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HYBRID FUNCTIONS IN THE CALCULUS OF VARIATIONS

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Abstract

The solution of problems in the calculus of variations is obtained by using hybrid functions. The properties of the hybrid functions which consist of block-pulse functions plus Legendre polynomials and block-pulse functions plus Chebyshev polynomials are presented. Two examples are considered, in the first example the brachistochrone problem is formulated as a nonlinear optimal control problem, and in the second example an application to a heat conduction problem is given. The operational matrix of integration in each case is introduced and is utilized to reduce the calculus of variations problems to the solution of algebraic equations. The method is general, easy to implement and yields very accurate results.

Keywords: Brachistochrone problem, Calculus of variations, Numerical methods, Hybrid functions.

1 INTRODUCTION

There has been a considerable renewal of interest in the classical problems of the calculus of variations both from the point of view of mathematics and of applications in physics, engineering, and applied mathematics.

Finding the brachistochrone, or path of quickest decent, is a historically interesting problem that is discussed in virtually all textbooks dealing with the calculus of variations. In 1696, the brachistochrone problem was posed as a challenge to mathematicians by John Bernoulli. The solution of the brachistochrone problem is often cited as the origin of the calculus of variations as suggested in [1]. The classical brachistochrone problem deals with a mass moving along a smooth path in a uniform gravitational field. A mechanical analogy is the motion of a bead sliding down a frictionless wire. The solution to this problem has been obtained by various methods such as the gradient method [2], successive sweep algorithm in [3-4] , the classical Chebyshev method [5] and multistage Monte Carlo method [6].

Orthogonal functions (OF's) have received considerable attention in dealing with various problems of dynamic systems. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations; thus greatly simplifying the problem . The approach is based on converting the underlying differential equations into an integral equation through integration, approximating various signals involved in the equation by truncated orthogonal series and using the operational matrix of integration P, to eliminate the integral operations. The form of P depends on the particular choice of the orthogonal functions. Special attention has been given to applications of Walsh functions [7], block-pulse functions [8], Laguerre series [9], Legendre polynomials [10] and Chebyshev polynomials [11].

There are three classes of sets of OF's which are widely used. The first includes sets of piecewise constant basis functions (PCBF'S) (e.g., Walsh, block-pulse, etc.). The second consists of sets of orthogonal polynomials (OP's) (e.g., Laguerre, Legendre, Chebyshev, etc.). The third is the widely used sets of sine-cosine functions

(SCF's) in Fourier series. While OP's and SCF's together form a class of continuous basis functions , PCBF's have inherent discontinuities or jumps. The inherent features(continuity or discontinuity) of a set of OF's largely determine their merit for application in a given situation. References [12] and [13] have demonstrated the advantages of PCBF spectral methods over Fourier spectral techniques. If a continuous function is approximated by PCBF's, the resulting approximation is piecewise constant. On the other hand if a discontinuous function is approximated by continuous basis functions the discontinuities are not properly modelled. Signals frequently have mixed features of continuity and jumps. These signals are continuous over certain segments of time, with discontinuities or jump occurring at the transitions of the segments. In such situations, neither the CBF's nor PCBF's taken alone would form an efficient basis in the representation of such signals.

The direct method of Ritz and Galerkin in solving variational problems has been of considerable concern and is well covered in many textbooks [14], [15]. Chen and Hsiao [7] introduced the Walsh series method to variational problems. Due to the nature of the Walsh functions, the solutions obtained were piecewise constant. Hwang and Shih [9], Chang and Wang [10] and Horng and Chou [11], used Laguerre polynomials, Legendre polynomials and Chebyshev polynomials respectively to derive continuous solutions for the first example in [7]. Furthermore, Razzaghi and Razzaghi [16], [17] applied Fourier series and Taylor series respectively to derive continuous solution for the second example in [7] which is an application to the heat conduction problem. It is shown in Razzaghi and Razzaghi [17] that, to obtain the Taylor series coefficient, an ill-conditioned matrix commonly known as the Hilbert matrix is used. Hence the Taylor series is not suitable for the solution of the second example in [7].

