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A topic related to infinite-dimensional Lie superalgebras

Minoru Wakimoto
Graduate School of Mathematics,
Kyushu University
Fukuoka 812-8581, Japan

As it is well known, the theory of infinite-dimensional Lie algebras has been beautifully and quite excitingly developed in recent two or three decades. A Lie algebra $\mathfrak{g}$ is a linear space with a bilinear map

$$[\ ,\ ] \ : \ \mathfrak{g} \times \mathfrak{g} \ni (X, Y) \ \longmapsto \ [X, Y] \in \mathfrak{g}$$

satisfying the conditions

(i) (skew-symmetry) $[Y, X] \ = \ -[X, Y],$

(ii) (Jacobi identity) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] \ = \ 0$

for all $X, Y, Z \in \mathfrak{g}$. The map $[\ ,\ ]$ is called the "bracket" or "bracket product" or "Lie algebra structure" on $\mathfrak{g}$. In this note, a "linear space" means always a linear space over the complex field $\mathbb{C}$.

The most interesting and important class of infinite-dimensional Lie algebras are affine Lie algebras and the Virasoro algebra, whose representation theory has deep connections with various areas in mathematics and mathematical physics such as invariant theory, modular functions, soliton equations, conformal field theory, particle physics, lattice models in statistical mechanics and so on, and has produced so many fruitful results. These results are explained in many literatures (see e.g., [5], [6], [11] and their bibliographies).

There are yet some important algebraic systems which are related to or derive their origin from Lie algebras. One among them is a Lie superalgebra. In our physical world, elementary particles are usually classified into two classes; bosons and fermions, where bosons behave under the commutation relations (i.e., $xy - yx = \cdots$) and fermions obey the anti-commutation relations (i.e., $xy + yx = \cdots$). The superalgebra is a natural algebraic structure including both commutation and anti-commutation relations in terms of $\mathbb{Z}_2$ structure, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ is the additive group such that

$$\bar{0} + \bar{0} = \bar{0}, \quad \bar{0} + \bar{1} = \bar{1}, \quad \bar{1} + \bar{1} = \bar{0}.$$

A $\mathbb{Z}_2$-graded vector space is a vector space V equipped with a (given) direct sum decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $V_{\bar{0}}$ (resp. $V_{\bar{1}}$) is called the "even" (resp. "odd") part of V. When $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a $\mathbb{Z}_2$-graded vector space, an element $v \in V$ is called "homogeneous" if v belongs to $V_{\bar{0}}$ or $V_{\bar{1}}$. If $v \in V_j$ ($j = \bar{0}$ or $\bar{1}$), the number $|v| := j \in \{\bar{0}, \bar{1}\}$ is called the "parity" of v.

A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bilinear map

$$[\ ,\ ] \ : \ \mathfrak{g} \times \mathfrak{g} \ni (X, Y) \ \longmapsto \ [X, Y] \in \mathfrak{g},$$

called a "Lie superbracket", satisfying the conditions

(i) (grade preserving) $\quad [\mathfrak{g}_j, \mathfrak{g}_k] \ \subset \ \mathfrak{g}_{j+k},$

(ii) (super-skew-symmetry) $\quad [Y, X] \ = \ -(-1)^{|X||Y|}[X, Y],$

(iii) (super-Jacobi identity) $\quad [X, [Y, Z]] - (-1)^{|X||Y|}[Y, [X, Z]] \ = \ [[X, Y], Z]$

for all homogeneous elements $X, Y, Z \in \mathfrak{g}$.

A linear subspace $\mathfrak{p}$ of $\mathfrak{g}$ is called a $\mathbb{Z}_2$-graded subspace if the direct sum decomposition $\mathfrak{p} = (\mathfrak{p} \cap \mathfrak{g}_{\bar{0}}) \oplus (\mathfrak{p} \cap \mathfrak{g}_{\bar{1}})$ holds. A $\mathbb{Z}_2$-graded linear subspace $\mathfrak{p}$ of $\mathfrak{g}$ is called a Lie sub-superalgebra if the condition

$$X, Y \in \mathfrak{p} \implies [X, Y] \in \mathfrak{p}$$

is satisfied, then $\mathfrak{p}$ itself is a Lie superalgebra. Moreover if $\mathfrak{p}$ satisfies a stronger condition

$$X \in \mathfrak{p}, \ \ Y \in \mathfrak{g} \implies [X, Y] \in \mathfrak{p},$$

$\mathfrak{p}$ is called an "ideal" of $\mathfrak{g}$. Notice that $\{0\}$ and $\mathfrak{g}$ itself are clearly ideals of $\mathfrak{g}$, called the "trivial ideals". A finite-dimensional Lie superalgebra $\mathfrak{g}$ is called "simple" if it does not have ideals other than trivial ones.

Each element X in a Lie superalgebra $\mathfrak{g}$ defines a linear operator $\mathrm{ad}X$ on $\mathfrak{g}$ by

$$\mathrm{ad}X(Y) \ := \ [X, Y] \qquad \text{for all } Y \in \mathfrak{g},$$

called the "adjoint action" of X on $\mathfrak{g}$.

A Lie superalgebra $\mathfrak{g}$ is called "abelian" or "super-commutative" in particular when $[X, Y] = 0$ holds for all elements $X, Y \in \mathfrak{g}$. Other typical and important examples of Lie superalgebras are provided by the space of linear transformations $\mathrm{End}(V)$ where $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a $\mathbb{Z}_2$-graded vector space. The space $\mathrm{End}(V)$ carries a natural $\mathbb{Z}_2$-gradation

$$\mathrm{End}(V) = (\mathrm{End}(V))_{\bar{0}} \oplus (\mathrm{End}(V))_{\bar{1}}$$

defined by

$$(\mathrm{End}(V))_j := \{f \in \mathrm{End}(V) \ ; \ f(V_k) \subset V_{j+k} \text{ for all } k \in \mathbb{Z}_2\},$$

and is a Lie superalgebra with the Lie superbracket defined by

$$[f, g] := f \circ g - (-1)^{|f||g|} g \circ f$$

for all homogeneous elements $f, g \in \mathrm{End}(V)$, where $f \circ g$ is the usual notation of the composition of maps. Notice that $\mathfrak{g}_{\bar{0}}$ is a Lie algebra, and that $[f, g] = f \circ g + g \circ f$ if both f and g are odd elements. In the case when $\dim(V_{\bar{0}}) =: m$ and $\dim(V_{\bar{1}}) =: n$ are finite, this Lie superalgebra is denoted by $\mathfrak{gl}(m|n)$. Choosing a basis of $V_{\bar{0}}$ and $V_{\bar{1}}$, elements f in $\mathfrak{gl}(m|n)$ are expressed by matrices

$$f = \begin{array}{c} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \begin{array}{l} \}\, m \\ \}\, n\,. \end{array} \\ \begin{array}{cc} \underbrace{}_{m} & \underbrace{}_{n} \end{array} \end{array} \tag{1}$$

In this matrix expression, even part and odd part are given by

$$\mathfrak{gl}(m|n)_{\bar{0}} = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right) \right\} \quad \text{and} \quad \mathfrak{gl}(m|n)_{\bar{1}} = \left\{ \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) \right\},$$

and the Lie superbracket of $\mathfrak{gl}(m|n)$ is written as follows:

$$\left[\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right), \left(\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right) \right]$$
$$= \left(\begin{array}{c|c} AA' - A'A + BC' + B'C & AB' - A'B + BD' - B'D \\ \hline CA' - C'A + DC' - D'C & DD' - D'D + CB' + C'B \end{array} \right).$$

We notice that, in this matrix expression (1) of f, the complex number

$$\mathrm{str}(f) := \mathrm{supertrace}(f) := \mathrm{trace}(A) - \mathrm{trace}(D)$$

is well-defined independent of a choice of a basis of $V_{\bar{0}}$ and $V_{\bar{1}}$, and that

$$\mathrm{supertrace}([f, g]) = 0 \qquad \text{for all } f, g \in \mathrm{End}(V),$$

and so

$$\mathfrak{sl}(m|n) \ := \ \{f \in \mathfrak{gl}(m|n) \ ; \ \text{supertrace}(f) = 0 \ \}$$

is a Lie sub-superalgebra. It is easy to see that, if $m \neq n$, $\mathfrak{sl}(m|n)$ does not have non-trivial ideals and so is a simple Lie superalgebra. In the case $m = n$, $\mathfrak{sl}(m|m)$ contains the identity matrix I_{2m} which spans a one-dimensional ideal, and the quotient Lie superalgebra $\mathfrak{sl}(m|m)/\mathbb{C}I_{2m}$ is simple. The Lie superalgebra $\mathfrak{sl}(m|n)$ ($\mathfrak{sl}(m|m)/\mathbb{C}I_{2m}$ when $m = n$) is called the simple Lie superalgebra of type $A(m-1, n-1)$.

