(s)f(s) = \sqrt{W(s)}f(s)$. The second derivative is computed as

$$F''(s) = \sqrt{W} f'' + \tfrac{W'}{\sqrt{W}} f' + f\sqrt{W}''.$$

The differential equation for f allows us to replace the first two terms by $-\sqrt{W}(s)f(s)$ and to rewrite the differential equation as

$$F'' + (1 - \tfrac{\sqrt{W}''}{\sqrt{W}})F = 0$$

This completes the second stage of the reduction.

Example 11 *Complete the second stage of the reduction for Airy's equation $y'' + ty = 0$.*

Solution. We found in the previous example that for this equation $W(s) = (3s/2)^{1/3}$, so that $\sqrt{W} = (3s/2)^{1/6}$, $\frac{\sqrt{W}''}{\sqrt{W}} = -(5/36s^2)$ and the equation $F'' + (1 + \frac{5}{36s^2})F = 0$ $\bullet$

Having completed the second stage of the reduction, we can apply the theory for the oscillator with almost constant frequency. If the function $\frac{\sqrt{W}''}{\sqrt{W}}$ tends to zero with a convergent improper integral, we can apply the previous results to obtain the following.

Theorem 7 *Suppose that $\omega(t) > 0$ is a continuous function defined for $t \geq t_0$ with a continuous second derivative. Define a new integration variable and a new function by*

$$s(t) = \int_{t_0}^{t} \omega(\tau)d\tau, \qquad W(s) = \omega(t)$$

Suppose that $s(\infty) = \infty$ and that the improper integral $\int_0^\infty |\frac{\sqrt{W}''(s)}{\sqrt{W(s)}}|ds < \infty$. Let $y(t)$ be any solution of the second order equation $y'' + \omega(t)^2 y = 0$ defined for $t \geq t_0$. Then we have the asymptotic formula when $t \longrightarrow \infty$:

$$y(t) = \frac{R\cos(s-\alpha) + \epsilon(s)}{\sqrt{W(s)}}$$

For suitable constants R, α and where the function $\epsilon(s) \longrightarrow 0$ when $s \longrightarrow \infty$ so that $\epsilon(s) \leq \int_s^\infty \frac{\sqrt{W}''}{\sqrt{W}} d\tau$.

Example 12 *Find an asymptotic formula for the solution of Airy's equation $y'' + ty = 0$.*

Solution. Referring to the previous two examples, we have $\omega(t) = \sqrt{t}, s(t) = (2/3)t^{3/2}, W(s) = (3s/2)^{1/3}$ and the asymptotic formula

$$y(t) = \frac{R\cos(s-\alpha) + O(s^{-1})}{s^{1/6}} = \frac{R\cos(\frac{2}{3}t^{3/2} - \alpha) + O(\frac{1}{t^{3/2}})}{t^{1/4}} \quad \bullet$$

Exercises

Find a suitable asymptotic formula for the solution of the following second-order equations when $t \to \infty$.

1. $y'' + t^2 y = 0$
2. $y'' + t^3 y = 0$

3.5 Damped Harmonic Oscillator with external force

In this section we extend the discussion of asymptotic solutions to the equation

$$my'' + cy' + ky = g(t).$$

where $m > 0, c > 0, k > 0$ and $g(t)$ is a continuous function which tends to zero when $t \longrightarrow \infty$. Since all solutions of the corresponding homogeneous equation tend to zero, we expect that the same will be true of the given (nonhomogenous) equation. This will be supplemented by an asymptotic expansion, as described in the following theorem.

Theorem 6 *Suppose that $m > 0, c > 0, k > 0$ and $g(t) \to 0$ when $t \to \infty$. Then any solution of the equation $my'' + cy' + ky = g(t)$ satisfies $\lim_{t\to\infty} y(t) = 0$. If in addition $g(t)$ has an asymptotic expansion*

$$g(t) = \sum_{k=1}^{N} \frac{g_k}{t^k} + O(t^{-(N+1)}) \qquad t \longrightarrow \infty,$$

then $y(t)$ has a corresponding asymptotic expansion

$$y(t) = \sum_{k=1}^{N} \frac{y_k}{t^k} + O(t^{-(N+1)}) \qquad t \longrightarrow \infty,$$

where the coefficients y_k can be obtained by direct substitution.