In the present paper we introduce a new direct computational method to solve problems of the calculus of variations. The method consists of reducing the variational problems into a set of algebraic equations by first expanding the candidate functions as hybrid functions with unknown coefficients. The hybrid functions, which consists of block-pulse functions plus

a) Legendre polynomials and

b) Chebyshev polynomials

are first introduced. The operational matrix of integrations in each case is given. The operational matrix of integration is then used to evaluate the coefficients of hybrid functions in such a way that the necessary conditions for extremization are imposed. Two examples are considered. In example 1, the brachistochrone problem is formulated as an optimal control problem and in the second example we will demonstrate the application of operational matrix of integration for hybrid functions by considering the second example in [7]. It is shown that the hybrid functions of block-pulse and Legendre polynomials approach produces an exact solution for the heat conduction problem.

2 Properties of Hybrid Functions of Block-Pulse and Legendre Polynomials

Hybrid functions $b(n, m, t), n = 1, 2, \cdots, N, m = 0, 1, \cdots, M-1$, have three arguments; n is the order of block-pulse functions, m is the order of Legendre polynomials, and t is the normalized time. They are defined on the interval $[0, t_f)$ as

$$b(n,m,t) = \begin{cases} P_m(\frac{2N}{t_f}t - 2n + 1), & t \in [(\frac{n-1}{N})t_f, \frac{n}{N}t_f) \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Here $P_m(t)$ are the well-known Legendre polynomials of order m which are orthogonal with respect to the weight function $w(t) = 1$ and satisfy the following recursive formula.

$$P_0(t) = 1, \quad P_1(t) = t$$

$$P_{m+1}(t) = \left(\frac{2m+1}{m+1}\right) t P_m(t) - \left(\frac{m}{m+1}\right) P_{m-1}(t), \quad m = 1, 2, 3, \cdots$$

Since $b(n, m, t)$ consists of block-pulse functions and Legendre polynomials, which are both complete and orthogonal, the set of hybrid functions of block-Pulse and Legendre polynomials is a complete orthogonal set.

2.1 Function Approximation

A function $f(t)$, defined over the interval 0 to t_f may be expanded as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c(n,m)b(n,m,t), \tag{2}$$

where

$$c(n,m) = (f(t), b(n,m,t))$$

in which $(.,.)$ denotes the inner product. If the infinite series in Eq. (2) is truncated, then Eq. (2) can be written as

$$f(t) \simeq \sum_{n=1}^{N}\sum_{m=0}^{M-1} c(n,m)b(n,m,t) = C^T B(t), \tag{3}$$

where

$$C = [c(1,0), \cdots, c(1,M-1), c(2,0), \cdots, c(2,M-1), \cdots, c(N,0), \cdots, c(N,M-1)]^T, \tag{4}$$

and

$$\begin{aligned} B(t) = [&b(1,0,t), \cdots, b(1,M-1,t)|b(2,0,t), \cdots, b(2,M-1,t)| \\ &\cdots|b(N,0,t), \cdots, b(N,M-1,t)]^T. \end{aligned} \tag{5}$$

2.2 The Operational Matrix of the Hybrid of Block-Pulse and Legendre Polynomials

The integration of the vector $B(t)$ defined in Eq. (5) can approximated by

$$\int_0^t B(t')dt' \simeq PB(t) \tag{6}$$

where P is the $NM \times NM$ operational matrix for integration and is given by

$$P = \begin{pmatrix} E & H & H & \cdots & H \\ 0 & E & H & \cdots & H \\ 0 & 0 & E & \cdots & H \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & E \end{pmatrix}. \tag{7}$$

In Eq. (7)

$$H = \frac{t_f}{N} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and E is operational matrix of integration for Legendre polynomials on the interval $[(\frac{n-1}{N})t_f, \frac{n}{N}t_f]$ given in [18] by

$$E = \frac{t_f}{2N} \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{1}{5} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{-1}{2M-3} & 0 & \frac{1}{2M-3} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2M-1} & 0 \end{pmatrix}.$$

2.3 The Approximation of $B(t)B^T(t)C$

The following property of the product of two Legendre polynomial vectors will also be used.

