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Quantization and Pseudo-differential Operators

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Abstract

We revise the main definitions of quantization and in this frame we treat in particular Weyl and anti-Wick Operators. The paper can also be ragarded as an introductory review of some aspects of the pseudo-differential calculus in $\mathbb{R}^n$.

Contents

1 Introduction: from Classical Physics to Quantum Systems

At the beginning of this century classical physics seemed to be able to explain practically every physical phenomenon of nature. On one hand the second Newton's law $F = ma$ had been reelaborated and lead to the formulation of the powerful theories of Lagrangean and Hamiltonian Mechanics that provided a satisfactory mathematical model for phenomena involving particles. On the other hand wave-like phenomena found an interpretation by the wave equation

$$\partial_t^2 u(t,x) = \Delta_x u(t,x)$$

(with $x \in \mathbb{R}^n, t \in \mathbb{R}, \Delta_x = \sum_{j=1}^n \partial_{x_j}^2$) and Maxwell laws of electromagnetism. This illusion found its end with a number of experiments in the years around 1900. Among the most famous of them we just cite the photo-electric effet, experiences on diffraction of light and the problem of the stability of the electron. It is beyond the scope of this paper to discribe precisely in which sense these discoveries disregarded the above-mentioned theories. In one word, one could say that in the case of microscopic systems involving particles such as atoms or electrons, nature seemed to bahave in quite a different way as it should have done according to the known classical theories of Physics. The most striking fact was that many of these new discovered features of nature seemed to be (and are today) beyond the possibility of perception of the world we are able to. The dual nature of matter both as particle and as wave and the uncertainty principle of Heisenberg are just the two most famous examples. It becomes then clear that a deep explanation of nature must renounce to the help of intuition and practical experience due to our senses and it appears then evident the importance of constructing an abstract mathematical model that at best fits the experiments.

To be more precise let us consider the case of a single particle in the space $\mathbb{R}^n$ (note that for $n = 3k$ it includes the case of k particles in the usual space $\mathbb{R}^3$). The state of the system is then completely determined by the $2n$ variables position $x \in \mathbb{R}^n$ and momentum (i.e., in the simplest case, velocity times mass) $\xi \in \mathbb{R}^n$. Each physical quantity relevant for the description of the system, for instance position, velocity, potential energy, kinetik energy, angular momentum, etc, is expressed by a real number depending on the state of the system and, from the mathematical point of view, is then a real function of (x,ξ). The space $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$ is called the *phase space* of the system. It not difficult to immagine that in more complicated cases, where the particles have to satisfy some constraints, the position variables describe a surface or more generally a manifold in $\mathbb{R}^n$. The phase space is then the cotangent bundle of this manifold and (x,ξ) play the role of local coordinates. However this is not essential for the subject we want to deal with. The evolution of the system is then a curve $t \in \mathbb{R} \to (x(t),\xi(t)) \in \mathbb{R}^{2n}$ in the phase space that satisfies the Hamiltonian system of equations

$$\begin{cases} \frac{dx_i}{dt} &= \partial_{\xi_i} H(x,\xi) \\ \frac{d\xi_i}{dt} &= -\partial_{x_i} H(x,\xi) \end{cases}$$

where $H = E - V$, E is the kinetik energy and V the potential. H is called *Hamiltonian function*. The Hamiltonian equations are nothing else than a rewriting of the second Newton law $F = ma$ in the frame of phase space.

The phase space model is no longer suitable for the description of quantum systems. Instead of this, Schrödinger and Heisenberg presented two different models that were later proved by Schrödinger to be equivalent. According to the model of Schrödinger, the space $L^2(\mathbb{R}^n_x)$ of square-integrable functions on $\mathbb{R}^n_x$ substitutes of the phase space $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$ as underlying space. More precisely the state of the system at a given time is represented by a function in $L^2(\mathbb{R}^n_x)$ modulo a complex multiplicative constant. That is, if $f \in L^2(\mathbb{R}^n_x)$, then for every complex constant c, the function cf represents the same state. Equivalently we can say that the states of a quantum system are points of the complex projective space $\mathcal{P}L^2(\mathbb{R}^n_x)$ and we can take as representative for each point in this space a unitary function of $L^2(\mathbb{R}^n)$. Observables are now identified with (unbounded) self-adjoint operators T on $L^2(\mathbb{R}^n)$ with dense domain and the rules used to associate an operator with each classical observable are called *quantization rules*. The description of the system is in general no more deterministic, as in classical mechanics, but probabilistic in the sense as follows. The probability that the value assumed by an observable T when the system is in the state ϕ lies in the set $I \in \mathbb{R}$ is given by $||E_T(I)\phi||^2$, where E_T is the *spectral projection* associated with the operator T. Only occasionally this probability distribution is concentrated in one point, recovering then as particular case a deterministic description. The evolution of the system is given by a curve $t \in \mathbb{R} \to \phi \in L^2(\mathbb{R}^n)$ that, instead of the Hamiltonial equations, satisfies the Schrödinger equation:

$$\mathcal{H}\phi = -\frac{h}{2\pi i}\partial_t \phi$$

where h is the Plank constant and $\mathcal{H}$ is the *Hamilton Operator*, i.e. the operator corresponding with the Hamiltonian function by the quantization rules. It becomes then clear that these rules play a foundamental role in the construction of the mathematical model for the quantum system. The two foundamental requirements that must be satisfied by these rules are the following.

1. The operator associated with the *position* observable $f(x, \xi) = x_j$ is the *multiplication* operator $M_j\phi = x_j\phi$

2. The operator associated with the *momentum* observable $f(x, \xi) = \xi_j$ is the *differentiation* operator $D_j\phi = \frac{h}{2\pi i}\partial_{x_j}\phi$

The quantization of general observables $f(x, \xi)$ is generally performed by replacing in the expression of f the variables x_j and ξ_j with the corresponding operators. This unfortunately leads to a non trivial commutativity problem as we have of course $x_j\xi_j = \xi_j x_j$ but $M_j D_j \neq D_j M_j$. This ambiguity makes it necessary to make a "choise" giving rise to different types of quantizations.

Here we shall fix mainly attention on the Weyl and the anti-Wick quantization, addressing to Berezin [2] and Robert [11] for a more detailed discussion of the general concept of quantization. Namely, the contents of the paper are the following. In the next Section 2 we begin to study Weyl quantization for partial differential operators. To treat more general symbols we organize in Sections 3 and 4 a calculus on $\mathbf{R}^n$, based on a fixed weight function $\Lambda(x,\xi)$. The corresponding operators are in principle a particular case of the so-called Weyl-Hörmander classes, cf. [9], however the peculiarities of the properties of $\Lambda(x,\xi)$ allow here stronger results. In particular, in Section 5 we introduce the τ-quantization of Λ-symbols, extending Shubin [12] and Boggiatto-Buzano-Rodino [3]; for $\tau = 1/2$ we recapture the Weyl quantization. General results on symbolic calculus are presented in Section 6. In Section 7 we treat the anti-Wick quantization in strict connection with Weyl operators, cf. Lerner [10], Wong [13]. The final Sections concern hypoelliptic operators, Sobolev spaces and Fredholm property.

We end these introductory remarks by fixing some notation.

$$\mathbf{Z}^+ = \{1,2,\dots\}, \quad \mathbf{N} = \{0,1,\dots\}, \quad \mathbf{R}^+ = \{x\in\mathbf{R} \mid x>0\}, \quad \mathbf{R}_0^+ = \{x\in\mathbf{R} \mid x\geq 0\}.$$

Given two multi-indices α, $\beta \in \mathbf{N}^n$ and $x\in\mathbf{R}^n$, we set

$$|\alpha| = \alpha_1+\dots+\alpha_n, \quad \alpha! = \alpha_1!\cdots\alpha_n!, \quad \alpha\leq\beta \iff \alpha_j\leq\beta_j, \qquad \text{for } j=1,\dots,n,$$

$$\alpha<\beta \iff \alpha\neq\beta \quad\text{and}\quad \alpha\leq\beta,$$

$$\binom{\alpha}{\beta} = \begin{cases} \dfrac{\alpha!}{\beta!(\alpha-\beta)!}, & \text{if } 0\leq\beta\leq\alpha,\\ 0, & \textit{otherwise}, \end{cases}$$

$$x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}, \qquad \partial^\alpha = \partial_x^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}, \qquad D^\alpha = D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n},$$

with $D_{x_j} = -i\partial_{x_j}$, for $j=1,\dots,n$, and $i^2=-1$.

Given x, $y\in\mathbf{R}^n$, we set

$$x\cdot y = x_1y_1+\dots+x_ny_n, \qquad |x|_{\mathbf{R}^n} = |x| = \sqrt{x\cdot x} = (x_1^2+\dots+x_n^2)^{1/2},$$

$$\langle x\rangle = \sqrt{1+|x|^2}, \qquad \langle\langle (x,y)\rangle\rangle = \sqrt{1+|x|^2+|y|^2}.$$

We employ standard notation of distribution theory: $\mathcal{S}(\mathbf{R}^n)$, $\mathcal{S}'(\mathbf{R}^n)$, $\mathcal{E}'(\Omega)$, $\mathcal{D}'(\Omega)$. Moreover $(u,v)_{L^2} = \langle u,\overline{v}\rangle = \int u\overline{v}\,dx$, for u, $v\in\mathcal{S}(\mathbf{R}^n)$.
If the domain of integration is not specified it is intended to be the whole space, i.e. for $x\in\mathbf{R}^n$: $\int u\,dx = \int_{\mathbf{R}^n} u\,dx$.

If $\xi \in \mathbf{R}^n$, we set $đ\xi = (2\pi)^{-n}d\xi$. so that $\int u\, đ\xi = (2\pi)^{-n} \int u\, d\xi$.
Finally, given two functions $f, g : X \to \mathbf{R}$ and $A \subset X$, we write $f(x) \prec g(x)$, for all $x \in A$, if there exists $C \in \mathbf{R}^+$ such that $f(x) \leq Cg(x)$, for all $x \in A$. Here the constant C may depend on parameters, indices, etc. possibly appearing in the expression of f and g, but not on $x \in A$. We write $f(x) \sim g(x)$ if $f(x) \prec g(x)$ and $g(x) \prec f(x)$.

2 Some Remarks on Weyl Quantization

Let $p(x,\xi) = \sum_{|\alpha|\leq m} c_\alpha(x)\xi^\alpha$ be a polynomial in ξ with coefficients $c_\alpha \in C^\infty(\mathbf{R}^n)$. If in correspondence with $p(x,\xi)$ we consider the differential operator $P(x,D) = \sum_{|\alpha|\leq m} c_\alpha(x)D^\alpha$, with $D^\alpha = (-i)^{|\alpha|}\partial_x^\alpha$, we have that the two above-mentioned rules of quantization are satisfied (apart from the inessential constant factor $\frac{h}{2\pi}$). The polynomial $p(x,\xi)$ is said *symbol* of the operators $P(x,D)$.

For $u \in \mathcal{S}(\mathbf{R}^n)$, we have $u = \int e^{ix\xi}\hat{u}(\xi)đ\xi$ and it is then convenient to rewrite the expression of $P(x,D)u$ as

$$P(x,D)u = \int e^{ix\xi}p(x,\xi)\hat{u}(\xi)đ\xi = \int e^{i(x-y)\xi}p(x,\xi)u(y)dyđ\xi$$

From this formula it appears natural to replace the polynomial $p(x,\xi)$ with functions $a(x,\xi)$ belonging to some more general space; the operators

$$A_a u(x) = \int e^{ix\xi}a(x,\xi)\hat{u}(\xi)đ\xi = \int e^{i(x-y)\xi}a(x,\xi)u(y)dyđ\xi$$

obtained in this way are said *pseudo-differential*, the function $a(x,\xi)$ is still called *symbol* of the operator.