Proof. The general solution of this equation can be written in terms of the solution of the homogeneous equation and an integral from the method of variation of parameters; in detail

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + \int_{t_0}^{t} K(t-s)\,g(s)\,ds$$

where $K(t)$ is the solution of the homogeneous equation with $K(0) = 0, K'(0) = 1$. The precise formula depends on whether we are in the overdamped, critically damped or underdamped case. In detail: If $c^2 > 4mk$, then $y_i(t) = e^{r_i t}$ and $K(t) = \left(\frac{e^{r_1 t} - e^{r_2 t}}{r_1 - r_2}\right)$, where $mr_i^2 + cr_i + k = 0$.
If $c^2 = 4mk$, then $y_1(t) = e^{rt}, y_2(t) = te^{rt}$ and $K(t) = te^{rt}$, where $r = -c/2m$.
If $c^2 < 4mk$, then $y_1(t) = e^{-\lambda t}\cos\mu t$, $y_2(t) = e^{-\lambda t}\sin\mu t$, and $K(t) = e^{-\lambda t}\frac{\sin\mu t}{\mu}$, where $\lambda = c/2m, \mu = \sqrt{4mk - c^2}/2m$.

The first two terms are solutions of the homogeneous equation and clearly tend to zero when $t \to \infty$. To estimate the integral terms, we fix $T = T(\epsilon)$ so large that $|g(t)| \le \epsilon$ when $t > T$. We write

$$\int_{t_0}^{t} K(t-s)g(s)\,ds = \int_{t_0}^{T} K(t-s)g(s)ds + \int_{T}^{t} K(t-s)g(s)ds.$$

The first integral is over an interval of fixed length, independent of t. From the form of the function $K(t)$ in each of the three cases we see that $K(t) \to 0$, hence the integral tends to zero. For the second integral, we have $|g(s)| < \epsilon$ throughout the region of integration, while the integral of the function K is bounded by a constant, independent of t. This proves that the second integral tends to zero when $t \to \infty$, hence the entire solution tends to zero.

To prove the existence of an asymptotic expansion, we first note that, by a single integration-by-parts, we have for $\lambda > 0$

$$\int_T^t \frac{e^{\lambda s}}{s^k}\,ds = \int_T^t \frac{1}{\lambda s^k}\,de^{\lambda s} = \left(\frac{1}{\lambda t^k}e^{\lambda t}\right) - \left(\frac{1}{\lambda T^k}e^{\lambda T}\right) + \frac{k}{\lambda}\int_T^t \frac{1}{s^{k+1}}e^{\lambda s}\,ds$$
$$\le \frac{1}{\lambda t^k}e^{\lambda t} + \frac{k}{\lambda T}\int_T^t \frac{1}{s^k}e^{\lambda s}\,ds.$$

Moving the last integral to the left side, we have

$$\left(1 - \frac{k}{\lambda T}\right)\int_T^t \frac{e^{\lambda s}}{s^k}\,ds \le \frac{e^{\lambda t}}{\lambda t^k}. \tag{5}$$

which is a useful inequality. If we continue the integration-by-parts, we can obtain the asymptotic result that for any $N = 1, 2, \dots, k = 1, 2, \dots$ and $\lambda > 0$

$$\int_T^t \frac{e^{\lambda s}}{s^k}\,ds = e^{\lambda t}\left(\frac{1}{\lambda t^k} + \frac{k}{\lambda^2 t^{k+1}} + \frac{k(k+1)}{\lambda^3 t^{k+2}} + \cdots + \frac{k(k+1)\cdots(k+N-1)}{\lambda^{N-1} t^{k+N}} + O\left(\frac{1}{t^{k+N+1}}\right)\right). \tag{6}$$

To prove the asymptotic expansion, first consider the overdamped case $c^2 > 4mk$. In this case we must analyze the integral

$$\int_{t_0}^{t} \frac{e^{-\lambda_1(t-s)} - e^{-\lambda_2(t-s)}}{\lambda_1 - \lambda_2}\,g(s)\,ds$$

where $g(s)$ is a sum of terms proportional to s^{-k} and another term which is $O(t^{-(N+1)}), t \to \infty$. From (6) it follows that

$$\int_T^t \frac{e^{-\lambda(t-s)}}{s^k}\,ds = \left(\frac{1}{\lambda t^k} + \frac{k}{\lambda^2 t^{k+1}} + \frac{k(k+1)}{\lambda^3 t^{k+2}} + \cdots + \frac{k(k+1)\cdots(k+N-1)}{\lambda^{N-1} t^{k+N}}\right) + O\left(\frac{1}{t^{k+N+1}}\right)$$

from which we obtain the existence of the asymptotic expansion. The remainder term is estimated by application of (5).

