Cubo Matemática Educacional
Vol. 4, Nº1, MAYO 2002

FUNCTIONAL DIFFERENTIAL EQUATIONS: AN INTRODUCTION

George Seifert
Iowa State University,
Ames, IA 50011
e-mail: gseifert@iastate.edu

1 Introduction

This paper is an attempt to acquaint the non-specialist with a topic in ordinary differential equations that has been becoming quite prominent in the last half-century. By using simple examples it hopes to show that, first, realistic models for many physical applications can be represented in terms of such functional differential equations, especially of delay type. Then it tries to show that standard mathematical analysis of not much more than undergraduate level can be used to get much important information about solutions of such equations.

As will be evident in sections 2 and 3, one of the important properties of solutions of such equations is that they often exhibit oscillatory properties as functions of the independent variable t, usually associated with time in physical applications modeled by these equations. In the opinion of the author, a considerable number of natural phenomena exhibit oscillatory behavior in terms of the time, and one of the interesting proper times of delay differential equations as they usually have solutions with just such oscillatory properties, while corresponding differential equations without delays do not.

Since this is an expository paper and not a survey, the author has included many straight forward computational details at a fairly unsophisticated level, but has omitted many areas in the field of functional differential equations in which important advances have been made.

2 Linear and Almost Linear Cases

One of the simplest differential equations is one that models the so called simple harmonic oscillator. With t the time variable and $x(t)$ the state variable representing displacement from equilibrium, it is just

$$\ddot{x}(t) + k^2 x(t) = 0; \tag{1}$$

here k is a positive constant, and the dot indicates derivative with respect to t, so $\ddot{x}(t) = d^2x/dt^2$. This of course simply states one of Newton's laws of motion: force equals mass times acceleration. For example, it models the small oscillations about equilibrium of a simple pendulum and is a simple special case of a linear homogeneous ordinary differential equation with constant coefficients. A well known elementary method for finding solutions for such equations is to look for solutions of the form $x(t) = ce^{\lambda t}$, where c and λ can be complex constants; we can allow (1) to have complex valued solutions, and note that their real and imaginary parts are respectively real valued solutions. Substitution of $x(t) = ce^{\lambda t}$ into (1) easily yields $\lambda^2 + k^2 = 0$ for any case where $c \neq 0$. We recall that for a general nth order linear homogeneous ordinary differential equation, this same method leads to an equation of the form $P(\lambda) = 0$ where P is a polynomial of degree n. Thus if $\lambda_1, \lambda_2, \ldots \lambda_k$ are distinct roots of this equation, a solution, in general complex valued, of (1) can be found of the form $x(t) = \sum_{j=1}^{k} c_j e^{\lambda_j t}$. The equation $P(\lambda) = 0$ is usually referred to as the characteristic equation for the given o.d.e., and P is the associated characteristic polynomial. We can easily show that if λ_j is a root of P of multiplicity $l_j \geq 1$, $t^{l_j-1}e^{\lambda_j t}$ is also a solution. We also recall the fact that all solutions of such an o.d.e. can be expressed in terms of such solutions.

As is well known from sufficiently long observation, the oscillations of a simple pendulum decrease in amplitude with time. So over a long time interval (1) is not an accurate model for such a pendulum. It is also true that for sufficiently large oscillations from equilibrium, the period of the oscillation is amplitude dependent, unlike the solutions of (1); this is because a better model for larger oscillations is given by the nonlinear equation $\ddot{x}(t)+k^2 \sin x = 0$, which also has periodic solutions, but the period of these are amplitude dependent.

If we are concerned only with small oscillations over a long period of time, we usually assume a frictional drag on the pendulum proportional to its speed and are led to a more realistic equation.

$$\ddot{x}(t) + b\dot{x}(t) + k^2 x(t) = 0 \tag{2}$$

where b is a positive constant. Since our characteristic equation for (2) is $\lambda^2 + \lambda b + k^2 = 0$, it follows easily that any solution of the form $ce^{i\lambda t}$, where λ is a root of this characteristic equation, approaches 0 as $t \to \infty$. Are all solutions of (2) linear combinations of such solutions? If $b = 2k$ the answer is no, since in this case $te^{\lambda t}$ is also a solution; in this case the characteristic equation has a non-simple (i.e. double) root. The author suggests that such a possibility for (2), or for any linear constant coefficient equation, is from the point of view of applications not as important; since the constants b and k in (2) are usually results of measurements, the chances of having $b = 2k$ is rather remote.