Unfortunately this correspondence is not a quantization because the operator A_a needs not to be self-adjoint also when the symbol $a(x,\xi)$ is real. To overcome this difficulty Weyl proposed another type of association namely

$$a(x,\xi) \to W_a u(x) = \int e^{i(x-y)\xi}a\left(\frac{x+y}{2},\xi\right)u(y)dyđ\xi$$

By this correspondence one has that the L^2-adjoint of the operator W_a is given by the formula

$$W_a^* = W_{\bar{a}}$$

from which it clear that in case a is a real function one obtains a self-adjoint operator. With respect to the ambiguity of quantization due to the non commutativity named in the previuos section, we remark that the Weyl quantization amounts to the "intermediate" choice $\frac{1}{2}(M_jD_j + D_jM_j)$ in the case of the observables $a(x,\xi) = x_j\xi_j$, we omit the easy verification. If $Pu = \sum_{|\alpha|\le m} c_\alpha D^\alpha u$ is a differential operator with C^∞ coefficients, then the adjoint is given by $Pu = \sum_{|\alpha|\le m} D^\alpha(\overline{c_\alpha}u)$ so, it is worth remarking that

$$W_{x_i\xi_j} = \frac{1}{2}(M_jD_j + D_jM_j) = \frac{1}{2}(A_a + A_a^*) \tag{2.1}$$

that is the Weyl quantization coincides with the Feymann quantization for which the observable a corresponds to the operator

$$F_a = \frac{1}{2}(A_a + A_a^*).$$

It is then natural to ask to what extend the Weyl and the Feynmann quantization coincide. We remark at once that for polynomial with complex coefficients we have in general no coincidence. Consider for example $a(x,\xi) = cx_j\xi_j$ with $c \in \mathbf{C}$. Then by linearity $W_a = \frac{c}{2}(M_jD_j + D_jM_j)$ but the Feynmann quantization is not $\mathbf{C}$ linear and we have $(cM_jD_j)^* = \bar{c}D_jM_j \neq cD_jM_j$. However, as observed before, meaningful physical observables are real functions, so for polynomials it is reasonable to consider real coefficients. In this case we have the following basic case where Weyl and Feynmann quantizations coincide.

Proposition 2.1 *Let* $a(x,\xi) = \sum_{\alpha,\beta} a_{\alpha,\beta}x^\beta\xi^\alpha$ *be a polynomial with real coefficients* $a_{\alpha,\beta}$ *of degree 1 either with respect to* x *or* ξ, *then:*

$$W_a u = \frac{1}{2}(A_a + A_a^*)\,u.$$

Proof. It is sufficient to consider $a(x,\xi) = a_{\alpha,\beta}x^\beta\xi^\alpha$. If $|\beta| = 0$ then obviously the thesis holds. Let $|\beta| = 1$, then $a(x,\xi) = x_j\xi^\alpha$ for some $j = 1, ..., n$ and we have

$$\begin{aligned} W_a u(x) &= \frac{1}{2}\int e^{i(x-y)\xi}x_j\xi^\alpha u(y)dyd\xi + \frac{1}{2}\int e^{i(x-y)\xi}y_j\xi^\alpha u(y)dyd\xi \\ &= \frac{1}{2}\left(A_{x_j\xi^\alpha}u + \mathcal{F}^{-1}\left[\xi^\alpha\widehat{y_ju}\right]\right) \\ &= \frac{1}{2}\left(A_{x_j\xi^\alpha}u + D^\alpha\left[M_ju\right]\right) \\ &= \frac{1}{2}\left(A_{x_j\xi^\alpha} + A^*_{x_j\xi^\alpha}\right)u. \end{aligned}$$

If $|\alpha| = 0$ and $|\beta| = 1$ the thesis is obviously true. By induction, suppose it holds for $|\alpha| = 0$ and some β then

$$W_{x_j x^\beta}u = \frac{1}{2}x_j W_{x^\beta}u + \frac{1}{2}W_{x^\beta}(y_j u) = \frac{1}{2}x_j x^\beta u + \frac{1}{2}x^\beta x_j u$$
$$= A_{x_j x^\beta}u = \frac{1}{2}\left(A_{x_j x^\beta}u + A^*_{x_j x^\beta}u\right).$$

If $|\alpha| = 1$ then $a(x,\xi) = x^\beta \xi_k$ for some $k = 1, ..., n$. For $|\beta| = 1$ the thesis is (2.1). Suppose now it holds for some β, we prove it for $a(x,\xi) = x_j x^\beta \xi_k$ with $j = 1, ..., n$. We have

$$W_{x_j x^\beta \xi_k}u(x) = \frac{1}{2}\int e^{i(x-y)\xi}(x_j + y_j)\left(\frac{x+y}{2}\right)^\beta \xi_k u(y)dyd\xi$$
$$= \frac{1}{2}x_j W_{x^\beta \xi_k} + \frac{1}{2}\int e^{i(x-y)\xi}y_j\left(\frac{x+y}{2}\right)^\beta \xi_k u(y)dyd\xi$$
$$= \frac{1}{2}x_j W_{x^\beta \xi_k}u + \frac{1}{2}W_{x^\beta \xi_k}(y_j u(y))$$
$$= \frac{x_j}{4}\left(A_{x^\beta \xi_k}u + A^*_{x^\beta \xi_k}u\right) + \frac{1}{4}\left(A_{x^\beta \xi_k}(y_k u) + A^*_{x^\beta \xi_k}(y_k u)\right)$$
$$= \frac{1}{4}x_j x^\beta D_k u + \frac{1}{4}x_j D_k(x^\beta u) + \frac{1}{4}x^\beta D_k(x_j u) + \frac{1}{4}D_k(x_j x^\beta u)$$

But $x^\beta D_k(x_j u) = D_k(x_j x^\beta u) - (D_k x^\beta)x_j u$ and $x^j D_k(x^\beta u) = x_j(D_k(x_j)u + x_j x^\beta D_k u$ so that

$$W_{x_j x^\beta \xi_k}u(x) = \frac{1}{2}x_j x^\beta D_k u + \frac{1}{2}D_k(x_j x^\beta u)$$
$$= \frac{1}{2}\left(A_{x_j x^\beta \xi_k}u + A^*_{x_j x^\beta \xi_k}u\right)$$

so the proposition is completely proved. ■

From second part of the proof of the above Proposition we also get the following Corollary.

Corollary 2.2 *Let* $a(x,\xi) = \sum_\beta a_\beta x^\beta$ *be a polynomial with complex coefficients* $a_{\alpha,\beta}$, *then:*

$$W_a u = A_a$$

(that is the Weyl quantization still yields multiplication operators for polynomial symbol independent on ξ*).*

We remark that if $|\alpha| > 1, |\beta| > 1$ the Proposition 2.1 is in general false. Consider infact for instance in dimension $n = 1$ the symbol $a(x,\xi) = x^2\xi^2$. We have

$$W_{x^2\xi^2}u = \frac{1}{4}\int e^{i(x-y)\xi}(x^2 + 2xy + y^2)\xi^2 u(y)dyd\xi$$
$$= \frac{1}{4}x^2D^2u + \frac{1}{2}xD^2(xu) + \frac{1}{4}D^2(x^2u)$$

Replacing $xD^2(xu) = -2ixDu + x^2D^2u$ and $-ixDu = \frac{1}{4}D^2(x^2u) + \frac{1}{4}x^2D^2u + \frac{1}{2}u$ yields

$$W_{x^2\xi^2}u = \frac{1}{2}\left(x^2D^2u + D^2(x^2u) + u\right) = \frac{1}{2}\left(A_{x^2\xi^2}u + A^*_{x^2\xi^2}u + Id\ u\right).$$

This suggests that one can in general try to express the difference between the two quantizations as a differential operator of lower order. This is in effect true in the sense of the following proposition.

Proposition 2.3 *Let $a(x,\xi) = \sum_{\alpha,\beta} a_{\alpha,\beta}x^\beta\xi^\alpha$ be a polynomial with real coefficients $a_{\alpha,\beta}$, then:*

$$W_a u = \frac{1}{2}\left(A_a + A_a^*\right)u + Ru$$

where

$$Ru = \sum_{\gamma,\delta} c_{\gamma,\delta}x^\gamma D^\delta u$$

and the sum extends to the multiindex $\gamma \leq \alpha$, $|\gamma| < |\alpha| - 2$, $\delta < \beta$, $|\delta| \leq |\beta| - 2$.

Proof. We have from Proposition (2.1) that the proposition is true for $|\beta| = 1$. As usual we suppose it true for some $|\beta|$ and shall proved it for $x_jx^\beta\xi^\alpha$, we have

$$W_{x_jx^\beta\xi^\alpha}u = \int e^{i(x-y)\xi}\frac{x_j + y_j}{2}(x+y)^\beta\xi^\alpha u(y)dyd\xi$$
$$= \frac{1}{2}x_jW_{x^\beta\xi^\alpha}u + \frac{1}{2}W_{x^\beta\xi^\alpha}(y_ju)$$
$$= \frac{1}{4}x_j\left(x^\beta D^\alpha u + D^\alpha(x^\beta u) + Ru\right)$$
$$+ \frac{1}{4}\left(x^\beta D^\alpha(x_ju) + D^\alpha(x_jx^\beta u) + Ru(x_ju)\right) \tag{2.2}$$

Using Leibnitz rule we have

$$D^\alpha(x_jx^\beta u) = \sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\partial^\gamma x_jD^{\alpha-\gamma}(x^\beta u) = x_jD^\alpha(x^\beta u) + \alpha_jD^{\alpha-e_j}(x^\beta u) \tag{2.3}$$

where the last term can be rewritten as

$$D^{\alpha-e_j}(x^\beta u) = x^\beta D^{\alpha-e_j}u + \sum_{0\neq\gamma\leq\alpha-e_j} \binom{\alpha-e_j}{\gamma}\partial^\gamma x^\beta D^{\alpha-e_j-\gamma}u$$

We have also

$$x^\beta D^\alpha(x_j u) = x^\beta \sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\partial^\gamma x_j D^{\alpha-\gamma}u = x^\beta x_j D^\alpha u + \alpha x^\beta D^{\alpha-e_j}u. \tag{2.4}$$

Substituting (2.3) and (2.4) into (2.2) we get

$$\begin{aligned} W_{x_j x^\beta \xi^\alpha} u =& \frac{1}{2}\left(A_{x_j x^\beta\xi^\alpha} + A^*_{x_j x^\beta \xi^\alpha}\right)u \\ &+ x_j Ru + R(x_j u) + \sum_{0\neq\gamma\leq\alpha-e_j}\binom{\alpha-e_j}{\gamma}\partial^\gamma x^\beta D^{\alpha-e_j-\gamma}u \end{aligned}$$

that is the thesis is proved. ■

Finally it follows from the above considerations that in the case of the harmonic oscillator of quantum mechanics, i.e. in the case $a(x,\xi) = x^2+\xi^2$ we have

$$W_{x^2+\xi^2} = \frac{1}{2}\left(A_{x^2+\xi^2} + A^*_{x^2+\xi^2}\right) = A_{x^2+\xi^2}$$

3 Weight Functions

Our classes of symbols will be defined in terms of general weight functions $\Lambda(z)$ in $\mathbf{R}^{2n}$. As an introduction, let us consider first the symbol of a differential operator with polynomial coefficients in $\mathbf{R}^n$:

$$a(x,\xi) = \sum_{|\alpha+\beta|\leq m} c_{\alpha,\beta}x^\beta\xi^\alpha.$$

Because $a(x,\xi)$ is a polynomial in (x,ξ), and $\partial_\xi^\alpha\partial_x^\beta a(x,\xi)$ is identically zero for $|\alpha+\beta|>m$, we have that for every $(\alpha,\beta)\in\mathbf{N}^n\times\mathbf{N}^n$

$$\left|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)\right| \prec \langle(x,\xi)\rangle^{m-|\alpha+\beta|}, \qquad \text{for all } (x,\xi)\in\mathbf{R}^n\times\mathbf{R}^n.$$

These inequalities suggest the definition of the following class of symbols, introduced by Shubin, Berezin and others authors. Let $z=(x,\xi)\in\mathbf{R}^{2n}$ and denote by $S^m = S^m(\mathbf{R}^{2n})$ the class of functions $a(z) \in C^\infty(\mathbf{R}^{2n})$ satisfying the following estimates for each $\gamma \in \mathbf{N}^{2n}$:

$$|\partial^\gamma a(z)| \prec \langle z\rangle^{m-|\gamma|}, \qquad \text{for all } z,$$

with $m \in \mathbf{R}$.

Of course one may substitute to $\langle z\rangle$ any *weight function* $\Lambda(z)$, continuous in $\mathbf{R}^{2n}$, for which there exists $\epsilon > 0$ such that

$$\langle z\rangle^\epsilon \prec \Lambda(z) \prec \langle z\rangle, \qquad \text{for all } z, \tag{3.1}$$

and define the class of symbols $a \in S^m_\Lambda(\mathbf{R}^{2n})$ such that for each $\gamma \in \mathbf{N}^{2n}$ we have

$$|\partial^\gamma a(z)| \prec \Lambda(z)^{m-|\gamma|}, \qquad \text{for all } z. \tag{3.2}$$

One easily verifies that $S^k \subset S^m_\Lambda$ with $m = \max\left\{\frac{k}{\epsilon}, k\right\}$, and $S^k_\Lambda \subset S^m$ with $m = \max\{k\epsilon, k\}$. This shows that taking into account more general weights does not essentially change the class of symbols under consideration. The reason for considering a more general weight $\Lambda(z)$ resides in the fact that sometimes it may be chosen in such a way as to yield better estimates in the applications, see for example the boundedness properties in the next Sections.