In the critically damped case $c^2 = 4mk$, we must analyze the integral $\int_{t_0}^t (t-s)e^{r(t-s)} g(s)\,ds$, where $g(t)$ is a sum of powers of t^{-1} and a remainder term. Writing $r = -\lambda$, a typical term can be integrated-by-parts in the following form:

$$\int_{t_0}^t \frac{1}{s^k}(t-s)e^{-\lambda(t-s)}\,ds = \int_{t_0}^t \frac{1}{s^k}\frac{d}{ds}\Big(\frac{1+\lambda(t-s)}{\lambda}e^{-\lambda(t-s)}\Big)\,ds$$

$$= \frac{1}{\lambda t^k} + \frac{k}{\lambda}\int_{t_0}^t \frac{1}{s^{k+1}}(1+\lambda(t-s))e^{-\lambda(t-s)}\,ds + O(te^{-\lambda t}),\, t \to \infty.$$

Repeated integration-by-parts gives the necessary asymptotic expansion. The remainder term is again estimated by (5).

Finally, we consider the underdamped case $c^2 < 4mk$. We note that the right side consists of terms proportional to t^{-k} and another term which is less than a constant multiplied by $t^{-(N+1)}$. Thus we must analyze an integral of the form $\int_{t_0}^t e^{-\lambda(t-s)}\frac{\sin\omega(t-s)}{\omega}\frac{1}{s^k}\,ds$.

From (5), any such integral is less than a constant times $t^{-k}, t \to \infty$. To obtain the asymptotic expansion, we write

$$\int_{t_0}^t e^{-\lambda(t-s)}\frac{\sin\omega(t-s)}{\omega}\frac{1}{s^k}\,ds = \int_{t_0}^t \frac{1}{s^k}\frac{d}{ds}\Big(e^{-\lambda(t-s)}\frac{\lambda\sin\omega(t-s)+\omega\cos\omega(t-s)}{\omega(\lambda^2+\omega^2)}\Big)\,ds$$

$$= \frac{1}{(\lambda^2+\omega^2)t^k} + k\int_{t_0}^t e^{-\lambda(t-s)}\frac{\lambda\sin\omega(t-s)+\omega\cos\omega(t-s)}{\lambda^2+\omega^2}\frac{1}{s^{k+1}}\,ds + O(e^{-\lambda t}),\, t \to \infty.$$

The new integral is of the same form as the original integral and can be integrated-by-parts again to obtain the terms of order $\frac{1}{t^{k+1}}, \frac{1}{t^{k+2}}, \ldots$ and to determine the remainder term.

Having proved the *existence* of the asymptotic expansion in each of the three cases, we now show how the coefficients may be obtained in a more direct computational fashion without dealing with integration. To do this, we look for a formal expansion of the desired form and equate the corresponding coefficients of t^{-1}, t^{-2}, etc. This leads to the formal equality

$$\begin{aligned} my'' + c' + ky &= \frac{ky_1}{t} + \frac{ky_2 - cy_1}{t^2} + \frac{ky_3 - 2cy_2 + 2my_1}{t^3} + \cdots \\ &+ \frac{ky_{n+1} - ncy_n + n(n-1)my_{n-1}}{t^{n+1}} \\ &= \frac{g_1}{t} + \frac{g_2}{t^2} + \cdots + \frac{g_{n+1}}{t^{n+1}} \end{aligned}$$

and to the chain of equations

$$ky_1 = g_1, ky_2 - cy_1 = g_2, ky_3 - 2cy_2 + 2my_1 = g_3, \ldots$$
$$ky_{n+1} - ncy_n + n(n-1)my_{n-1} = g_{n+1} \qquad n = 1, 2, \ldots .$$