Let

$$\begin{aligned} P(t) &= [P_0(t), P_1(t), \cdots, P_{M-1}(t)]^T, \\ A &= [a_0, a_1, \cdots, a_{M-1}]^T. \end{aligned}$$

Then we have

$$P(t)P^T(t)A = \tilde{A}P^T(t), \tag{8}$$

where $\tilde{A}$ is an $M \times M$ matrix given in [18].

Let

$$\begin{aligned} B_n(t) &= [b(n,0,t), b(n,1,t), \cdots, b(n,M-1,t)]^T, \quad n = 1,2,\cdots,N, \\ \bar{C}_n &= [c(n,0), c(n,1), \cdots, c(n,M-1)]^T, \quad n = 1,2,\cdots,N. \end{aligned}$$

Then using Eqs. (4) and (5) we get

$$B(t) = [B_1(t), B_2(t), \cdots, B_N(t)]^T, \tag{9}$$

$$C = [\bar{C}_1, \bar{C}_2, \cdots, \bar{C}_N]^T. \tag{10}$$

By using Eqs. (9) and (10) we obtain

$$B(t)B^T(t)C = \begin{pmatrix} B_1(t)B_1^T(t)\bar{C}_1 & 0 & \cdots & 0 \\ 0 & B_2(t)B_2^T(t)\bar{C}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_N(t)B_N^T(t)\bar{C}_N \end{pmatrix}. \tag{11}$$

Similarly to Eq. (8) we have

$$B_n(t)B_n^T(t)\bar{C}_n = \tilde{\bar{C}}_n B_n(t), \qquad n = 1,2,\cdots,N. \tag{12}$$

Using Eqs. (11) and (12), we get

$$B(t)B^T(t)C = \tilde{C}B(t), \tag{13}$$

where $\tilde{C}$ is an $NM \times NM$ diagonal matrix given by

$$\tilde{C} = \begin{pmatrix} \tilde{\bar{C}}_1 & 0 & \cdots & 0 \\ 0 & \tilde{\bar{C}}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\bar{C}}_N \end{pmatrix}.$$

2.4 Integration of $B(t)B^T(t)$

The integration of the cross product of two hybrid Legendre vectors can be obtained as

$$D = \int_0^{t_f} B(t)B^T(t)dt. \tag{14}$$

where D is a diagonal matrix, given by

$$D = \begin{pmatrix} L & 0 & \cdots & 0 \\ 0 & L & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L \end{pmatrix}, \tag{15}$$

with L the $M \times M$ diagonal matrix given by

$$L = \frac{t_f}{N} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2M-1} \end{pmatrix}.$$

3 Properties of Hybrid Functions of Block-Pulse and Chebyshev Polynomials

Hybrid functions $\hat{b}(n,m,t), n = 1,2,\cdots,N, m = 0,1,\cdots,M-1$, have three arguments; n is the order of block-pulse functions, m is the order of Chebyshev polynomials, and t is the normalized time. They are defined on the interval $[0,t_f)$ as

$$\hat{b}(n,m,t) = \begin{cases} T_m(\frac{2N}{t_f}t - 2n + 1), & t \in [(\frac{n-1}{N})t_f, \frac{n}{N}t_f) \\ 0, & \text{otherwise}. \end{cases} \tag{16}$$

Here $T_m(t)$ are the well-known Chebyshev polynomials of order m which are orthogonal with respect to the weight function $w(t) = \dfrac{1}{\sqrt{1-t^2}}$ and satisfy the following recursive formula.

$$T_o(t) = 1, \quad T_1(t) = t$$

$$T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t) \ , \quad m = 1, 2, 3, \cdots$$

Since $\hat{b}(n, m, t)$ consists of block-pulse functions and Chebyshev polynomials, which are both complete and orthogonal, the set of the hybrid functions of block-pulse and Chebyshev polynomials is a complete orthogonal set.

