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Generalized functions and convolutions

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1. Introduction

The goal of this article is to show how convolutions are used as a tool for constructing spaces of generalized functions. We provide all details of the construction but do not develop further properties or applications. References are provided for the interested reader.

What distinguishes generalized functions from ordinary functions is that they do not always have values at points. If F is a generalized function, then it may not make sense to ask what is the value of F at x. Of course, it may happen that a generalized function has a value at a point. For example, all generalized functions that correspond to ordinary functions have values at points. The difference is that while an ordinary function is defined by its values at points, a generalized function is not.

What does then define a generalized function? This question does not have a simple answer. It depends on the approach to a particular theory of generalized functions. In this article we concentrate on theories of generalized functions that use convolutions. Roughly speaking, these generalized functions are objects that can be convolved with some functions and the convolutions are continuous functions.

The guiding principle in any theory of generalized functions is the expectation that differentiation can always be performed. In particular, we expect that continuous functions will have generalized derivatives of any order. The most common approach to generalized functions is based on the following simple observation. For an arbitrary smooth function φ with compact support, the right hand side of the formula

$$\int_{-\infty}^{\infty} f'(x)\varphi(x)\,dx = -\int_{-\infty}^{\infty} f(x)\varphi'(x)\,dx \tag{1}$$

makes sense even if f is not differentiable. Thus, even though we cannot say what the value of $f'(x)$ is, we can say what the value of the integral $\int_{-\infty}^{\infty} f'(x)\varphi(x)\,dx$ is. If we think of φ as a probability density concentrated around some point x_0, then the integral can be interpreted as the average value of f' around x_0. Writing $\int_{-\infty}^{\infty} f'(x)\varphi(x)\,dx$ is an obvious abuse of notation, because if f' is not a function, the integral is meaningless. For this reason we will write $\langle f', \varphi\rangle$ instead. In general,

$$\langle f^{(n)}, \varphi\rangle = (-1)^n \int_{-\infty}^{\infty} f(x)\varphi^{(n)}(x)\,dx.$$

In the case of functions defined on $\mathbb{R}^N$ we have

$$\langle D^\alpha f, \varphi\rangle = (-1)^{|\alpha|} \int_{\mathbb{R}^N} f(x) D^\alpha \varphi(x)\,dx,$$

where $\alpha = (k_1, \ldots, k_N)$ is a multi-index and $|\alpha| = k_1 + \cdots + k_N$.

This idea leads to the following approach to generalized functions: First we choose a space $\mathcal{T}$ of smooth *test functions* and we equip this space with a topology (or a convergence). Then we define a space of generalized functions as the space of all "objects" F such that $\langle F, \varphi\rangle$ is well-defined (as a number) for every $\varphi \in \mathcal{T}$ and that the following two conditions are satisfied:

(a) For any $a, b \in \mathbb{R}^N$ and any $\varphi, \psi \in \mathcal{T}$ we have

$$\langle F, a\varphi + b\psi\rangle = a\langle F, \varphi\rangle + b\langle F, \psi\rangle.$$

(b) If $\varphi_n \to \varphi$ in $\mathcal{T}$, then $\langle F, \varphi_n\rangle \to \langle F, \varphi\rangle$.

By choosing different spaces $\mathcal{T}$ we obtain different spaces of generalized functions (see, for example, [3], [4], [10], [11], [30], [31]). The most common choice is the space considered by L. Schwartz in his pioneering work [31]. It is the space of all smooth functions with compact support in $\mathbb{R}^N$, usually denoted by $\mathcal{D}$. A sequence $\varphi_n \in \mathcal{D}$ is said to converge to some $\varphi \in \mathcal{D}$ if the following two conditions are satisfied:

(i) There exists a compact set $K \subset \mathbb{R}^N$ such that all φ_n's and φ vanish outside of K.

(ii) $D^\alpha\varphi_n \to D^\alpha\varphi$ uniformly for every multi-index α.

The obtained space of generalized functions is denoted by $\mathcal{D}'$. Elements of $\mathcal{D}'$ are called *Schwartz distributions* (or simply distributions).