The classification of all finite-dimensional simple Lie superalgebras was given by Kac [4]. It is never so easy as in the case of Lie algebras. Assuming the existence of a Cartan subalgebra and a non-degenerate super-invariant super-symmmetric even bilinear form $(\ |\ )$,

$$A(m,n), \ \mathfrak{osp}(M|N), \ D(2,1;a) \ (a \neq 0, -1), \ F(4), \ G(3)$$

is the complete list of finite-dimensional Lie superalgebras with $\mathfrak{g}_{\bar{1}} \neq \{0\}$ satisfying these two conditions (see [4]), called of Cartan type. Among the above list, $A(m,n)$ and $\mathfrak{osp}(M|N)$ are called of classical type, since they can be expressed in terms of matrix forms as follows :

$\mathfrak{osp}(2m+1|2n) :=$

$$\left\{ \begin{array}{cl} \begin{array}{c} \left(\begin{array}{ccc|cc} A & B & u & x_1 & x_2 \\ C & -{}^tA & v & y_1 & y_2 \\ -{}^tv & -{}^tu & 0 & z_1 & z_2 \\ \hline {}^ty_2 & {}^tx_2 & {}^tz_2 & D & E \\ -{}^ty_1 & -{}^tx_1 & -{}^tz_1 & F & -{}^tD \end{array} \right) \begin{array}{l} \}m \\ \}m \\ \}1 \\ \}n \\ \}n \end{array} \\ \begin{array}{ccccc} \underbrace{\quad}_{m} & \underbrace{\quad}_{m} & \underbrace{\quad}_{1} & \underbrace{\quad}_{n} & \underbrace{\quad}_{n} \end{array} \end{array} ; & \begin{array}{ll} \text{(i)} & A \text{ is an } m \times m \text{ matrix} \\ \text{(ii)} & D \text{ is an } n \times n \text{ matrix} \\ \text{(iii)} & B, C \text{ are skew-symmetric} \\ \text{(iv)} & E, F \text{ are symmetric} \end{array} \end{array} \right\},$$

$\mathfrak{osp}(2m|2n) :=$

$$\left\{ \begin{array}{cl} \begin{array}{c} \left(\begin{array}{cc|cc} A & B & x_1 & x_2 \\ C & -{}^tA & y_1 & y_2 \\ \hline {}^ty_2 & {}^tx_2 & D & E \\ -{}^ty_1 & -{}^tx_1 & F & -{}^tD \end{array} \right) \begin{array}{l} \}m \\ \}m \\ \}n \\ \}n \end{array} \\ \begin{array}{cccc} \underbrace{\quad}_{m} & \underbrace{\quad}_{m} & \underbrace{\quad}_{n} & \underbrace{\quad}_{n} \end{array} \end{array} ; & \begin{array}{ll} \text{(i)} & A \text{ is an } m \times m \text{ matrix} \\ \text{(ii)} & D \text{ is an } n \times n \text{ matrix} \\ \text{(iii)} & B, C \text{ are skew-symmetric} \\ \text{(iv)} & E, F \text{ are symmetric} \end{array} \end{array} \right\}.$$

Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra from the above list. A maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\bar{0}}$ is called a Cartan subalgebra of $\mathfrak{g}$ if $\mathrm{ad}H$ is a diagonalizable linear transformation of $\mathfrak{g}$ for all $H \in \mathfrak{h}$. Notice that a Cartan subalgebra is not unique, but there is usually a "suitable" or "standard" choice of a Cartan subalgebra. For $A(m,n)$ or $\mathfrak{osp}(M|N)$, the subalgebra $\mathfrak{h}$ consisting of all diagonal matrices is a Cartan subalgebra.

Let $\mathfrak{h}$ be a Cartan subalgebra. Then, since all $\mathrm{ad}H$ ($H \in \mathfrak{h}$) are diagonalizable and mutually commutative, one can decompose $\mathfrak{g}$ as the direct sum of simultaneous eigenspaces of $\mathrm{ad}H$ ($H \in \mathfrak{h}$). Each simultaneous eigenspace is given, by its definition, in the following form

$$\begin{aligned} \mathfrak{g}_\alpha &:= \{X \in \mathfrak{g} \ ; \ \mathrm{ad}H(X) = \alpha(H)X \quad \text{for all } H \in \mathfrak{h}\} \\ &= \{X \in \mathfrak{g} \ ; \ [H, X] = \alpha(H)X \quad \text{for all } H \in \mathfrak{h}\}, \end{aligned}$$

where $\alpha \in \mathfrak{h}^*$ is a linear function on $\mathfrak{h}$. A non-zero linear function $\alpha \in \mathfrak{h}^*$ is called a "root" or more precisely a "root of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$" if $\mathfrak{g}_\alpha \neq \{0\}$, and then $\mathfrak{g}_\alpha$ is called the "root space" of α. A root α is called "even" (resp. "odd") if its root space $\mathfrak{g}_\alpha$ is contained in even (resp. odd) part of $\mathfrak{g}$, namely $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\bar{0}}$ (resp. $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\bar{1}}$). Let Δ denote the set of all roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and Δ_{even} (resp. Δ_{odd}) be its subset consisting of all even (resp. odd) roots. Then $\mathfrak{g}$ decomposes into the direct sum of the Cartan subalgebra and its root spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

Notice that, for $\alpha, \beta \in \Delta$ such that $\alpha + \beta \in \Delta$, $\alpha + \beta$ is an even root if both α and β are even or both are odd, and is an odd root if one among α and β is even and the other is odd.

Let us look at an example where $\mathfrak{g} = \mathfrak{sl}(m|n)$ and $\mathfrak{h}$ is its Cartan subalgebra consisting of all diagonal matrices. In this case, we define a linear form $\alpha^{(i,j)} \in \mathfrak{h}^*$ ($1 \le i, j \le m+n$) by

$$\alpha^{(i,j)}(H) = x_i - x_j \qquad \text{for all } H = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_{m+n} \end{pmatrix} = \sum_{i=1}^{m+n} x_i E_{i,i} \in \mathfrak{h}.$$

Then

$$\Delta = \{\alpha^{(i,j)} ; \ 1 \le i, j \le m+n \text{ and } i \neq j\}$$

is the set of all roots, and

$$\mathfrak{g}_{\alpha^{(i,j)}} = \mathbb{C} \cdot E_{i,j}$$

is the root space of $\alpha^{(i,j)}$ because

$$[H, E_{i,j}] = \sum_{k=1}^{m+n} x_k [E_{k,k}, E_{i,j}] = (x_i - x_j) E_{i,j}.$$

Putting

$$\Delta^+ := \{\alpha^{(i,j)}\ ;\ i<j\} \qquad \text{and} \qquad \Delta^- := \{\alpha^{(i,j)}\ ;\ i>j\},$$

one sees that $\Delta^- = -\Delta^+$ and the set Δ is decomposed into the disjoint union $\Delta = \Delta^+ \cup \Delta^-$. This decomposition has the property

$$\alpha, \beta \in \Delta^+ \text{ and } \alpha+\beta \in \Delta \implies \alpha+\beta \in \Delta^+,$$

which means

$$\mathfrak{n}^+ \ := \ \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$$

is a Lie sub-superalgebra of $\mathfrak{g}$. In our case of $\mathfrak{sl}(m|n)$, $\mathfrak{n}^+$ is the sub-superalgebra consisting of all upper triangular matrices.

In general case, one can decompose Δ as the disjoint union of $\Delta^\pm$ satisfying the above conditions. A root α is called a "positive root" (resp. "negative root") if it belongs to Δ^+ (resp. Δ^-). A positive root α is called "simple" if it does not decompose into the sum of two positive roots, namely there exist no positive roots β and γ satisfying $\alpha = \beta + \gamma$. Then a positive root is written as a linear combination of simple roots with coefficients in $\mathbb{Z}_{\geq 0}$. It is known that the number of simple roots is equal to $\dim\mathfrak{h}$, called the "rank" of $\mathfrak{g}$, and the set of simple roots is usually denoted by $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ where $\ell = \dim\mathfrak{h}$. In the case $\mathfrak{sl}(m|n)$, $\alpha_i := \alpha^{(i,i+1)}$ $(i = 1, \cdots, m+n-1)$ are simple roots and all positive roots $\alpha^{(i,j)}$ $(i<j)$ are written as $\alpha^{(i,j)} = \sum_{k=i}^{j-1} \alpha_k$.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra of Cartan type. Then, by its assumption, there exists a non-degenerate bilinear form $(\ |\ )$ on $\mathfrak{g}$ satisfying the following conditions:

(i) (super-invariance) $([X,Y]|Z) = (X|[Y,Z])$,

(ii) (super-symmetry) $(Y|X) = (-1)^{|X||Y|}(X|Y)$,

(iii) (even) $(\mathfrak{g}_{\bar{0}}|\mathfrak{g}_{\bar{1}}) = \{0\}$,

which is uniquely determined up to scalar multiples since $\mathfrak{g}$ is simple. Notice that, for $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(M|N)$, one can take $(X|Y) = \text{supertrace}(XY)$.

Given such inner product $(\ |\ )$ on $\mathfrak{g}$, we consider the Lie superalgebra

$$\widehat{\mathfrak{g}} \ = \ (\mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]) \oplus \mathbb{C}K \oplus \mathbb{C}d \tag{2}$$

with the $\mathbb{Z}_2$-gradation

$$(\widehat{\mathfrak{g}})_{\bar{0}} = (\mathfrak{g}_{\bar{0}} \otimes \mathbb{C}[t,t^{-1}]) \oplus \mathbb{C}K \oplus \mathbb{C}d, \qquad (\widehat{\mathfrak{g}})_{\bar{1}} = \mathfrak{g}_{\bar{1}} \otimes \mathbb{C}[t,t^{-1}]$$

and with the Lie superbracket $[\ ,\ ]$ defined by

$$\begin{array}{lcl} [X \otimes t^m, Y \otimes t^n] & := & [X, Y] \otimes t^{m+n} + m(X|Y)\delta_{m+n,0} \cdot K, \\ [d, X \otimes t^m] & := & mX \otimes t^m, \\ [K, \widehat{\mathfrak{g}}] & := & \{0\} \qquad (i.e., \ \mathbb{C}K \text{ is the center of } \widehat{\mathfrak{g}}) \end{array}$$

for all $X, Y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$, where $\mathbb{C}[t, t^{-1}]$ denotes the ring of polynomials in t and t^{-1}, namely the ring of all Laurent polynomials in t. This infinite-dimensional Lie superalgebra $\widehat{\mathfrak{g}}$ is usually called the "affinization" of $\mathfrak{g}$ or the "affine Lie superalgebra" over $\mathfrak{g}$. Then the finite-dimensional Lie superalgebra $\mathfrak{g}$ is called the "underlying" finite-dimensional Lie superalgebra of $\widehat{\mathfrak{g}}$ and is naturally identified with the sub-superalgebra $\mathfrak{g} \otimes t^0$ of $\widehat{\mathfrak{g}}$ by the map $\mathfrak{g} \ni X \longmapsto X \otimes t^0 \in \mathfrak{g} \otimes t^0$.