A basic example, which we shall consider in the sequel, is given by

$$\Lambda(z) = 1 + \sum_{j=1}^{2n} |z_j|^{1/q_j},$$

$q = (q_1, \dots, q_{2n})$ with $q_j \geq 1$, adapted to the study of the operator having polynomial symbol of the form $a(z) = \sum_{\alpha\cdot q\leq m} c_\alpha z^\alpha$.

Turning to the general case, to obtain a good pseudo-differential calculus, we have to impose to $\Lambda(z)$ some additional conditions. Namely, beside (3.1), we shall assume that $\Lambda(z)$ is *slowly varying*, i.e. there exists $\epsilon > 0$ such that

$$\Lambda(z) \sim \Lambda(\zeta), \text{ for } |\zeta - z| \leq \epsilon\Lambda(z). \tag{3.3}$$

Let us observe that, starting from (3.1), (3.3), one can always find $\tilde\Lambda(z) \in C^\infty(\mathbf{R}^{2n})$, with $\tilde\Lambda(z) \sim \Lambda(z)$, satisfying (3.1), (3.3) and the additional property

$$|\partial^\gamma \tilde\Lambda(z)| \prec \tilde\Lambda(z)^{1-|\gamma|}. \tag{3.4}$$

In the next section we shall then be allowed to assume that (3.4) is also satified by $\Lambda(z)$.

From (3.3) it also easily follows that $\Lambda(z)$ is *temperate*, in the sense that

$$\Lambda(z) \prec \Lambda(\zeta)\langle z - \zeta\rangle. \tag{3.5}$$

Another property we shall require for $\Lambda(z)$ is the following. We shall assume that for all t we have

$$\Lambda(tz) \prec \Lambda(z), \tag{3.6}$$

where we may take $t \in \mathbb{R}$, or more generally $t = (t_1, ..., t_{2n}) \in \mathbb{R}^{2n}$ with $tz = (t_1 z_1, ..., t_{2n} z_{2n})$. Combining (3.3) with (3.6) we obtain for $t' \in \mathbb{R}^{2n}$, $t'' \in \mathbb{R}^{2n}$

$$\Lambda(t'z + t''\zeta) \prec \Lambda(\zeta)\langle z - \zeta\rangle. \tag{3.7}$$

From (3.5) it also follows that for any $s \in \mathbb{R}$

$$\Lambda(z)^s \prec \Lambda(\zeta)^s \langle z - \zeta\rangle^{|s|} \tag{3.8}$$

and more precisely for $s < 0$

$$\Lambda(z)^s \prec \left(1 + \Lambda(\zeta)\langle z - \zeta\rangle^{-1}\right)^s \tag{3.9}$$

The next proposition, used in the sequel, combines the preceding estimates in a more general form.

Proposition 3.1 *Define*

$$\tilde{\lambda}_s(x, y, \xi) = \begin{cases} \Lambda(x,\xi)^s \langle x - y\rangle^s, & \text{for } s \geq 0 \\ (1 + \Lambda(x,\xi)\langle x - y\rangle^{-1})^s, & \text{for } s < 0, \end{cases} \tag{3.10}$$

then

$$\Lambda(v'x + v''y, \xi)^s \prec \min\left\{\tilde{\lambda}_s(x, y, \xi), \tilde{\lambda}_s(y, x, \xi)\right\}, \tag{3.11}$$

for all $s \in \mathbb{R}$, $v', v'' \in \mathbb{R}^n$, provided $(x, y) \to (v'x + v''y, x - y)$ is an isomorphism on $\mathbb{R}^{2n}$.

Summing up, we shall assume in the next section that $\Lambda(z)$ satifies (3.1), (3.3), (3.6), which give all the other preceding estimates.

Addressing finally to the readers acquainted with the work of Hörmander [9], Chapter 18, or Beals [1], on general pseudo-differential operators, we want to regard Λ in the frame of the more general weights considered there. Precisely we observe that $\zeta^2\Lambda(z)^2 = \frac{y^2}{\Lambda(x,\xi)^2} + \frac{\eta^2}{\Lambda(x,\xi)^2}$ gives a metric g in the sense of [9]. In fact g is slowly varying, in view of (3.3). Moreover the symplectic dual metric g_σ is given by $\Lambda(z)^2\zeta^2 = \Lambda(x,\xi)^2 y^2 + \Lambda(x,\xi)^2\eta^2$ and the temperance property of [9] reads $\Lambda(z) \prec \Lambda(\zeta)(1 + \Lambda(\zeta)|z - \zeta|)^N$, for some $N > 0$, which obviously follows from (3.5). Since $g \leq g^\sigma$ in view of (3.1), the uncertainty principle is also satisfied.

The Weyl calculus of Hörmander [9] applies then to pseudo-differential operators related to our weight function Λ. Let us also refer to Bony-Chemin [6], concerning Sobolev spaces associated with a generic metric, and Lerner [10] about Wick quantization. The calculus of [9] allows also linear symplectic changes of variables.

On the other hand, Λ presents pecularities, coming from the previous properties, which will be essential in the development of our theory. So, rather then appealing to the above-mentioned general calculus, in the next sections we shall prefer to start from scratch, emphasizing, for a fixed system of coordinates, the results related to the quantization problem.

4 Symbols and Amplitudes

First we want to be more precise about notations for classes of symbols.

Definition 4.1 *The symbol class S^m_Λ, denoted for short S^m in the sequel, $m \in \mathbf{R}$, consists of the functions $a(z) \in C^\infty(\mathbf{R}^{2n})$ which satisfy the estimates*

$$|\partial^\gamma a(z)| \prec \Lambda(z)^{m-|\gamma|}.$$

Let us assume, without loss of generality, that $\Lambda(z)$ satisfies (3.4); then we may write $\Lambda(z) \in S^1$.

We list in the following some basic propositions; proofs are omitted.

Proposition 4.2 *We have $S^{-\infty} = \bigcap_m S^m = \mathcal{S}(\mathbf{R}^{2n})$.*

Proposition 4.3 (i) *$S^m \subset S^{m'}$, if $m \leq m'$.*

(ii) *If $a \in S^m$, and $b \in S^{m'}$, then $ab \in S^{m+m'}$ and $a + b \in S^{\max\{m,m'\}}$.*

(iii) *If $a \in S^m$, then $D^\alpha a \in S^{m-|\alpha|}$ for all α.*

(iv) *If $a \in S^m$, then $T_w a(z) = a(z - w) \in S^m$ for all $w \in \mathbf{R}^{2n}$.*

Let us observe that S^m is a Fréchet space with respect to the seminorms $|a|_{k,S^m} = \sup_{|\gamma|\leq k} \sup_{z\in\mathbf{R}^{2n}} \Lambda(z)^{-m+|\gamma|}|\partial^\gamma a(z)|$.

The preceding Proposition 4.3 can be reconsidered in the corresponding topology; we have in particular continuity of the linear map $D^\alpha : S^m \to S^{m-|\alpha|}$.

Definition 4.4 *Let $a_j \in S^{m_j}$, $j = 1, 2, \ldots, m_j \to -\infty$ with $m_{j+1} \leq m_j$ for all j, and let $a \in S^{m_1}$. We write $a \sim \sum_{j=1}^\infty a_j$ if for all integer $r \geq 2$ $a - \sum_{1\leq j<r} a_j \in S^{m_r}$. We say also in this case that $\sum_{j=1}^\infty a_j$ is an asymptotic expansion for a.*

Proposition 4.5 *Let $a_j \in S^{m_j}$, $j = 1, 2, \ldots, m_j \to -\infty$ with $m_{j+1} \le m_j$ for all j. Then there exists $a \in S^{m_1}$ such that $a \sim \sum_{j=1}^{\infty} a_j$. If another symbol a' has the same property, then $a - a' \in \mathcal{S}(\mathbf{R}^{2n})$.*

In the next Section 5 we shall associate to the symbol $a(z) \in S^m$ the pseudo-differential operator of generalized Weyl type

$$u \to \int e^{i(x-y)\cdot\xi} a\left((1-\tau)x + \tau y, \xi\right) u(y)\, dy d\xi, \tag{4.1}$$

where $\tau = (\tau_1, \ldots, \tau_n) \in \mathbf{R}^n$ is fixed and, for short, we denote by τy the vector $(\tau_1 y_1, \ldots, \tau_n y_n)$ etc.. It is then natural to consider from the very beginning more general operators of the form

$$Au(x) = \int e^{i(x-y)\cdot\xi} a(x, y, \xi) u(y)\, dy d\xi, \tag{4.2}$$

where the function $a(x, y, \xi) \in C^\infty(\mathbf{R}^{3n})$, called *amplitude*, satisfies the following estimates suggested by (3.9).

Definition 4.6 *We define $\overline{S}^m$ to be the class of all $a(x, y, \xi) \in C^\infty(\mathbf{R}^{3n})$ satisfying*

$$\left|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x, y, \xi)\right| \prec \lambda_{m, m', \alpha, \beta, \gamma}(x, y, \xi), \tag{4.3}$$

with

$$\lambda_{m, m', \alpha, \beta, \gamma}(x, y, \xi) = \Lambda(x, \xi)^m \langle x - y\rangle^{m'} \left(1 + \Lambda(x, \xi)\langle x - y\rangle^{-m'}\right)^{-|\alpha+\beta+\gamma|}, \tag{4.4}$$

for a suitable $m' \in \mathbf{R}$ depending on $a(x, y, \xi)$, but independent of α, β, γ.

Note that (4.3) implies the somewhat weaker estimate

$$|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x, y, \xi)| \prec \Lambda(x, \xi)^{m - |\alpha+\beta+\gamma|} \langle x - y\rangle^{m' + m'|\alpha+\beta+\gamma|},$$

which will be often used in the sequel.

In (4.3) the variables x and y seem to play a different role; the subsequent property, consequence of Proposition 3.1 gives however a symmetric form to Definition 4.6.

Proposition 4.7 *The estimate (4.3) is equivalent to each one of the following two, for suitable values of $m' \in \mathbf{R}$:*

$$\left|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x, y, \xi)\right| \prec \lambda_{m, m', \alpha, \beta, \gamma}(y, x, \xi), \tag{4.5}$$

$$\left|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x, y, \xi)\right| \prec \min\left\{\lambda_{m, m', \alpha, \beta, \gamma}(x, y, \xi), \lambda_{m, m', \alpha, \beta, \gamma}(y, x, \xi)\right\}.$$

Proposition 4.8 *If $a(x,\xi) \in S^m$, then for every $\tau \in \mathbb{R}^n$ the amplitude $b(x,y,\xi) = a((1-\tau)x + \tau y, \xi)$, from (4.1), belongs to $\overline{S}^m$. In particular $b(x,y,\xi) = a(x,\xi)$ and $b(x,y,\xi) = a(y,\xi)$ are in $\overline{S}^m$.*

Proposition 4.9 *If $b(x,y,\xi) \in \overline{S}^m$, then $b(y,x,\xi) \in \overline{S}^m$ and $a(x,\xi) = b(x,x,\xi) \in S^m$.*

Finally we observe that $\overline{S}^m$ is a Fréchet space with respect to the seminorms

$$|a|_{k,\overline{S}^m} = \sup_{|\alpha+\beta+\gamma|\le k} \sup_{x,y,\xi} \lambda_{m,m',\alpha,\beta,\gamma}(x,y,\xi)^{-1} \left|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x,y,\xi)\right|.$$

5 Pseudo-differential operators

We begin by considering a general operator A of the form (4.2) i.e.

$$Au(x) = \int e^{i(x-y)\cdot\xi} a(x,y,\xi)u(y)\,dyd\xi, \tag{5.1}$$

with $a \in \overline{S}^m$. We may regard (5.1) as oscillatory integral, and define it for $u \in \mathcal{S}(\mathbb{R}^n)$. Otherwise, $Au \in \mathcal{S}'(\mathbb{R}^n)$ shall be defined on $v \in \mathcal{S}(\mathbb{R}^n)$ by taking $(x-y)\cdot\xi$ as phase, and $a(x,y,\xi)u(y)v(x)$ as amplitude, i.e.:

$$\langle Au, v\rangle = \int e^{i(x-y)\cdot\xi} a(x,y,\xi)u(y)v(x)\,dxdyd\xi.$$

Using Definition 4.6 and subsequent properties, we obtain:

Theorem 5.1 *The operator A in (5.1) defines a continuous map from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$, and it extends to a continuous map from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.*

Definition 5.2 *We shall write $\mathcal{L}^m$ for the class of all the operators of the form (5.1) with amplitudes $a(x,y,\xi) \in \overline{S}^m$.*

The Schwartz kernel K_A of A, defined by $\langle K_A, v\otimes u\rangle = \langle Au, v\rangle = \langle u, {}^tAv\rangle$, is the distribution in $\mathcal{S}'(\mathbb{R}^{2n})$ given by

$$\langle K_A(x,y), \phi(x,y)\rangle = \int e^{i(x-y)\cdot\xi} a(x,y,\xi)\phi(x,y)\,dxdyd\xi, \qquad \phi \in \mathcal{S}(\mathbb{R}^{2n}).$$

We have for K_a the following relevant properties.