Since $k > 0$ these can be solved uniquely to determine the necessary coefficients $y_1 \ldots, y_{n+1}$. In order to prove that this expansion agrees with the solution to within $O(t^{-(n+1)})$ we recall the first part of the proof. If $Y(t) = \sum_{k=1}^{N} \frac{y_k}{t^k}$ denotes the approximate solution just obtained, then by construction we have $L[Y] = \sum_{k=1}^{N} \frac{g_k}{t^k} + O(t^{-(N+1)}), t \to \infty$. But the given solution $y(t)$ also satisfies this same relation, by hypothesis; hence the difference $y - Y$ satisfies $L[y - Y] = O(t^{-(N+1)}), t \to \infty$. Now $y(t) - Y(t)$ is the sum of a homogeneous solution and the integral obtained from the method of variation of parameters, which we break into two integrals as before. The homogeneous solution tends to zero exponentially fast, as does the first part of the integral of the variation-of-parameters solution. In the second integral the integrand $g(t) = O(t^{-(n+1)}), t \to \infty$, while the integral of $K(t)$ is finite. This allows us to conclude that the second integral $= O(t^{-(n+1)}), t \to \infty$ •

Exercises

Find a suitable asymptotic formula for the solution of the following second-order equations when $t \to \infty$.

1) $y'' + 4y' + 5y = \frac{3}{t}$ 2) $y'' + 4y' + 5y = 20 + \frac{3}{t}$

3. Use repeated integration-by-parts to prove the asymptotic estimate (6) in the text.

4. Suppose that the function $g(t)$ has an asymptotic expansion of the form
$$g(t) = \sum_{k=1}^{N} \frac{g_k \cos t}{t^k}.$$
Show how to obtain a corresponding asymptotic expansion of the solution.

Asymptotic behavior of nonoscillatory equations

In the previous sections we determined the asymptotic behavior of the solutions of second-order linear equations based on the simple harmonic oscillator. In these cases we found that all solutions have the same oscillatory behavior. In this section we consider the case of equations which admit exponential growth or decay, corresponding to the second-order equation $y'' - (b^2 + h(t))y = 0$, where $b > 0$ and $h(t) \to 0$ when $t \to \infty$. We expect that the solution can be asymptotically approximated by a combination of e^{bt} and e^{-bt}, which are the solutions in case $h(t) \equiv 0$. We will prove the following precise result

Theorem 9 *Suppose that $b > 0$ and $\int_a^\infty |h(t)|\,dt < \infty$ for some $a > 0$. Then there is a fundamental set of solutions $\{y_1, y_2\}$ of the differential equation $y'' - (b^2 + h(t))y = 0$ so that*

$$y_1(t)e^{bt} = 1 + \epsilon_1(t), \qquad y_2(t)e^{-bt} = 1 + \epsilon_2(t)$$

where $\lim_{t\to\infty} \epsilon_i(t) = 0$, with $\epsilon_i(t) = O(e^{-bt} + \int_t^\infty h(s)ds)$, $i = 1, 2$.

Proof. For the first part, we define $x(t) = y(t)e^{bt}$, which solves the equation $x'' - 2bx' = h(t)x$. The general solution is obtained from the variation-of-parameters formula as

$$x(t) = C_1 e^{2bt} + C_2 + \tfrac{1}{2b}\int_{t_0}^t \left(e^{2b(t-s)} - 1\right)x(s)\,h(s)\,ds.$$

Here t_0 is chosen so that $\int_{t_0}^\infty |h(t)|\,dt \le b$. We need to choose the constants C_1, C_2 so that the solution satisfies $\lim_{t\to\infty} x(t) = 1$. This is done by solving the integral equation

$$x(t) = 1 - \tfrac{1}{2b}\int_t^\infty \left(e^{2b(t-s)} - 1\right)x(s)\,h(s)\,ds \tag{7}$$

by the method of successive approximations, by defining $x_0(t) = 1$ and

$$x_{n+1}(t) = 1 - \tfrac{1}{2b}\int_t^\infty \left(e^{2b(t-s)} - 1\right)x_n(s)\,h(s)\,ds.$$

We will show that $|x_n(t) - x_{n-1}(t)| \le 2^{-n}$ for $n \ge 1$ and $t \ge t_0$. Clearly

$$|x_1(t) - x_0(t)| \le \tfrac{1}{2b}\int_t^\infty |(e^{2b(t-s)} - 1)h(s)|ds \le \tfrac{1}{2b}\int_t^\infty |h(s)|ds \le \tfrac{1}{2}.$$