The root structure of an affine Lie superalgebra $\widehat{\mathfrak{g}}$ is easily seen from the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ of $\mathfrak{g}$. From this decomposition and the definition (2) of $\widehat{\mathfrak{g}}$, the affine Lie superalgebra $\widehat{\mathfrak{g}}$ decomposes as follows :

$$\widehat{\mathfrak{g}} = (\mathfrak{h} \otimes t^0) \oplus \mathbb{C}K \oplus \mathbb{C}d \oplus \Big(\bigoplus_{0 \neq m \in \mathbb{Z}} \mathfrak{h} \otimes t^m \Big) \oplus \Big(\sum_{\alpha \in \Delta} \sum_{n \in \mathbb{Z}} \mathfrak{g}_\alpha \otimes t^n \Big).$$

Then one can see easily that

$$\mathfrak{h}^{aff} := (\mathfrak{h} \otimes t^0) \oplus \mathbb{C}K \oplus \mathbb{C}d$$

is a Cartan subalgebra and $\mathfrak{h} \otimes t^m$ and $\mathfrak{g}_\alpha \otimes t^n$ are root spaces. To see it, one needs only to compute the following brackets for $\alpha \in \Delta$, $X \in \mathfrak{g}_\alpha$, $H, H' \in \mathfrak{h}$ and $n, m \in \mathbb{Z}$:

$$\begin{array}{l} [H \otimes t^0, X \otimes t^n] = [H, X] \otimes t^n = \alpha(H) \cdot X \otimes t^n, \\ [d, X \otimes t^n] = n \cdot X \otimes t^n, \\ [K, X \otimes t^n] = 0, \end{array} \tag{3}$$

and

$$\begin{array}{l} [H \otimes t^0, H' \otimes t^m] = [H, H'] \otimes t^m = 0, \\ [d, H' \otimes t^m] = m \cdot H' \otimes t^m, \\ [K, H' \otimes t^m] = 0. \end{array} \tag{4}$$

These formulas (3) and (4) imply that $\mathfrak{g}_\alpha \otimes t^n$ and $\mathfrak{h} \otimes t^m$ are simultanious eigenspaces of all ad$h$ ($h \in \mathfrak{h}^{aff}$) and so they are root spaces.

Identifying $\mathfrak{h} \otimes t^0$ with $\mathfrak{h}$, one can write $\mathfrak{h}^{aff}$ as

$$\mathfrak{h}^{aff} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d.$$

Then $\mathfrak{h}$ is a linear subspace of $\mathfrak{h}^{aff}$, and a linear form $\alpha \in \mathfrak{h}^*$ extends uniquely to a linear form $\alpha \in (\mathfrak{h}^{aff})^*$ by letting $\alpha(K) = \alpha(d) := 0$. Introduce a linear form $\delta \in (\mathfrak{h}^{aff})^*$ defined by

$$\delta(d) := 1, \quad \delta(K) := 0 \qquad \text{and} \qquad \delta(H) := 0 \ (\forall H \in \mathfrak{h}).$$

Using this linear form δ, (3) and (4) are written as follows for all $h \in \mathfrak{h}^{aff}$:

$$\begin{aligned} [h, X \otimes t^n] &= (n\delta + \alpha)(h) \cdot X \otimes t^n, \\ [h, H' \otimes t^m] &= m\delta(h) \cdot H' \otimes t^m, \end{aligned} \tag{5}$$

which shows that the set of all roots of $\widehat{\mathfrak{g}}$ is

$$\Delta^{aff} = \{n\delta + \alpha, \ m\delta \ ; \ \alpha \in \Delta, \ n \in \mathbb{Z}, \ m \in \mathbb{Z} \setminus \{0\}\}$$

and that $\mathrm{mult}(m\delta) = \dim \mathfrak{h}$ $(0 \neq m \in \mathbb{Z})$ since the root space of $m\delta$ is $\mathfrak{h} \otimes t^m$. When viewed from an affine Lie superalgebra $\widehat{\mathfrak{g}}$, a Cartan subalgebra $\mathfrak{h}$ of the underlying finite-dimensional Lie superalgebra $\mathfrak{g}$ is sometimes called the "finite Cartan subalgebra" or the "finite part" of $\mathfrak{h}^{aff}$.

It is usual to choose the positive root system of $\widehat{\mathfrak{g}}$ as follows :

$$\Delta^{aff+} = \{n\delta, \ n\delta + \alpha \ ; \ n \geq 1, \alpha \in \Delta\} \cup \Delta^+,$$

so that the sum of positive root spaces of $\widehat{\mathfrak{g}}$ is equal to

$$\Big(\bigoplus_{n \geq 1} \mathfrak{g} \otimes t^n\Big) \oplus \Big(\bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \otimes t^0\Big).$$

A non-degenerate super-invariant supersymmetric bilinear form $(\ | \ )$ on $\mathfrak{g}$ extends naturally to the one on the affine Lie superalgebra $\widehat{\mathfrak{g}}$ requiring that $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ is orthogonal to K and d, and

$$(X \otimes t^m | Y \otimes t^n) := (X|Y)\delta_{m+n,0}, \quad (K|d) := 1, \quad (K|K) = (d|d) := 0.$$

This inner product is non-degenerate on $\mathfrak{h}^{aff}$, and so induces a linear isomorphism

$$(\mathfrak{h}^{aff})^* \ni \lambda \ \longleftrightarrow \ h_\lambda \in \mathfrak{h}^{aff}$$

by $\lambda(h) = (h_\lambda | h)$ for all $h \in \mathfrak{h}^{aff}$. Then the dual space $(\mathfrak{h}^{aff})^*$ carries a non-degenerate inner product $(\ | \ )$ defined by

$$(\lambda|\mu) := \lambda(h_\mu) = (h_\lambda|h_\mu) \qquad \text{for all } \lambda, \mu \in (\mathfrak{h}^{aff})^*.$$

We remark here that, under this isomorphism $\mathfrak{h}^{aff} \cong (\mathfrak{h}^{aff})^*$, the *primitive imaginary root* δ is identified with the *canonical central element* K, so in particular

$$(\delta|\delta) = 0 \qquad \text{and} \qquad (\delta|\alpha) = 0 \ \text{ for all } \alpha \in \Delta, \tag{6}$$

since the element corresponding to $\alpha \in \Delta$ belongs to the finite Cartan subalgebra $\mathfrak{h} \cong \mathfrak{h} \otimes t^0$. When α is a root such that $|\alpha|^2 := (\alpha|\alpha) \neq 0$, the element $\alpha^\vee := \dfrac{2\alpha}{|\alpha|^2}$ is called the "coroot" corresponding to α.

It is known that, given a positive root system Δ^+ of $\mathfrak{g}$, there exists a unique element $\rho \in (\mathfrak{h}^{aff})^*$, called the "Weyl vector" of $\widehat{\mathfrak{g}}$, satisfying the conditions

(i) $(\rho|\alpha) = \dfrac{(\alpha|\alpha)}{2}$ for all simple roots α of Δ^+,

(ii) $(\rho|\delta)(\lambda|\mu) = \displaystyle\sum_{\alpha\in\Delta^+_{\text{even}}} (\alpha|\lambda)(\alpha|\mu) - \sum_{\alpha\in\Delta^+_{\text{odd}}} (\alpha|\lambda)(\alpha|\mu)$ for all $\lambda, \mu \in \mathfrak{h}^*$,

(iii) $\rho(d) = 0$.