Suppose we wish to model another system where there is again a restoring force proportional to the displacement from equilibrium, but where the force due to the speed is not proportional to the speed at the time of observation, but at a time just a constant amount before this time. We are led to an equation of the form

$$\ddot{x}(t) + b\dot{x}(t-r) + x(t) = 0 \tag{3}$$

where by a rescaling of the t variable we can always assume the coefficient of $x(t)$ to be 1. To find solutions of (3) we can again look for complex solutions of the form

$x(t) = ce^{\lambda t}$ where now λ must be a root of the characteristic equation

$$\lambda^2 + \lambda b e^{-r\lambda} + 1 = 0. \tag{4}$$

It is clearly not a polynomial equation in λ, and in fact will have an infinite number of distinct roots. If all roots of (4) have negative real parts, it can be shown that all solutions of (3) satisfy $x(t) \to 0$ as $t \to \infty$, although the proof of this is far from simple. On the other hand, if there exists a root λ of (4) with $Re\lambda > 0$, then there exists a solution $x(t)$ which becomes unbounded as $t \to \infty$, as is easy to see. (For $\lambda = a + ib$, we use the notation $Re\lambda = a, Im\lambda = b$).

Before undertaking an analysis of all possible roots of (4) let us say something about the concept of general solution for equations like (3), so called delay differential equations (d.d.e.s for short). As in the ordinary differential equation case, we formulate an initial value problem (i.v.p. for short) for a more general second order d.d.e. of the form

$$\ddot{x}(t) = f(x(t), \dot{x}(t-r)) \tag{3.1}$$

where $f(x, y)$ is continuously differentiable in the (x, y) plane.

If we are interested in solutions of (3.1) for $t \geq 0$, we must clearly know $\dot{x}(t)$ for $-r \leq t \leq 0$. Let $\psi(t)$ be a function continuous on $[-r, 0]$, and x_0 be any real number. We can then assert the existence of a unique solution $\dot{x}(t) = x(t; x_0, \psi)$ of (3.1) on some interval $[0, d_1), d_1 \leq r$, such that $x(0) = x_0$ and $\dot{x}(t) = \psi(t)$ for $-r \leq t \leq 0$, by defining it to be the unique solution of the o.d.e.

$$\ddot{x}(t) = f(x(t), \psi(t))$$

such that $x(0) = x_0$ and $\dot{x}(0) = \psi(0)$ on the interval $[0, d_1)$. If $d_1 = r$ we can repeat this procedure on $[r, d_2)$ for some $d_2 \leq 2r$, and continue in this way to some maximal interval $[0, d_m)$; here $d_m = \infty$ is possible. Thus is called using the method of steps. Thus for d.d.e.s our initial conditions for solutions are necessarily in terms of given functions on the delay interval $[-r, 0)$. Later we will formulate the i.v.p. for d.d.e.s in a more general manner.

Returning to equation (3) we can find solutions on $(-\infty, \infty)$ of the form $ce^{\lambda t}$, c a constant, λ a root of (4). However not all solutions on $[0, \infty)$ as stated above for

(3) can be defined on $(-\infty, \infty)$; the initial function given on $[-r, 0]$ is continuous, but not necessarily differentiable and if $\dot{x}(t) = \psi(t)$, $-r \leq t \leq 0$, $\ddot{x}(t)$ need not exist on this interval. In fact, it is also easy to see that not all solutions on $[0, \infty)$ are linear combinations of $c_j e^{\lambda_j t}$ on that interval; clearly unless the initial function $\psi(t)$ is of that form on $[-r, 0]$, this cannot be true.

We now state some properties of the roots of (4); these will in fact also hold for characteristic roots of all linear d.d.e.s with constant coefficients of any order, or even systems of such linear d.d.e.s.

(i) There exists a constant c such that if λ is a root of (4), $Re\lambda \leq c$.

(ii) The set of all roots of (4) is infinite, and has no limit point in the complex plane.