In every construction of a space of generalized functions it is necessary to describe how ordinary functions are identified with the defined objects. In the case of this approach a locally integrable function f on $\mathbb{R}^N$ is identified with the distribution F whose "action" on a test function φ is defined by

$$\langle F, \varphi\rangle = \int_{\mathbb{R}^N} f(x)\varphi(x)\,dx.$$

The purpose of this note is to describe theories of generalized functions based on the convolution product of functions. The constructions presented in the following sections produce spaces that are different from spaces of generalized functions obtained by the method described above. Before we discuss those constructions, we would like to mention a rather striking result of R. A. Struble.

The convolution of two functions f and g on $\mathbb{R}^N$, if meaningful, is defined by

$$(f * g)(x) = \int_{\mathbb{R}^N} f(z)g(x-z)\,dz.$$

If f is a locally integrable function and φ is a test function with compact support, then the convolution $f*\varphi$ is well defined. Moreover, if f is differentiable, we have

$$D^\alpha(f*\varphi) = D^\alpha f * \varphi = f * D^\alpha\varphi$$

(Note that this is a version of (1).) As in the case of (1), $f * D^\alpha\varphi$ is well defined even if f is not differentiable. This suggests a possibility of defining generalized functions as objects that can be convolved with test functions. Obviously, we need to assume more than that. Struble in [32] shows that surprisingly little is necessary. It turns out that if we consider all mappings Λ from $\mathcal{D}$ into the space of continuous functions such that

$$\Lambda(\varphi * \psi) = \Lambda(\varphi) * \psi \quad \text{for all } \varphi, \psi \in \mathcal{D}, \tag{2}$$

then we obtain a space isomorphic to $\mathcal{D}'$. Note that not only no continuity of Λ is assumed, but we do not even need a topology on $\mathcal{D}$. Moreover, linearity of Λ is not assumed, it is a consequence of (2).

Struble's idea can be used to define other spaces of generalized functions mentioned above. All we need is to specify the space of test functions (see [25]).

2. Mikusiński's Operational Calculus

Now we turn to the Operational Calculus of Jan Mikusiński. The main idea appears first in *Hyperliczby* (Hypernumbers), a little booklet published in Poland in 1944. Publication by Poles was forbidden at that time. The work was handwritten by Jan Mikusiński on X-ray film and printed with homemade ink. Only seven copies were made. The complete theory was first published in Polish in 1953 [12]. The first English translation was published in 1959 [13].

Consider the space of continuous complex-valued functions defined on $[0, \infty)$. This space will be denoted by $\mathcal{C}$. The convolution in $\mathcal{C}$ is defined by

$$(f * g)(x) = \int_0^x f(x-y)g(y)dy$$

Note that the convolution of any two functions from $\mathcal{C}$ exists and is an element of $\mathcal{C}$. Moreover, $\mathcal{C}$ with ordinary addition of functions and convolution is a commutative ring. From a theorem proved by Titchmarsh [33] it follows that $\mathcal{C}$ is an integral domain, i.e., $f * g = 0$ only if $f = 0$ or $g = 0$. This allows us to construct a quotient field. This field will be denoted by $\mathcal{M}$ and its elements will be called *operators*.

An element of $\mathcal{M}$ can be represented as $\frac{f}{g}$ where $f, g \in \mathcal{C}$, $g \neq 0$, and the quotient indicates "division with respect to convolution". In other words,

$$\frac{f_1}{g_1} = \frac{f_2}{g_2} \quad \text{if and only if} \quad f_1 * g_2 = f_2 * g_1.$$

The operations in $\mathcal{M}$ are defined in the usual way:

$$\frac{f_1}{g_1} + \frac{f_2}{g_2} = \frac{f_1 * g_2 + f_2 * g_1}{g_1 * g_2}$$

and

$$\frac{f_1}{g_1} * \frac{f_2}{g_2} = \frac{f_1 * f_2}{g_1 * g_2}.$$

The zero of the field $\mathcal{M}$ is the operator $\frac{0}{f}$ and the unit of the field is the operator $\frac{f}{f}$, where f is any nonzero element of $\mathcal{C}$. One can easily see that in both cases the choice of a particular f is irrelevant. The inverse of a nonzero operator $\frac{f}{g}$ is $\frac{g}{f}$. The product of an operator $\frac{f}{g}$ by a complex number α is defined as $\frac{\alpha f}{g}$.

