Cubo Matemática Educacional
Vol. 4, Nº1, MAYO 2002

A BRIEF LOOK AT CONTROL THEORY THROUGH ITS HISTORY

Jack W. Macki
Department of Mathematical Sciences
University of Alberta, Edmonton, Canada T2G 2G1

Introduction

Control is as old as civilization. Among the first control devices of which we have definite evidence are regulators which were used for automatically controlling the intake for water storage tanks, dating from about the third century B.C.(Figure 1).

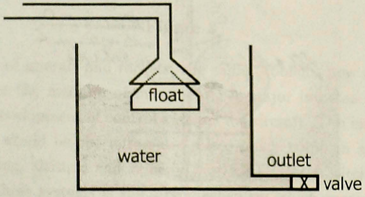


Figure 1

As civilization developed, so did the need for control devices, that is, mechanisms for controlling the **state of a system**. For the storage regulator the system is the valve, the float, and the tank with its water. The state of this system could be a list of two items: the amount (or height) of water in the tank, and whether the valve was open or closed. Undoubtedly similar control devices have also been used throughout history for agricultural irrigation.

When the industrial age began in the 1700's, the need for control devices increased dramatically. One of the best known is James Watt's governor for a steam engine, which regulated the supply of fuel to an engine to keep it running at a constant rate of rpm (Figure 2). The vertical shaft is attached to the motor; as the motor speed increases, centrifugal force moves the ball outward, and the attached levers close the throttle.

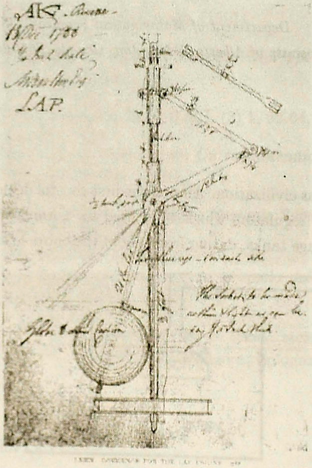


Figure 2

Of course warfare made use of any new developments in technology, and in fact was a driving force in the development of technology. One of the first weapons to need a fairly sophisticated control system was the torpedo. The first torpedoes, developed during the Civil War in the United States, were crude devices which travelled on the surface. This made them vulnerable to marksmen, since a direct hit from a rifle round could easily detonate the explosive. To run under the water, the torpedo needed a depth sensor and control vanes to force it down or up. It also needed a control system for the speed of the screw propeller. But there is much more: the torpedo can oscillate about three axes, the vertical (yaw), horizontal perpendicular to the direction of motion (pitch) and horizontal parallel to the direction of motion (roll). The mathematical modelling of the control of such a complex device was far beyond the methods of the era, and solutions were arrived at by hit and miss until well into the Second World War.

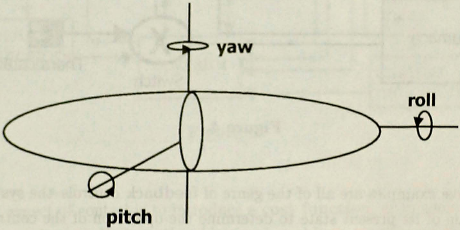


Figure 3

With the rise of aircraft and rockets, the same problems now arose with water replaced by air as the medium of movement. A major impetus to the theory of control was the development of control systems for aircraft. The most sophisticated of these systems would be the automatic pilot, which holds an aircraft to a predetermined heading, altitude and velocity (and also controls pitch, roll and yaw). The simplest of these systems is the servomechanism which controls, for example, the setting of the ailerons. It must compare the aileron position with the setting of the pilot's control lever, and use a servo motor to adjust the actual position to fit the pilot's desired position. From the mathematical point of view this simple servo driven control is of the same conceptual form as a thermostat in a dwelling, which compares the actual temperature (state) at a point in the dwelling with the desired temperature (target state) as set by the thermostat. The system allows power to the furnace/air conditioning to make changes as needed. The entire system can be modelled as a set of boxes with lines showing the use of information to control the furnace (Figure 4). With both ailerons and thermostats one must avoid "chattering" — roughly the rapid turning on and off of the control mechanism. For example, you don't want your furnace turning on and off every 2 minutes.

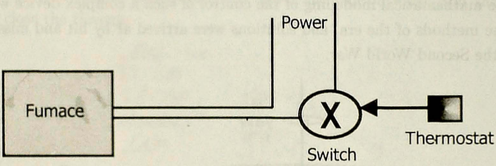


Figure 4

These three examples are all of the genre of **feedback** controls–the system uses only the value of its present state to determine the operation of the control apparatus. In engineering language, the control system "feeds back" observations based on the state to decide what the control device should do. There is no external interference with the system once the parameters (headings, temperature) are set, and the time of at which action is taken is of no consequence.

Nuclear reactors introduced a need for reliable fail-safe feedback control systems. One must continually monitor the temperature of the core to prevent accidents. The core temperature can be controlled by two methods: circulation of cooling fluid, and insertion of neutron moderating control rods (equivalently, withdrawal of some fuel rods.). Such systems get very complex, here is a diagram of part of one such control system (Figure 5).