Remark 1. *In the case when $\mathfrak{g}$ is an affine Lie algebra or an affine Lie superalgebra, it is usual to denote its underlying finite-dimensional algebra by $\overline{\mathfrak{g}}$ and also to denote all objects of $\overline{\mathfrak{g}}$ by putting "bar" on the top. Namely, in this notation, the Cartan subalgebra of $\overline{\mathfrak{g}}$ is denoted by $\overline{\mathfrak{h}}$ and the set of roots is denoted by $\overline{\Delta}$ and so on, while the objects of an affine Lie superalgebra $\mathfrak{g}$ are denoted by usual notation without extra accessories.*

In the sequel of this paper, we shall make use of this notation. So, when $\mathfrak{g}$ is an affine Lie superalgebra, we denote its Cartan subalgebra by $\mathfrak{h}$ in place of $\mathfrak{h}^{aff}$ and the set of all roots of $\mathfrak{g}$ by Δ ; namely $\mathfrak{h} = \overline{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}d$ and

$$\Delta = \{n\delta + \alpha, m\delta \ ; \ \alpha \in \overline{\Delta}, \ n \in \mathbb{Z}, \ m \in \mathbb{Z} \setminus \{0\}\}.$$

In particular when $\mathfrak{g}$ has no odd part namely $\mathfrak{g}_{\bar{1}} = \{0\}$, then $\mathfrak{g}$ is a usual Lie algebra, and its affinization is the usual affine Lie algebra. The first remarkable result of the representation theory of affine Lie algebras was the denominator identity:

$$\prod_{\alpha\in\Delta^+} (1 - e^{-\alpha})^{\text{mult}(\alpha)} = \sum_{w\in W} \varepsilon(w) e^{w\rho - \rho}, \tag{7}$$

which was discovered by Macdonald [9] by the analysis on the structure of the affine root systems, and is called Macdonald identity or Weyl-Kac denominator identity. In this formula (7), Δ^+ is the set of all positive roots of the affine Lie algebra $\widehat{\mathfrak{g}}$, and $\text{mult}(\alpha)$ is the dimension of the root space $\mathfrak{g}_\alpha$, and W is the

Weyl group of $\widehat{\mathfrak{g}}$, and $\varepsilon(w)$ is the signature of $w \in W$. For detail explanation on these terminologies and proof and related materials, refer to books [5] and [11]. The simplest example of Macdonald's identitity, for $\widehat{\mathfrak{sl}}(2,\mathbb{C})$, is the Jacobi triple product identity (see [5] and [11]):

$$\prod_{n=1}^{\infty}(1-u^n v^n)(1-u^{n-1}v^n)(1-u^n v^{n-1}) = \sum_{n\in\mathbb{Z}}(-1)^n u^{\frac{n^2+n}{2}} v^{\frac{n^2-n}{2}}. \tag{8}$$

This formula (8) produces a lot of identities related to the Euler's function

$$\varphi(q) := \prod_{n=1}^{\infty}(1-q^n)$$

by letting $u = \pm q^a$ and $v = \pm q^b$, called the "specialization" of variables, where a and b are some suitable rational numbers. For example, one easily sees the following :

$$u = q^2,\ v = q \implies \varphi(q) = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{3n^2+n}{2}} \tag{9}$$

$$u = q,\ v = q \implies \frac{\varphi(q)^2}{\varphi(q^2)} = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} \tag{10}$$

$$u = -q,\ v = -1 \implies \frac{\varphi(q^2)^2}{\varphi(q)^2} = \frac{1}{2}\sum_{n\in\mathbb{Z}} q^{\frac{n^2+n}{2}} = \sum_{n=0}^{\infty} q^{\frac{n^2+n}{2}}. \tag{11}$$

The formula (10) shows that $\frac{\varphi(q)^2}{\varphi(q^2)}$ is the generating function of squares. Making the k-products of (10), one has

$$\begin{aligned}
\left(\frac{\varphi(q)^2}{\varphi(q^2)}\right)^k &= \Big(\sum_{n_1\in\mathbb{Z}}(-1)^{n_1}q^{n_1^2}\Big)\cdots\Big(\sum_{n_k\in\mathbb{Z}}(-1)^{n_k}q^{n_k^2}\Big) \\
&= \sum_{n_1,\cdots n_k\in\mathbb{Z}} (-1)^{n_1+\cdots n_k} q^{n_1^2+\cdots+n_k^2} \\
&= \sum_{m=0}^{\infty}\Bigg(\sum_{\substack{n_1,\cdots n_k\in\mathbb{Z} \\ \text{s.t.} \\ n_1^2+\cdots+n_k^2=m}} (-1)^m\Bigg) q^m = \sum_{m=0}^{\infty}(-1)^m \square_k(m) q^m,
\end{aligned}$$

where

$$\square_k(m) := \sharp\{(n_1,\cdots,n_k)\in\mathbb{Z}^k\ ;\ n_1^2+\cdots+n_k^2 = m\}$$

is the number of representations of m as a sum of k squares of integers taking into account the order of summands. For example $\square_2(m)$'s, for small m, look as follows:

$$\begin{array}{ll} \square_2(0) = \sharp\{(0,0)\} = 1, & \square_2(1) = \sharp\{(0,\pm 1), (\pm 1, 0)\} = 4, \\ \square_2(2) = \sharp\{(\pm 1, \pm 1)\} = 4, & \square_2(3) = 0, \qquad \cdots. \end{array}$$

The numbers $(n^2+n)/2$, where $n = 0, 1, 2, \cdots$, are called "triangular numbers" because they appear as the number of nodes in the following sequence :

$$\bigcirc, \quad \begin{array}{c}\bigcirc\\ \bigcirc\bigcirc\end{array}, \quad \begin{array}{c}\bigcirc\\ \bigcirc\bigcirc\\ \bigcirc\bigcirc\bigcirc\end{array}, \quad \begin{array}{c}\bigcirc\\ \bigcirc\bigcirc\\ \bigcirc\bigcirc\bigcirc\\ \bigcirc\bigcirc\bigcirc\bigcirc\end{array}, \quad \cdots.$$

Then the formula (11) means that $\frac{\varphi(q^2)^2}{\varphi(q)}$ is the generating function of triangular numbers, so one has

$$\left(\frac{\varphi(q^2)^2}{\varphi(q)}\right)^k = \sum_{m=0}^{\infty} \triangle_k(m) q^m,$$

where

$$\triangle_k(m) := \sharp\left\{(n_1, \cdots, n_k) \in (\mathbb{Z}_{\geq 0})^k \, ; \begin{array}{ll} \text{(i)} & n_1, \cdots, n_k \text{ are triangular numbers} \\ \text{(ii)} & n_1 + \cdots + n_k = m \end{array}\right\}.$$

The denominator identity for affine Lie superalgebras is much more complicated than in the case of usual affine Lie algebras and is written in the following form

$$\frac{\prod\limits_{\alpha \in \Delta^+_{\text{even}}} (1 - e^{-\alpha})^{\text{mult}(\alpha)}}{\prod\limits_{\alpha \in \Delta^+_{\text{odd}}} (1 + e^{-\alpha})^{\text{mult}(\alpha)}} = e^{-\rho} \sum_{w \in W} \varepsilon(w) w\left(\frac{e^{\rho}}{\prod\limits_{i=1}^{k}(1 + e^{-\beta_i})}\right), \tag{12}$$

where $\{\beta_1, \cdots, \beta_k\}$ is a maximal set of simple odd roots satisfying $(\beta_i|\beta_j) = 0$ for all $i, j = 1, \cdots, k$ (see [7] and [8] for complete explanation and details). This formula (12) holds except for $\widehat{A}(n,n)$, $\widehat{osp}(N+2|N)$ and $D(2,1;a)$ where $(\rho|\delta)$ happens to be 0. Usually we call the left side of (12) the "denominator", and denote it by R.

To explain the Weyl group W, we introduce the following linear transformations r_α and t_β of $\mathfrak{h}^*$ defined for $\alpha, \beta \in \mathfrak{h}^*$ such that $(\alpha|\alpha) \neq 0$:

$$r_\alpha(\lambda) := \lambda - \frac{2(\lambda|\alpha)}{(\alpha|\alpha)}\alpha, \tag{13}$$

$$t_\beta(\lambda) := \lambda + (\lambda|\delta)\beta - \left\{\frac{(\beta|\beta)}{2}(\lambda|\delta) + (\lambda|\beta)\right\}\delta, \tag{14}$$

for all $\lambda \in \mathfrak{h}^*$. It is easy to see that these transformations satisfy

$$r_\alpha t_\beta r_\alpha = t_{r_\alpha \beta} \qquad \text{and} \qquad t_\beta t_{\beta'} = t_{\beta+\beta'} \tag{15}$$

and preserve the inner product, namely

$$(r_\alpha(\lambda)|r_\alpha(\mu)) = (\lambda|\mu) \qquad \text{and} \qquad (t_\beta(\lambda)|t_\beta(\mu)) = (\lambda|\mu)$$

for all $\lambda, \mu \in \mathfrak{h}^*$. Notice that r_α is the reflection with respect to the hyperplane $\{H\,;\,\alpha(H) = 0\}$, and so in particular $r_\alpha(\alpha) = -\alpha$ and $(r_\alpha)^2$ is the identity transformation.

The Weyl group W is a subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by r_α's and t_β's with all positive even roots α and all positive even coroots β of $\bar{\mathfrak{g}}$ of positive (resp. negative) square length if $(\rho|\delta) > 0$ (resp. $(\rho|\delta) < 0$). The signature function ε is the group homomorphism $W \longrightarrow \{\pm 1\}$ defined by $\varepsilon(r_\alpha) := -1$ and $\varepsilon(t_\beta) := 1$. Notice that the primitive imaginary root δ is fixed by all elements in the Weyl group since $(\delta|\alpha) = 0$ for all roots α.