(iii) For any c_2 and $c_2 > c_1$, the set of all roots λ of (4) for which $c_1 \leq Re\lambda \leq c_2$ is finite.

These properties can be established by using the particular form of the characteristic function $f(\lambda) = \lambda^2 + b\lambda e^{-r\lambda} + 1$, and also some classical results for entire functions like $f(\lambda)$; cf. [7] for example, where entire functions are called integral functions.

We now study the roots of (4) in some detail. First, there are obviously no nonnegative real roots of (4), and a simple analysis of (4) shows that there is at least one real negative root since $f(0) = 1$ and $f(\lambda)$ is negative for $\lambda < \infty$ and sufficiently negative. We then see that there exist solutions of (3) on $(-\infty, \infty)$ that tend to 0 strictly monotonically as $t \to \infty$. Another reasonably simple analysis of (4) shows that it can have no more than 2 real roots. So in a sense, most of the real solutions on $(-\infty, \infty)$ oscillate as t increases. It can also be shown that if $br > 1$, (4) can have no more than one real root; we leave the proof of this to the reader.

However, (4) can have roots λ with $Re\lambda > 0$, and so (3) can have solutions which are unbounded as $t \to \infty$. We will show this by showing that for any $r = r_0, 0 < r_0 < \pi/2$, there exists a $b_0 = b(r_0) > 0$ such that (4) has a pure imaginary root $i\beta_0$, $\beta_0 \neq 0$, and that the real part of this root changes sign as b is varied from b_0.

Let $\lambda = \alpha + i\beta$; then (4) becomes

$$
\begin{aligned}
F(\alpha,\beta,b,r) &\equiv \alpha^2 - b^2 + be^{-\alpha r}(\alpha\cos\beta r + \beta\sin\beta r) + 1 = 0 \\
G(\alpha,\beta,b,r) &\equiv 2\alpha\beta + be^{-\alpha r}(\beta\cos\beta r - \alpha\sin\beta r) = 0
\end{aligned} \tag{5}
$$

If we choose $\beta = \beta_0 = \pi/2r_0$, $\alpha = \alpha_0 = 0$, $b = b_0 = (\pi^2 - 4r_0^2)/2\pi r_0$, where $0 < r_0 < \pi/2$, then $\lambda_0 = i\beta_0$ is a solution of (5) for the values of $\alpha_0, \beta_0, r_0, b_0$ as above. We will use the Implicit Function Theorem to show first that there exist functions $\alpha(b,r)$, $\beta(b,r)$, and a positive constant δ_0 such that for $|b - b_0| < \delta_0$, $|r - r_0| < \delta_0$, these functions are differentiable, and in fact, the partial derivative $\alpha_b(b_0,r_0)$ is nonzero. Thus $\alpha(b,r_0)$ changes sign at $b = b_0$, as asserted above. By direct computation we get

$$F_\alpha = -(\pi^2 - 4r_0^2)/4r_0, \quad F_\beta = -(\pi^2 + 4r_0^2)/2\pi r_0$$

at $(\alpha_0,\beta_0,b_0,r_0)$, and using the Cauchy-Riemann equations, or by direct computation,

$$G_\alpha = (\pi^2 + 4r_0^2)/2\pi r_0, \quad G_\beta = -(\pi^2 - 4r_0^2)/4r_0$$

So the Implicit Function Theorem holds. We could in fact, have used a well-known result from complex analysis to get this by showing that if $f(\lambda) = \lambda^2 - be^{-r\lambda} + 1$, then $f(\lambda_0) = 0$ and Im $(\lambda_0) = 0$ implies $f'(\lambda_0) \neq 0$, which is not difficult to verify. However, we want to show that $\alpha_b \neq 0$ at (b_0, r_0). So we compute the partials F_b and G_b and use the fact that

$$\begin{pmatrix}\alpha_b \\ \beta_b\end{pmatrix} = D_0^{-1}\begin{pmatrix}-G_\beta & G_\alpha \\ F_\beta & -F_\alpha\end{pmatrix}\begin{pmatrix}F_b \\ G_b\end{pmatrix}$$

where $D_0 = F_\alpha G_\beta - G_\alpha F_\beta = F_\alpha^2 + G_\alpha^2 > 0$ at $(\alpha_0,\beta_0,b_0,r_0)$. Since $F_b = \pi/2r_0$, and $G_b = 0$ at this point, it follows that $\alpha_b = (-G_\beta F_b + G_\alpha G_b)D_0^{-1} = (-F_\alpha F_b)D_0^{-1}$ at this point, and so there exists $\delta_0 > 0$, such that for any $b, 0 < b - b_0 < \delta_0$, $r - r_0 < \pi/2$, (3) will have a solution which becomes unbounded as $t \to \infty$.