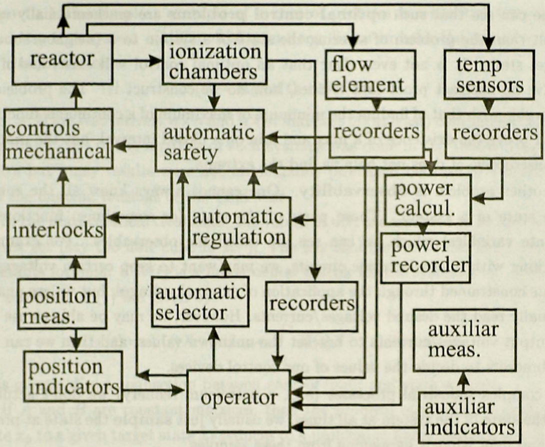


Figure 5

Another aspect of control is to introduce a **cost function**, or in some contexts, a performance criterion. Perhaps the oldest such problems are those of the calculus of variations. A prototype of such problems is the **brachistrchrone** problem, which we can state in a simplified form as the problem in the plane of designing a curve linking the points $P = (0,1)$ and $Q = (1,0)$ which allows a particle sliding down the curve under gravity from P to Q to descend in the least possible time. At first glance this seems far removed from control theory, but let's rephrase it: You need to move a particle from P to Q under the force of gravity. Obviously any curve which is always decreasing (and even many which are not always decreasing) will do. Each choice of curve is a control for moving from your initial state (particle at P) to your desired **target state** (particle at Q). Now you must choose the control (the curve) in such a way as to minimize your **cost**, which in this case is the elapsed time.

One can see that such **optimal control problems** are mathematically more difficult than the problem of steering the state of a system to a (neighborhood of a)target state. It is not even clear that an optimal control will exist, and if we can give an abstract proof that it does, how do we construct it? The problem is comparable with that of finding the minimum or maximum of a continuous function. We can prove the existence of a max and min on a closed interval, but the proof is not constructive–it does not help us find the extrema.

Another problem is **observability**. One cannot always know all the entries in the state of a system. Those parts of the state (or sometimes, functions of the state variables) which we can see are called the observables. For example, in working with microelectronic circuits, we may want to keep certain voltages or currents constrained through the application of external voltages, but we are unable to actually read the desired voltages/currents. However, we may be able to use the chip output voltages/currents to bracket the unknown values, and then we can use these brackets to decide the values of our control devices.

In complex industrial processes (e.g., a petroleum refinery) we don't actually know the state of the system at all times, we usually just **sample** the state at preset times and infer what is happening from these samples.

On top of all this, there is the problem of **noise** and uncertainty. In the control of a spacecraft, for example, we get data on the state of the system which is full of noise–static from solar wind particles, losses of data streams for various reasons. We need to devise filters which give us a reasonably accurate picture of the state, and we need a control algorithm which is **robust** in the face of errors induced by random noise and imperfect modelling.

The Mathematical Formulation

Now we can present some of the mathematics that goes with the above general outline.

We denote the state of the system by an n-vector $x(t)$, and the control mechanism by an m-vector $u(t)$. The interaction of the control apparatus with the state of the system is usually governed by a difference or differential equation (ordinary or partial) The simplest example of a control problem is when the state evolves according to a system of differential equations, linear in both the state vector and the control vector:

$$x'(t) = A(t)x(t) + B(t)u(t). \tag{1}$$

Here A, B are respectively $n \times n$ and $n \times m$ matrices of (at least) continuous functions. The major part of the analysis of such systems was done in the 1940's, 1950's, and 1960's but many results were not immediately published because of World War II and the ongoing tensions of the Cold War.

In the mathematical treatment, the values of $u(t)$ are assumed to lie in a compact convex set (say the unit cube in R^m). The key tool here is to use the variation of constants formula: if $X(t)$ is a fundamental matrix for the system $x' = A(t)x$, then the solutions of (1) are given by

$$x(t) = X(t)X^{-1}(0)x_0 + \int_0^t X(t)X^{-1}(s)B(s)u(s)ds. \tag{2}$$

This gives a direct connection between control input and state output.

If A and B are constant matrices, then the problem of steering a given initial state x_0 to a given target state x_f reduces to a problem in linear algebra, and formula (2) leads almost directly to the **bang-bang principle**:

If you can steer from an initial state x_0 to a target x_f using controls which take values in a compact strictly convex set, then you can accomplish the same result using only controls which take values at the extreme points of this set.

The reason it is called the *bang-bang* principle is that if you were using a scalar control with values from the interval $[-1, 1]$, then the principle says that you need only use the control values -1 and $+1$; in other words, you don't need a throttle, you just "bang" the system with maximum control power until you get to your target.

If we make a simple translation of the state variable, we can assume that $x_f = 0$.

Theorem: Define the $n \times mn$ **controllability matrix**

$$M = B,\ AB,\ A^2B,\ \ldots\ A^{n-1}B.$$

i) We can steer any state in some small ball about the origin in R^n to the 0-state if and only if $\text{rank} M = n$.

ii) We can steer any initial state to the 0-state if and only if

$$\operatorname{rank} M = n \text{ and } \operatorname{Re}(\lambda) < 0$$

for every eigenvalue of A.

The analysis of control problems got a major boost in the 1950's and 1960's with the appearance of major results by the Soviet school, the seminal work of the Romanian mathematician Popov, important work by engineers in England, and results obtained by the RIAS group in the eastern U.S. and the Rand Corporation group in California. Much previously secret international work was studied and improved by these groups and incorporated into monographs and research papers in the open literature. The major center for the Soviet school was Leningrad (now St. Petersburg) centered on A.I. Lur'e. RIAS was a research center funded primarily by the U.S. Defence Department, and located in Maryland, and the Rand Corporation was a government-funded think-tank based in the Los Angeles area. There were close contacts between the RIAS group and Princeton University, and several seminal works were published by Princeton University Press, including translations of key Russian monographs and papers. At Rand, Richard Bellman was publishing his results in dynamic programming and optimal control theory. In addition, he edited an important series of monographs published by Academic Press, and published (with Kenneth Cooke) a fundamental monograph on finite difference equations, which play a large part in practical applications of control theory. Last, but certainly not least, we have the invention by Kalman of the Kalman filter for uncertain systems.