Theorem 5.3 *If $A \in \mathcal{L}^m$, then $K_A \in C^\infty(\mathbb{R}^{2n}\backslash\Delta)$, where $\Delta = \{(x,x), x \in \mathbb{R}^n\}$.*

Theorem 5.4 *Let A be a continuous map from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}'(\mathbf{R}^n)$, with kernel $K_A \in \mathcal{S}'(\mathbf{R}^{2n})$. The following properties are equivalent:*
(i) A is regularizing, i.e. it extends to a continuous map from $\mathcal{S}'(\mathbf{R}^n)$ to $\mathcal{S}(\mathbf{R}^n)$;
(ii) $K_A \in \mathcal{S}(\mathbf{R}^{2n})$;
(iii) $A \in \mathcal{L}^{-\infty} = \bigcap_m \mathcal{L}^m$.

Take note that if the amplitude a is in $\mathcal{S}(\mathbf{R}^{3n})$ then certainly $a \in \overline{S}^{-\infty}$, but the opposite is not true; take for example $\phi \in \mathcal{S}(\mathbf{R}^n)$ and define $a(x,y,\xi) = \phi(x+y)\phi(\xi) \in \overline{S}^{-\infty}$, which is not rapidly decreasing at $x = -y$.

The next theorem has a crucial role when developing the symbolic calculus; it states that every $A \in \mathcal{L}^m$ can be represented in the special form (4.1) for any fixed $\tau \in \mathbf{R}^n$, with a suitable symbol in S^m.

Theorem 5.5 *Let $A \in \mathcal{L}^m$ be given, with kernel K_A and amplitude $a \in \overline{S}^m$, and let $\tau \in \mathbf{R}^n$ be fixed. There exists one and only one symbol $b_\tau \in S^m$ such that*

$$Au(x) = \int e^{i(x-y)\cdot\xi} b_\tau\left((1-\tau)x + \tau y, \xi\right) u(y)\, dy d\xi. \tag{5.2}$$

For $b_\tau(x,\xi)$ we have the following expression

$$\begin{aligned} b_\tau(x,\xi) &= \mathcal{F}_{y\to\xi} K_A\left(x + \tau y, x - (1-\tau)y\right) \\ &= \int e^{-iy\cdot\eta} a\left(x+\tau y, x-(1-\tau)y, \xi-\eta\right) dy d\eta, \end{aligned} \tag{5.3}$$

and the following asymtotic expansion

$$b_\tau(x,\xi) \sim \sum_{\beta,\gamma} \frac{(-1)^{|\beta|}}{\beta!\gamma!} \tau^\beta (1-\tau)^\gamma \left(\partial_\xi^{\beta+\gamma} D_x^\beta D_y^\gamma a\right)(x,x,\xi), \tag{5.4}$$

where we may assume the terms are re-arranged with decreasing orders.

Proof. We may give the standard oscillatory meaning to (5.3) and, changing the order of the integrations, obtain

$$\begin{aligned} \mathcal{F}_{y\to\xi} K_A(x+\tau y, x-(1-\tau)y) &= \int e^{iy\cdot(\eta-\xi)} a(x+\tau y, x-(1-\tau)y, \eta)\, dy d\eta \\ &= \int e^{-iy\cdot\eta} a(x+\tau y, x-(1-\tau)y, \xi-\eta)\, dy d\eta. \end{aligned}$$

This proves the second identity in (5.3). Let us show now that the function $b_\tau(x,\xi)$ defined by (5.3) is in S^m. In fact, integrating by parts we have for arbitrary M and N

$$\partial_\xi^\alpha \partial_x^\beta b_\tau(x,\xi) = \int e^{-iy\cdot\eta}\langle y\rangle^{-2M}(1-\Delta_\eta)^M \left\{\langle\eta\rangle^{-2N}(1-\Delta_y)^N a_{\alpha\beta\tau}(x,y,\xi,\eta)\right\} dy d\eta,$$

where

$$a_{\alpha\beta\tau}(x,y,\xi,\eta) = \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\left(\partial_\xi^\alpha \partial_x^{\beta-\gamma}\partial_y^\gamma a\right)(x+\tau y, x-(1-\tau)y, \xi-\eta).$$

Using (4.3), we can estimate under the integral sign by

$$\begin{aligned}\Lambda(x+\tau y,\xi-\eta)^m\langle y\rangle^{m'}&\left(1+\Lambda(x+\tau y,\xi-\eta)\langle y\rangle^{-m'}\right)^{|\alpha+\beta|}\langle y\rangle^{-2M}\langle\eta\rangle^{-2N}\\ &\le \Lambda(x+\tau y,\xi-\eta)^{m-|\alpha+\beta|}\langle y\rangle^{m'+m'|\alpha+\beta|-2M}\langle\eta\rangle^{-2N}.\end{aligned} \tag{5.5}$$

Using (3.8), we may further estimate in (5.5)

$$\Lambda(x+\tau y,\xi-\eta)^{m-|\alpha+\beta|} \prec \Lambda(x,\xi)^{m-|\alpha+\beta|}\langle(\tau y,\eta)\rangle^{|m|\mu+|\alpha+\beta|\mu}$$

and choosing sufficiently large M and N we conclude $b_\tau \in S^m$. It remains to prove that b_τ satisfies (5.2) and admits the asymptotic expansion (5.4). Let us denote for a moment K_τ the kernel of the operator in the right-hand side of (5.2); we have for $\phi \in \mathcal{S}(\mathbf{R}^{2n})$

$$\begin{aligned}\langle K_\tau,\phi\rangle &= \int e^{i(x-y)\cdot\xi} b_\tau((1-\tau)x+\tau y,\xi)\,\phi(x,y)\,dxdyd\xi\\ &= \int e^{iw\cdot\xi} b_\tau(v,\xi)\phi(v+\tau w, v-(1-\tau)w)\,dvdwd\xi,\end{aligned}$$

by setting $x = v+\tau w$, $y = v-(1-\tau)w$. From (5.3) we then get

$$\begin{aligned}\langle K_\tau,\phi\rangle &= \langle \mathcal{F}_{w\to\xi}K_A(v+\tau w, v-(1-\tau)w), \mathcal{F}^{-1}_{w\to\xi}\phi(v+\tau w, v-(1-\tau)w)\rangle\\ &\langle K_A(v+\tau w, v-(1-\tau)w), \phi(v+\tau w, v-(1-\tau)w)\rangle = \langle K_A,\phi\rangle.\end{aligned} \tag{5.6}$$

This gives (5.2) and proves also the uniqueness of the symbol b_τ. Finally, to obtain (5.4), we argue on the expression

$$b_\tau(x,\xi) = \int e^{iy\cdot(\eta-\xi)} a(x+\tau y, x-(1-\tau)y, \eta)\, dy d\eta$$

and expand with respect to y at $y=0$:

$$\begin{aligned} &a(x+\tau y, x-(1-\tau)y, \eta) \\ &\quad = \sum_{|\beta+\gamma|<N} \frac{(-1)^{|\gamma|}}{\beta!\gamma!} \tau^\beta (1-\tau)^\gamma y^{\beta+\gamma} \left(\partial_x^\beta \partial_y^\gamma a\right)(x,x,\eta) + r_N(x,y,\eta). \end{aligned}$$

Since

$$\int e^{iy\cdot(\eta-\xi)} y^{\beta+\gamma} \left(\partial_x^\beta \partial_y^\gamma a\right)(x,x,\eta)\, dy d\eta = (-1)^{|\beta+\gamma|} \left(\partial_\xi^{\beta+\gamma} D_x^\beta D_y^\gamma a\right)(x,x,\xi),$$

it will be sufficient to prove that $R_N(x,\xi) = \int e^{iy\cdot(\eta-\xi)} r_N(x,y,\eta)\, dy d\eta$ is in S^{m-2N}. This is easily obtained by giving to the remainder $r_N(x,y,\eta)$ the standard integral form and repeating the arguments in first part of the proof. ■

Definition 5.6 *Let $A \in \mathcal{L}^m$. With the notations of the preceding Theorem 5.5 we call $b_\tau(x,\xi)$ the τ-symbol of A. The symbol $b_0(x,\xi)$, corresponding to $\tau = (0,\dots,0)$, is also called the left symbol of A; $b_1(x,\xi)$ corresponding to $\tau = (1,\dots,1)$ is called the right symbol and $b_{1/2}$ corresponding to $\tau = (1/2,\dots,1/2)$ the Weyl symbol. In the sequel we shall also write b_W for the Weyl symbol.*

From (5.3) we get the kernel in terms of the τ-symbol: $K_A(x,y) = \mathcal{F}^{-1}_{\xi\to x-y} b_\tau((1-\tau)x+\tau y, \xi)$. It is worth to write down explicitly the three expressions

$$\begin{aligned} Au(x) &= \int e^{i(x-y)\cdot\xi} b_0(x,\xi) u(y)\, dy d\xi \\ &= \int e^{ix\cdot\xi} b_0(x,\xi) \hat{u}(\xi)\, d\xi, \end{aligned} \tag{5.7}$$

$$Au(x) = \int e^{i(x-y)\cdot\xi} b_1(y,\xi) u(y)\, dy d\xi, \tag{5.8}$$

$$Au(x) = \int e^{i(x-y)\cdot\xi} b_W\left(\frac{x+y}{2},\xi\right) u(y)\, dy d\xi. \tag{5.9}$$

and list the corresponding asymptotic expansions in terms of the amplitude $a(x,y,\xi)$:

$$b_0(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} \left(\partial_\xi^\alpha D_y^\alpha a\right)(x,x,\xi), \tag{5.10}$$

$$b_1(x,\xi) \sim \sum_\alpha \frac{(-1)^{|\alpha|}}{\alpha!} \left(\partial_\xi^\alpha D_x^\alpha a\right)(x,x,\xi), \tag{5.11}$$

$$b_W(x,\xi) \sim \sum_{\beta,\gamma} \frac{(-1)^{|\beta|}}{\beta!\gamma!} \left(\frac{1}{2}\right)^{|\beta+\gamma|} \left(\partial_\xi^{\beta+\gamma} D_x^\beta D_y^\gamma a\right)(x,x,\xi). \tag{5.12}$$

6 Symbolic calculus

We give now some applications of Theorem 5.5. The following theorem allows us to deduce the expression of the τ_2-symbol of $A \in \mathcal{L}^m$ from that of the τ_1-symbol, $\tau_1 \neq \tau_2$, in terms of an asymptotic series.

Theorem 6.1 *If $b_{\tau_1}(x,\xi), b_{\tau_2}(x,\xi)$ are respectively τ_1 and τ_2-symbols of $A \in \mathcal{L}^m$, then:*

$$b_{\tau_2}(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}(\tau_1 - \tau_2)^\alpha \partial_\xi^\alpha D_x^\alpha b_{\tau_1}(x,\xi). \tag{6.1}$$

For a given $A \in \mathcal{L}^m$ with amplitude $a(x,y,\xi)$, let us now consider the *transposed* operator ${}^tA$ and the *formal adjoint* operator A^*. Coming back to the proof of Theorem 5.1, we have that ${}^tA$, defined by

$$\langle {}^tAv, u\rangle = \langle v, Au\rangle, \qquad u,v \in \mathcal{S}(\mathbf{R}^n), \tag{6.2}$$

belongs to $\mathcal{L}^m$ with amplitude

$${}^ta(x,y,\xi) = a(y,x,-\xi). \tag{6.3}$$

Let us also introduce the *conjugate operator* $\overline{A}$:

$$\langle \overline{A}u, v\rangle = \overline{\langle A\overline{u}, \overline{v}\rangle}, \qquad u,v \in \mathcal{S}(\mathbf{R}^n)$$

belonging to $\mathcal{L}^m$ with amplitude

$$\overline{a}(x,y,\xi) = \overline{a(x,y,-\xi)}. \tag{6.4}$$

Then we define $A^* = {}^t\overline{A} = \overline{({}^tA)}$, satisfying

$$(A^*v, u)_{L^2} = (v, Au)_{L^2}, \qquad u,v \in \mathcal{S}(\mathbf{R}^n), \tag{6.5}$$

Therefore also A^* belongs to S^m, and it has amplitude

$$a^*(x,y,\xi) = \overline{a(y,x,\xi)}. \tag{6.6}$$

Let us express the τ-symbols of ${}^tA, \overline{A}, A^*$, which we shall denote by ${}^tb_\tau(x,\xi), \overline{b}_\tau(x,\xi)$, $b^*_\tau(x,\xi)$, in terms of $b_\tau(x,\xi)$. We first observe that, in view of (5.3) and (6.3),

$${}^tb_\tau(x,\xi) = \int e^{-iy\cdot\eta}a(x-(1-\tau)y, x+\tau y, \eta-\xi)\,dyd\eta = b_{1-\tau}(x,-\xi).$$