More generally, it follows by direct computation that for each $r > 0$ and $b = b_n = (-1)^n[(2n+1)^2\pi^2 - 4r^2]/2r(2n+1)\pi$, $n = 0, 1, 2, \ldots$, (4) will have a pure imaginary root $\lambda = i(2n+1)\pi/2r$. Since we want positive damping in (3); i.e., $b > 0$, we can for any integer $n \geq 0$ for which $(-1)^n((2n+1)\pi - 2r) > 0$ get values of b_n for which (4) will have a pure imaginary root $i(2n+1)\pi/2r$, and proceeding as in the case of $n = 0$, $r < \pi/2$ above can show that again for b near b_n, (4) will have a root with positive real part; we omit the details.

In dealing with a general system of delay-differential equations and the associated i.v.p.s., it is convenient to introduce the set of n-dimensional vector functions continuous on $[-r, 0]$, say C_r, and associate with any n-dimensional vector valued function $x(t)$ continuous on $[-r, b], b > 0$, a function on $[0, b)$ to C_r denoted by x_t and defined by $x(t+\theta) : -r \leq \theta \leq 0$. We can then express the right side of such an equation as a vector valued function on C_r. For example, let $n = 2$, and consider $(f(\phi, \psi), g(\phi, \psi))$ where f and g are continuous for (ϕ, ψ) in C_r where the continuity is in terms of the norm

$$\|(\phi, \psi)\| = \sup\{(\phi^2(\theta) + \psi^2(\theta))^{\frac{1}{2}} : -r \leq \theta \leq 0\}.$$

Consider the system

$$\begin{aligned} \dot{x}(t) &= f(x_t, y_t) \\ \dot{y}(t) &= g(x_t, y_t) \end{aligned} \tag{3.2}$$

where for any function $(x(t), y(t))$ on the real interval $[-r, b)$, $b > 0$, we use the notation $(x_t, y_t) = x(t+\theta, y(t+\theta)), -r \leq \theta \leq 0$; i.e., (x_t, y_t) is in C_r. Then the i.v.p. for (3.2) is to find $(x(t), y(t))$ continuous on $[-r, b)$ and which satisfies (3.2) for $0 < t < b$, and $(x(\theta), y(\theta)) = (\phi(\theta), \psi(\theta))$ for $-r \leq \theta \leq 0$ for any given (ϕ, ψ) in C_r. Note that for any solution $(x(t), y(t))$ of (3.2), we can associate the function $(x_t, y_t) : t \to C_r$, but if we regard this as a solution to (3.2), we would have to specify the concept of derivative for such C_r-valued functions. This can be done, but leads to complications which we prefer not to get into.

To illustrate how (3) can be formulated in this general manner, define $f(\phi, \psi) = \psi(0), g(\phi, \psi) = -\phi(0) - \psi(-r)$ for (ϕ, ψ) in C_r. Then (3.2) becomes

$$\begin{aligned} \dot{x}(t) &= y(t) \\ \dot{y}(t) &= -x(t) - by(t-r), \end{aligned}$$

a system clearly equivalent to (3). Note that in the earlier formulation of the i.v.p.

for (3) we need only $\phi(0) = x(0)$, but $\psi(t) = \dot{x}(t)$ on all of $[-r, 0]$.

We now give an indication of why all solutions of a general i.v.p. for (3) can not be expressed in terms of exponential functions $e^{\lambda_j t}$, where λ_j are roots of (4). consider $\phi(t) = 0, \psi(t) = r + t$ for $-r \leq t \leq 0$. We use the method of steps. For $0 \leq t \leq r$, (3) becomes $\ddot{x}(t) + bt + x(t) = 0$, and since $x(0) = \phi(0) = 0$, we get $x(t) = -bt$ on this interval. For $r \leq t \leq 2r$, (3) becomes $\ddot{x}(t) - b^2 + x(t) = 0$, with $x(r) = -br$ and $\dot{x}(r) = -b$. This has a solution of the form $x(t) = b^2 + A\cos t + B\sin t$ where A and B are uniquely determined. Thus the solution on $[0, 2r]$ can not be expressed in terms of $e^{\lambda t}$ where λ is a root of (4).