We now compute explicitly this formula (12) in the case $\widehat{\mathfrak{sl}}(2|1)$. For this sake, we recall the root system of $\bar{\mathfrak{g}} = \mathfrak{sl}(2|1)$:

$$\overline{\Delta}^+ = \{\alpha^{(1,2)}, \alpha^{(1,3)}, \alpha^{(2,3)}\} = \{\alpha_1, \ \alpha_2, \ \alpha_1 + \alpha_2\}$$

where $\alpha_1 := \alpha^{(1,2)}$ is an even simple root and $\alpha_2 := \alpha^{(2,3)}$ is an odd simple root. The elements H_i $(i = 1, 2)$ corresponding to these simple roots in the finite Cartan subalgebra

$$\bar{\mathfrak{h}} = \left\{ \left(\begin{array}{cc|c} a & & \\ & b & \\ \hline & & c \end{array} \right) \ ; \ a + b - c = 0 \right\}$$

are given by

$$H_1 = \left(\begin{array}{cc|c} 1 & & \\ & -1 & \\ \hline & & 0 \end{array} \right), \qquad H_2 = \left(\begin{array}{cc|c} 0 & & \\ & 1 & \\ \hline & & 1 \end{array} \right),$$

since these elements satisfy

$$\begin{aligned} \mathrm{str}(H_1 H) &= \mathrm{str}\left(\left(\begin{array}{cc|c} 1 & & \\ & -1 & \\ \hline & & 0 \end{array} \right) \left(\begin{array}{cc|c} a & & \\ & b & \\ \hline & & c \end{array} \right) \right) = \mathrm{str} \left(\begin{array}{cc|c} a & & \\ & -b & \\ \hline & & 0 \end{array} \right) \\ &= a - b = \alpha_1(H) \end{aligned}$$

and

$$\begin{aligned}
\mathrm{str}(H_2H) &= \mathrm{str}\left(\left(\begin{array}{cc|c} 0 & & \\ & 1 & \\ \hline & & 1 \end{array}\right)\left(\begin{array}{cc|c} a & & \\ & b & \\ \hline & & c \end{array}\right)\right) = \mathrm{str}\left(\begin{array}{cc|c} 0 & & \\ & b & \\ \hline & & c \end{array}\right) \\
&= b - c = \alpha_2(H)
\end{aligned}$$

for all $H = \left(\begin{array}{cc|c} a & & \\ & b & \\ \hline & & c \end{array}\right) \in \bar{\mathfrak{h}}$. Then the inner products of simple roots are given as follows :

$$\begin{aligned}
(\alpha_1|\alpha_1) &= \mathrm{str}(H_1H_1) = \mathrm{str}\left(\left(\begin{array}{cc|c} 1 & & \\ & -1 & \\ \hline & & 0 \end{array}\right)\left(\begin{array}{cc|c} 1 & & \\ & -1 & \\ \hline & & 0 \end{array}\right)\right) \\
&= \mathrm{str}\left(\begin{array}{cc|c} 1 & & \\ & 1 & \\ \hline & & 0 \end{array}\right) = 2, \\
(\alpha_1|\alpha_2) &= \mathrm{str}(H_1H_2) = \mathrm{str}\left(\left(\begin{array}{cc|c} 1 & & \\ & -1 & \\ \hline & & 0 \end{array}\right)\left(\begin{array}{cc|c} 0 & & \\ & 1 & \\ \hline & & 1 \end{array}\right)\right) \\
&= \mathrm{str}\left(\begin{array}{cc|c} 0 & & \\ & -1 & \\ \hline & & 0 \end{array}\right) = -1, \\
(\alpha_2|\alpha_2) &= \mathrm{str}(H_2H_2) = \mathrm{str}\left(\left(\begin{array}{cc|c} 0 & & \\ & 1 & \\ \hline & & 1 \end{array}\right)\left(\begin{array}{cc|c} 0 & & \\ & 1 & \\ \hline & & 1 \end{array}\right)\right) \\
&= \mathrm{str}\left(\begin{array}{cc|c} 0 & & \\ & 1 & \\ \hline & & 1 \end{array}\right) = 0.
\end{aligned}$$

Then one has $(\rho|\alpha_1) = \dfrac{(\alpha_1|\alpha_1)}{2} = 1$ and $(\rho|\alpha_2) = \dfrac{(\alpha_2|\alpha_2)}{2} = 0$, and $(\rho|\delta)$ is computed from the condition (ii) of the definition of ρ, in particular by letting $\lambda = \mu = \alpha_1$, as follows :

$$(\rho|\delta)\cdot|\alpha_1|^2 = (\alpha_1|\alpha_1)^2 - \{(\alpha_2|\alpha_1)^2 + (\alpha_1+\alpha_2|\alpha_1)^2\} = 4 - \{(-1)^2 + 1^2\} = 2,$$

and so one obtains $(\rho|\delta) = 1$ since $|\alpha_1|^2 = 2$.

Since α_1 is the unique positive even root with positive square length and $\alpha_1^\vee = \dfrac{2\alpha_1}{(\alpha_1|\alpha_1)} = \alpha_1$, the Weyl group W of $\widehat{\mathfrak{sl}}(2|1)$ is generated by r_{α_1} and t_{α_1} and so

$$W = \{t_{n\alpha_1},\ t_{n\alpha_1} r_{\alpha_1}\ ;\ n \in \mathbb{Z}\}.$$

The denominator identity for $\widehat{s\ell}(2|1)$ is

$$e^\rho R = \sum_{w\in W} \varepsilon(w) w \left(\frac{e^\rho}{1+e^{-\alpha_2}}\right), \tag{16}$$

where

$$R = \prod_{n=1}^{\infty} \frac{(1-e^{-n\delta})^2(1-e^{-(n-1)\delta-\alpha_1})(1-e^{-n\delta+\alpha_1})}{(1+e^{-(n-1)\delta-\alpha_2})(1+e^{-n\delta+\alpha_2})(1+e^{-(n-1)\delta-\alpha_1-\alpha_2})(1+e^{-n\delta+\alpha_1+\alpha_2})}.$$

We now calculate the right side of (16). Since

$$r_{\alpha_1}(\rho) = \rho - \frac{2(\rho|\alpha_1)}{(\alpha_1|\alpha_1)}\alpha_1 = \rho - \alpha_1 \quad \text{and} \quad r_{\alpha_1}(\alpha_2) = \alpha_2 - \frac{2(\alpha_2|\alpha_1)}{(\alpha_1|\alpha_1)}\alpha_1 = \alpha_2 + \alpha_1,$$

one has

$$\text{right side of (16)} = \sum_{n\in\mathbb{Z}} t_{n\alpha_1}\left(\frac{e^\rho}{1+e^{-\alpha_2}}\right) - \sum_{n\in\mathbb{Z}} t_{n\alpha_1}\left(\frac{e^{\rho-\alpha_1}}{1+e^{-\alpha_1-\alpha_2}}\right). \tag{17}$$

where the action $t_{n\alpha_1}$ is given as follows:

$$\begin{aligned}
t_{n\alpha_1}(\rho) &= \rho + (\rho|\delta)n\alpha_1 - \{n^2(\rho|\delta) + (\rho|n\alpha_1)\}\delta &&= \rho + n\alpha_1 - (n^2+n)\delta,\\
t_{n\alpha_1}(\alpha_1) &= \alpha_1 + (\alpha_1|\delta)n\alpha_1 - \{n^2(\alpha_1|\delta) + (\alpha_1|n\alpha_1)\}\delta &&= \alpha_1 - 2n\delta,\\
t_{n\alpha_1}(\alpha_2) &= \alpha_2 + (\alpha_2|\delta)n\alpha_1 - \{n^2(\alpha_2|\delta) + (\alpha_2|n\alpha_1)\}\delta &&= \alpha_2 + n\delta.
\end{aligned}$$

Then, by putting $q := e^{-\delta}$ and $x := e^{-\alpha_1}$ and $y := e^{-\alpha_2}$ for simplicity, (17) is calculated as follows:

$$\begin{aligned}
\text{right side of (16)} &= \sum_{n\in\mathbb{Z}} \frac{e^\rho x^{-n} q^{n(n+1)}}{1+yq^n} - \sum_{n\in\mathbb{Z}} \frac{e^\rho x^{-n} q^{n(n+1)}\cdot xq^{-2n}}{1+xyq^{-n}}\\
&= e^\rho \left\{\sum_{n\in\mathbb{Z}} \frac{x^{-n}q^{n(n+1)}}{1+yq^n} - \sum_{n\in\mathbb{Z}} \frac{x^{-n+1}q^{n(n-1)}}{1+xyq^{-n}}\right\},
\end{aligned}$$

so (16) becomes as follows:

$$R = \sum_{n\in\mathbb{Z}} \frac{x^{-n}q^{n(n+1)}}{1+yq^n} - \sum_{n\in\mathbb{Z}} \frac{x^{n+1}q^{n(n+1)}}{1+xyq^n}. \tag{18}$$

To expand the right side into the Taylor series in the domain $|y|$, $|xy|$, $|q| < 1$, we note the following:

$$\frac{1}{1+yq^n} = \begin{cases} \displaystyle\sum_{m=0}^{\infty}(-1)^m y^m q^{mn} & \text{if } n \geq 0, \\ \displaystyle\frac{1}{yq^n(1+y^{-1}q^{-n})} = y^{-1}q^{-n}\sum_{m=0}^{\infty}(-1)^m (y^{-1}q^{-n})^m & \\ \displaystyle = -\sum_{m<0}(-1)^m y^m q^{mn} & \text{if } n < 0. \end{cases} \tag{19}$$

Using this, the equality (18) is rewritten as follows:

$$\begin{aligned} R &= \Big\{\sum_{m,n\geq 0} - \sum_{m,n<0}\Big\}(-1)^m x^{-n} q^{n(n+1)} \cdot y^m q^{mn} \\ &\quad - \Big\{\sum_{m,n\geq 0} - \sum_{m,n<0}\Big\}(-1)^m x^{n+1} q^{n(n+1)} \cdot (xy)^m q^{mn} \\ &= \Big\{\sum_{m,n\geq 0} - \sum_{m,n<0}\Big\}(-1)^m x^{-n} y^m q^{n(m+n+1)} \\ &\quad - \Big\{\sum_{m,n\geq 0} - \sum_{m,n<0}\Big\}(-1)^m x^{m+n+1} y^m q^{n(m+n+1)}. \end{aligned} \tag{20}$$