Using then Theorem 6.1 we conclude:

Theorem 6.2 *If $A \in \mathcal{L}^m$, then ${}^tA \in \mathcal{L}^m$ and the τ-symbol ${}^tb_\tau(x,\xi)$ of ${}^tA$ can be expressed in terms of the τ-symbol $b_\tau(x,\xi)$ of A by the formula*

$${}^tb_\tau(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}(1-2\tau)^\alpha \partial^\alpha_\xi D^\alpha_x b_\tau(x,-\xi).$$

From (6.4), (6.6) and (5.3) we have similarly $\overline{b}_\tau(x,\xi) = \overline{b_\tau(x,-\xi)}$, $b^*_\tau(x,\xi) = \overline{b_{1-\tau}(x,\xi)}$ and, summing up, from Theorem 6.1 we obtain:

Theorem 6.3 *If $A \in \mathcal{L}^m$ then $A^* \in \mathcal{L}^m$. Moreover the τ-symbol $b^*(x,\xi)$ of A^* is related to the $(1-\tau)$-symbol $b_{1-\tau}(x,\xi)$ of A via the relation*

$$b^*_\tau(x,\xi) = \overline{b_{1-\tau}(x,\xi)},$$

and can be expressed in terms of the τ-symbol $b_\tau(x,\xi)$ of A via the asymptotic series

$$b^*_\tau(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}(1-2\tau)^\alpha \partial^\alpha_\xi D^\alpha_\xi \overline{b_\tau(x,\xi)}.$$

Corollary 6.4 *If $A \in \mathcal{L}^m$, then $b^*_W(x,\xi) = \overline{b_W(x,\xi)}$. In particular, the condition $A = A^*$ is equivalent to the real-valuedness of the Weyl symbol $b_W(x,\xi)$.*

Finally, we consider the composition $A'A''$ of two pseudo-differential operators, $A' \in \mathcal{L}^{m'}, A'' \in \mathcal{L}^{m''}$. Let us write A' in terms of its 0-symbol $b'_0(x,\xi)$ and A'' in terms of its 1-symbol $b''_1(x,\xi)$; we have from (5.7), (5.8)

$$A'v(x) = \int e^{ix\xi}b'_0(x,\xi)\hat{v}(\xi)\,d\xi,$$

$$\mathcal{F}(A''u)(\xi) = \int e^{-iy\xi}b''_1(y,\xi)u(y)\,dy.$$

This allows to write

$$A'A''u(x) = \int e^{i(x-y)\cdot\xi} b'_0(x,\xi) b''_1(y,\xi) u(y)\, dy d\xi.$$

Since

$$c(x,y,\xi) = b'_0(x,\xi) b''_1(y,\xi) \tag{6.7}$$

belongs to $\overline{S}^{m'+m''}$ in view of Proposition 4.8, we conclude $A'A'' \in \mathcal{L}^{m'+m''}$. Applying further (5.4) to the amplitude (6.7), we conclude that $b_\tau(x,\xi)$, the τ-symbol of $A = A'A''$, has the asymptotic expansion:

$$b_\tau(x,\xi) \sim \sum_{\beta,\gamma} \frac{(-1)^{|\beta|}}{\beta!\gamma!} \tau^\beta (1-\tau)^\gamma \partial_\xi^{\beta+\gamma} \left(D_x^\beta b'_0(x,\xi) D_x^\gamma b''_1(x,\xi) \right).$$

Applying Leibniz rule, we obtain then the following result.

Theorem 6.5 *If $A' \in \mathcal{L}^{m'}$, $A'' \in \mathcal{L}^{m''}$, then $A = A'A'' \in \mathcal{L}^{m'+m''}$. The τ-symbol $b_\tau(x,\xi)$ of the product operator A can be expressed in terms of the 0-symbol $b'_0(x,\xi)$ of A' and the 1-symbol b''_1 of A'' by*

$$b_\tau(x,\xi) \sim \sum_{\substack{\beta,\gamma,\delta,\epsilon \\ \delta+\epsilon=\beta+\gamma}} \frac{(-1)^{|\beta|}(\beta+\gamma)!}{\beta!\gamma!\delta!\epsilon!} \tau^\beta (1-\tau)^\gamma (\partial_\xi^\delta D_x^\beta b'_0)(\partial_\xi^\epsilon D_x^\gamma b''_1). \tag{6.8}$$

Combining (6.8) with Theorem 6.1, we may in principle express b_τ in terms of b'_{τ_1}, b''_{τ_2}, for any $\tau_1, \tau_2 \in \mathbf{R}^n$. A natural choice is, of course, to set $\tau = \tau_1 = \tau_2$; let us consider for example the cases $\tau = (\frac{1}{2}, \ldots, \frac{1}{2})$ and $\tau = (0, \ldots, 0)$.

Theorem 6.6 *Let $b'_W(x,\xi)$, $b''_W(x,\xi)$ be the Weyl symbols of $A' \in \mathcal{L}^{m'}$, $A'' \in \mathcal{L}^{m''}$. Then the Weyl symbol $b_W(x,\xi)$ of the product $A = A'A''$ has asymptotic expansion*

$$b_W(x,\xi) \sim \sum_{\alpha,\beta} \frac{(-1)^{|\beta|}}{\alpha!\beta!} 2^{-|\alpha+\beta|} \partial_\xi^\alpha D_x^\beta b'_W(x,\xi) \partial_\xi^\beta D_x^\alpha b''_W(x,\xi). \tag{6.9}$$

Theorem 6.7 *Let $b'_0(x,\xi)$, $b''_0(x,\xi)$ be the 0-symbols of $A' \in \mathcal{L}^{m'}$, $A'' \in \mathcal{L}^{m''}$. Then the 0-symbol $b_0(x,\xi)$ of $A = A'A''$ has asymptotic expansion*

$$b_0(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha b'_0(x,\xi) D_x^\alpha b''_0(x,\xi).$$

7 Anti-Wick symbols

Let $\Phi(z) = \pi^{-n/4}e^{-\frac{1}{2}|x|^2}$ be the normalized gaussian function on $\mathbb{R}^{2n}$, i.e. $||\Phi||_{L^2(\mathbb{R}^{2n})} = 1$. We begin by remarking that for $j = 1, ..., 2n$

$$z_j * \Phi(z) = z_j. \tag{7.1}$$

In terms of quantization this means that the map

$$a(z) \mapsto W_{a*\Phi} \tag{7.2}$$

satisfies rule (1.) and (2.) of Section 2. The map (7.2) is called *anti-Wick* quantization and we shall usually write $A_a = W_{a*\Phi}$. Further in this section we shall give a deeper interpretetion of this quantization in terms of L^2-projections and examine it into the frame of the pseudo-differential calculus. For the moment we want to compare Weyl and anti-Wick quantization on a more informal level concerning differential operator. From (7.1) we expect that, at least for polynomial observables, the anti-Wick quantization do differs too much from the Weyl quantization. More precisely we remark that if the observable $a(z)$ is a polynomial in $z \in \mathbb{R}^{2n}$ then the anti-Wick operator $W_{a*\Phi}$ is a differential operator. This is immediate since (7.1) can be easily generalized to $z^\alpha = z^\alpha * \Phi(z) + R_\alpha(z)$ where $R_\alpha(z) = \sum_{\gamma<\alpha} c_\gamma z^\gamma$. Then $a * \Phi$ is also a polynomial, but by Theorem 6.1, $W_{a*\Phi}$ can written by means of the usual 0-symbol $b_0(z)$ which is still a polynomial in view of (6.1). As the operators that arise in physics are essentially differential operators it is then natural to ask if the converse is also true, that is if polynomial symbols yield through anti-Wick quantization all differential operators (with polynomial coefficients). The answer is affirmative as we see from the next proposition.

Proposition 7.1 *For each multiindex α there exists a polynomial $P_\alpha(z)$ such that*

$$P_\alpha(z) * \Phi(z) = z^\alpha \tag{7.3}$$

and $P(z) = z^\alpha + \sum_{\gamma<\alpha} c_\gamma z^\gamma$.

Proof. Set $P(z) = \sum_{\gamma\leq\alpha} c_\gamma z^\gamma$, then

$$P_\alpha(z) * \Phi(z) = \sum_{\beta\leq\alpha} c_\beta \int (z-w)^\beta \Phi(z)dw = \sum_{\beta,\gamma:\ \gamma\leq\beta\leq\alpha} z^\gamma c_\beta \binom{\beta}{\gamma} \int w^{\beta-\gamma}\Phi(z)dw$$
$$= \sum_{\gamma\leq\alpha} k_\gamma(c_\beta) z^\gamma$$

where

$$k_\gamma(c_\beta) = \sum_{\gamma\le\beta\le\alpha} c_\beta \binom{\beta}{\gamma} \int w^{\beta-\gamma}\Phi(z)dw. \tag{7.4}$$

As we require that (7.3) is satisfied, i.e. $z^\alpha = \sum_{\beta\le\alpha} k_\gamma(c_\beta)z^\gamma$, we get the conditions

$$\begin{cases} k_\alpha(c_\beta) = 1 \\ k_\gamma(c_\beta) = 0, \quad \text{for } \gamma < \alpha \end{cases} \tag{7.5}$$

Let N denote the number of multiindices $\gamma \le \alpha$, then (7.5) is a system of N linear equations in c_β, $\beta \le \alpha$. We show that it has always a solution.

From the first equation of (7.5) we get immediately $c_\alpha = 1$. Consider now the equations in (7.5) corresponding to multiindices γ with $|\gamma| = |\alpha| - 1$; they are n independent equations of the form

$$c_\gamma + b = 0 \tag{7.6}$$

with $b = 1$, so all c_β with $|\beta| = |\alpha| - 1$ are determined. We can consider now the $2n$ equations in (7.5) for which $|\gamma| = |\alpha| - 2$, they are again of the form (7.6), where now $b = \sum_{\beta:\gamma<\beta\le\alpha} c_\beta\binom{\beta}{\gamma}\int w^{\beta-\gamma}\Phi(w)dw$ (c_β are already determined for $\gamma < \beta$), and we see that also all c_β with $|\beta| = |\alpha| - 2$ are determined. Repeating the argument we get then for the sistem (7.5) the solution:

$$\begin{cases} c_\alpha = 1 \\ c_\gamma = -\sum_{\beta:\gamma<\beta\le\alpha} c_\beta\binom{\beta}{\gamma}\int w^{\beta-\gamma}\Phi(w)dw \quad \text{for} \quad \gamma < \alpha \end{cases} \tag{7.7}$$

■

We pass now to consider the definition of the anti-Wick quantization under a different point of view. We shall see further in this section that they are equivalent.