However, in more general theory for i.v.p.s for linear autonomous d.d.e.s (cf. [3] for example) it follows that if $\lambda_j : j = 1, 2, \ldots, m$ are simple roots of (4), and if $\lambda_j : j = m{+}1, m{+}2, \ldots$ are the remaining roots of (4) such that Re $\lambda_j < 0, j \geq m{+}1$, then the solution of any i.v.p. for (3) can be expressed in the form

$$x(t) = \sum_{j=1}^{m} c_j e^{\lambda_j t} + v_m(t)$$

where c_j are suitable complex constants, and $v_m(t) \to 0$ as $t \to \infty$.

Before turning to a special type of nonhomogeneous linear d.d.e, we give an answer to a question: are all the roots of (4) simple? The answer is in the affirmative if $be < r < 2/3\sqrt{5}$. This can be done by a direct analysis of the pair of equations $f(\lambda) = 0,\ f'(\lambda) = 0$ where $f(\lambda) = \lambda^2 + b\lambda e^{-r\lambda} + 1$, some of the details of which are given in the appendix of this paper.

We will now discuss a very special case of a nonhomogeneous linear d.d.e.; specifically an equation of the form

$$\ddot{x}(t) + b\dot{x}(t - r) + x(t) = g(t) \tag{6}$$

where as before b and r and positive constants, and

$$g(t) = \sum_{t=1}^{\infty} a_j \cos \mu_j t + b_j \sin \mu_j t \text{ where } \sum_{j=p}^{\infty} (a_j^2 + b_j^2)^{\frac{1}{2}} < \infty. \tag{7}$$

Note that $g(t)$ is not necessarily periodic; for example it is not difficult to show that $\cos t + \cos \pi t$ is not periodic. In fact, it belongs to a larger class of functions, called almost periodic (a.p. for short) functions. Although there are other definitions of a.p. functions, we find the following perhaps the simplest: $g(t)$ is a.p. if for any $\varepsilon > 0$ there exist sequences of real numbers $\{a_j\}, \{b_j\}$ and $\{\mu_j\}, j = 0, 1, 2, \ldots$, and a $N(\varepsilon) > 0$ such that $n > N(\varepsilon)$ implies

$$\left|\sum_{j=0}^{n} a_j \cos \mu_j t + b_j \sin \mu_j t - g(t)\right| < \varepsilon,$$

uniformly for t in $(-\infty, \infty)$.

Clearly, if $\mu_j = j\omega, j = 0, 1, 2, \ldots$, for some constant $\omega > 0$, any $g(t)$ as given in (7) is periodic with period $2\pi/\omega$.

We can then look for solutions of (6) of the form

$$x(t) = \sum_{j=0}^{\infty} \widetilde{a}_j \cos \mu_j t + \widetilde{b}_j \sin \mu_j t, \tag{6.1}$$

and as for the homogeneous problem, we look for complex solutions $z(t)$

$$\ddot{z}(t) + b\dot{z}(t - r) + z(t) = G(t) \tag{6.2}$$

where

$$G(t) = \sum_{j=0}^{\infty} c_j e^{i\mu_j t}, \; c_j = a_j - ib_j.$$

Clearly, the real part of a solution $z(t)$ of (6.2) is a solution $x(t)$ of (6). So substituting

$$z(t) = \sum_{j=0}^{\infty} \widetilde{c}_j e^{i\mu_j t} \tag{8}$$

into (6.2) we get easily that

$$\widetilde{c}_j = (-\mu_j^2 - ib\mu_j e^{-i\mu_j r} + 1)^{-1} c_j, j = 0, 1, 2, \ldots. \tag{9}$$

provided $i\mu_j$ is not a root of (4).