A function $f \in \mathcal{C}$ can be identified with any quotient of the form $\frac{f*g}{g}$, $g \neq 0$. In fact, any locally integrable function f vanishing to the left of some real number a can be identified with an operator. If $a \geq 0$ then we can use $\frac{f*g}{g}$, where g is any nonzero function from $\mathcal{C}$. Now consider the case when $a < 0$. Note that, for any continuous function g vanishing to the left of $-a$, the convolution $f * g$ is in C. Thus f can be identified with $\frac{f*g}{g}$, where g is any nonzero element of C vanishing to the left of $-a$. Following the standard convention for quotient fields, we will identify a function $f \in C$ with the corresponding operator. For example, we will write $\frac{f}{g} * g = f$ which is a clear abuse of notation.

The constant function 1 (or the characteristic function of $[0, \infty)$) is not the unit of the field. Actually, the unit of $\mathcal{M}$ is not a function. Since it can be interpreted as the Dirac delta distribution, we will adopt the notation $\delta = \frac{f}{f}$. It is important to remember that the inverse of a function f is not $\frac{1}{f}$, at least not if 1 is interpreted as a constant function. Obviously, $\frac{\delta}{f}$ is the inverse of f since $\frac{\delta}{f} * f = \delta$. However, $\frac{\delta}{f}$ is not a representation of the inverse of f as a quotient in $\mathcal{M}$ because δ is not an element of C. We can represent the inverse of f by $\frac{g}{f*g}$, where $g \in C$ is any nonzero function, but this is somewhat artificial. If necessary, we will use the notation f^{-1} to indicate the inverse of f in $\mathcal{M}$. More generally, we will write

$$\left(\frac{f}{g}\right)^{-1} = \frac{g}{f}.$$

It is convenient to identify a complex number a with the operator $a\delta$ or $\frac{af}{f}$, where f is any nonzero function from C. The advantage of this identification is that there is no necessity to distinguish between multiplication by a scalar and the convolution.

At this point it may still be unclear why $\mathcal{M}$ can be called a space of generalized functions. We will address this issue now. We need to show that every continuous function has well defined derivatives of all orders which are operators.

Following [13] we will denote by l the characteristic function of $[0, \infty)$. Note that for any $f \in C$ we have

$$(l * f)(x) = \int_0^x f(y)dy.$$

Since the result is the integral of f, the operator associated with this function is called the *integral operator*. Following our convention, we will use the same symbol l to denote the function and the operator. Denote by s the inverse of l, i.e, $s = l^{-1}$. Obviously, $s * l = \delta$ and $s * (l * F) = F$ for any $F \in \mathcal{M}$. For this reason it is natural to call s the *differential operator*. If $f \in C$, then

$$f' = s * f - f(0)$$

This formula should be interpreted with caution. Since this is an equality of operators, f' represents the operator $\frac{f'*g}{g}$ and $f(0)$ the operator $\frac{f(0)g}{g}$.

In general, for any positive integer n, we have

$$f^{(n)} = s^n * f - s^{n-1} * f(0) - s^{n-2} * f'(0) - \cdots - s * f^{n-2}(0) - f^{n-1}(0).$$

The right hand side of this formula reminds us of the familiar formula for the Laplace transform of a derivative. This is not a coincidence. Mikusiński's operational calculus provides a justification for Heaviside's operational calculus different from the Laplace transform approach. The main advantage of this approach is that there are no restrictions on the rate of growth of functions. It is also simpler, because it is based on the familiar algebra of quotients.

We will not proceed any farther with the development of the theory of Mikusiński's operational calculus. For the most complete presentation of the theory see [14] and [15].

Before we move to the next topic we would like to mention two modifications of the definition of operators. Let

$$\mathcal{C}_0 = \{f \in \mathcal{C} \text{ and } f^{(n)}(0) = 0 \text{ for all } n = 0, 1, 2, \dots\}$$

It turns out that if we use $\mathcal{C}_0$ instead of $\mathcal{C}$, then the constructed quotient field $\mathcal{M}$ is the same. It suffices to note that if $f \in \mathcal{C}$ and $g \in \mathcal{C}_0$, then $f * g \in \mathcal{C}_0$. Thus, an arbitrary element of $\mathcal{M}$ represented by a quotient $\frac{f}{g}$, with $f, g \in \mathcal{C}$, can also be represented by $\frac{f*h}{g*h}$ where h is any nonzero function from $\mathcal{C}_0$.