3.1 Function Approximation

A function $f(t)$, defined over the interval 0 to t_f may be expanded as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c(n, m)\hat{b}(n, m, t), \tag{17}$$

where

$$c(n, m) = (f(t), \hat{b}(n, m, t))$$

in which (.,.) denotes the inner product. If the infinite series in Eq. (17) is truncated then Eq. (17) can be written as

$$f(t) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} c(n, m)\hat{b}(n, m, t) = C^T \hat{B}(t), \tag{18}$$

where

$$C = [c(1, 0), \cdots, c(1, M-1)|c(2, 0), \cdots, c(2, M-1)| \cdots |c(N, 0), \cdots, c(N, M-1)]^T,$$

$$\begin{aligned} \hat{B}(t) = \ & [b(1, 0, t), \cdots, b(1, M-1, t)|b(2, 0, t), \cdots, b(2, M-1, t)| \\ & \cdots |b(N, 0, t), \cdots, b(N, M-1, t)]^T. \end{aligned} \tag{19}$$

3.2 The Operational Matrix of the Hybrid of Block-Pulse and Chebyshev Polynomials

The integration of the vector $B(t)$ defined in Eq. (19) can approximated by

$$\int_0^t B(t')dt' \simeq \hat{P}\hat{B}(t)$$

where $\hat{P}$ is the $NM \times NM$ operational matrix for integration and is given by

$$\hat{P} = \begin{pmatrix} \hat{E} & \hat{H} & \hat{H} & \cdots & \hat{H} \\ 0 & \hat{E} & \hat{H} & \cdots & \hat{H} \\ 0 & 0 & \hat{E} & \cdots & \hat{H} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \hat{E} \end{pmatrix}.$$

In the above matrix

$$\hat{H} = \frac{t_f}{N}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \frac{-1}{3} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & 0 \\ \frac{(-1)^{M-1}}{2M(M-2)} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and E is operational matrix of integration for Chebyshev polynomials on the interval $[(\frac{n-1}{N})t_f, \frac{n}{N}t_f]$ given in [11] by

$$\hat{E} = \frac{t_f}{N}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{8} & 0 & \frac{1}{8} & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{6} & \frac{-1}{4} & 0 & \frac{1}{12} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \frac{(-1)^{M-1}}{2(M-1)(M-3)} & 0 & 0 & 0 & \cdots & \frac{-1}{4(M-3)} & 0 & \frac{1}{4(M-1)} \\ \frac{(-1)^{M}}{2M(M-2)} & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{4(M-2)} & 0 \end{pmatrix}$$

4 Hybrid Functions Direct Method

For now, we will use hybrid of block-pulse and Legendre polynomials, similar results can be obtained by using hybrid of block-pulse and Chebyshev polynomials. Consider the problem of finding the extremum of the functional

$$J(x) = \int_0^1 F[t, x(t), \dot{x}(t)]dt. \tag{20}$$

The necessary condition for $x(t)$ to extremize $J(x)$ is that it should satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \tag{21}$$

with appropriate boundary conditions. However, the above differential equation can be integrated easily only for simple cases. Thus numerical and direct methods such as the well-known Ritz and Galerkin methods have been developed to solve variational problems. Here we consider a Ritz direct method for solving Eq. (21) using the hybrid functions.