Assuming the inequality for the value n, we have

$$\begin{aligned}
|x_{n+1}(t) - x_n(t)| &= \frac{1}{2b}\left|\int_t^\infty \left(e^{2b(t-s)} - 1\right)\left(x_n(s) - x_{n-1}(s)\right)h(s)\,ds\right| \\
&\le \frac{1}{2b}\int_t^\infty |x_n(s) - x_{n-1}(s)|\,|h(s)|\,ds \\
&\le \frac{1}{2b}2^{-n}\int_{t_0}^\infty |h(s)|ds \\
&\le 2^{-(n+1)}
\end{aligned}$$

which was to be proved. Therefore the sequence $x_n(t)$ is uniformly convergent to a limit function $x(t)$ which satisfies the desired integral equation and satisfies $|x(t)| \le 2$ for all $t \ge t_0$. To prove the required convergence, we have

$$|x(t) - 1| \le \tfrac{1}{2b}\int_t^\infty |e^{2b(t-s)} - 1|x(s)h(s)\,ds \le \tfrac{1}{b}\int_t^\infty |h(s)|\,ds \to 0.$$

To prove the existence of the solution $y_2(t)$, we define $z(t) = y(t)e^{-bt}$, which satisfies the equation $z'' + 2bz' = h(t)z$. From the variation of parameters, the general solution is written

$$z(t) = C_1 + C_2 e^{-2bt} + \tfrac{1}{2b}\int_{t_0}^t \left(1 - e^{-2b(t-s)}\right)h(s)z(s)\,ds$$

where t_0 is again chosen so that $\int_{t_0}^\infty |h(t)|\,dt \le b$. Because of the exponential increase of the integrand, we cannot simply replace all of the integrals by improper integrals in choosing the correct solution. This will be obtained as the unique solution of the integral equation

$$z(t) = 1 - \frac{1}{2b}\int_{t_0}^{t} e^{-2b(t-s)}h(s)z(s)\,ds - \frac{1}{2b}\int_{t}^{\infty} h(s)z(s)\,ds \qquad (8)$$

by setting $z_0(t) = 1$ and

$$z_{n-1}(t) = 1 - \frac{1}{2b}\int_{t_0}^{t} e^{-2b(t-s)}h(s)z(s)\,ds - \frac{1}{2b}\int_{t}^{\infty} h(s)z_n(s)\,ds.$$

As above, we have

$$|z_1(t) - z_0(t)| \le \frac{1}{2b}\int_{t_0}^{t} e^{-2b(t-s)}|h(s)|\,ds + \frac{1}{2b}\int_{t}^{\infty} |h(s)|\,ds \le \frac{1}{2b}\int_{t_0}^{\infty} |h(s)|ds \le \frac{1}{2}$$

and similarly one obtains $|z_n(t) - z_{n-1}(t)| \le 2^{-n}$ for all $n \ge 1, t \ge t_0$. Therefore the sequence $z_n(t)$ is uniformly convergent to a limit function $z(t)$ which satisfies the desired integral equation and satisfies $|z(t)| \le 2$ for all $t \ge t_0$. To prove the required convergence, we have

$$|z(t) - 1| \le \frac{1}{2b}\int_{t_0}^{t} e^{-2b(t-s)}|h(s)z(s)|\,ds + \frac{1}{2b}\int_{t}^{\infty} |h(s)z(s)|ds$$

The second integral clearly tends to zero at the required rate when $t \to \infty$. For the first integral, we have for $t \ge 2t_0$,

$$\begin{aligned} \left|\int_{t_0}^{t} e^{-2b(t-s)}h(s)z(s)\,ds\right| &\le \int_{t_0}^{\frac{t}{2}} e^{-2b(t-s)}|h(s)z(s)|ds + \int_{\frac{t}{2}}^{t} e^{-2b(t-s)}|h(s)z(s)|ds \\ &\le e^{-bt}\int_{t_0}^{\infty} |h(s)|\,ds + \int_{\frac{t}{2}}^{\infty} |h(s)|\,ds \to 0 \end{aligned}$$

at the required rate, which completes the proof •

The above result can be used to determine the asymptotic behavior of the solution of the Airy equation $y'' - ty = 0$, when $t \to \infty$. Following the discussion in section 4.3.4, we define a new independent variable s and a new function $f(s)$ by $s(t) = (2/3)t^{3/2}$ and $y(t) = f(s)$, resulting in the equation

$$f''(s) + \frac{1}{3s}f'(s) - f(s) = 0.$$

The first-derivative term is removed by the further substitution $F(s) = (3s/2)^{1/6} f(s)$, resulting in the equation