Under the condition that (4) has no pure imaginary roots, the $\widetilde{c}_j, j = 0, 1, \ldots$ are well defined, but does the series (8) converge for all real t? Since there are a finite number of roots of (4) in any strip of the complex λ-plane defined by $|Re\lambda| < \rho$, $\rho > 0$ and fixed, it follows easily that if (4) has no pure imaginary roots, there exists a $\rho_1 > 0$ such that

$$|f(i\mu_j)| = |-\mu_j^2 + i\mu_j b e^{-i\mu_j r} + 1| \geq \rho_1 |\mu_j|^2; j = 0, 1, \ldots, \tag{9.1}$$

if $|\mu_j| \to \infty$ as $j \to \infty$, which shows, using (9), that (8) converges absolutely.

If $|u_j| \leq B < \infty, j = 0, 1, \ldots$, we may assume without loss of generality that $\mu_j \to \beta_0$ as $j \to \infty$. But $|f(i\beta_0)| > 0$, and so

$$\begin{aligned} |f(i\mu_j)| &= |f(i\mu_j) - f(i\beta_0) + f(i\beta_0))| \\ &\geq |f(i\beta_0)| - |f(i\mu_j) - f(i\beta_0))|1 \geq \varepsilon_0/2 \end{aligned}$$

for all j sufficiently large; so from (9) we get

$$|\widetilde{c}_j| \leq |c_j|(2/\varepsilon_0),$$

and since

$$|c_j| = (a_j^2 + b_j^2)^{\frac{1}{2}},\ j = 0, 1, 2, \ldots,$$

we see that $\sum_{j=0}^{\infty} \widetilde{c}_j e^{i\mu_j t}$ converges absolutely for all t. Does this last series converge to a solution of (6)? If we can show that

$$\sum_{j=0}^{\infty} \mu_j^2 \widetilde{c}_j e^{i\mu_j t} \tag{10}$$

converges absolutely, then its sum is just $\ddot{z}(t)$, where $z(t)$ is given by (8). If $|\mu_j|, j = 1, 2, \ldots$, is bounded, it follows easily that (10) converges absolutely; a similar argument will show that in this case $\sum_{j=0}^{\infty} \mu_j \widetilde{c}_j e^{i\mu_j (t-r)}$ will also converge absolutely to $\dot{z}(t-r)$, and so in this case $z(t)$ solves (6.2). If $|\mu_j|, j = 0, 1, 2, \ldots$ is unbounded, it has already been noted that (9.1) holds which with (9) shows that (10) converges absolutely, and a similar argument show that $\dot{z}(t)$ exists if $z(t)$ is given by (8). So we have proved that if (4) has no pure imaginary roots, we can find a series solution for (6), and in fact this series solution will be a.p.

We can also try to find solutions of a certain type for so called almost linear equations with delays, which in a sense, are perturbations of linear homogeneous delay equations, using the method of undetermined coefficients. Let us consider the example:

$$\ddot{x}(t) + b\dot{x}(t-r) + x(t) = \varepsilon g(t, x(t), x(t-r)) \tag{11}$$

where ε is a small positive constant, g is sufficiently smooth and $g(t+T, x, y) = g(t, x, y)$ for all t, x, y, where $T > 0$ is constant. Suppose we look for a solution $x(t)$ of (8) which also satisfies $x(t+T) = x(t)$ for all t. We can proceed as follows: take any function T-periodic function $y(t)$, possibly suitably restricted, and notice that since $g(t, y(t), y(t-r))$ is also T-periodic, it will have a series $\sum_{k=-\infty}^{\infty} \widetilde{c}_k e^{ik\omega t}$ where $\omega = 2\pi/T$. Then one can use the method of undetermined coefficient to find a series solution of (8) with $g(t, x(t), x(t-r))$ replaced by this series. We then try to impose additional conditions that this mapping of $y(t)$ to this series solution $x(t, y)$ is unique and that it has a fixed point. The fixed point will then clearly solve (11).