Suppose, the rate variable $\dot{x}(t)$ can be expressed as

$$\dot{x}(t) = C^T B(t). \tag{22}$$

Using Eq. (6), $x(t)$ can be represented as

$$\begin{aligned} x(t) &= \int_0^t \dot{x}(t')dt' + x(0) \\ &= C^T P B(t) + [x(0), 0, \cdots, 0, x(0), 0, \cdots, 0, \cdots, x(0), 0, \cdots, 0]^T B(t). \end{aligned} \tag{23}$$

We can also express t in terms of $B(t)$ as

$$\begin{aligned} t = \Big[&\frac{1}{2N}, \frac{1}{2N}, 0, \cdots, 0, \frac{3}{2N}, \frac{1}{2N}, \cdots, 0, \cdots, \\ &0, \cdots, \frac{2N-1}{2N}, \frac{1}{2N}, 0, \cdots, 0\Big] B(t) = d^T B(t) \end{aligned} \tag{24}$$

Substituting Eqs. (22-24) in Eq. (20), the functional $J(x)$ becomes a function of

$c(n,m)$, $n = 1, 2, \cdots, N$, $m = 0, 1, 2, \cdots, M-1$. Hence to find the extremum of $J(x)$ we solve

$$\frac{\partial J}{\partial c(n,m)} = 0, \qquad n = 1, 2, \cdots, N, \qquad m = 0, 1, \cdots, M-1. \tag{25}$$

The above procedure is now used to solve the following examples.

5 Illustrative Examples

In this section two problems of the calculus of variations are considered. Example 1 is the classical brachistochrone problem, where as example 2 is an application to the heat conduction problem taken from [7].

5.1 Example 1: The Brachistochrone Problem

5.1.1 The Brachistochrone Problem as an Optimal Control Problem

As an optimal control problem, the brachistochrone problem may be formulated as [5].

Minimize the performance index J,

$$J = \int_0^1 \left[\frac{1 + U^2(t)}{1 - X(t)}\right]^{\frac{1}{2}} dt, \tag{26}$$

subject to

$$\dot{X}(t) = U(t), \tag{27}$$

with

$$X(0) = 0, \quad X(1) = -0.5. \tag{28}$$

Eqs. (26), (27) and (28), describe the motion of a bead sliding down a frictionless wire in a constant gravitational field. The minimal time transfer expression is obtained from the law of conservation of energy. Here X and t are dimensionless and they represent respectively the vertical and horizontal coordinates of the sliding bead.

As is well known the exact solution to the brachistochrone problem is the cycloid defined by the parametric equations

$$x = 1 - \frac{\beta}{2}(1 + \cos 2\alpha), \qquad t = \frac{t_0}{2} + \frac{\beta}{2}(2\alpha + \sin 2\alpha), \tag{29}$$

where

$$\tan \alpha = \frac{dX}{dt} = U.$$

With the given boundary conditions, the integration constants are found to be

$$\beta = 1.6184891, \quad t_0 = 2.7300631.$$

5.1.2 The Numerical Method

Suppose, the rate variable $\dot{X}(t)$ can be expressed approximately as

$$\dot{X}(t) = C^T B(t). \tag{30}$$

Using Eqs. (6) and (28), $X(t)$ can be represented as

$$\begin{aligned} X(t) &= \int_0^t \dot{X}(t')dt' + X(0) \\ &= C^T P B(t), \end{aligned} \tag{31}$$

and by using Eqs. (27) and (30) we have

$$U^2(t) = C^T B(t) B^T(t) C. \tag{32}$$

Equation (32) can be simplified by using the property of the product of two hybrid Legendre function vectors given in Eq. (13).