$$F'' - (1 - \frac{5}{36s^2})F = 0.$$

According to the above theorem, this equation has a fundamental set of solutions $F_1(s), F_2(s)$ with $F_1(s) = e^s(1 + O(s^{-1}))$, $F_2(s) = e^{-s}(1 + O(s^{-1})), s \to \infty$. Re-writing this in terms of the original notations, we have

$$\begin{aligned} y_1(t) &= \frac{e^s(1 + O(s^{-1}))}{s^{1/6}} = \frac{e^{(2/3)t^{3/2}}(1 + O(t^{-3/2}))}{t^{1/4}}, \\ y_2(t) &= \frac{e^{-s}(1 + O(s^{-1}))}{s^{1/6}} = \frac{e^{-(2/3)t^{3/2}}(1 + O(t^{-3/2}))}{t^{1/4}}, \end{aligned}$$

in particular

$$\lim_{t\to\infty} \frac{\log y_1(t)}{t^{3/2}} = \frac{2}{3}, \quad \lim_{t\to\infty} \frac{\log y_2(t)}{t^{3/2}} = -\frac{2}{3}.$$

The standard notations in the literature are $y_1(t) = Bi(t), y_2(t) = Ai(t)$ for the particular solutions with the indicated asymptotic behavior.

Returning to the theory, we now discuss the solutions when the function $h(t) \to 0$ more slowly. We can still assert the existence of solutions with suitable exponential growth and decay, in a slightly weaker sense, according to the following result. The main steps of the proof are outlined.

Theorem 10 *Suppose that $b > 0$ and $h(t) \longrightarrow 0$ when $t \longrightarrow \infty$. Then there is a fundamental set of solutions $\{y_1, y_2\}$ of the differential equation $y'' - (b^2 + h(t))y = 0$ so that*

$$\lim_{t\to\infty} \frac{\log y_1(t)}{t} = b, \quad \lim_{t\to\infty} \frac{\log y_2(t)}{t} = -b.$$

This is proved by *factoring* the differential equation, by writing $D = d/dt$ and the differential operator as

$$D^2 - (b^2 + h(t)) = (D + k(t))(D - k(t)).$$

This is possible if and only if the function $k(t)$ is a solution of the *Ricatti equation*

$$k'(t) + k(t)^2 = b^2 + h(t)$$

We first develop some information about the Ricatti equation

Lemma 2 *Suppose that $h(t) \longrightarrow 0$ when $t \longrightarrow \infty$. Then there are solutions $k_1(t), k_2(t)$ of the Ricatti equation for which*

$$\lim_{t\to+\infty} k_1(t) = b, \quad \lim_{t\to+\infty} k_2(t) = -b.$$

Proof. By hypothesis, for any $\epsilon > 0$ there is a T_ϵ so that $|h(t)| < \epsilon^2$ for $t > T_\epsilon$. Let $k_1(t)$ be the solution for which $k_1(T_{b/2}) = b$. It follows that $b\sqrt{3}/2 \le k_1(t) \le b\sqrt{5}/2$ for all $t > T_{b/2}$, otherwise we obtain a contradiction by the mean-value theorem, since $k'(t) < 0$ if $k(t) > b\sqrt{5}/2$ (resp. $k'(t) > 0$ if $k(t) < b\sqrt{3}/2$). Let S be the set of accumulation points of $k_1(t), t \to \infty$. If S contains two different points $\alpha > \beta > b$, then by taking t sufficiently large we can arrange that $k'(t) < 0$ whenever $k_1(t) \in [\alpha, \beta]$, again a contradiction. Therefore, $\lim_{t\to\infty} k_1(t) = k_\infty$ exists. If $k_\infty \neq b$, then from the Riccati equation, we see that $k'(t) \to b^2 - k_\infty^2 \neq 0$, a contradiction. A corresponding analysis can be used to show that there exists a solution $k_2(t)$ for which $\lim_{t\to\infty} k_2(t) = -b$, as required •

Proof of the theorem. Let $y_1(t)$ be defined by $y_1(t) = \exp\left(\int_0^t k_1(s)\,ds\right)$. Clearly $y_1' - k_1 y_1 = 0$ and so $D^2 - (b^2 + h(t))y = (D + k_1)(D - k_1)y_1 = 0$. It is immediate that

$$\frac{\log y_1(t)}{t} = \frac{1}{t}\int_0^t k_1(s)\,ds \longrightarrow b.$$

Similarly $y_2(t)$ is defined as $y_2(t) = \exp\left(\int_0^t k_2(s)\,ds\right)$ and satisfies

$$\frac{\log y_2(t)}{t} = \frac{1}{t}\int_0^t k_2(s)\,ds \longrightarrow -b \quad \bullet$$