We first consider the orthogonal projection $P_{y,\eta}$ in $L^2(\mathbf{R}^n)$ on the vector

$$\Phi_{y,\eta}(x) = \pi^{-n/4}e^{ix\cdot\eta}e^{-\frac{1}{2}|y-x|^2}, \tag{7.8}$$

where (y, η) are parameters in $\mathbf{R}^{2n}$; that is, for $u \in L^2(\mathbf{R}^n)$:

$$P_{y,\eta}u(x) = \left(\int \overline{\Phi_{y,\eta}(t)}u(t)\,dt\right)\Phi_{y,\eta}(x). \tag{7.9}$$

We want to regard $P_{y,\eta}$ as pseudo-differential operator with symbol in the classes of the preceding sections. To this end, we introduce the multiplication operator and the shift operator in $\mathbf{R}^n$

$$M_\eta u(x) = e^{ix\cdot\eta}u(x), \tag{7.10}$$

$$T_y u(x) = u(x-y) \tag{7.11}$$

and set then

$$U_{y,\eta} = M_\eta T_y. \tag{7.12}$$

Since $M_\eta^{-1} = M_{-\eta} = M_\eta^*$ and $T_y^{-1} = T_{-y} = T_y^*$, the operator $U_{y,\eta}$ is unitary, i.e. $U_{y,\eta}^{-1} = U_{y,\eta}^*$. Consider now $\Phi_{0,0}(x) = \pi^{-n/4} e^{-\frac{1}{2}|x|^2}$, for which $\|\Phi_{0,0}\|_{L^2}^2 = \pi^{-n/2} \int e^{-|x|^2}\, dx = 1$, and the orthogonal projection $P_{0,0}$ on $\Phi_{0,0}$, with Schwartz kernel

$$K_{0,0}(x,y) = \pi^{-n/2} e^{-\frac{1}{2}(|x|^2+|y|^2)}. \tag{7.13}$$

We have $\Phi_{y,\eta}(x) = U_{y,\eta}\Phi_{0,0}(x)$ and therefore summing up, the orthogonal projection onto the vector with unitary norm $\Phi_{y,\eta}$ can be written as $P_{y,\eta} = U_{y,\eta} P_{0,0} U_{y,\eta}^{-1}$. Let us begin by computing $\sigma_{0,0}$, the Weyl symbol of $P_{0,0}$; from (5.3) and (7.13) we have

$$\sigma_{0,0}(x,\xi) = \mathcal{F}_{t\to\xi} K_{0,0}\left(x + \frac{1}{2}t, x - \frac{1}{2}t\right) = 2^n e^{-(|x|^2+|\xi|^2)}.$$

Looking finally for $\sigma_{y,\eta}$, Weyl symbol of $P_{y,\eta}$, we may write

$$\begin{aligned} P_{y,\eta}u(x) &= U_{y,\eta} \int e^{i(x-t)\cdot\xi} \sigma_{0,0}\left(\frac{x+t}{2}, \xi\right) (U_{y,\eta}^{-1}u)(t)\, dt d\xi \\ &= \int e^{i(x-t)\cdot\xi} \sigma_{0,0}\left(\frac{x+t}{2} - y, \xi - \eta\right) u(t)\, dt d\xi, \end{aligned}$$

in view of (7.10), (7.11), (7.12). Hence

$$P_{y,\eta}u(x) = \int e^{i(x-t)\cdot\xi} \sigma_{y,\eta}\left(\frac{x+t}{2}, \xi\right) u(t)\, dt d\xi, \tag{7.14}$$

with real-valued Weyl symbol

$$\sigma_{y,\eta}(x,\xi) = \sigma_{0,0}(x-y, \xi-\eta) = 2^n e^{-(|x-y|^2+|\xi-\eta|^2)}. \tag{7.15}$$

Using Corollary 6.4 and summarizing, we have the following result.

Proposition 7.2 *The orthogonal projection operator $P_{y,\eta}$ on the unitary vector (7.8) can be written as a self-adjoint pseudo-differential operator, with Weyl symbol $\sigma_{y,\eta} \in \mathcal{S}(\mathbf{R}^{2n})$ given by (7.15).*

Consider now $a \in S^m$; we may define

$$A = \int a(y,\eta) P_{y,\eta}\, dy d\eta$$

as integral of operators. To be definite, if $u \in \mathcal{S}(\mathbf{R}^n)$, then $P_{y,\eta}u(x)$, as a function of x, y, η, belongs to $\mathcal{S}(\mathbf{R}^{3n})$. Since $a(y,\eta) \in S^m$ is a multiplier of $\mathcal{S}(\mathbf{R}^{2n})$ in view of Proposition 4.2 and Proposition 4.3, (ii), then the mapping

$$u \in \mathcal{S}(\mathbf{R}^n) \to a(y,\eta) P_{y,\eta} u(x) \in \mathcal{S}(\mathbf{R}^{3n})$$

is continuous. Considering

$$Au(x) = \int a(y,\eta)(P_{y,\eta}u)(x)\, dy d\eta, \tag{7.16}$$

we conclude that A acts continuously from $\mathcal{S}(\mathbf{R}^n)$ into $\mathcal{S}(\mathbf{R}^n)$.

Definition 7.3 *Let $a \in S^m$. We call pseudo-differential operator with anti-Wick symbol a the map $A : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^n)$ defined by* (7.16).

Inserting (7.14), (7.15) into (7.16), we obtain for $u \in \mathcal{S}(\mathbf{R}^n)$

$$Au(x) = \int e^{i(x-t)\cdot\xi} b\left(\frac{x+t}{2}, \xi\right) u(t)\, dt d\xi, \tag{7.17}$$

with

$$\begin{aligned} b(x,\xi) &= 2^n \int a(y,\eta) e^{-(|x-y|^2+|\xi-\eta|^2)}\, dy d\eta \\ &= (2\pi)^{-n}(a * \sigma_{0,0})(x,\xi). \end{aligned} \tag{7.18}$$

It follows in particular that the operator with anti-Wick symbol $a(x,\xi) = 1$ is the identity operator. Before analysing in detail (7.18), we list some direct consequences of (7.16); they are actually valid under much more general assumptions than $a \in S^m$.

Proposition 7.4 *Assume that A has anti-Wick symbol $a(x,\xi) \in S^m$. Then the (formal) adjoint A^* has anti-Wick symbol $\overline{a(x,\xi)}$; in particular if $a(x,\xi)$ is real valued then A is self-adjoint.*

Proposition 7.5 *Assume that A has anti-Wick symbol $a \in S^m$. If $a(x,\xi) \geq 0$ for $(x,\xi) \in \mathbf{R}^{2n}$, then $A \geq 0$, that is $(Au,u)_{L^2} \geq 0$ for all $u \in \mathcal{S}(\mathbf{R}^n)$. Moreover if $a(x,\xi) > 0$ for $(x,\xi) \in \mathbf{R}^{2n}$, then $A > 0$, i.e. $(Au,u) > 0$ for $u \neq 0$.*

Proposition 7.6 *Every operator A with anti-Wick symbol in S^0 is bounded in $L^2(\mathbf{R}^n)$.*

We now return to (7.17), (7.18), and show that A is actually a pseudo-differential operator in the classes of the preceding sections.

Theorem 7.7 *Let A be an operator with anti-Wick symbol $a(x,\xi) \in S^m$. Then $A \in \mathcal{L}^m$. Precisely, its Weyl symbol $b(x,\xi)$ defined by (7.18) belongs to S^m with asymptotic expansion $b(x,\xi) \sim \sum_{\alpha,\beta} \frac{c_{\alpha\beta}}{\alpha!\beta!} \partial_\xi^\alpha \partial_x^\beta a(x,\xi)$, where*

$$c_{\alpha\beta} = 2^n \int \eta^\alpha y^\beta e^{-(|y|^2+|\eta|^2)}\, dy d\eta; \tag{7.19}$$

in particular $c_{00} = 1$ and $c_{\alpha\beta} = 0$ for odd $|\alpha + \beta|$.

Proof. It will be sufficient to prove that, for $b(x,\xi)$ defined as in (6.16), we have

$$R_N = b - \sum_{|\alpha+\beta| \le N} \frac{c_{\alpha\beta}}{\alpha!\beta!} \partial_\xi^\alpha \partial_x^\beta a \in S^{m-\rho N}_{\mathcal{P},\rho}. \tag{7.20}$$

In fact, we may Taylor expand

$$a(y,\eta) = \sum_{|\alpha+\beta|<N} \frac{1}{\alpha!\beta!} \partial_\xi^\alpha \partial_x^\beta a(x,\xi)(\eta-\xi)^\alpha (y-x)^\beta + r_N(x,y,\xi,\eta).$$

Inserting in the expression $b(x,\xi) = 2^n \int a(y,\eta) e^{-(|x-y|^2+|\xi-\eta|^2)}\, dy d\eta$, we obtain (7.20) with $c_{\alpha\beta}$ given by (7.19) and

$$R_N(x,\xi) = 2^n \int r_N(x,y,\xi,\eta) e^{-(|x-y|^2+|\xi-\eta|^2)}\, dy d\eta.$$

To prove $R_N(x,\xi) \in S^{m-N}$ we appeal to the temperance property, and precisely to (3.8). The theorem is therefore proved. ∎

Not every operator $A \in \mathcal{L}^m$ admits anti-Wick symbol $a \in S^m$; observe in fact that the Weyl symbol $b(x,\xi)$ in (7.18) must be an analytic function of $(x,\xi) \in \mathbf{R}^{2n}$. From the symbolic calculus we have however a converse of Theorem 7.7 modulo regularizing operators, cf. Theorem 5.4.

Theorem 7.8 *For every $A \in \mathcal{L}^m$ there exists $a'(x,\xi) \in S^m$ such that, writing A' for the operator with anti-Wick symbol $a'(x,\xi)$, we have $A - A' \in \mathcal{L}^{-\infty}$. If $a(x,\xi)$ is the Weyl symbol of A, then $a' \sim \sum_{\alpha,\beta} \tilde{c}_{\alpha\beta} \partial_x^\alpha \partial_\xi^\beta a$ for suitable constants $\tilde{c}_{\alpha\beta} \in \mathbf{R}$, with $\tilde{c}_{00} = 1$, $\tilde{c}_{\alpha\beta} = 0$ for $|\alpha + \beta| = 1$.*

We may now draw some important conclusions, concerning boundedness and compactness of pseudo-differential operators. Namely from Theorem 7.8, Proposition 7.6 and Proposition 7.5 we have:

Theorem 7.9 *Every $A \in \mathcal{L}^0$ extends to a bounded operator on $L^2(\mathbf{R}^n)$. Moreover, if $A \in \mathcal{L}^m$ with $m < 0$, then A is compact in $L^2(\mathbf{R}^n)$.*

Theorem 7.10 *Let $A \in \mathcal{L}^2$ have Weyl symbol $a(x,\xi) \geq 0$; then for a suitable $C > 0$*

$$(Au, u)_{L^2} \geq -C\|u\|_{L^2}, \qquad \text{for all } u \in \mathcal{S}(\mathbf{R}^n). \tag{7.21}$$

It is possible to improve Theorem 7.10, precisely the conclusion (7.21) keeps valid if $a \in S^4$ and $a(x,\xi) \geq 0$, cf. Hörmander [9], Chapter 18.

However pseudo-differential operators with positive Weyl symbol are not positive in general. Consider for example in dimension $n = 1$ the symbol $a(x,\xi) = x^2\xi^2 \geq 0$. The operator A with Weyl symbol a can be written as $A = \frac{1}{4}(xD_x + D_x x)^2 - \frac{1}{4}$ and therefore $(Au, u)_{L^2} = \frac{1}{4}\|(xD_x + D_x x)u\|^2_{L^2} - \frac{1}{4}\|u\|^2_{L^2}$, where the right-hand side is negative for suitable $u \in \mathcal{S}(\mathbf{R})$.

As in Quantum Mechanics the smoothness of the observables is not required, we remark finally that it would be interesting to extend the pseudo-differential calculus to some classes of non-smooth symbols. In this regard we recall next some results where the anti-Wick symbol lies in the $L^p(\mathbb{R}^{2n})$ spaces. As at the beginning of this section, let A_a be the operator with anti-Wick symbol $a(z)$ and $\Phi(z)$ the gaussian function.

Theorem 7.11 *Let $a \in L^p(\mathbb{R}^{2n})$, $1 \leq p \leq \infty$. Then the linear map $T : a \in L^p(\mathbb{R}^{2n}) \to A_a \in B(L^2(\mathbb{R}^n))$ is continuous.*

A proof of this result making use of the Closed Graph Theorem can be found in [4]. Other results are the following.

Theorem 7.12 *Let $a \in L^p(\mathbb{R}^{2n})$, $p \in [1,2]$, then A_a is a Hilbert-Schmidt operator and for the Hilbert-Schmidt norm $\|A_a\|_2$ we have $\|A_a\|_2 = \|b\|_{L^2(\mathbb{R}^{2n})}$, where $b = (2\pi)^{-n} a * \Phi \in L^2(\mathbb{R}^{2n})$ is the Weyl symbol of A_a.*

Theorem 7.13 *Let $a \in L^1(\mathbb{R}^{2n})$, then A_a is a Trace Class operator and:*

$$Tr(A_a) = (2\pi)^{-3n} \int a(y,\eta)\, \Phi(x-y, \xi-\eta)\, dx dy d\xi d\eta.$$

For a deeper investigation in this direction see Wong [14].

8 Hypoelliptic Operators

Definition 8.1 *A symbol $a \in S^m$ is called Λ-hypoelliptic if there exist $l \leq m$ and $R > 0$ such that*

$$\Lambda(z)^l \prec |a(z)|, \qquad \textit{for } |z| \geq R \tag{8.1}$$

and for each $\gamma \in \mathbf{N}^{2n}$ we have

$$|\partial^\gamma a(z)| \prec |a(z)|\Lambda(z)^{-|\gamma|}, \qquad \textit{for } |z| \geq R. \tag{8.2}$$

We denote $HS^{m,l}$ the corresponding class of symbols. We say that $a \in S^m$ is Λ-elliptic if (8.1), (8.2) are satisfied with $l = m$ in (8.1) and we use the notation ES^m instead of $HS^{m,m}$.

When checking Λ-ellipticity, it is convenient to use the following propositions.