5.1.3 The Performance Index Approximation

Using Eqs. (26), (31) and (32) the performance index J can be approximated as follows:

$$J = \int_0^1 \left(\frac{1 + C^T \tilde{C} B(t)}{1 - C^T P B(t)} \right)^{\frac{1}{2}} dt. \tag{33}$$

Divide the interval $[0,1]$ into N equal subintervals, we have

$$J=\sum_{n=1}^{N}\int_{\frac{n-1}{N}}^{\frac{n}{N}}\left(\frac{1+C^{T}\tilde{C}B(t)}{1-C^{T}PB(t)}\right)^{\frac{1}{2}}dt. \tag{34}$$

In order to use the Gaussian integration formula we transform the t-interval $(\frac{n-1}{N},\frac{n}{N})$ into the τ-interval $(-1,1)$ by means of the transformation

$$t=\frac{1}{2}\left(\frac{1}{N}\tau+\frac{2n-1}{N}\right). \tag{35}$$

The optimal control problem in Eqs. (26-28) is then restated as follows:

Minimize

$$J=\frac{1}{2}\int_{-1}^{1}\left[\frac{1+u^{2}(\tau)}{1-x(\tau)}\right]^{\frac{1}{2}}d\tau, \tag{36}$$

subject to

$$\frac{dx}{d\tau}=\frac{1}{2}u(\tau), \tag{37}$$

with

$$x(-1)=0,\quad x(1)=-0.5. \tag{38}$$

Using Eqs. (34) and (35) we get

$$J=\sum_{n=1}^{N}\frac{1}{2N}\int_{-1}^{1}\left(\frac{1+C^{T}\tilde{C}B(\frac{1}{2}(\frac{1}{N}\tau+\frac{2n-1}{N}))}{1-C^{T}PB(\frac{1}{2}(\frac{1}{N}\tau+\frac{2n-1}{N}))}\right)^{\frac{1}{2}}d\tau. \tag{39}$$

Using the Gaussian integration formula, Eq.(39) can be approximated as

$$J\approx\sum_{n=1}^{N}\frac{1}{2N}\sum_{j=0}^{k}\left(\frac{1+C^{T}\tilde{C}B(\frac{1}{2}(\frac{1}{N}\tau_j+\frac{2n-1}{N}))}{1-C^{T}PB(\frac{1}{2}(\frac{1}{N}\tau_j+\frac{2n-1}{N}))}\right)^{\frac{1}{2}}w_j, \tag{40}$$

where τ_j, $j=0,1,\ldots,k$ are the $k+1$ zeros of Legendre polynomial P_{k+1}, and w_j are the corresponding weights, given in [19]. The idea behind the above approximation is the exactness of the Gaussian integration formula for polynomials of degree not exceeding $2k+1$.

5.1.4 Evaluating the Vector C

The optimal control problem has now been reduced to a parameter optimization problem which can be stated as follows.

Find $c(n,m)$, $n = 1,2,\cdots,N$, $m = 0,1,\ldots,M-1$ that minimizes Eq.(40) subject to

$$x(-1) = 0, \quad x(1) = -0.5. \tag{41}$$

We now minimize Eq. (40) subject to Eq. (41) using the Lagrange multiplier technique. Suppose

$$J^* = J + \lambda_1 x(-1) + \lambda_2[x(1) + 0.5].$$

The necessary conditions for a minimum are

$$\frac{\partial J^*}{\partial c(n,m)} = 0 \quad n = 1,2,\cdots,N, \quad m = 0,1,\ldots,M-1 \tag{42}$$

and

$$\frac{\partial J^*}{\partial \lambda_1} = 0, \qquad \frac{\partial J^*}{\partial \lambda_2} = 0. \tag{43}$$

Eqs. (42) and (43) give $(NM+2)$ non-linear equations with $(NM+2)$ unknowns which can be solved for $c(n,m)$, λ_1 and λ_2 using Newton's iterative method. The initial values required to start Newton's iterative method have been chosen by taking $x(\tau)$ as a linear function between $x(-1) = 0$ and $x(1) = -0.5$.

In Table 1 the results for hybrid Legendre approximation with $N = 2$, $k = 5$ and $M = 3, 4, 5$ together with $N = 2$, $k = 8$ and $M = 5$ are listed, we compare the solution obtained using the proposed method with other solutions in the literature together with the exact solution.