Summary of Techniques Introduced

The asymptotic behavior of the solution of many second-order linear equations can often be obtained by suitable application of the method of variation of parameters. In the case of equations modelled on the simple harmonic oscillator, this method yields the asymptotic solution in the form of a suitable trigonometric function together with a suitable remainder (error) term. Other second-order equations that can be reduced to this include equations with a first-order term and equations with a positive zero-order term. For equations that are modelled on the damped oscillator, all solutions tend to zero and a suitable asymptotic expansion can be obtained. For homogeneous equations with a negative zero-order term, one obtains two linearly independent solutions one of which exhibits exponential increase, the other exponential decrease.

Exercises

1. Find the possible asymptotic behavior of the solutions of the equation $y'' = t^2 y$.

2. Find the possible asymptotic behavior of the solutions of the equation $y'' = t^3 y$.

3. Show that if $x(t)$ is a solution of the integral equation (4), then $x(t)$ is also a solution of the differential equation $x'' - 2bx = h(t)x$ and that $y(t) = x(t)e^{bt}$ solves the equation $y'' = (b^2 + h(t))y$.

4. Show that if $z(t)$ is a solution of the integral equation (5), then $z(t)$ is also a solution of the differential equation $z'' + 2bz = h(t)z$ and that $y(t) = z(t)e^{-bt}$ solves the equation $y'' = (b^2 + h(t))y$.

4 Asymptotic Solutions of Linear Systems

In the previous sections we studied the asymptotic behavior for single linear equations of the first and second order. We now briefly consider the corresponding questions for systems of linear equations.

4.1 Nonhomogenous systems

In detail, we will begin with a system of the form

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}(t)$$

where A is a fixed $n \times n$ matrix and the vector function $\mathbf{g}(t)$ tends to zero when $t \to \infty$. In detail, we make the following hypotheses:

i) The matrix A has purely imaginary eigenvalues $r = i\mu$ with n linearly independent eigenvectors

ii) $\mathbf{g}(t)$ satisfies the property that for some $t_0 > 0$, $\int_{t_0}^{\infty} |\mathbf{g}(t)|\, dt < \infty$.

We can write the solution by the method of variation-of-parameters as

$$\mathbf{x}(t) = e^{(t-t_0)A}\mathbf{x}_0 + \int_{t_0}^{t} e^{(t-s)A}\mathbf{g}(s)\, ds.$$

By hypothesis A is diagonalizable with purely imaginary eigenvalues. Writing $A = U\Lambda U^{-1}$, we see that for any power $m \geq 1$, $A^m = U\Lambda^m U^{-1}$ and so the matrix exponential can be computed as

$$e^{tA} = \sum_{m=0}^{\infty} \frac{A^m}{m!} = \sum_{m=0}^{\infty} \frac{U\Lambda^m U^{-1}}{m!} = Ue^{t\Lambda}U^{-1}.$$

But the diagonal matrix $e^{t\Lambda}$ contains the terms $e^{i\mu t}$ along the diagonal, which are bounded functions of t. Therefore the terms of the matrix exponential are bounded by a constant. In detail, if the terms of the matrices U, U^{-1} satisfy $|U_{ij}| \leq M, |U_{ij}^{-1}| \leq M$, then

$$|(e^{tA})_{ij}| \leq nM^2 \qquad 1 \leq i, j \leq n, -\infty < t < \infty.$$

Using this observation, we can re-write the solution for $t \geq t_0$ as

$$\mathbf{x}(t) = e^{(t-t_0)A}\mathbf{x}_0 + \int_{t_0}^{\infty} e^{(t-s)A}\mathbf{g}(s)\, ds - \int_{t}^{\infty} e^{(t-s)A}\mathbf{g}(s)\, ds$$

$$= e^{tA}\left(e^{-t_0 A}\mathbf{x}_0 + \int_{t_0}^{\infty} e^{-sA}\mathbf{g}(s)\, ds\right) - \int_{t}^{\infty} e^{(t-s)A}\mathbf{g}(s)\, ds.$$

The first term is solution of the homogeneous system corresponding to the initial condition $e^{-t_0 A}\mathbf{x}_0 + \int_{t_0}^{\infty} e^{-sA}\mathbf{g}(s)\, ds$. The second term tends to zero, since by the above estimations, we have

$$|\int_{t}^{\infty} e^{(t-s)A}\mathbf{g}(s)\, ds| \leq nM^2 \int_{t}^{\infty} |\mathbf{g}(s)|\, ds.$$

The above computations are summarized as follows.