Note that while in the Introduction the convolution was defined as an integral from $-\infty$ to ∞, in this section it was defined as an integral from 0 to x. This difference is not essential. If $\mathcal{C}$ is defined as the space of all complex-valued functions defined on $\mathbb{R}$ that vanish on $(-\infty, 0)$ and are continuous on $[0, \infty)$, then both definitions are equivalent.

From the above remarks it should also be clear that we could start the construction from the space of all continuous functions with the support bounded from the left.

3. Boehmians on $\mathbb{R}^N$

One of the major limitations of the operational calculus is the restriction on the support. Functions whose support is not bounded on the left cannot be identified with operators. Boehmians were introduced in [16] as a modification of the construction of operators that allows functions (and generalized functions) with supports unbounded on the left while preserving the nature of operators as convolution quotients.

In order to understand better the origin of Boehmians we first discuss regular operators introduced by T. K. Boehme in [5]. Mikusiński's operators are defined globally. For a general operator it does not make sense to talk about the support. It is also meaningless to ask whether two operators are equal in a neighborhood of a point. Boehme's motivation was to identify a subclass of operators which exhibit local properties. Regular operators have a well defined support and can be compared locally. In order to define them we need the notion of a delta sequence.

By a *delta sequence* we mean a sequence of smooth functions $\varphi_1, \varphi_2, \ldots \in \mathcal{C}$ satisfying the following conditions:

(a) $\varphi_n \geq 0$ for all $n \in \mathbb{N}$,

(b) $\int \varphi_n = 1$ for all $n \in \mathbb{N}$,

(c) For every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $\operatorname{supp} \varphi_n \subset [0, \varepsilon]$ for all $n > n_0$.

An operator $F \in \mathcal{M}$ is called a *regular operator* if there exists a delta sequence $\varphi_1, \varphi_2, \ldots$ and functions $f_1, f_2, \ldots \in \mathcal{C}$ such that

$$F = \frac{f_1}{\varphi_1} = \frac{f_2}{\varphi_2} = \frac{f_3}{\varphi_3} = \ldots$$

Clearly, any function $f \in \mathcal{C}$ is a regular operator. Indeed,

$$f = \frac{f * \varphi_1}{\varphi_1} = \frac{f * \varphi_2}{\varphi_2} = \frac{f * \varphi_3}{\varphi_3} = \ldots$$

The same representation works for Schwartz distributions with support in $[0, \infty)$. Boehme shows in [5] that there are regular operators that are not distributions. However, since regular operators are defined as a subclass of operators, they inherit the restriction on the support.

There are two basic reasons for the restriction on the support in the construction of operators. First, it ensures that the convolution is well-defined. Second, we have Titchmarsh's theorem which makes the construction of the quotient field possible.

When dealing with operators as convolution quotients it is not necessary to convolve the numerators unless one wants to work with inverse operators. It is necessary to convolve denominators with denominators and denominators with numerators. If we restrict the functions used in denominators to test functions with compact support and give up the possibility of inverting all nonzero operators, then it is no longer necessary to restrict the support of functions in the numerators since the convolution of an arbitrary continuous function on $\mathbb{R}$

with a test function with compact support always exists. However, Titchmarsh's theorem is no longer true. Indeed, consider $f(x) = \sin x$ and φ equal to the characteristic function of the interval $[0, 2\pi]$. If ψ is a nonzero test function with compact support, then $\varphi * \psi$ is a nonzero test function and $f * (\varphi * \psi) = 0$. It turns out that we can overcome this difficulty by replacing quotient of functions by *quotients of sequences of functions*.

The name *Boehmians* is used for any space obtained by the construction presented below. The minimal structure necessary for the construction is the following:

I A nonempty set $\mathcal{X}$,

II A commutative semigroup $(\mathcal{S}, *)$,

III An operation $\odot : \mathcal{X} \times \mathcal{S} \to \mathcal{X}$ such that for every $x \in \mathcal{X}$ and $s_1, s_2 \in \mathcal{S}$ we have $x \odot (s_1 * s_2) = (x \odot s_1) \odot s_2$,

IV A nonempty collection $\Delta \subset \mathcal{S}^{\mathbb{N}}$ such that

(i) If $x, y \in \mathcal{X}$, $(s_n) \in \Delta$, and $x \odot s_n = y \odot s_n$ for all $n \in \mathbb{N}$, then $x = y$,

(ii) If $(s_n), (t_n) \in \Delta$, then $(s_n * t_n) \in \Delta$.