The case where g is a.p. in t independently of x, y can be addressed the same way, but is much more complicated. To see this let us consider the simple case where $g(t, x, y) = p(t)x$, where $p(t) = 2(\cos t + \cos \pi t) = \sum_{j=-2}^{2} c_j e^{i\lambda_j t}$ where $c_0 = 0, c_j = 1, j \neq 0$, and $\lambda_1 = \lambda_{-1} = 1$, $\lambda_2 = \lambda_{-2} = \pi$. If $y(t) = \sum_{j=-2}^{2} \widetilde{c}_j e^{i\lambda_j t}$, then the series for $p(t)y(t)$ will involve terms such as $e^{\pm it}, e^{\pm i\pi t}, e^{\pm i2t}, e^{\pm i(1+\pi)t}, e^{\pm i(1-\pi)t}, e^{\pm i2\pi t}$. Thus the method of undetermined coefficients gives us a series involving these terms. But then the mapping $y(t) \to x(t)$ is not "into". So the "input" $y(t)$ into g, and subsequently the solution $x(t)$, must involve all $\lambda_j \in \{m + n\pi : m, n = 0, \pm 1, \pm 2, \dots\}$, a countable set, to get an "into" map. This creates complications that can to some extent be overcome; cf. [1] for a general method of formal series solutions of equations such as (6) and (11).

On the other hand, by another much less complicated method, one need only impose conditions on the roots of the characteristic equation of the linear part of (8) and use the fact that $g(t, y(t), y(t-r))$ is a.p. if $y(t)$ is to obtain the existence of a unique solution $x(t; y)$ of (11) with $x(t)$ on the right side replaced by $y(t)$, and show that for $|\varepsilon|$ sufficiently small, this map defines a contraction on a suitable metric space of a.p. functions; from the theory of a.p. function, the fact that uniform on $(-\infty, \infty)$ limits of a.p. functions are again a.p. is very important.

3 Nonlinear Delay-Differential Equations

We will confine ourselves to a specific and fairly simple nonlinear d.d.e., an equation that can be used to model the variation in size of population of a single biological space growing in a constant environment with food supplied at a constant rate usually referred to as a logistic equation. Such an equation without delay is given by

$$\dot{N}(t)/N(t) = a - bN(t), \; N(0) = N_0 > 0 \tag{12}$$

here a and b are positive constants, N_0 is the initial size of the population, whose size at $t > 0$ is given by $N(t)$. Eq. (12) says that the rate of change of the population in time is exponential if the size is small (near zero), but as the population increases, the food supply, being supplied at a constant rate, decreases for each particular individual, and reduces the exponential rate of increase by an amount proportional to the size of the population.

The i.v.p. for (12) can be explicitly solved. However, even without an explicit solution we can obtain some important properties of such solutions. First, they are all positive for all $t > 0$; then each solution is either strictly increasing, strictly decreasing, or constant valued. If it is observed, that the population size exhibits substantial oscillatory properties, it follows that (12) cannot therefore be a correct model for such population process. Suppose it can be assumed that the rate at which food is consumed depends not only on the size of the population at t, i.e., $N(t)$, but also on its size at a previous time, say at $t - r, r > 0$. We then could model this by a d.d.e. with an i.v.p. such as

$$\dot{N}(t)/N(t) = a - bN(t) - N(t-r), \; N(t) \geq 0, \; -r \leq t < 0, \; N(0) > 0. \tag{13}$$

By a suitable change of time scale and a rescaling of population variable $N(t)$, we can assume that in (13) $r = 1$; i.e., define $\bar{N}(t) = rN(rt)$ and by a routine calculation we get

$$\dot{\bar{N}}(t)/\bar{N}(t) = \bar{a} - b\bar{N}(t) - \bar{N}(t-1), \bar{a} = ra \tag{14}$$

and it can be shown that under certain conditions, all solutions of (14) sufficiently near the equilibrium solution $\bar{N}(t) = \bar{a}/(b+1)$ for $-r \leq t \leq 0$ will oscillate about this equilibrium point; i.e. there will exist a sequence $t_j \to \infty$ as $j \to \infty, b_{j+1} > t_j$, such that

$$(-1)^j(\bar{N}(t_j) - \bar{a}/(b+1)) > 0, j = 0, 1, 2, \ldots.$$

This is proved in [6], and the method, loosely speaking, is to look at the linearization of (14) about $\bar{a}/(b+1)$, impose conditions that this linear part has characteristic roots with nonzero imaginary and negative real parts, and to show that under further conditions, the oscillatory behavior of the corresponding solutions of the linearization is transferred to certain solutions of the nonlinear equation; for details, cf. [6].