In a popular article, it would be a major digression to try to list all of the players and their contributions. Instead we focus on a few individuals and a small number of landmark results and monographs.

In the English-languate literature, Lefschetz' 1965 monograph "Stability of Nonlinear Control Systems" was a major advance in the propagation of the latest nonlinear results. This monograph served to bring control theory to the attention of English speaking mathematicians. In it, Lefschetz explained, extended, and systematized results which had recently appeared in the world literature. The monograph appeared in the series "Mathematics in Science and Engineering" founded by Richard Bellman and published by Academic Press. In this same series appeared "Optimization Techniques: with Applications to Aerospace Systems" by George

Leitmann (1962), "Optimum Design of Digital Control Systems" by Julius Tou (1963), "Dynamic Programming in Chemical Engineering and Process Control" by Sanford M. Roberts (1965), "Random Processes in Nonlinear Control Systems" by A. A. Pervozvanskii (1965), "Adaptive Processes in Economic Systems" by R.E. Murphy (1965), and "Control Systems Functions and Programming Approaches", by Dimitris N. Chorafas (1966). By 1975 there were over 70 volumes in this series, of which more that 45 were directly related to control theory. In one decade, this series of monographs established control theory as a flourishing mathematical discipline.

A first major goal of engineers and theoreticians was to establish some usable results for nonlinear systems. English researchers and the Romanian Popov pioneered the use of Laplace and Fourier transforms for systems that were linear in the state. As a simple example, if in equation (1) the matrices A and B are constant, then we can take the Laplace transform of this equation and get (using Greek letters for transforms):

$$\xi(s) = [sI - A]^{-1}[x_0 + B\mu(s)]. \tag{3}$$

This is known as the **frequency domain** formulation of the control problem. If all the eigenvalues of A have negative real part (note the leading term on the right side of (4)) then we can "steer" the state to $x_f = 0$ exponentially fast by just setting $u(t) \equiv 0$ (it may be however, that we can help things along by a judicious choice of $u(\cdot)$). Note also that in this case the system would be robust—small errors in the coefficients would not change the outcome.

In one dimension, one of the first nonlinear problems to get a thorough treatment was the **Problem of Lur'e**:

$$x' = -kx + u, \; u' = \phi(s), \; s = cx - \rho u, \text{ with } k > 0.$$

Here the control parameters are c and ρ and the problem is to find necessary and sufficient conditions (n.a.s.c.) for solutions $(x(t), u(t))$ of this system to go to $(0,0)$ as time goes to infinity, independent of the initial conditions and regardless of the choice of function ϕ (this is called absolute stability). This is a typical indirect control problem: the evolution of the state $x(t)$ is influenced by the control function $u(t)$, but it is *the rate of change* of $u(t)$ which is determined by the feedback $cx - \rho u$. The Soviet school made effective use of Liapunov functions to determine appropriate n.a.s.c. for absolute stability of Lure's problem. For this system, one can use $V = py^2 + \Phi(s)$, where $\Phi(s) = \int^s \phi(r)dr$. One can then develop simple sufficient conditions for the trajectories of solutions to cross the level curves of this function in the direction of its decrease, giving stability. Their work was applied to systems of higher dimension as well, with a commensurate increase in the linear algebra needed.

Popov, followed by Yakubovich and by Kalman, applied Fourier transform methods to make major advances in the analysis of control systems. Consider the vector system:

$$x' = Ax - \phi(s)b, \; u' = \phi(s), \; s = c^T x - \gamma u,$$

where x, b are n-vectors, A is a matrix and all other symbols represent scalars (T denotes transpose).

If the Fourier transform of a scalar, vector or matrix function $f(t)$ is defined as $F(\omega) = \int_0^{+\infty} e^{-i\omega t} f(t)dt$ where the integration is carried out for each component of a vector or matrix, then the critical matrix function in Popov's theory is the **transfer function**:

$$G(i\omega) = c^T F(e^{At})b + \frac{\gamma}{i\omega}.$$

He showed that the inequality

$$Re(1 + i\omega q)G(i\omega) \geq 0 \text{ for some } q \geq 0 \text{ and all real } \omega$$

implies the system is absolutely stable. Stability is important because it implies that disturbances die out naturally.

However, it is the **synthesis** of controls which is the most important problem facing engineers in practice. How do we design a control which will force the state to arrive at and remain in a neighborhood of the given target state, to within some error tolerance? Again, we can translate the desired state to the origin, so we are asking how to steer the state of the system to a small neighborhood of the origin in state $\times$ control-space. The analysis for linear problems is a beautiful piece of mathematical theory, using linear algebra, linear functional analysis and convexity. As described above, if we want to send a particular initial state x_0 to a neighborhood of the origin in state space, we convert the initial value problem (1), $x(0) = x_0$ to a single integral equation using the variation of parameters formula (2) for systems of differential equations. This simple formula connects our choice of $u(\cdot)$ to the corresponding output $x(\cdot)$. It is clear from this formula, for example, that if the functions $u(\cdot)$ come from a convex set, the possible outputs $x(\cdot)$ also form a convex set. In fact, if the values of $u(t)$ lie in a convex set, then this formula combines with a famous theorem of Liapunov (not the stability Liapunov) on the range of a vector measure to assure us that we need only use controls which take their values at the extreme points of this convex set–i.e., the bang-bang principle holds.