Sequences in Δ are called delta sequences.

If these ingredients are available, then the construction of a space of Boehmians is possible. First we define a collection $\mathcal{A}$ of all pairs (x_n, s_n) such that $x_n \in \mathcal{X}$, $(s_n) \in \Delta$, and $x_n \odot s_m = x_m \odot s_n$ for all $m, n \in \mathbb{N}$. If $(x_n, s_n), (y_n, t_n) \in \mathcal{A}$ and $x_n \odot t_m = y_m \odot s_n$ for all $m, n \in \mathbb{N}$, then we write $(x_n, s_n) \sim (y_n, t_n)$. The relation $\sim$ is an equivalence in $\mathcal{A}$. The space of Boehmians $\mathcal{B}(\mathcal{X})$ is the space of equivalence classes in $\mathcal{A}$. To simplify the notation, the equivalence class of (x_n, s_n) will be denoted by $\frac{x_n}{s_n}$.

There is a canonical embedding of $\mathcal{X}$ into $\mathcal{B}(\mathcal{X})$:

$$x \mapsto \frac{x \odot s_n}{s_n}.$$

The operation $\odot$ can be extended to $\mathcal{B}(\mathcal{X}) \times \mathcal{S}$:

$$\frac{x_n}{s_n} \odot t = \frac{x_n \odot t}{s_n}.$$

Now by taking different spaces $\mathcal{X}$, $\mathcal{S}$, different operations $*$ and $\odot$, and different families of delta sequences Δ, we obtain a variety of spaces of Boehmians. In the most important applications, $\mathcal{X}$ and $\mathcal{S}$ are function spaces. Sometimes different choices of $\mathcal{X}$ produce the same space of Boehmians. Roughly speaking, the choice of $\mathcal{X}$ is responsible for the "rate of growth", while the choice of Δ is responsible for regularity (or irregularity) of Boehmians.

In the standard example we take $\mathcal{X} = \mathcal{C}^\infty(\mathbb{R}^N)$ (the space of smooth functions on $\mathbb{R}^N$), $\mathcal{S} = \mathcal{D}(\mathbb{R}^N)$, and the ordinary convolution on $\mathbb{R}^N$ for both $*$ and $\odot$. Finally, for Δ we take the collection of all sequences of functions $\varphi_1, \varphi_2, \dots \in \mathcal{D}$ satisfying the following conditions:

(A) $\varphi_n \geq 0$ for all $n \in \mathbb{N}$,

(B) $\int \varphi_n = 1$ for all $n \in \mathbb{N}$,

(C) For every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $\varphi_n(x) = 0$ if $\|x\| > \varepsilon$ and $n > n_0$.

The obtained space of Boehmians $\mathcal{B}(\mathcal{C}^\infty)$ contains all Schwartz distributions and all regular operators. Indeed, if F is a distribution, then $F * \varphi \in \mathcal{C}^\infty(\mathbb{R}^N)$ for any test function φ. Thus, $\frac{F*\varphi_n}{\varphi_n}$ represents a Boehmian, for any delta sequence (φ_n). Now, if F is a regular operator, then there exists a delta (φ_n) sequence such that $F = \frac{f_1}{\varphi_1} = \frac{f_2}{\varphi_2} = \frac{f_3}{\varphi_3} = \dots$. If $f_1, f_2, f_3, \dots \in \mathcal{C}^\infty(\mathbb{R})$, then we identify F with $\frac{f_n}{\varphi_n}$. If $f_1, f_2, f_3, \dots$ are not all in $\mathcal{C}^\infty(\mathbb{R}^N)$, then we use $\frac{f_n*\varphi_n}{\varphi_n*\varphi_n}$.