We now rewrite the second term in the right side of the above by putting $m' := -(m+1)$ and $n' := -(n+1)$. Then, since $m, n \geq 0 \Longleftrightarrow m', n' < 0$ and $m, n < 0 \Longleftrightarrow m', n' \geq 0$, the second term in (20) is rewritten as follows:

$$\begin{aligned} &\Big\{\sum_{m,n\geq 0} - \sum_{m,n<0}\Big\}(-1)^m x^{m+n+1} y^m q^{n(m+n+1)} \\ &= \Big\{\sum_{m',n'<0} - \sum_{m',n'\geq 0}\Big\}(-1)^{m'+1} x^{-m'-n'-1} y^{-m'-1} q^{(n'+1)(m'+n'+1)}, \end{aligned}$$

and so (20) becomes as follows:

$$\begin{aligned} R &= \Big\{\sum_{m,n\geq 0} - \sum_{m,n<0}\Big\}(-1)^m (x^{-1}y^{-1}q)^n y^{m+n} q^{n(m+n)} \\ &\quad - \Big\{\sum_{m',n'\geq 0} - \sum_{m',n'<0}\Big\}(-1)^{m'} (x^{-1}y^{-1}q)^{m'+n'+1} y^{n'} q^{n'(m'+n'+1)}. \end{aligned} \tag{21}$$

We put $j := n$ and $k := m+n$ in the first term, and $j := m'+n'+1$ and $k := n'$ in the second term. Then one has

$$\left.\begin{array}{lcl} m,n \geq 0 & \Longleftrightarrow & k \geq j \geq 0 \\ m,n < 0 & \Longleftrightarrow & k < j < 0 \end{array}\right\} \quad \text{in the first term}$$

and

$$\left.\begin{array}{lcl} m',n' \geq 0 & \Longleftrightarrow & j > k \geq 0 \\ m',n' < 0 & \Longleftrightarrow & j \leq k < 0 \end{array}\right\} \quad \text{in the second term,}$$

and so (21) is rewritten as

$$R = \left\{\sum_{j,k\geq 0} - \sum_{j,k<0}\right\}(-1)^{j+k}(x^{-1}y^{-1}q)^j y^k q^{jk}.$$

We now put $u := (xy)^{-1}q$ and $v := y$. Then, since $x = (uv)^{-1}q$ and $y = v$, the denominator R is written in terms of u, v and q as follows:

$$R = \prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-uvq^{n-1})(1-(uv)^{-1}q^n)}{(1+uq^{n-1})(1+u^{-1}q^n)(1+vq^{n-1})(1+v^{-1}q^n)}.$$

Thus we obtain the formula

$$\begin{aligned} &\prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-uvq^{n-1})(1-u^{-1}v^{-1}q^n)}{(1+uq^{n-1})(1+u^{-1}q^n)(1+vq^{n-1})(1+v^{-1}q^n)} \\ &= \left(\sum_{m,n\geq 0} - \sum_{m,n<0}\right)(-1)^{m+n}u^m v^n q^{mn}. \end{aligned} \tag{22}$$

Letting $u \to -u$ and $v \to -v$, one can write this formula also in the following form

$$\prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-uvq^{n-1})(1-u^{-1}v^{-1}q^n)}{(1-uq^{n-1})(1-u^{-1}q^n)(1-vq^{n-1})(1-v^{-1}q^n)} = \left(\sum_{m,n\geq 0} - \sum_{m,n<0}\right)u^m v^n q^{mn}. \tag{23}$$

Formulas (22) and (23) are called the $\widehat{\mathfrak{sl}}(2|1)$-identity.

From this formula (23) one can deduce a formula for $\square_2(m)$ and $\square_4(m)$. To see it, we start with rewriting the left side and the first term in the right side of (23) as follows:

$$\text{left side of (23)} = \frac{1-uv}{(1-u)(1-v)} \prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-uvq^n)(1-u^{-1}v^{-1}q^n)}{(1-uq^n)(1-u^{-1}q^n)(1-vq^n)(1-v^{-1}q^n)},$$

and

$$\begin{aligned}\sum_{m,n\geq 0} u^m v^n q^{mn} &= 1+\sum_{m\geq 1} u^m + \sum_{n\geq 1} v^n + \sum_{m,n\geq 1} u^m v^n q^{mn} \\ &= 1+\frac{u}{1-u}+\frac{v}{1-v}+\sum_{m,n\geq 1} u^m v^n q^{mn} \\ &= \frac{1-uv}{(1-u)(1-v)}+\sum_{m,n\geq 1} u^m v^n q^{mn}.\end{aligned}$$

Then the formula (23) is rewritten as follows:

$$\begin{aligned}&\prod_{n=1}^{\infty}\frac{(1-q^n)^2(1-uvq^n)(1-u^{-1}v^{-1}q^n)}{(1-uq^n)(1-u^{-1}q^n)(1-vq^n)(1-v^{-1}q^n)} \\ &= 1+\sum_{m,n\geq 1}\frac{(1-u)(1-v)}{1-uv}\left(u^m v^n-\frac{1}{u^m v^n}\right)q^{mn}. \end{aligned} \tag{24}$$

In particular letting $v=u$, one obtains the following:

$$\prod_{n=1}^{\infty}\frac{(1-q^n)^2(1-u^2q^n)(1-u^{-2}q^n)}{(1-uq^n)^2(1-u^{-1}q^n)^2} = 1+\sum_{m,n\geq 1}\frac{1-u}{1+u}\cdot\frac{u^{2(m+n)}-1}{u^{m+n}}\cdot q^{mn}. \tag{25}$$

Lemma 1.

1) $$\left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^2 = 1+4\sum_{\substack{j\geq 1\\ k\geq 0}}(-1)^k q^{j(2k+1)},$$

2) $$\left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^4 = 1+8\sum_{m,n\geq 1}(-1)^{(m-1)(n-1)} m q^{mn}.$$

Proof. To prove 1), we let $u=i$ in (25). Then

$$\begin{aligned}\text{left side of (25)} &= \prod_{n=1}^{\infty}\frac{(1-q^n)^2(1+q^n)^2}{(1+q^{2n})^2} = \left(\prod_{n=1}^{\infty}\frac{1-q^{2n}}{1+q^{2n}}\right)^2 \\ &= \left(\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2}{1-q^{4n}}\right)^2 = \left(\frac{\varphi(q^2)^2}{\varphi(q^4)}\right)^2,\end{aligned}$$

and

$$\text{right side of (25)} = 1+\frac{1+i}{1-i}\sum_{m,n\geq 1}\frac{(-1)^{m+n}-1}{i^{m+n}}q^{mn}$$

$$= 1 - i \sum_{\substack{m,n\geq 1 \\ \text{s.t.} \\ m+n=\text{odd}}} \frac{-2}{i^{m+n}} q^{mn} = 1 - 4 \sum_{\substack{m,n\geq 1 \\ m=\text{even} \\ n=\text{odd}}} \frac{1}{i^{m+n+1}} q^{mn}.$$

Putting $m = 2j$ and $n = 2k - 1$, this is rewritten as

$$= 1 - 4 \sum_{j,k\geq 1} \frac{1}{i^{2(j+k)}} \cdot q^{2j(2k-1)} = 1 - 4 \sum_{j,k\geq 1} (-1)^{j+k} q^{2j(2k-1)}.$$

So, putting $q^2 = x$, one has

$$\left(\frac{\varphi(x)^2}{\varphi(x^2)}\right)^2 = 1 - 4 \sum_{j,k\geq 1} (-1)^{j+k} x^{j(2k-1)},$$

namely

$$\left(\sum_{n\in\mathbb{Z}} (-1)^n x^{n^2}\right)^2 = 1 - 4 \sum_{j,k\geq 1} (-1)^{j+k} x^{j(2k-1)} = 1 + 4 \sum_{\substack{j\geq 1 \\ k\geq 0}} (-1)^{j+k} x^{j(2k+1)}.$$

Now letting $x = -q$ proves the formula in 1).

To show 2), we let $u \longrightarrow -1$ in (25). Then the left side becomes

$$\text{left side of (25)} = \prod_{n=1}^{\infty} \frac{(1-q^n)^4}{(1+q^n)^4} = \left(\prod_{n=1}^{\infty} \frac{(1-q^n)^2}{1-q^{2n}}\right)^4 = \left(\frac{\varphi(q)^2}{\varphi(q^2)}\right)^4,$$

and the right side is calculated, using $\displaystyle\lim_{u\to -1} \frac{u^{2(m+n)} - 1}{u+1} = -2(m+n)$, as follows:

$$\begin{aligned} \text{right side of (25)} &= 1 + 4 \sum_{m,n\geq 1} (-1)^{m+n+1} (m+n) q^{mn} \\ &= 1 + 8 \sum_{m,n\geq 1} (-1)^{m+n+1} m q^{mn}. \end{aligned}$$

Then (25) gives

$$\left(\sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}\right)^4 = 1 + 8 \sum_{m,n\geq 1} (-1)^{m+n+1} m q^{mn}.$$

Now replacing q by $-q$ proves 2). □

The formula 1) of this lemma, by putting $2k+1=d$, gives the famous formula of Jacobi [3]:

$$\Box_2(n) = 4 \sum_{\substack{d\in\mathbb{N}_{\text{odd}} \\ d|n}} (-1)^{\frac{d-1}{2}} \quad \text{for } n \in \mathbb{N} = \mathbb{Z}_{>0}. \tag{26}$$

As to the right side of 2), we notice the following :