The situation with genuinely nonlinear problems is much more challenging. Basically, engineers went ahead and developed intuitively plausible methods, without much help or input from the mathematical community. One of the most important ideas was to truncate Fourier expansions of controls and states. The method goes by various names—the method of harmonic balance, the describing function method, equivalent linearization, equivalent gains. Mathematicians recognize all of these techniques as being variants of the Ritz/Galerkin method. As a simple example, consider the scalar nonlinear problem:

$$y'(t) = -y(t) + u^2(t), \;\; y(0) = 0, \;\; y(\pi) = 0. \tag{4}$$

We assume that $u(t) = \sum_{k=-\infty}^{\infty} a_k e^{ikt}$, and $y(t) = \sum_{k=-\infty}^{\infty} b_k e^{ikt}$, and for simplicity assume that we use an even extension of each to the interval $(-\pi, \pi)$, so $a_{-1} = a_1$ and $b_{-1} = b_1$. Then we plug these expressions into (4) and throw away all but the terms in $k = 0$ and $k = \pm 1$. This results in the following (note that $b_0 = 0$) :

$$\begin{gathered} \sum_{k=-\infty}^{\infty} ikb_k e^{ikt} = -\sum_{k=-\infty}^{\infty} b_k e^{ikt} + \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-k}^{k} a_j a_{k-j}\Big) e^{ikt}, \\ -2b_1 \sin(t) = -2b_1 \cos(t) + a_0^2 + 2a_0 a_1 \cos(t). \end{gathered} \tag{5}$$

So this simple *algebraic* equation connects the control coefficients a_0, a_1 to the response coefficient b_1. Nyquist and Bode developed graphical methods for analyzing the stability of nonlinear systems after such a truncation, and these were quickly extended to control problems. Such methods are surprisingly effective for a single systems, but if one has a large number of interacting control systems, or a problem with a high dimension to the state and control variables, then they can be extremely difficult to apply.

Optimal Control

When one puts in a **cost functional**, the situation becomes even more complex. For example, consider the system (x is the state, u the control): $x'(t) = f(t, x(t), u(t))$, with desired target state x_f. Note that we don't necessarily specify the time of arrival t_f at the target. Now include a cost functional for a given successful control:

$$C[u(\cdot)] = \int_0^{t_f} f_0(t, x(t), u(t))dt.$$

The idea is to minimize C over all successful controls. For example, if we take $f^0 \equiv 1$, then we are minimizing the time to get to the target state. The connection with the calculus of variations is clear, but in fact was not noted in the early days of control theory. In the late 1950's, the Pontryagin group at Moscow State University stunned the control theory world by announcing the famous "maximum principle".

To understand the significance of this result, consider the problem addressed in standard calculus courses of finding the absolute max of a real-valued differentiable function f(x) defined on an interval $[a, b]$. A sufficient condition that this max exist is that f be continuous (a condition not always met in engineering practice). But this <u>sufficient</u> condition does nothing to help us find the point(s) where the max occurs (or its value), for this we need to use the <u>necessary</u> condition $f'(c) = 0$. Pontryagin and his group gave us a necessary condition for a control to provide a solution to an optimal control problem.

To formulate this powerful principle, we consider for simplicity the **autonomous** problem

$$x'(t) = f(x(t),\ u(t)), u(t)\epsilon U,\ C[u(\cdot)] = \int_0^{t_f} f^0(x(t), u(t))dt.$$

Here $x(t)$ and f are n-vectors, $u(t)$ is an m-vector with values in a convex closed bounded set U. Given a particular control and response pair $(x(t), u(t))$, we first form the extended state $\hat{x}(t)$ by adding the new component $x_0(t) = \int_0^t f_0(r, x(r), u(r))dr$ (the running cost), so $x_0'(t) = f_0(x(t), u(t))$. We add this differential equation to our original system, writing the extended system as $\hat{x}'(t) = \hat{f}'(\hat{x}(t), u(t))$. For a given (assumed optimal) control and associated extended state we can linearize our extended differential equation about $\hat{x}(t)$ to get the associated linearized equation and its adjoint:

$$\hat{y}'(t) = \frac{\partial \hat{f}}{\partial x}\hat{y}, \;\; \hat{w}'(t) = -\left[\frac{\partial \hat{f}}{\partial x}\right]^T \hat{w}(t).$$

The solutions of the linearized adjoint are called *costates*. Finally, we need to introduce an associated Hamiltonian: $\mathcal{H}(\hat{w}, \hat{x}, u) = \hat{w}(t) \cdot \hat{f}(\hat{x}, u)$, the inner product of $\hat{w}(t)$ and $\hat{x}'(t)$. This is indeed a genuine Hamiltonian in the sense of physics for the linearized system and its adjoint—that is, these two equations can be written respectively as:

$$y'(t) = \frac{\partial H}{\partial w}, \;\; w'(t) = -\frac{\partial H}{\partial y}.$$

The Hamiltonian is important because the optimal control will maximize it at each value of t, and in fact this maximum value of H will be zero. That is, if $\mathcal{M}(\hat{w}(\cdot), \hat{x}(\cdot))(t) = max_{v \epsilon U} \mathcal{H}(\hat{w}(t), \hat{x}(t), v)$, then the maximum principle states:

If $(\hat{x}_*(\cdot), u_*(\cdot))$ is an optimal control-response pair minimizing C for our problem, then there exists a solution $\hat{w}(t)$ of the associated adjoint equation such that almost everywhere

$$\mathcal{H}(\hat{w}(t), \hat{x}_*(t), u_*(t)) = \mathcal{M}(\hat{w}(t), \hat{x}_*(t)) = 0, \text{ and } w_0(t) \equiv w_0(0) \leq 0.$$

Therefore we need only find the general solution (!) of the linear adjoint equation and try to find solutions which make $\mathcal{H}$ identically zero. During this search (keep in mind we don't know $u_*(t)$) we have to carry $u(\cdot)$ along, and -if we are lucky- we will end up with a set of constraints on $u(\cdot)$ which will characterize it. In fact, one may not need to find the general solution of the adjoint equation.