The space $\mathcal{B}(\mathcal{C}^\infty)$ is essentially larger than the space of distributions $\mathcal{D}'$. For instance, there are Boehmians of infinite order with a single point support (see [5] and [17]), which is impossible in $\mathcal{D}'$. All ultradistributions defined by Beurling or Romieu can be identified with elements of $\mathcal{B}(\mathcal{C}^\infty)$ (see [18]). An unusual property of $\mathcal{B}(\mathcal{C}^\infty)$, in comparison with other spaces of generalized functions, is that there are non-harmonic solutions of the Laplace equation in $\mathcal{B}(\mathcal{C}^\infty)$ (see [19]).

If we choose for $\mathcal{X}$ the space of all continuous functions on $\mathbb{R}^N$ or the space of all locally integrable functions on $\mathbb{R}^N$ and leave the other components unchanged, we obtain a space of Boehmians isomorphic to $\mathcal{B}(\mathcal{C}^\infty)$. However, the space becomes larger if the family of delta sequences is defined as follows:

(1) $\int \varphi_n = 1$ for all $n \in \mathbb{N}$,

(2) The exists a constant M such that $\int |\varphi_n| \leq M$ for all $n \in \mathbb{N}$,

(3) For every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $\varphi_n(x) = 0$ if $\|x\| > \varepsilon$ and $n > n_0$.

For special applications we can also take $\mathcal{X}$ to be the space $\mathcal{D}$ and obtain Boehmians with compact support, or the space of integrable functions and obtain integrable Boehmians, or the space of tempered functions (bounded by polynomials) and obtain tempered Boehmians, and so on. One of the advantages of using smooth functions in the definition of Boehmians is the simplicity of the definition of the derivative of a Boehmian. In order to differentiate a Boehmian it suffices to differentiate the functions in the numerator, i.e., $D^\alpha\left(\frac{f_n}{\varphi_n}\right)=\frac{D^\alpha f_n}{\varphi_n}$.

For more detail and some applications of different spaces of Boehmians see for example [1], [2], [6], [8], [9], [20], [23], [26], [28], [29], [34].

4. Boehmians on manifolds

In the definition of Boehmians it is necessary that $(\mathcal{S},*)$ is a commutative semigroup. In all cases considered in the previous section $*$ and $\odot$ denoted the ordinary convolution on $\mathbb{R}^N$. Since this convolution is commutative, we never encounter any problems with that requirement. In this section we will consider some examples where the situation is more complicated. We first define Boehmians on the sphere S^d, where $d \geq 2$, and then on more general manifolds. Since there is no natural convolution on S^d, we need some technical preparations.

Let $e_N=(0,\dots,0,1)\in\mathbb{R}^N$. We shall refer to e_N as the north pole of S^{N-1}, the unit sphere in $\mathbb{R}^N$. Let $\mathcal{T}$ denote the set of all real orthogonal $N\times N$ matrices of determinant one. Note that if $T\in\mathcal{T}$, then T corresponds to a rotation of S^{N-1}. We will use freely the same symbol to denote the matrix and the corresponding transformation of $\mathbb{R}^n$. Let $\mathcal{Z}=\{Z\in\mathcal{T}\mid Ze_N=e_N\}$. By a zonal function we mean a function that is invariant under all rotations $Z\in\mathcal{Z}$. The collection of all zonal functions will be denoted by $\mathcal{A}$:

$$\mathcal{A}=\left\{\varphi\in L^\infty(S^{N-1})\mid \varphi\circ Z=\varphi \text{ for all } Z\in\mathcal{Z}\right\}.$$

Let $f\in L^1(S^{N-1})$ and $\varphi\in\mathcal{A}$. For $x\in S^{N-1}$, we define

$$(f*\varphi)(x)=(f*\varphi)(T_xe_N)=\int_{S^{N-1}}f(z)\varphi\left(T_x^{-1}z\right)\,dz,$$

where $T_x\in\mathcal{T}$ is such that $T_xe_N=x$. It turns out that this operation is well defined. Indeed, if $f\in L^1(S^{N-1})$, $\varphi\in\mathcal{A}$, $T,R\in\mathcal{T}$, and $Te_N=Re_N$, then one prove that

$$\int_{S^{N-1}}f(z)\varphi\left(T^{-1}z\right)\,dz=\int_{S^{N-1}}f(z)\varphi\left(R^{-1}z\right)\,dz,$$

(see [24]). It can also be shown that $(\mathcal{A},*)$ is a commutative semi-group. The family of delta sequences Δ is defined as the collection of all sequences $\varphi_1,\varphi_2,\dots\in\mathcal{A}\cap C(S^{N-1})$ such that the following conditions are satisfied:

(a) $\varphi_n\geq 0$ for all $n\in\mathbb{N}$,

(b) $\int_{S^{N-1}} \varphi_n = 1$ all $n \in \mathbb{N}$,

(c) supp $\varphi_n \to e_N$ as $n \to \infty$, i.e., for every neighborhood $\mathcal{V}$ of e_N there exists an $n_0 \in \mathbb{N}$ such that supp $\varphi_n \subset \mathcal{V}$ for all $n \geq n_0$.