Lemma 2. *Let N be a positive integer. Then*

$$\sum_{\substack{(m,n)\in\mathbb{N}\times\mathbb{N} \\ \text{s.t. } mn=N}} (-1)^{(m-1)(n-1)} m = \sum_{\substack{d\in\mathbb{N} \\ \text{s.t. } 4\nmid d|N}} d. \tag{27}$$

Proof. To prove this lemma, we look at the left side of (27). Writing N as $N=2^kM$ where M is an odd integer, all of the divisors of N are

$$\{2^r a \ ; \ 0 \le r \le k, \text{ and } a|M\}.$$

Let (m,n) be a pair of positive integers satisfying $mn=N$. Then one can rewrite m and n as follows:

$$m = 2^r a \quad \text{and} \quad n = 2^s b, \qquad \text{where } r+s=k \text{ and } ab=M.$$

So the left side of (27) can be rewritten as follows:

$$\text{left side of (27)} = \sum_{\substack{0\le r,s\le k \\ \text{s.t.} \\ r+s=k}} \sum_{\substack{a,b|M \\ \text{s.t.} \\ ab=M}} (-1)^{(2^r a-1)(2^s b-1)} \cdot 2^r a. \tag{28}$$

Here we notice that

$$(-1)^{(2^r a-1)(2^s b-1)} = \begin{cases} 1 & \text{if } r=0 \text{ or } s=0 \quad (i.e., \ r=0 \text{ or } r=k), \\ -1 & \text{if } r,s \ge 1 \quad (i.e., \ 0<r<k), \end{cases} \tag{29}$$

since both a and b in (28) are odd.

In the case when $k=0$ or 1, one sees, by (29), that $(-1)^{(2^r a-1)(2^s b-1)}=1$ and so the right side of (28) is equal to

$$\sum_{0\le r\le k} \sum_{a|M} 2^r a = \begin{cases} \sum_{a|M} a & \text{if } k=0 \ (i.e., \ N=M : \text{odd}) \\ \sum_{a|M} a + \sum_{a|M} 2a & \text{if } k=1 \ (i.e., \ N=2M) \end{cases} = \sum_{d|N} d,$$

hence (27) holds. So we have only to consider the case when $k \geq 2$. In this case, the right side of (28) is the sum of the following three components:

$$\text{right side of (28)} = (\text{term } r=0) + (\text{terms for } 0<r<k) + (\text{term } r=k).$$

Then, using (29), this becomes as follows:

$$\begin{aligned}\text{right side of (28)} &= \sum_{a|M} a - \sum_{r=1}^{k-1}\sum_{a|M} 2^r a + \sum_{a|M} 2^k a \\ &= \sum_{a|M} a - (2^k-2)\sum_{a|M} a + \sum_{a|M} 2^k a \ = \ \sum_{a|M} a + \sum_{a|M} 2a,\end{aligned}$$

which is equal to $\displaystyle\sum_{4\nmid d|N} d$ because the set $\{a,\, 2a\ ;\ a|M\}$ is just the set of divisors d of N satisfying $4 \nmid d$. Thus the lemma is proved. □

From Lemma 1.2) and Lemma 2, we arrive at the beautiful formula of Jacobi [3]:

$$\square_4(N) = 8\sum_{4\nmid d|N} d \qquad \text{if } N \text{ is a positive integer.} \tag{30}$$

Turning back to the $\widehat{\mathfrak{sl}}(2|1)$-identity (23), we let $u = \xi q^{\frac{1}{2}}$ and $v = \eta q^{\frac{1}{2}}$ in (23). Then we have

$$\begin{aligned}&\prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-\xi\eta q^n)(1-\xi^{-1}\eta^{-1}q^{n-1})}{(1-\xi q^{n-\frac{1}{2}})(1-\xi^{-1}q^{n-\frac{1}{2}})(1-\eta q^{n-\frac{1}{2}})(1-\eta^{-1}q^{n-\frac{1}{2}})} \\ &= \Big(\sum_{m,n\geq 0} - \sum_{m,n<0}\Big)\xi^m\eta^n q^{mn+\frac{m+n}{2}} \\ &= q^{-\frac{1}{4}}\Big(\sum_{m,n\geq 0} - \sum_{m,n<0}\Big)\xi^m\eta^n q^{(m+\frac{1}{2})(n+\frac{1}{2})}.\end{aligned} \tag{31}$$

We rewrite the second term of the right side in the above by putting $m = -(m'+1)$ and $n = -(n'+1)$. Then, since

$$\sum_{m,n<0} \xi^m\eta^n q^{(m+\frac{1}{2})(n+\frac{1}{2})} = \sum_{m',n'\geq 0} \xi^{-m'-1}\eta^{-n'-1}q^{(m'+\frac{1}{2})(n'+\frac{1}{2})},$$

the above formula (31) is rewritten as follows :

$$\prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-\xi\eta q^n)(1-\xi^{-1}\eta^{-1}q^{n-1})}{(1-\xi q^{n-\frac{1}{2}})(1-\xi^{-1}q^{n-\frac{1}{2}})(1-\eta q^{n-\frac{1}{2}})(1-\eta^{-1}q^{n-\frac{1}{2}})}$$

$$= q^{-\frac{1}{4}} \sum_{m,n\geq 0} \left(\xi^m \eta^n - \frac{1}{\xi^{m+1}\eta^{n+1}} \right) q^{(m+\frac{1}{2})(n+\frac{1}{2})}$$
$$= \sum_{m,n\geq 0} \frac{\xi^{2m+1}\eta^{2n+1} - 1}{\xi^{m+1}\eta^{n+1}} \cdot q^{\frac{(2m+1)(2n+1)-1}{4}}.$$

Dividing both sides by $1 - \xi^{-1}\eta^{-1}$, this formula is written as follows :

$$\prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-\xi\eta q^n)(1-\xi^{-1}\eta^{-1}q^n)}{(1-\xi q^{n-\frac{1}{2}})(1-\xi^{-1}q^{n-\frac{1}{2}})(1-\eta q^{n-\frac{1}{2}})(1-\eta^{-1}q^{n-\frac{1}{2}})}$$
$$= \sum_{m,n\geq 0} \frac{\xi^{2m+1}\eta^{2n+1} - 1}{(\xi\eta - 1)\xi^m\eta^n} \cdot q^{\frac{(2m+1)(2n+1)-1}{4}}. \tag{32}$$

Letting $\xi = \eta = 1$ in (32) and then replacing q by q^2, one has

$$\left(\prod_{n=1}^{\infty} \frac{1-q^{2n}}{1-q^{2n-1}} \right)^4 = \sum_{m,n\geq 0} (2m+1) q^{\frac{(2m+1)(2n+1)-1}{2}}. \tag{33}$$

And also letting $\xi = 1$ and $\eta = -1$ in (32) gives

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^{2n-1})^2} = \sum_{m,n\geq 0} (-1)^n q^{\frac{(2m+1)(2n+1)-1}{4}}. \tag{34}$$

From these formulas (33) and (34), one obtains

$$\Delta_4(N) = \sum_{\substack{k\in\mathbb{N} \\ k|(2N+1)}} k \quad \text{and} \quad \Delta_2(N) = \sum_{\substack{k\in\mathbb{N} \\ k|(4N+1)}} (-1)^{\frac{k-1}{2}},$$

for $N \in \mathbb{Z}_{\geq 0}$. It will be interesting to compare these formulas with (26) and (30).

We now look at one more example $\widehat{\mathfrak{osp}}(3|2)$. In this case, the data which we need to compute the denominator identity are as follows :

(i) root system of $\mathfrak{osp}(3|2)$: $\begin{cases} \overline{\Delta}^+_{\text{odd}} = \{\alpha_1,\ \alpha_1+\alpha_2,\ \alpha_1+2\alpha_2\}, \\ \overline{\Delta}^+_{\text{even}} = \{\alpha_2,\ 2(\alpha_1+\alpha_2)\}. \end{cases}$

(ii) positive root system of $\widehat{\mathfrak{osp}}(3|2)$:
$\{n\delta,\ n\delta - \alpha,\ (n-1)\delta + \alpha\ ;\ \alpha \in \overline{\Delta}^+, n \geq 1\}$.

(iii) multiplicity of roots :
$\text{mult}(m\delta) = 2, \quad \text{mult}(n\delta + \alpha) = 1 \quad \text{for all} \quad \alpha \in \overline{\Delta} \quad \text{and} \quad m, n.$

(iv) inner product in $\bar{\mathfrak{h}}^*$: $\begin{pmatrix} (\alpha_1|\alpha_1) & (\alpha_1|\alpha_2) \\ (\alpha_2|\alpha_1) & (\alpha_2|\alpha_2) \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$

(v) Weyl vector ρ : $(\rho|\alpha_1) = 0$, $(\rho|\alpha_2) = \frac{1}{4}$ and $(\rho|\delta) = -\frac{1}{2}$.

(vi) Weyl group : $W = \{t_{2n(\alpha_1+\alpha_2)},\ t_{2n(\alpha_1+\alpha_2)}r_{2(\alpha_1+\alpha_2)}\ ;\ n \in \mathbb{Z}\}$.