4 Some Final Remarks

For a discussion of functional differential equations in general we refer to books by Hale [3], Hale and Verdun Lunel [4] and Kolmanovskii and Myshkis [5]. They use mathematical analysis at a rather sophisticated level; i.e., at least at the graduate level. Loosely speaking, functional differential equations in general include not only those of retarded type, to which we have confined ourselves in the simple examples above, but of advanced type, mixed type, and neutral type; the latter are, roughly speaking, equations where there are time delays in the highest order derivative as well as possibly in the lower derivative terms. It seems to the author that most applications can be modeled in terms of retarded type equations. However, there are examples of applications leading to models of these other types. Again, we refer to well known books on the general subject, such as [3] and [4].

Finally, it can be suggested that ordinary differential equations are limits of d.d.e.s as the delay approaches zero. In fact one can devise a proof for the existence of solutions of initial value problems for ordinary differential equations by using this idea; cf. the book by P. Hartman: "Ordinary Differential equations" J. Wiley & Sons, Inc. N.Y., London, Sydney, (1964). Also, an excellent monograph, using fairly elementary analysis, on d.d.e.s is by R.D. Driver, "Ordinary and Delay Equations", Appl. Math. Sciences 20, Springer-Verlag, N.Y., Heidelberg, Berlin (1977).

5 Appendix

We show that if $eb < r < 2/3\sqrt{5}$, then $f(\lambda) = 0$ implies $f'(\lambda) \neq 0$. Suppose not, i.e., $f'(\lambda) = 2\lambda + b(1 - \lambda r)e^{-r\lambda} = 0$. Solving this last equation for $be^{-r\lambda}$ and substituting it into $f(\lambda) = 0$ yields eventually $P(\lambda) \equiv \lambda^3 r + \lambda^2 + r\lambda - 1 = 0$ provided $\lambda \neq 0$. But $f(0) = 1$, so we must have $\lambda \neq 0$. This last equation has 3 roots, $\lambda_1, \lambda_2, \lambda_3$ where one, say λ_1, must be real. Since $P(0) = -1$ and $P(\lambda) > 0$ for λ sufficiently large, we must have one, say $\lambda_1 > 0$. But $f(\lambda_1) > 0$ so λ_1 cannot be a root of $f(\lambda) = 0$ and $f'(\lambda) = 0$. We next show that under the above condition, λ_2 and λ_3 are both real. This follows easily; $P\left(\frac{-2}{3r}\right) = \frac{4}{27r^2} - \frac{5}{3} > 0$, since $r < 2/3\sqrt{5}$, and since $P(0) = -1$, we must have, say, λ_2 real, and clearly $\lambda_2 > -\frac{2}{3r} > -\frac{1}{r}$. But since $P\left(-\frac{1}{r}\right) < 0$, we also have $\lambda_3 > -\frac{1}{r}$. So $f(\lambda_j) = \lambda_j^2 + b\lambda_j e^{-\lambda_j r} + 1 > \frac{-b}{r}e + 1 \geq 0, j = 2, 3$, since $\lambda e^{-\lambda r}$ is increasing for $\lambda < 0$, and $eb < r$. This shows that λ_2 and λ_3 cannot solve $f(\lambda) = 0$, which completes our proof.

References

[1] **Fischer, A.**, *Almost periodic solutions with prescribed spectrum of systems of linear and quasilinear differential equations with almost periodic coefficients with constant time lag*, Math. Bohem 124(4), 351–379, 1999.

[2] **Fink, A.M.**, *Almost Periodic Differential Equations*, Lecture Notes in Math. 337, Springer-Verlag, 1974.

[3] **Hale, J.K.**, *Theory of Functional Differential Equations*, Appl. Math. Sciences 3, Springer-Verlag, 1977.

[4] **Hale, J.K. and Verdun Lunel, S.M.**, *Introduction to Functional Differential Equations*, Springer-Verlag, Berlin, New York, 1993.

[5] **Kolmanovskii, N. and Myshkis, A.** *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer Acad. Press, 1999.

[6] **Seifert, G.**, *Oscillatory solutions for certain delay-differential equations*, Proc. Am. Math. Soc., 104(2), 553–557, 1988.

[7] **Titschmarsh, E.C.**, *The Theory of Functions*, Oxford U. Press, 1944.