Methods	$x(1)$	$u(-1)$	J
Dynamic programming gradient method[2]	-0.5	-0.7832273	0.9984988
Dynamic programming successive sweep method[3,4]	-0.5	-0.7834292	0.9984989
Chebyshev solutions[5]			
$M = 4$	-0.5	-0.7844893	0.9984982
$M = 7$	-0.5	-0.7864215	0.99849815
$M = 10$	-0.5	-0.7864406	0.9984981483
Hybrid Legendre, $N = 2, k = 5$			
$M = 3$	-0.5	-0.7852418	0.9984989
$M = 4$	-0.5	-0.7864397	0.9984983
$M = 5$	-0.5	-0.7864402	0.9984981
Hybrid Legendre $N = 2$, $k = 8$ and $M = 5$	-0.5	-0.7864408	0.99849814829
Exact Solution[4]	-0.5	-0.7864408	0.99849814829

Table 1. The hybrid Legendre and other solutions in the literature.

5.2 Example 2: Application to the Heat Conduction Problem

Consider the extremization of

$$J = \int_0^1 \left[\frac{1}{2}\dot{x}^2 - xg(t)\right] dt = \int_0^1 F(t, x, \dot{x})dt, \tag{44}$$

where $g(t)$ is a known function satisfying

$$\int_0^1 g(t)dt = -1,$$

with the boundary conditions

$$\dot{x}(0) = 0 \quad , \quad \dot{x}(1) = 0. \tag{45}$$

Schechter [20] gave a physical interpretation for this problem by noting an application in heat conduction and Chen and Hsiao [7] considered the case where $g(t)$ is given by

$$g(t) = \begin{cases} -1, & 0 \le t < \frac{1}{4}, \quad \frac{1}{2} \le t < 1, \\ 3, & \frac{1}{4} \le t < \frac{1}{2}, \end{cases} \tag{46}$$

and gave an approximate solution using Walsh functions. The exact solution is

$$x(t) = \begin{cases} \frac{1}{2}t^2, & 0 \le t < \frac{1}{4} \\ -\frac{3}{2}t^2 + t - \frac{1}{8}, & \frac{1}{4} \le t < \frac{1}{2} \\ \frac{1}{2}t^2 - t + \frac{3}{8}, & \frac{1}{2} \le t < 1. \end{cases}$$

Here of we solve the same problem using hybrid of Legendre and block-pulse functions with $M = 3$ and $N = 4$. First we assume

$$\dot{x}(t) = C^T B(t). \tag{47}$$

In view of Eq. (46), we write Eq. (44) as

$$J = \frac{1}{2}\int_0^1 \dot{x}^2(t)dt + 4\int_0^{\frac{1}{4}} x(t)dt - 4\int_0^{\frac{1}{2}} x(t)dt + \int_0^1 x(t)dt,$$

or

$$J = \frac{1}{2}\int_0^1 C^T B(t)B^T(t)C dt + 4C^T P\int_0^{\frac{1}{4}} B(t)dt - 4C^T P\int_0^{\frac{1}{2}} B(t)dt + C^T P\int_0^1 B(t)dt.$$

Let

$$w(t) = \int_0^t B(t')dt',$$

then using Eq. (20), we have

$$J = \frac{1}{2}C^T DC + C^T P\left[4w\left(\frac{1}{4}\right) - 4w\left(\frac{1}{2}\right) + w(1)\right]. \tag{48}$$

where

$$D = \int_0^1 B(t)B^T(t)dt.$$

The boundary conditions in Eq.(45) can be expressed in terms of hybrid of Legendre and block-pulse functions as

$$C^T B(0) = 0 \quad , \quad C^T B(1) = 0\,. \tag{49}$$

We now minimize Eq. (48) subject to Eq. (49) using the Lagrange multiplier technique. Suppose

$$J^* = J + \lambda_1 C^T B(0) + \lambda_2 C^T B(1), \tag{50}$$

where λ_1 and λ_2 are the two multipliers. Using Eq. (25) we obtain

$$\frac{\partial J^*}{\partial C} = DC + P\left[4w\left(\frac{1}{4}\right) - 4w\left(\frac{1}{2}\right) + w(1)\right] + \lambda_1 B(0) + \lambda_2 B(1) = 0. \tag{51}$$