As an example, consider the following scalar problem, with $u(t) \epsilon [-1, 1]$:

$$x'(t) = -x^2(t) + u(t), \; x(0) = 0, \; x(t_f) = 1, \; C(u(\cdot)) = \int_0^{t_f} u(r)dr.$$

The extended linearized system is (with $y_0(\cdot)$ as defined above, $y_1(\cdot) = x(\cdot)$) :

$$y_0'(t) = 0, \;\; y_1'(t) = -2x(t)y_1(t).$$

This can be written in system form as

$$\hat{y} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}, \;\; \hat{y}'(t) = \begin{bmatrix} 0 & 0 \\ 0 & -2x(t) \end{bmatrix} \hat{y}(t).$$

The associated (extended) adjoint system and Hamiltonian are

$$\hat{w}'(t) = \begin{bmatrix} 0 & 0 \\ 0 & 2x(t) \end{bmatrix} \hat{w}(t),\ \mathcal{H} = w_0(t)u + w_1(t)(-x^2(t) + u),$$

$$\mathcal{M} = \max_{v\epsilon[-1,1]} (w_0(0)v + w_1(t)(-x_*^2(t) + v)) = [w_0(0) + w_1(t)]v - x^2 w_1(t).$$

The linear function $\mathcal{M}(v)$ must be nonpositive for $v\epsilon[-1,1]$. It will have a maximum for $v\epsilon[-1,1]$ at $v = \text{sgn}[w_0(0) + w_1(t)]$, so we know our optimal control must be bang-bang. In this case we can find $\hat{w}(t)$ explicitly, and an elementary argument shows that the control will have at most one *switch* from ± 1 to ∓ 1. Knowing this, we can return to the original equation for $x(t)$, replace $u(t)$ by 1 and -1 respectively, and solve the resulting equations, piecing together successful responses with at most one switch. We leave the details to the reader and turn to other issues.

When the Pontryagin maximum principle was announced, and in particular when the monograph of Pontryagin, Bol'tanskii, Gamkrelidze and Mischenko appeared in English in 1964 (Russian version in 1961),.the connection with the calculus of variations was clearly pointed out. It was eventually noted that a paper of E.J. McShane published in 1937 foreshadowed the maximum principle. These connections are thoroughly explained in the 1966 monograph of Hestenes. Here is a very simplified explanation (based on that given in the monograph of Lee and Marcus) of the connection, with no attempt made to carefully state hypotheses. Consider a control process

$$x'(t) = f(x(t), u(t)),\ x(0) = x^0, x(1) = x_1, C(u(\cdot)) = \int_0^1 h(x,u)dt,$$

with $x(t)\epsilon R^n$, $u(t)\epsilon R^m$. If we could solve the equation $p = f(x,u)$ for $u = g(x,p)$ then we could use the differential equation to write $u(t) = g(x(t), x'(t))$. This means that we could replace $u(t)$ in the cost functional by $g(x(t), x'(t))$,, so we can restate our control problem as

$$\min \int_0^1 h(x(t), g(x(t), x'(t)))dt,$$

where the minimum is taken over all functions $x(\cdot)$ which satisfy $x(0) = x_0$, $x(1) = x_1$. This is just a classic problem in the calculus of variations, and the standard necessary conditions correspond to the maximum principle.

The maximum principle provided for the first time a method for finding optimal controls, or at least reducing the search to a smaller subclass of the original set of optimal controls. However, its use can be very difficult in higher dimensions, and in problems involving several interacting control systems.

Dynamic Programming and Nonsmooth Analysis

About the same time that the Pontryagin maximum principle was being developed, Richard Bellman was inventing a very effective method for a broad range of optimization problems, a method ideally suited to the computer. He christened it **dynamic programming** and described its uses in detail in his monograph "Applied Dynamic Programming" in 1962, which was followed by a monograph of Dreyfus in 1965. The foundation of this approach was the following observation: suppose that you are trying to carry out a process so as to minimize some functional–i.e., to do things optimally. If you carry out an optimal policy up to a certain time, arriving at a some intermediate state, then your policies after that time must be optimal for initially starting at this intermediate state.

To clarify this somewhat obscure statement, consider the problem of minimizing the functional

$$J(y(\cdot)) = \int_0^T [(y')^2 + y^2]dt, \quad y(0) = c, \quad y' \epsilon L^2(0,T).$$

Instead of applying the standard calculus of variations analysis, we treat the problem as a decision process: for each t, assuming that $y(t)$ is known, we must choose $y'(t)$ so as to minimize J. To translate this into something practical, we write the minimum value as $f(c,T) = \min J(y(\cdot))$, and assume that it is well defined for all $T \geq 0$ and $c \epsilon R$. We describe this minimum function as the min of $J(y(\cdot))$ for an interval of length T with initial condition c (notice that the problem is autonomous, so there is no dependence on where our interval is located on the t-axis).