Proposition 8.2 *Let $a \in S^m$. We have $a \in ES^m$ if and only if there exists $R > 0$ such that*

$$\Lambda(z)^m \prec |a(z)|, \qquad \textit{for } |z| \geq R. \tag{8.3}$$

Proof. In fact, from the very definition of class S^m we have $|\partial^\gamma a(z)| \prec \Lambda(z)^{m-|\gamma|}$ and therefore (8.3) implies (8.2). ■

Example 8.3 *(Standard elliptic polynomials) Consider the weight function $\Lambda(z) = \left(1 + \sum_{j=1}^{2n} z_j^{2m}\right)^{1/2}$, which is asymptotically equivalent to $|z|$. Consider a polynomial $a(z) = \sum_{|\alpha|\leq m} c_\alpha z^\alpha$ and its principal part $a_\mu(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha$. We have that $a \in ES^1$ if and only if $|z|^m \prec |a_\mu(z)|$; because of the homogeneity, this is equivalent to the standard ellipticity condition*

$$a_\mu(z) \neq 0, \qquad \textit{for } z \neq 0. \tag{8.4}$$

Example 8.4 *(Quasi-elliptic polynomials)*

Fix $M = (M_1, \dots, M_{2n})$, $2n$-tuple of positive integers. We write $\mu = max_j M_j$ and

$$m = (m_1, \dots, m_{2n}), \qquad \textit{with } m_j = \mu/M_j,\ j = 1, \dots, 2n \tag{8.5}$$

and consider consequently $a(z) = \sum_{\alpha\cdot m\leq\mu} c_\alpha z^\alpha$, with quasi-principal part

$$a_\mu(z) = \sum_{\alpha\cdot m=\mu} c_\alpha z^\alpha. \tag{8.6}$$

Define

$$\Lambda(z) = \left(1 + \sum_{j=1}^{2n} z_j^{2M_j}\right)^{1/\mu}, \tag{8.7}$$

which is asympotically equivalent to

$$[z]_M = \sum_{\alpha\cdot m} c_\alpha z^\alpha \tag{8.8}$$

We have that $a \in ES^\mu$ *if and only if*

$$[z]_M \prec |a_\mu(z)|. \tag{8.9}$$

Let us observe that $a_\mu(z)$ *is quasi-homogeneous of degree* μ *with respect to the weight* $m = (m_1, \ldots, m_{2n})$ *in* (8.5), *i.e.*

$$a_\mu(t^{m_1} z_1, \ldots, t^{m_{2n}} z_{2n}) = t^\mu a_\mu(z_1, \ldots, z_{2n}), \qquad \textit{for all } t > 0.$$

Since $[z]_M$ *in* (8.8) *is also quasi-homogeneous of degree* μ, *we have that* (8.9) *is equivalent to the quasi-ellipticity condition:*

$$a_\mu(z) \neq 0, \qquad \textit{for } z \neq 0. \tag{8.10}$$

For $z = (x, \xi) \in \mathbf{R}^2$, *a simple example of quasi-elliptic polynomial is given by*

$$\xi^h + rx^k, \qquad \operatorname{Im} r \neq 0 \tag{8.11}$$

where h *and* k *are positive integers;*

We begin by starting without proof some preliminary propositions.

Proposition 8.5 (i) *If* $a \in HS^{m,l}$, *then* $a^{-1} \in HS^{-l,-m}$.
(ii) *If* $a \in HS^{m,l}$, *then* $a^{-1}\partial^\gamma a \in S^{-|\gamma|}$ *for every* γ.
(iii) *If* $a \in HS^{m,l}$, $b \in HS^{m',l'}$ *then* $ab \in HS^{m+m',l+l'}$.
(iv) *If* $a \in HS^{m,l}$ *and* $r \in S^{m'}$ *with* $m' < l$, *then* $a + r \in HS^{m,l}$.

Definition 8.6 *Let* $A \in \mathcal{L}^m$ *and* $\tau \in \mathbf{R}^n$. *We write* $A \in H\mathcal{L}^{m,l}$ *if* $b_\tau(x,\xi)$, *the* τ*-symbol of* A, *belongs to* $HS^{m,l}$. $E\mathcal{L}^m$ *are correspondingly defined to be the classes of all the pseudo-differential operators with* τ*-symbol in* ES^m.

The next result shows that the definition of $H\mathcal{L}^{m,l}$, $E\mathcal{L}^m$, does not depend on $\tau \in \mathbf{R}^n$.

Proposition 8.7 *Let $\tau_1, \tau_2 \in \mathbf{R}^n$ be given. If $b_{\tau_1}(x,\xi)$, the τ_1-symbol of $A \in \mathcal{L}^m$, is in $HS^{m,l}$, then also the τ_2-symbol $b_{\tau_2}(x,\xi)$ belongs to $HS^{m,l}$.*

Remark 8.8 *The preceeding proposition, jointly with Theorems 7.7, 7.8, gives that A belongs to $H\mathcal{L}^{m,l}$ if and only if it admits, possibly modulo regularizing operators, anti-Wick symbol in $HS^{m,l}$.*

Proposition 8.9 (i) *If $A \in H\mathcal{L}^{m,l}$, $B \in H\mathcal{L}^{m',l'}$, then $AB \in H\mathcal{L}^{m+m',l+l'}$.*

(ii) *If $A \in H\mathcal{L}^{m,l}$, then ${}^tA \in H\mathcal{L}^{m,l}$ and $A^* \in H\mathcal{L}^{m,l}$.*

(iii) *If $A \in H\mathcal{L}^{m,l}$ and $R \in \mathcal{L}^{m'}$ with $m' < l$, then $A + R \in H\mathcal{L}^{m,l}$.*

Theorem 8.10 *If $A \in H\mathcal{L}^{m,l}$, then there exists $B \in H\mathcal{L}^{-l,-m}$ such that*

$$BA = I + S_1, \qquad AB = I + S_2, \tag{8.12}$$

where $S_j \in \mathcal{L}^{-\infty}$, $j = 1, 2$. Such a map B is said parametrix of A.

Proof. Let us begin by considering the operator B_1 with Weyl symbol $b_1(z) = a(z)^{-1}$, belonging to $S^{-l,-m}$ in view of Proposition 8.5 (i). From Theorem 6.6 we have

$$B_1A = I + R_1, \tag{8.13}$$

where R_1 has Weyl symbol r_1 with asymptotic expansion

$$r_1(x,\xi) \sim \sum_{|\alpha+\beta|>0} \frac{(-1)^{|\beta|}}{\alpha!\beta!} 2^{-|\alpha+\beta|} \partial_\xi^\alpha D_x^\beta a^{-1} \partial_\xi^\beta D_x^\alpha a. \tag{8.14}$$

From Proposition 8.5 (ii), (iii) we have for all α, β

$$\partial_\xi^\alpha D_x^\beta a^{-1} \partial_\xi^\beta D_x^\alpha a = \left(a^{-1}\partial_\xi^\alpha D_x^\beta a^{-1}\right)\left(a\partial_\xi^\beta D_x^\alpha a\right) \in S^{-2|\alpha+\beta|},$$

which implies $r_1 \in S^{-2}$, i.e. $R_1 \in \mathcal{L}^{-2}$. From (8.13) we get for every N the identity

$$\left(\sum_{0\le j<N} (-1)^j R_1^j\right) B_1A = I - (-1)^N R_1^N. \tag{8.15}$$

Let us write r_j for the Weyl symbol of R_1^j, which one can compute by applying repeatedly Theorem 6.6 to r_1 in (8.14). We have $r_j \in S^{-2j}$, and using Proposition 4.5 we may construct

$$r \sim \sum_{j\ge 0}(-1)^j r_j \in S^0. \tag{8.16}$$

The operator $R \in S^0$ with Weyl symbol r satisfies, in view of (8.15), (8.16), $RB_1A = I + S_1$, where $S_1 \in \mathcal{L}^{-\infty}$. Therefore the first identity in (8.12) is valid for $B = RB_1$. Similarly we construct $\tilde{B} \in H\mathcal{L}^{-l,-m}$ satisfying the second identity. ∎

Note that the parametrix is unique, modulo terms in $\mathcal{L}^{-\infty}$. Hence operators in $H\mathcal{L}^{m,l}$ are *globally hypoelliptic*, in the following sense.

Corollary 8.11 *Let $A \in H\mathcal{L}^{m,l}$. If $u \in \mathcal{S}'(\mathbf{R}^n)$, and $Au \in \mathcal{S}(\mathbf{R}^n)$, then $u \in \mathcal{S}(\mathbf{R}^n)$.*

The solvability of the equation $Au = v$, with $A \in H\mathcal{L}^{m,l}$, will be discussed in Section 10. We begin here to give the following special result of existence and uniqueness, which will be basis for the definition of the Sobolev spaces in the next Section 9.

Theorem 8.12 *Let $a \in ES^m$ and assume $a(x,\xi) > 0$ for every $(x,\xi) \in \mathbf{R}^{2n}$. Denote by A the operator with anti-Wick symbol $a(x,\xi)$. The map $A : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^n)$ is an isomorphism, extending to an isomorphism on $\mathcal{S}'(\mathbf{R}^n)$. Moreover, the inverse A^{-1} belongs to $E\mathcal{L}^{-m}$.*

Proof. Observe that A actually belongs to $E\mathcal{L}^m$, in view of Remark 8.8, and is self-adjoint in view of Proposition 7.4. Since $a(x,\xi) > 0$ implies $A > 0$ by virtue of the second part of Proposition 7.5, the injectivity of A on $\mathcal{S}(\mathbf{R}^n)$ is granted. We want to show that the equation $Af = g$ admits a solution $f \in \mathcal{S}(\mathbf{R}^n)$ for every given $g \in \mathcal{S}(\mathbf{R}^n)$. To this end, consider $b = a^{-1}$, which belongs to ES^{-m} in view of Proposition 8.5 (i), and denote by B the operator with anti-Wick symbol b. Since $b(x,\xi) > 0$ for every $(x,\xi) \in \mathbf{R}^{2n}$, we have that also B is injective on $\mathcal{S}(\mathbf{R}^n)$. Therefore we may reduce ourselves to find a solution f of the equivalent equation

$$BAf = Bg \in \mathcal{S}(\mathbf{R}^n). \tag{8.17}$$

On the other hand we have

$$BA = I + R, \qquad \text{with } R \in \mathcal{L}^{-2}. \tag{8.18}$$

This can be proved by observing that the Weyl symbols of A and B are respectively, in view of Theorem 7.7: $a_W = a + \tilde{a}$, with $\tilde{a} \in S^{m-2}$ and $b_W = a^{-1} + \tilde{b}$, with $\tilde{b} \in S^{-m-2}$. Since B is in $E\mathcal{L}^{-m}$ by virtue of Remark 8.8, from Proposition 8.9 (i), Theorem 6.6 and formula (6.9) we then deduce $BA \in E\mathcal{L}^0$ with Weyl symbol $1 + r$, where $r \in S^{-2}$. The claim (8.18) is therefore proved, and equation (8.17) reads

$$f + Rf = Bg \in \mathcal{S}(\mathbf{R}^n). \tag{8.19}$$

Now we can apply the standard Fredholm theory to the map $I + R : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$, since R is compact in $L^2(\mathbf{R}^n)$ in view of the second part of Theorem 7.9. Namely, the equation (8.19) admits a solution $f \in L^2(\mathbf{R}^n)$ for every right-hand side in $L^2(\mathbf{R}^n)$, provided $u + Ru = 0$ has no non-trivial solution in $L^2(\mathbf{R}^n)$. On the other hand we know from Corollary 8.11 that every $u \in L^2(\mathbf{R}^n)$ solving $u + Ru = 0$ belongs actually to $\mathcal{S}(\mathbf{R}^n)$, whereas $I + R = BA$ is injective on this space as product of injective operators.

We can therefore solve (8.19) by a certain $f \in L^2(\mathbf{R}^n)$, which actually belongs to $\mathcal{S}(\mathbf{R}^n)$ in view of Corollary 8.11. Since such f also solves $Af = g$, we have proved that the continuous map $A : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^n)$ is an isomorphism.

Finally we observe that the inverse A^{-1} on the Fréchet space $\mathcal{S}(\mathbf{R}^n)$ is also continuous, and then by transposition A extends to an isomorphism from $\mathcal{S}'(\mathbf{R}^n)$ to $\mathcal{S}'(\mathbf{R}^n)$. The remark at the end of the proof of Theorem 8.10 shows $A^{-1} \in E\mathcal{L}^{-m}$, that gives the conclusion. ■

Example 8.13 *We may actually find a symbol a satisfying the assumptions of Theorem 8.12, i.e. $a \in ES^m$ and $a(z) > 0$ for every $z = (x, \xi) \in \mathbf{R}^{2n}$. An obvious example is*

$$a(z) = \Lambda(z)^m. \tag{8.20}$$

9 Sobolev spaces

By using anti-Wick quantization, a scale of Sobolev spaces, depending on the parameter $s \in \mathbf{R}$, can be associated in a natural way to any given weight function according to the following definition.