This gives us all the necessary ingredients for a construction of Boehmians. For more detail of this construction and some other properties of Boehmians on the sphere see [24].

Note that in this example, as well as in any other construction considered so far, $\mathcal{S}$ is a subspace of $\mathcal{X}$ and $*$ and $\odot$ are the same operation. We will now take another look at the construction of Boehmians on the sphere S^{N-1}. In this approach $\mathcal{S}$ is not a subspace of $\mathcal{X}$. Actually, functions in $\mathcal{S}$ and in $\mathcal{X}$ are defined on different spaces.

We first note that $\mathcal{T}$ is a compact group (with respect to composition). We denote by I the identity of the group, by ST the product of S and T in $\mathcal{T}$, and by S^{-1} the inverse of S in $\mathcal{T}$.

We denote by $\mathcal{C}(\mathcal{T})$ the space of continuous complex-valued functions on $\mathcal{T}$. The convolution of $\varphi, \psi \in \mathcal{C}(\mathcal{T})$ is defined by

$$(\varphi * \psi)(T) = \int_{\mathcal{T}} \varphi(S)\psi(S^{-1}T)\, dS,$$

where dS is the normalized Haar measure on $\mathcal{T}$. Since $\mathcal{T}$ is compact, the Haar measure is bi-invariant. $\mathcal{C}(\mathcal{T})$ is an algebra under the defined convolution, but it is not commutative. Since commutativity in $\mathcal{S}$ is necessary for the construction, we will use the center of $\mathcal{C}(\mathcal{T})$ for $\mathcal{S}$. Thus $\varphi \in \mathcal{S}$ if and only if $\varphi * \psi = \psi * \varphi$ for all $\psi \in \mathcal{C}(\mathcal{T})$.

For $\varphi \in \mathcal{C}(\mathcal{T})$ and $f \in \mathcal{C}(S^{N-1})$, we define the $\odot$-convolution as follows:

$$(f \odot \varphi)(x) = \int_{\mathcal{T}} f(T^{-1}x)\varphi(T)\, dT.$$

Then $f \odot \varphi \in \mathcal{C}(S^{N-1})$. Moreover, for any $f \in \mathcal{C}(S^{N-1})$ and $\varphi, \psi \in \mathcal{S}$, we have

$$f \odot (\varphi * \psi) = (f \odot \varphi) \odot \psi.$$

Finally, Δ is defined as the collection of all sequences of functions $\varphi_n \in \mathcal{S}$ for which the following conditions are satisfied:

(i) $\int_{\mathcal{T}} \varphi_n(T)\, dT = 1$ for all $n \in \mathbb{N}$,

(ii) There exists a constant M such that $\int_{\mathcal{T}} |\varphi_n(T)|\, dT \leq M$ for all $n \in \mathbb{N}$,

(iii) For every neighborhood $\mathcal{V}$ of I there exists an $n_0 \in \mathbb{N}$ such that $\operatorname{supp} \varphi_n \subset \mathcal{V}$ for all $n \geq n_0$.

Since we insist that delta sequences are formed from functions in $\mathcal{S}$, it may not be entirely obvious that such delta sequences exist. One can show that this is not a problem (see [22]). We now have all the elements necessary for a construction of Boehmians. We proceed as described in Section 3.

This second construction of Boehmians on the sphere is more complicated and requires more abstract tools than the first one. The main advantage of this approach is that it lends itself to generalizations. To this goal we will use the framework of locally compact groups. This approach was already used in the early work of D. Nemzer [27].