Here is a highly intuitive derivation of the socalled Bellman equation for this problem: Let $q = x'(0)$ be the initial slope. If we break the interval $[0,\ T]$ into $[0,\ \Delta] \bigcup [\Delta,\ T]$, with Δ infinitesimal, then $y(\Delta) \approx c + q\Delta$, and

$$J(y) = \left(\int_0^\Delta + \int_\Delta^T\right) [(y')^2 + y^2]dt.$$

The optimality principle states that having arrived at $y(\Delta)$ at time Δ, we must continue to choose $y'(t)$ so that $\int_\Delta^T$ is minimized. Since Δ is infinitesimal, we can take $y(t) = c$, $y'(t) = q$ on $[0, \Delta]$. Thus the first integral in $J(y(\cdot))$ above becomes $(q^2 + c^2)\Delta$. If we choose $y'(t)$ optimally thereafter, then the second integral gives $m(c+q\Delta, T-\Delta)$, i.e., it produces the minimum of J for an interval of length $T-\Delta$ and initial state $c + q\Delta$. So we have

$$J(u) = (q^2 + c^2)\Delta + m(c + v\Delta, T - \Delta),$$

$$m(c, T) = \min_q[(q^2 + c^2)\Delta + m(c + q\Delta, T - \Delta)].$$

Now, continuing our intuitive analysis with the infinitesimal Δ, we have

$$m(c + q\Delta, T - \Delta) \approx m(c, T) + \frac{\partial m}{\partial c} q\Delta - \frac{\partial m}{\partial T}\Delta.$$

Substituting this into the equation preceding, simplifying and letting $\Delta \to 0$, we get

$$\frac{\partial m}{\partial T} = \min_q \left[c^2 + q^2 + q\frac{\partial m}{\partial c},\right]$$

with the initial condition $m(c, 0) = 0$.

The minimum on the right occurs for $q = -\frac{1}{2}\frac{\partial m}{\partial c}$, and so we have the following nonlinear partial differential equation for $m(c, T)$:

$$\frac{\partial m}{\partial T} = c^2 - \frac{1}{4}\left(\frac{\partial m}{\partial c}\right)^2, \quad m(c, 0) = 0.$$

The point is that one could now solve this equation by numerical methods, and thus obtain the optimal value function, without knowing the optimal control or associated optimal state. Then one can determine the optimal control by an incremental choice of $y'(t)$, choosing this function so the cost stays at $m(x(t), T - t)$.

As an aside, for this simple problem, we can actually determine the optimal cost function directly. We make the change of variable $y(t) = cz(t)$ in the original problem, so $z(0) = 1$. This converts the cost to $c^2 \int_0^T [(z'(t))^2 + z^2(t)]dt$, which shows that $J(u(\cdot)) = c^2\mathcal{J}(T)$. Plugging this into the partial differential equation above, we get a Ricatti equation for $\mathcal{J}$:

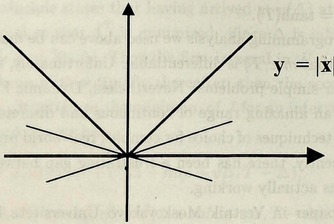
$$\mathcal{J}'(T) + \mathcal{J}^2(T) = 1, \ \mathcal{J}(0) = 0,$$

with solution $\mathcal{J}(T) = \tanh(T)$.

The Dynamic Programming analysis we used above can be made rigorous, under the key assumption that $m(c,T)$ is differentiable. Unfortunately, this requirement is seldom met, even for simple problems. Nevertheless, Dynamic Programming gives correct answers for an amazing range of continuous and discrete problems, and is still one of the main techniques of choice for complex real-world problems (e.g., space missions). Until recently, there has been a frustrating gap between what could be proved and what was actually working.

In a landmark paper in Vestnik Moskovskovo Universiteta in 1958, A.F. Fillipov showed how one could interpret a control problem involving the differential equation $x'(t) = f(x(t), u(t))$, $u(t) \epsilon U(t) \subseteq R^m$, as a differential equation with multivalued right hand side. Basically, one thinks of the differential equation as $x'(t) \epsilon \bigcup_{v \epsilon U} f(x(t), v)$, where one must choose a solution of this multivalued differential equation which reaches the target and (if there is a cost) minimizes the cost. This allowed the use of sophisticated new tools from multifunction theory to be used to prove existence, and made people aware of the importance of multivalued functions.

We mentioned above that a rigorous justification of Dynamic Programming is hindered by the nonsmoothness of the optimal value function as a function of the initial and or terminal conditions on the state. Although the function is often Lipschitz continuous, it is seldom differentiable. In the course of dealing with nonsmoothness for general problems in optimization, various multivalued extensions of the derivative of a convex or locally Lipschitz function have been considered. In general, the tangent line to a real-valued function of one variable is replaced by a set of tangent lines which form a cone. For example, the simple absolute value function $f(x) = |x|$ is assigned the interval $[-1,\ 1]$ as its derivative at the origin, or if you prefer, it is assigned the cone of straight lines through the point $(0,0)$ with slopes from this closed interval (Figure 6).



The generalized gradient at the origin is the cone of straight lines through $(0,0)$ with slope between -1 and $+1$.