Theorem 11 *Suppose that the $n \times n$ matrix A has purely imaginary eigenvalues with n linearly independent eigenvectors. Suppose further that the vector function $\mathbf{g}(t)$ satisfies the condition that for some $t_0 > 0$ $\int_{t_0}^{\infty} |\mathbf{g}(s)|\, ds < \infty$. Then any solution of the system of differential equations $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$, has the form* $\mathbf{x}(t) = e^{tA}\mathbf{y} + \epsilon(t)$, for some vector $\mathbf{y}$ where $\epsilon(t) \to 0$ when $t \to \infty$.

4.2 Homogenous systems with almost-constant coefficients

In this sub-section we use the results of the previous sub-section to treat a system of the form

$$\mathbf{x}'(t) = (A + E(t))\mathbf{x}(t)$$

where A is a fixed $n \times n$ matrix and the matrix $E(t)$ tends to zero when $t \to \infty$. In detail, we make the following hypotheses:

i) The matrix A has purely imaginary eigenvalues $r = i\mu$ with n linearly independent eigenvectors;

ii) The matrix elements $E_{ij}(t)$ satisfy the property that for some $t_0 > 0$, $\int_{t_0}^{\infty} |E_{ij}(t)|\, dt < \infty, 1 \le i, j \le n$.

We can write the solution by the method of variation-of-parameters as

$$\mathbf{x}(t) = e^{(t-t_0)A}\mathbf{x}_0 + \int_{t_0}^{t} e^{(t-s)A}\mathbf{g}(s)\, ds$$

where the nonhomogeneous term is $\mathbf{g}(t) = E(t)\mathbf{x}(t)$ Once we know that the solution $\mathbf{x(t)}$ remains bounded, then we can apply the result of the previous sub-section and obtain the following result.

Theorem 12 *Suppose that the $n \times n$ matrix A has purely imaginary eigenvalues with n linearly independent eigenvectors. Suppose further that the matrix function $E(t)$ satisfies the condition that for some $t_0 > 0$ $\int_{t_0}^{\infty} |E_{ij}(s)|\, ds < \infty, 1 \le i, j \le n$. Then any solution of the system of differential equations $\mathbf{x}' = (A + E(t))\mathbf{x}$, has the form $\mathbf{x}(t) = e^{tA}\mathbf{y} + \epsilon(t)$ for some vector $\mathbf{y}$ where $\epsilon(t) \to 0$ when $t \to \infty$.*

It remains to establish the boundedness of the solution $\mathbf{x}(t)$. This is easily done by refering to a system of coordinates in which the matrix A is diagonal, with diagonal elements $\sqrt{-1}\mu j, 1 \le j \le n$. Calling the new solution again $x_j(t)$, we have

$$x_j'(t) = \sqrt{-1}\mu_j x_j(t) + \sum_{k=1}^{n} E_{jk}(t) x_k(t) \qquad 1 \le j \le n.$$

Letting $y_j(t) = e^{-\sqrt{-1}t\mu_j} x_j(t)$ we define $F(t) = \frac{1}{2}\sum_{j=1}^{n} |y_j(t)|^2$. From the defining differential equation we have

$$F'(t) = \sum_{j,k=1}^{n} y_j(t) E_{jk}(t) e^{\sqrt{-1}(\mu_k - \mu_j)t} \bar{y}_k(t)$$

But the inequality $ab \le (a^2 + b^2)/2$ together with the assumptions on $E(t)$ shows that we have an inequality of the form $|F'(t)| \le \epsilon(t)F(t)$ where $\int_{t_0}^{\infty} \epsilon(t)\, dt < \infty$. Hence $\log F(t)$ is bounded, $t_0 < t < \infty$, thus also $F(t)$. This completes the proof that the solution $\mathbf{x}(t)$ remains bounded, $t_0 < t < \infty$ •

Referencias

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