Let G be a locally compact group, e the identity element of G, and $L^1(G)$ the convolution algebra of integrable functions with respect to the left Haar measure on G. The convolution in $L^1(G)$ is defined by

$$(\varphi * \psi)(x) = \int_G \varphi(z)\psi(z^{-1}x)\,dx. \tag{3}$$

By $\mathcal{Z}(G)$ we will denote the center of $L^1(G)$, i.e., $\varphi \in \mathcal{Z}(G)$ if and only if $f * \varphi = \varphi * f$ for all $f \in L^1(G)$. As in the case of the sphere, Δ is defined as the collection of all sequences $\varphi_1, \varphi_2, \ldots \in L^1(G)$ such that

(A) $\varphi_n \in \mathcal{Z}(G)$ for all $n \in \mathbb{N}$,

(B) $\varphi_n \geq 0$ for all $n \in \mathbb{N}$,

(C) $\int_G \varphi_n = 1$ for all $n \in \mathbb{N}$,

(D) For every neighborhood $\mathcal{V}$ of e there exists an $n_0 \in \mathbb{N}$ such that $\operatorname{supp} \varphi_n \subset \mathcal{V}$ for all $n \geq n_0$.

One can show that conditions IV(i) and IV(ii) in Section 3 are satisfied. However, we cannot expect that delta sequences will exist for an arbitrary group G. A locally compact group G will be called a B-group if there exists a delta sequence in $L^1(G)$. Clearly, every first countable locally compact commutative group is a B-group. We can also show that every first countable compact group is a B-group [21].

If G is a B-group, then we can take $\mathcal{X} = L^1(G)$, $S = \mathcal{Z}(G)$, the convolution in $L^1(G)$ for both $\odot$ and $*$, and Δ as defined above. Then all conditions necessary for the construction of Boehmians are satisfied.

Now we are ready to describe a method of construction of Boehmians that can be applied to a variety of manifolds.

Let $\mathcal{M}$ be a σ-compact manifold and let $\mathcal{T}$ be a locally compact group of transformations on $\mathcal{M}$ such that:

(I) Every $T \in \mathcal{T}$ is a homeomorphism of $\mathcal{M}$,

(II) For every $f \in \mathcal{C}(\mathcal{M})$ the mapping $T \mapsto f \circ T$ is continuous,

(III) For every $x, y \in \mathcal{M}$ there exists a $T \in \mathcal{T}$ such that $Tx = y$.

For $f \in \mathcal{C}(\mathcal{M})$ and $\varphi \in L^1(\mathcal{T})$ define

$$(f \odot \varphi)(x) = \int_{\mathcal{T}} f(T^{-1}x)\varphi(T)\,dT.$$

If $f \in \mathcal{C}(\mathcal{M})$ and $\varphi, \psi \in \mathcal{Z}(\mathcal{T})$ we have

$$f \odot (\varphi * \psi) = (f \odot \varphi) \odot \psi,$$

where $*$ is defined by (3).

If $\mathcal{T}$ is a B-group, then the construction of Boehmians is possible for $\mathcal{X} = \mathcal{C}(\mathcal{M})$, $\mathcal{S} = \mathcal{Z}(\mathcal{T})$, and $*$ and $\odot$ as defined above. For example, if we can find a locally compact group of transformations $\mathcal{T}$ on $\mathcal{M}$ that is first countable and commutative or first countable and compact, then we can construct Boehmians on $\mathcal{M}$.

5. Other convolutions

We would like to close this paper with a brief description of the approach to generalized functions defined by convolutions studied by I. H. Dimovski in [7].

The convolution introduced in Section 2, i.e.,

$$(f * g)(x) = \int_0^x f(x-y)g(y)dy,$$

(called the *Duhamel convolution* in [7]), is associated with the Volterra integration operator

$$(Lf)(x) = \int_0^x f(y)\,dy$$

via the following property

$$Lf = H * f,$$

where H is the characteristic function of $[0, \infty)$. Since the convolution product is associative, we have

$$L(f * g) = (Lf) * g \quad \text{for any } f, g \in \mathcal{C}.$$

This suggests the following generalization of the notion of the convolution:

A bilinear, commutative and associative operation $*: X \times X \to X$ in a linear space X is called a convolution of a given linear operator $L: X \to X$ if $L(f*g) = (Lf)*g$ holds for all $f, g \in X$.

If certain technical conditions are satisfied, one can construct an operational calculus for such a convolution similar to the construction presented in Section 2.

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