Figure 6

Analysis for such multivalued generalizations of the gradient has been developing since the 1950's. Major schools have developed in a multitude of countries, motivated by a wide range of applied and pure problems. R.T. Rockafellar of the United States is one of the leaders in this field, and his student Francis H. Clarke has developed a generalized gradient (the "Clarke gradient") which works very well for control theory. Clarke published a ground-breaking paper, "The maximum principle under minimal hypotheses" in 1976 (SIAM J. Control Optim. 14(1976), 1078-1091) which convincingly demonstrated the effectiveness of his new tools, and he followed this in 1983 with a monograph. In 1990 Clarke, Ledyaev, Stern and Wolenski published the monograph "Nonsmooth Analysis and Control Theory" which provides an up-to-date treatment. The generalized gradient allows a much "cleaner" and complete theoretical treatment of existence, uniqueness, necessary conditions and sufficient conditions for a wide range of optimization problems.

Robustness

From the engineering standpoint, the question of existence is often secondary. What really matters is that the solution be **robust**. This means that the system has a tolerance for disturbances and\or inaccuracies within specified bounds. This requirement reflects the fact that one never precisely knows the parameters in a modelling differential equation (and associated cost function) nor can one attain or measure exactly the given initial conditions. In addition, the successful control itself is an idealization which in practice can only be approximated.

Bode in 1945 established the criterion for a limited type of robustness for scalar systems.

Here is a simple example of how the question of robustness arises. Consider the control system whose governed by the scalar ordinary differential equation with constant coefficients:

$$ax''(t) + bx'(t) + cx(t) = u(t), \quad -1 \leq u(t) \leq +1, \quad x(0) = x_0, x'(0) = x_1,$$

with the aim of steering the state vector $(x(t), x'(t))$ to a given neighborhood of $(0,0)$ and keeping it there. For any given choice of control function $u(t)$ we take the Laplace transform of the equation and see that the transform of $x(t)$, call it $X(s)$, is given by:

$$X(s) = \frac{1}{as^2 + bs + c}(U(s) + bx_0 + ax_1 + asx_0),$$

where $U(s)$ is the transform of $u(t)$. The fraction $\frac{1}{as^2+bs+c}$ is called the *transfer function* for the problem. Now the function $x(t)$ will be the convolution of the inverse transform of the transfer function with the function $u(t)$, plus the inverse transform of

$$\frac{bx_0 + ax_1 + asx_0}{as^2 + bs + c}.$$

If the quadratic in the denominator of the transfer function has both roots with real part negative (such polynomials are called *Hurwitz polynomials*), then any easy argument shows that the state vector decays exponentially. In other words, this system attains the target neighborhood for any control. The coefficients in the equation can be perturbed by small amounts and this conclusion will still hold. In this case the system is robust in every sense.

Suppose we are dealing with a control problem involving an n^{th} order scalar constant coefficient differential operator:

$$\sum_{k=0}^{n} a_k \frac{d^k}{dt^k}.$$

Then we mimic the analysis in the above example, taking the Laplace transform, and conclude that we want the polynomial $\sum_{k=0}^{n} a_k s^k$ to be Hurwitz, that is, all of its roots should have negative real part. But unlike the above example, suppose that we are given *a priori* that each coefficient a_k is restricted to a (not necessarily tiny) interval: $m_k \le a_k \le M_k$. Then as the coefficients independently vary over their respective intervals, we get a family of polynomials, the roots of which will cover a region in the complex plane. At first glance, this is a very complicated situation. A beautiful result of Kharitonov from 1978 provides an amazingly simple resolution–it says you need only check **four** polynomials! The four Kharitonov polynomials are

$$k_{11}(s) = m_0 + m_1 s + M_2 s^2 + M_3 s^3 + m_4 s^4 + m_5 s^5 + ...$$

$$k_{12}(s) = m_0 + M_1 s + M_2 s^2 + m_3 s^3 + m_4 s^4 + M_5 s^5 + ...,$$

$$k_{21}(s) = M_0 + m_1 s + m_2 s^2 + M_3 s^3 + M_4 s^4 + m_5 s^5 ...,$$

$$k_{22}(s) = M_0 + M_1 s + m_2 s^2 + m_3 s^3 + M_4 s^4 + M_5 s^5 + ...,$$

Kharitonov's theorem states that the entire family is Hurwitz if and only if these four polynomials are Hurwitz. This result has been extended in many directions, for example, if the coefficients do not vary independently over intervals but instead each coefficient is a function of a vector parameter, then one can develop an analogous result under reasonable conditions (Anagnost, Desoer and Minnichelli, 1988). One can also prove a similar result for systems governed by difference equations (Mansour, Kraus and Anderson, 1988).

The robustness of systems governed by vector differential equations is a more difficult topic, particularly when the control is a vector, and especially when it is a feedback control. We cannot take the time and space to deal with that here, we just mention a few of the key ideas. In 1966 Zames introduced a highly geometric approach based on cones in an appropriate space of input-output relations. Safanov in 1980 published a monograph establishing robustness results for systems, using sophisticated topological methods.

Robustness is also a desired property for systems with statistical indeterminacy or some type of random disturbance. We restrict ourselves to perhaps the most fundamental problem: How to estimate parameters (for example, the rate of growth of bacteria, or the orientation angles of a space satellite) when you have limited samples of some observables, and these samples are possibly corrupted by noise. It is theoretically important, but not very useful from the practical point of view, to prove that as the uncertainties die out, the estimate converges to the true value. What is important is to get an idea of the probability distribution of the unknowns, given assumed distributions for the indeterminacies and the disturbances. One might also hope to discover the dynamics of the expected value and/or the variance of important parameters.

Robustness has understandably been a major subject of interest from the 1970's right up to the present, and is deserving of its own survey article. Fortunately such articles and monographs exist, and the reader is referred to the list of references in the Bibliography.

Acknowledgments

I would like to express my appreciation to the editor, Claudio Cuevas, and to the referee, for their patience and kind advice. I am especially indebted to the referee for providing several important references on robustness.