Definition 9.1 *Let $a \in ES^s$ satisfy $a(x, \xi) > 0$ for every $(x, \xi) \in \mathbf{R}^{2n}$ and A be the operator with anti-Wick symbol a, then we set:*

$$H^s = A^{-1}\left(L^2(\mathbf{R}^n)\right) = \left\{u \in \mathcal{S}'(\mathbf{R}^n) | Au \in L^2(\mathbf{R}^n)\right\}.$$

From Theorem 8.12, A is a bijection between H^s and $L^2(\mathbf{R}^n)$, therefore a natural Hilbert space structure is induced from $L^2(\mathbf{R}^n)$ on H^s via A.

Furthermore the definition of H^s as well as its topology are independent of the operator A. More precisely we have the following

Proposition 9.2 *H^s is a Hilbert space with respect to the scalar product $(u, v)_{H^s} = (Au, Av)_{L^2}$ and the corresponding norm $\|u\|_{H^s} = \|Au\|_{L^2}$, where A is an operator with strictly positive anti-Wick symbol $a \in ES^s$. Moreover if b is another* *strictly positive symbol in the same class ES^s, then the operator B with anti-Wick symbol b defines the same space H^s and the two norms $\|Au\|_{L^2}$ and $\|Bu\|_{L^2}$ are equivalent.*

The proof is obvious in view of Theorem 8.12. We remark that it could be sometimes useful to consider the Hilbert space topology of H^s defined through a fixed standard scalar product. We shall do this by considering as "standard" the scalar product associated to the operator W_s with anti-Wick symbol $\Lambda(x,\xi)^s$, cf. Example 8.13.

Proposition 9.3 *If $A \in \mathcal{L}^m$, then it defines for every $s \in \mathbf{R}$ a continuous operator $A : H^s \to H^{s-m}$.*
Proof. Referring to W_t, the operator in $E\mathcal{L}^t$ with anti-Wick symbol Λ^t, $t \in \mathbf{R}$, we have

$$\|Au\|_{H^{s-m}} = \|W_{s-m}Au\|_{L^2} = \|W_{s-m}AW_s^{-1}W_su\|_{L^2} \leq C\|W_su\|_{L^2} = C\|u\|_{H^s},$$

since $W_{s-m}AW_s^{-1} \in \mathcal{L}^0$ and we may apply Theorem 7.9. ■

Proposition 9.4 *For $t > s$ we have a compact immersion $j : H^t \to H^s$.*
Proof. Write $j = W_{-s}W_{-s}^{-1}W_t^{-1}W_t$, where $W_{-s}^{-1}W_t^{-1} \in L^{s-t} : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ is compact, in view of the second part of Theorem reft6.4. Since the maps $W_t : H^t \to L^2(\mathbf{R}^n)$, $W_{-s} : L^2(\mathbf{R}^n) \to H^s$ are continuous, the proposition is proved. ■

We have the following obvious consequence of Proposition 9.4.

Proposition 9.5 *An operator A in $\mathcal{L}^m$ defines a compact map $A : H^s \to H^t$, whenever $s - t > m$. In particular, a regularizing operator is continuous and compact from H^s to H^t for any s, $t \in \mathbf{R}$.*

Proposition 9.6 *For every $s \in \mathbf{R}$, we have continuous immersions $j : \mathcal{S}(\mathbf{R}^n) \to H^s$, $j : H^s \to \mathcal{S}'(\mathbf{R}^n)$; moreover $\bigcap_s H^s = \mathcal{S}(\mathbf{R}^n)$, $\bigcup_s H^s = \mathcal{S}'(\mathbf{R}^n)$.*

Proof. The first assertions follow from the very Definition 9.1. Moreover, since $\bigcup_m \mathcal{L}^m$ contains all the differential operators with polynomial coefficients, we have $\bigcap_s H^s = \mathcal{S}(\mathbf{R}^n)$. Similarly, using the structure theorem for $\mathcal{S}'(\mathbf{R}^n)$, we obtain $\bigcup_s H^s = \mathcal{S}'(\mathbf{R}^n)$. ■

Example 9.7 *An equivalent definition of the space H^s in the case when Λ is defined as in example 8.4 with weight $m = m_1, \dots, m_{2n}$ and s multiple of all m_j, is the following $H^s = \{u \in \mathcal{S}'(\mathbf{R}^n) \mid x^\alpha D^\beta u \in L^2(\mathbf{R}^n)$ for all $\gamma = (\alpha, \beta)$, such that $\gamma \cdot m \leq s\}$ with norm $\|u\|_{H^s} = \sum_{\gamma=(\alpha,\beta):\gamma\cdot m \leq s} \|x^\alpha D^\beta u\|_{L^2}$.*

10 Fredholm operators

In this section we deal with the Fredholm property of the hypoelliptic operators in $E\mathcal{L}^m$. For completeness we begin by recalling some elementary fact about Functional Analysis.

Let E and F be Banach spaces and let $\mathcal{B}(E,F)$ denote the space of the bounded linear operators from E to F.

Definition 10.1 *Let $A \in \mathcal{B}(E,F)$; we indicate with* $\operatorname{Coker} A$ *the quotient space* $F/\operatorname{Im} A$. *An operator $A \in \mathcal{B}(E,F)$ is said to be a Fredholm operator if both* $\operatorname{Ker} A$ *and* $\operatorname{Coker} A$ *are finite dimensional. We set* $\operatorname{Fred}(E,F)$ *to denote the set of Fredholm operators from E to F and for $A \in \operatorname{Fred}(E,F)$ we define the index of A as the integer number* $\operatorname{Ind} A = \dim \operatorname{Ker} A - \dim \operatorname{Coker} A$.

Proposition 10.2 *If $A \in \operatorname{Fred}(E,F)$ then there exist an operator $B \in \operatorname{Fred}(F,E)$ and two operators P_1 and P_2 such that:*

$$\begin{cases} BA = I_E - P_1, \\ AB = I_F - P_2 \end{cases}$$

and P_1 and P_2 have finite rank (I_E and I_F denote respectively the identity operators on E and F). More precisely P_1 is a projection operator onto $\operatorname{Ker} A$ *and $I_F - P_2$ is a projection onto* $\operatorname{Im} A$.

Let us denote by $K(E,F)$ the set of the compact operators from E to F; in the opposite direction we have the following result.

Proposition 10.3 *If $A \in \mathcal{B}(E,F)$ and there exist two operators $B_1, B_2 \in \mathcal{B}(F,E)$ such that*

$$\begin{cases} B_1 A = I_E - R_1 \\ AB_2 = I_F - R_2 \end{cases}$$

with $R_1 \in K(E,E)$ and $R_2 \in K(F,F)$, then $A \in \operatorname{Fred}(E,F)$.

Proposition 10.4 *If $A \in \operatorname{Fred}(E,F)$ and A' is its dual operator, then $A' \in \operatorname{Fred}(F',E')$ and $\operatorname{Ind} A' = -\operatorname{Ind} A$.*

The analogous property holds in the case of Hilbert spaces, that is, if H_1, H_2 are Hilbert spaces and $A \in \operatorname{Fred}(H_1,H_2)$, then $A^* \in \operatorname{Fred}(H_2,H_1)$ and $\operatorname{Ind} A^* = -\operatorname{Ind} A$. More precisely $\operatorname{Ker} A = (\operatorname{Im} A^*)^\perp$ and $(\operatorname{Im} A)^\perp = \operatorname{Ker} A^*$.

We come now to the case of pseudo-differential operators with symbols in the classes S^m, acting on the Sobolev spaces H^s with $s \in \mathbf{R}$.

As we know an operator $A \in \mathcal{L}^m$ can be viewed as a continuous operator both on $\mathcal{S}(\mathbf{R}^n)$ or on $\mathcal{S}'(\mathbf{R}^n)$. In the former case A turns out to be also a continuous operator from H^s to H^{s-m} with domain $\mathcal{S}(\mathbf{R}^n)$, so that it has a unique continuous extension to the whole space H^s. It is a straightforward matter that this extension coincides with the restriction to H^s of the operator A viewed as an operator on $\mathcal{S}'(\mathbf{R}^n)$.

Definition 10.5 *We shall denote with A_s (sometimes in the sequel again A by abuse) the restriction of $A : \mathcal{S}'(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ to H^s or equivalently the extension of $A : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^n)$ from $\mathcal{S}(\mathbf{R}^n)$ to H^s.*

Combining the existence of the parametrix with preceding proposition, with respect to the Fredholm property we have easily the following theorem.

Theorem 10.6 *Let $A \in E\mathcal{L}^m$, then:*

(i) $A_s \in \mathrm{Fred}(H^s, H^{s-m})$;

(ii) $\mathrm{Ind}\, A_s = \dim \mathrm{Ker}\, A - \dim \mathrm{Ker}\, A^*$, $\mathrm{Ind}\, A_s = \dim \mathrm{Ker}\, A - \dim \mathrm{Ker}\, {}^tA$, *where A^* is the formal adjoint in* (6.5) *and ${}^tA$ the transposed operator in* (6.2), *so that the index is independent of s;*

(iii) *If $T \in L^{m'}$ with $m' < m$ then $A_s + T_s \in \mathrm{Fred}(H^s, H^{s-m})$ and* $\mathrm{Ind}(A_s + T_s) = \mathrm{Ind}\, A_s$.

Corollary 10.7 *If $A \in E\mathcal{L}^m$ has real principal Weyl symbol then* $\mathrm{Ind}\, A = 0$.

We consider now the following question: Suppose we are given an invertible pseudo-differential operator A. Is the inverse operator A^{-1} still pseudo-differential in the same class? The answer to the question is affirmative for many classes of pseudo-differential operators of order zero and leads to the concept of the Spectral Invariance introduced by Gramsch-Ueberberg-Wagner. References on this subject can be found in Gramsch [8], Cordes [7]. For the case of operators in the $\mathcal{L}^0$ classes, we refer to Boggiatto-Schrohe [5] and state without proof the following result.

Proposition 10.8 *Let $A \in L^0$ and suppose that A is invertible in the space of the bounded operators on $L^2(\mathbf{R}^n)$, then $A^{-1} \in L^0$.*

Example 10.9 *Consider in* $\mathbf{R}$ $P = D_x + rx^k$, *with k positive integer and $r \in \mathbf{C}$. We regard P in the frame of Example 8.4, Example 9.7, with weight $(k, 1)$ and consequent definition of Λ. The operator P is hypoelliptic if and only if* $\mathrm{Im}\, r \neq 0$.

The transposed operator is given by ${}^tP = -D_x + rx^k$, *in the same class* $E\mathcal{L}^1$. *The classical solutions of* $Pu = 0$ *are the functions*

$$u(x) = Ce^{\left(-ir\frac{x^{k+1}}{k+1}\right)}, \qquad C \in \mathbf{C}, \tag{10.1}$$

which for $C \neq 0$ *belong to* $\mathcal{S}(\mathbf{R})$, *or* $\mathcal{S}'(\mathbf{R})$, *if and only if* k *is odd and* $\operatorname{Im} r < 0$. *Similarly,* ${}^tPu = 0$ *admits a non-trivial solution in* $\mathcal{S}(\mathbf{R})$ *if and only if* k *is odd and* $\operatorname{Im} r > 0$. *We may regard* P *as a Fredholm operator, setting for example* $P : H^k \to L^2(\mathbf{R})$, *with* $\operatorname{Ind} P = \dim \operatorname{Ker} P - \dim \operatorname{Ker} {}^tP$ *given by*

$$\operatorname{Ind} P = \begin{cases} 0 & \text{for } k \text{ even}, \\ 1 & \text{for } k \text{ odd and } \operatorname{Im} r < 0, \\ -1 & \text{for } k \text{ odd and } \operatorname{Im} r > 0, \end{cases}$$

as we compute from (10.1).

Remark 10.10 *Concerning hypoelliptic operators in* $H\mathcal{L}^{m,l}$, *they cannot be regarded in general as Fredholm on* H^s. *However, adapting the preceding arguments, we may easily show that* $A \in H\mathcal{L}^{m,l}$ *is a Fredholm operator on the topological spaces* $\mathcal{S}(\mathbf{R}^n)$, $\mathcal{S}'(\mathbf{R}^n)$ *in the following sense:*
(i) the map $A : \mathcal{S}'(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ *satisfies* $\operatorname{Ker} A \subset \mathcal{S}(\mathbf{R}^n)$ *and* $\dim \operatorname{Ker} A < \infty$;
(ii) $\operatorname{Im} A$ *is a closed subspace in* $\mathcal{S}'(\mathbf{R}^n)$;
(iii) for $\operatorname{Coker} A = \mathcal{S}'(\mathbf{R}^n)/\operatorname{Im} A$ *we may find representatives in* $\mathcal{S}(\mathbf{R}^n)$ *and we have* $\dim \operatorname{Coker} A < \infty$.

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