We also have

$$\begin{aligned}
w(1) &= \frac{1}{4}[1,0,0,1,0,0,1,0,0,1,0,0]^T, \\
w\left(\frac{1}{2}\right) &= \frac{1}{4}[1,0,0,1,0,0,0,0,0,0,0,0]^T, \\
w\left(\frac{1}{4}\right) &= \frac{1}{4}[1,0,0,0,0,0,0,0,0,0,0,0]^T, \\
B(0) &= [1,-1,1,0,0,0,0,0,0,0,0,0]^T, \\
B(1) &= [0,0,0,0,0,0,0,0,0,1,1,1]^T.
\end{aligned}$$

Equations (49) and (51) define a set of 14 simultaneous linear algebraic equations from which the coefficient vector C and the multipliers λ_1 and λ_2 can be found. The vector $C^T P$ is

$$C^T P = \frac{1}{64}\left[\frac{2}{3}, 1, \frac{1}{3}, 2, -1, -1, -\frac{10}{3}, -3, \frac{1}{3}, -\frac{22}{3}, -1, \frac{1}{3}\right]^T \tag{52}$$

Further, to define $x(t)$ for t in the interval $\left[0, \frac{1}{4}\right]$ we map $\left[0, \frac{1}{4}\right]$ into $[-1, 1]$ by mapping t into $8t - 1$ and similarly for the other intervals. Using the above equation and $P_0 = 1, P_1 = t$ and $P_2 = \frac{3}{2}t^2 - \frac{1}{2}$, we get

$$x(t) = \begin{cases} \frac{1}{64}\left[\frac{2}{3} + (8t-1) + \frac{1}{3}\left[\frac{3}{2}(8t-1)^2 - \frac{1}{2}\right]\right] = \frac{1}{2}t^2, & 0 \le t \le \frac{1}{4} \\ \frac{1}{64}\left[2 - (8t-3) - \left[\frac{3}{2}(8t-3)^2 - \frac{1}{2}\right]\right] = -\frac{3}{2}t^2 + t - \frac{1}{8}, & \frac{1}{4} \le t \le \frac{1}{2} \\ \frac{1}{64}\left[-\frac{10}{3} - 3(8t-5) + \frac{1}{3}\left[\frac{3}{2}(8t-5)^2 - \frac{1}{2}\right]\right] = \frac{1}{2}t^2 - t + \frac{3}{8}, & \frac{1}{2} \le t \le \frac{3}{4} \\ \frac{1}{64}\left[-\frac{22}{3} - (8t-7) + \frac{1}{3}\left[\frac{3}{2}(8t-7)^2 - \frac{1}{2}\right]\right] = \frac{1}{2}t^2 - t + \frac{3}{8}, & \frac{3}{4} \le t \le 1 \end{cases}$$

which is the exact solution. This exact solution can not be obtained either with CBF's or with PCBF's.

6 Conclusion

The aim of present work is to develop an efficient and accurate method for solving problems of the calculus of variations. The problem has been reduced to solving a system of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique. The advantages of using the hybrid Legendre method are:

(1) The operational matrix P contains many zeros which plays an important role in simplifying the performance index.

(2) The Gaussian integration formula is exact for polynomials of degree not exceeding $2k+1$.

(3) Only small values of k, N and M are needed to obtain very satisfactory results for the brachistochrone problem.

(4) Hybrid functions approach provides an exact solution for the heat conduction problem.

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