Bibliography

Control Theory in General

- **D.J. Bell, P.A. Cook and N. Munro editors**, Design of Modern Control Systems, Peter Peregrinus Ltd., Stevenage (England), 1982. Part of a series published by the Institution of Electrical Engineers. Covers the basics of virtually all of the techniques used by engineers.

- **Richard Bellman**, *Introduction to the Mathematical Theory of Control Processes*, Volumes I and II, Academic Press, New York, 1967 and 1971. Another classic, packed with examples and motivation.

- **Jay C. Hsu and Andrew U. Meyer**, *Modern Control Principles and Applications*, McGraw-Hill, New York, 1968. This is a classic which covers the

basics of virtually everything discussed. There is a short historical commentary in section 1.6.

- **A. Isidori**, *Nonlinear control Systems: An Introduction*, Springer, New York, 1985.
- **Solomon Lefschetz**, *Stability of Nonlinear Control Systems*, Academic Press New York, 1965.
- **E.D. Sontag**, *Mathematical Control Theory*, Springer, New York, 1990.
- **T.L. Vincent and W.J. Grantham**, *Nonlinear and Optimal Control Systems*, John Wiley and Sons, New York, 1997. A thorough treatment of modern techniques in control.
- **W. Murray Wonham**, *Linear Multivariable Control: a Geometric Approach*, Springer, New York, 1979. This is a highly original and very geometric (i.e., linear algebra) treatment of all of the important aspects of synthesis for linear systems.

Discrete Control Problems

- **Albert B. Bishop**, *Introduction to Discrete Linear Controls*, Academic Press, New York, 1975. Several of the general texts listed above cover some aspects of discrete problems. Bishop's monograph is devoted exclusively to the subject.

Optimal Control

- **Wendell H. Fleming and Raymond W. Rishel**, *Deterministic and Stochastic Optimal Control*, Springer, New York, 1975. this is a very original text, aimed at a thorough treatment of the maximum principle, carefully describing the connection with the calculus of variations, a precise presentation of dynamic programming, and then a thorough presentation of stochastic control problems. This is a demanding text which rewards the reader who puts in the effort to master it.
- **E.B. Lee and L. Markus**, *Foundations of Optimal Control Theory*, John Wiley and Sons, New York, 1967. Published as part of a series by SIAM, this collaboration between an engineer and a mathematician produced a classic. It

rigorously covers the known results in optimal control and stability up to the time of publication.

- **Jack Macki and Aaron Strauss**, *Introduction to Optimal Control Theory*, Springer, New York, 1982. Focusses on a modern mathematical treatment of linear theory and the maximum principle.

- **L.S. Pontryagin, V.G. Bol'tanskii, R.S. Gamkrelidze and E.f. Mischenko**, *The Mathematical Theory of Optimal Processes*, translated from the Russian, Pergamon, Oxford, 1964. The classic exposition of their discovery.

Stochastic Control

- **M.H.A. Davis**, *Linear Estimation and Stochastic Control*, Chapman and Hall, London, 1977. A very readable presentation of the basic ideas in stochastic estimation and filtering.

- **Robert J. Elliott**, *Stochastic Calculus and Applications*, Springer, New York, 1982. A very thorough and readable introduction to modern stochastic analysis, with applications to control. Requires a good background in probability and measure theory.

Robustness

- **R.A. Freeman and P.V. Kokotovic**, *Robust Nonlinear Control Design.* State-Space and Lyapounov Techniques, Birkhauser, Boston, 1996.

- **P.V. Kokotovic**, *Constructive nonlinear control: progress in the 90's*, Proc. 14th Triennial World Congress of the IFAC, Beijing, 1999.

- **G. Leitmann**, *On one approach to the control of uncertain systems*, J. Dynamical Systems, Measurement and Control 115 (1993).

- **M. Milanese, R. Tempo and A. Vicino editors**, *Robustness in Identification and Control*, Plenum Press, New York, 1989. This proceedings from a 1989 conference contains a nice range of articles on contemporary work in robustness.

- **Z. Qu**, *Robust Control of Nonlinear Uncertain Systems*, John Wiley and Sons, New York, 1998.

- **Michael George Safonov**, *Stability and Robustness of Multivariable Feedback Systems*, MIT Press, Cambridge (Massachusettsf), 1980. A mathematically sophisticated approach to robustness for both continuous and discrete feedback controls. One of the first to deal effectively with multivariable problems.

- **T.L. Vincent and W.J. Grantham**, *Nonlinear and Optimal Control Systems*, John Wiley and Sons, New York, 1997.

Nonsmooth Analysis

- **J.-P. Aubin and H. Frankowska**, *Set-Valued Analysis*, Birkhauser, Boston, 1990.

- **J.-P. Aubin and A. Cellina**, *Differential Inclusions*, Springer, Berlin, 1994.

- **Frank H. Clarke**, *Optimization and Nonsmooth Analysis*, John Wiley and Sons, New York, 1983.

- **F.H. Clarke, Yu.S. Ledyaev, R.J. Stern and P.R. Wolenski**, *Nonsmooth Analysis and Control Theory*, Springer, New York, 1998.

- **P.D. Loewen**, *Optimal Control via Nonsmooth Analysis*, CRM Proc. Lecture Notes, Vol. 2, Amer. Math. Soc., Providence, RI, 1993.

- **R.T. Rockefafellar**, *The Theory of Subgradients and Its Application to Problems of Optimization: Convex and Nonconvex Functions*, Helderman Verlag, Berlin, 1981.