These data are obtained from easy calculation as shown below. From the matrix expression

$$\mathfrak{osp}(3|2) = \left\{ \left(\begin{array}{ccc|cc} a & 0 & u & x_1 & x_2 \\ 0 & -a & v & y_1 & y_2 \\ -v & -u & 0 & z_1 & z_2 \\ \hline y_2 & x_2 & z_2 & d & e \\ -y_1 & -x_1 & -z_1 & f & -d \end{array} \right) \right\},$$

one sees that $\mathfrak{osp}(3|2)$ is a 12-dimensional Lie superalgebra with a basis $E_{1,1}-E_{2,2}$ and $E_{4,4}-E_{5,5}$ and the following elements :

$$\begin{array}{lllll} E_{1,4}-E_{5,2}, & E_{2,4}-E_{5,1}, & E_{3,4}-E_{5,3}, & E_{1,3}-E_{3,2}, & E_{2,3}-E_{3,1}, \\ E_{1,5}-E_{4,2}, & E_{2,5}-E_{4,1}, & E_{3,5}-E_{4,3}, & E_{4,5}, & E_{5,4}. \end{array} \tag{35}$$

Let

$$\bar{\mathfrak{h}} := \{\text{diagonal matrices}\} = \mathbb{C}\cdot(E_{1,1}-E_{2,2}) \oplus \mathbb{C}\cdot(E_{4,4}-E_{5,5})$$

be the Cartan subalgebra. Then the Lie superbrackets of elements in $\bar{\mathfrak{h}}$ with each element in the above basis are given as follows :

$$\begin{aligned} [a(E_{1,1}-E_{2,2})+d(E_{4,4}-E_{5,5}),\ E_{1,4}-E_{5,2}] &= (a-d)(E_{1,4}-E_{5,2}), \\ [a(E_{1,1}-E_{2,2})+d(E_{4,4}-E_{5,5}),\ E_{2,4}-E_{5,1}] &= (-a-d)(E_{2,4}-E_{5,1}), \\ [a(E_{1,1}-E_{2,2})+d(E_{4,4}-E_{5,5}),\ E_{3,4}-E_{5,3}] &= -d(E_{3,4}-E_{5,3}), \\ [a(E_{1,1}-E_{2,2})+d(E_{4,4}-E_{5,5}),\ E_{1,5}-E_{4,2}] &= (a+d)(E_{1,5}-E_{4,2}), \\ [a(E_{1,1}-E_{2,2})+d(E_{4,4}-E_{5,5}),\ E_{2,5}-E_{4,1}] &= (-a+d)(E_{2,5}-E_{4,1}), \\ [a(E_{1,1}-E_{2,2})+d(E_{4,4}-E_{5,5}),\ E_{3,5}-E_{4,3}] &= d(E_{3,5}-E_{4,3}), \\ [a(E_{1,1}-E_{2,2})+d(E_{4,4}-E_{5,5}),\ E_{1,3}-E_{3,2}] &= a(E_{1,3}-E_{3,2}), \\ [a(E_{1,1}-E_{2,2})+d(E_{4,4}-E_{5,5}),\ E_{2,3}-E_{3,1}] &= -a(E_{2,3}-E_{3,1}), \\ [a(E_{1,1}-E_{2,2})+d(E_{4,4}-E_{5,5}),\ E_{4,5}] &= 2d\cdot E_{4,5}, \\ [a(E_{1,1}-E_{2,2})+d(E_{4,4}-E_{5,5}),\ E_{5,4}] &= -2d\cdot E_{5,4}. \end{aligned}$$

This table shows that the elements in (35) are eigenvectors of $\mathrm{ad}\bar{\mathfrak{h}}$, namely they are root vectors, and that the set of all roots is $\overline{\Delta} = \overline{\Delta}^+ \cup (-\overline{\Delta}^+)$, where

$$\overline{\Delta}^+ = \{\alpha_1,\ \alpha_2,\ \alpha_1+\alpha_2,\ \alpha_1+2\alpha_2,\ 2(\alpha_1+\alpha_2)\}$$

and α_1 and α_2 are linear forms on $\bar{\mathfrak{h}}$ defined by

$$\begin{aligned} \alpha_1(a(E_{1,1}-E_{2,2})+d(E_{4,4}-E_{5,5})) &:= d-a, \\ \alpha_2(a(E_{1,1}-E_{2,2})+d(E_{4,4}-E_{5,5})) &:= a. \end{aligned} \tag{36}$$

Notice that α_1 is an odd simple root and α_2 is an even simple root, and that the corresponding elements H_{α_i} in $\bar{\mathfrak{h}}$ are given as follows :

$$H_{\alpha_1} = \frac{1}{2}\left(\begin{array}{ccc|cc} -1 & & & & \\ & 1 & & & \\ & & 0 & & \\ \hline & & & -1 & \\ & & & & 1 \end{array}\right) = -\frac{1}{2}(E_{1,1} - E_{2,2}) - \frac{1}{2}(E_{4,4} - E_{5,5}),$$

$$H_{\alpha_2} = \frac{1}{2}\left(\begin{array}{ccc|cc} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ \hline & & & 0 & \\ & & & & 0 \end{array}\right) = \frac{1}{2}(E_{1,1} - E_{2,2}), \tag{37}$$

since

$$\begin{aligned} \mathrm{str}(H_{\alpha_1}H) &= \frac{1}{2}\cdot\mathrm{str}\left(\left(\begin{array}{ccc|cc} -1 & & & & \\ & 1 & & & \\ & & 0 & & \\ \hline & & & -1 & \\ & & & & 1 \end{array}\right)\left(\begin{array}{ccc|cc} a & & & & \\ & -a & & & \\ & & 0 & & \\ \hline & & & d & \\ & & & & -d \end{array}\right)\right) \\ &= \frac{1}{2}\cdot\mathrm{str}\left(\begin{array}{ccc|cc} -a & & & & \\ & -a & & & \\ & & 0 & & \\ \hline & & & -d & \\ & & & & -d \end{array}\right) = -a + d = \alpha_1(H) \end{aligned}$$

and

$$\begin{aligned} \mathrm{str}(H_{\alpha_2}H) &= \frac{1}{2}\cdot\mathrm{str}\left(\left(\begin{array}{ccc|cc} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ \hline & & & 0 & \\ & & & & 0 \end{array}\right)\left(\begin{array}{ccc|cc} a & & & & \\ & -a & & & \\ & & 0 & & \\ \hline & & & d & \\ & & & & -d \end{array}\right)\right) \\ &= \frac{1}{2}\cdot\mathrm{str}\left(\begin{array}{ccc|cc} a & & & & \\ & a & & & \\ & & 0 & & \\ \hline & & & 0 & \\ & & & & 0 \end{array}\right) = a = \alpha_2(H) \end{aligned}$$

hold for all elements $H = a(E_{1,1} - E_{2,2}) + d(E_{4,4} - E_{5,5}) \in \bar{\mathfrak{h}}$. Then the inner products of simple roots are computed, using (36) and (37), as follows :

$$\begin{aligned}
(\alpha_1|\alpha_1) &= \alpha_1(H_{\alpha_1}) = \left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right) = 0,\\
(\alpha_1|\alpha_2) &= \alpha_1(H_{\alpha_2}) = 0 - \frac{1}{2} = -\frac{1}{2},\\
(\alpha_2|\alpha_2) &= \alpha_2(H_{\alpha_2}) = \frac{1}{2}.
\end{aligned}$$

Then, by a similar calculation as in $\widehat{\mathfrak{sl}}(2|1)$ using the condition (ii) of the definition of ρ, one has $(\rho|\delta) = -\frac{1}{2} < 0$. So, in this case, positive even roots of *negative* square length take a role in the Weyl group W. Since positive even roots are α_2 and $2(\alpha_1+\alpha_2)$ and their square lengths are $|\alpha_2|^2 = \frac{1}{2}$ and $|2(\alpha_1+\alpha_2)|^2 = 4(\alpha_1+\alpha_2|\alpha_1+\alpha_2) = -2$, one sees that $2(\alpha_1+\alpha_2)$ is the only positive even root with negative square length and the corresponding coroot is

$$(2(\alpha_1+\alpha_2))^\vee = \frac{2\cdot 2(\alpha_1+\alpha_2)}{|2(\alpha_1+\alpha_2)|^2} = \frac{4(\alpha_1+\alpha_2)}{-2} = -2(\alpha_1+\alpha_2).$$

All of the above data are thus obtained.

In this case, the denominator identity is

$$e^\rho R = \sum_{w\in W} \varepsilon(w) w \left(\frac{e^\rho}{1+e^{-\alpha_1}}\right) \tag{38}$$

since $(\rho|\alpha_1) = 0$ (cf. [7]). Then, using these data, one can easily compute both side of (38) to obtain the following :

$$\begin{aligned}
&\prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-vq^{n-1})(1-v^{-1}q^n)(1-(uv)^2q^{n-1})(1-(uv)^{-2}q^n)}{(1+uq^{n-1})(1+u^{-1}q^n)(1+uvq^{n-1})(1+(uv)^{-1}q^n)(1+uv^2q^{n-1})(1+u^{-1}v^{-2}q^n)}\\
&= \Bigg\{ \sum_{\substack{m,n\geq 0\\ \text{s.t.}\\ m\equiv n+1 \bmod 2}} - \sum_{\substack{m,n<0\\ \text{s.t.}\\ m\equiv n+1 \bmod 2}} \Bigg\} (-1)^{\frac{m-n+1}{2}} u^{\frac{m+n-1}{2}} v^m q^{\frac{mn}{2}},
\end{aligned} \tag{39}$$

where $u := e^{-\alpha_1}$, $v := e^{-\alpha_2}$ and $q := e^{-\delta}$. We rewrite the right side of (39) as follows. First decompose the first term in the right side into the sum of three components :

$$\sum_{\substack{m,n\geq 0\\ m+n=\text{odd}}} = \sum_{\substack{m=0\\ n\in\mathbb{N}_{\text{odd}}}} + \sum_{\substack{n=0\\ m\in\mathbb{N}_{\text{odd}}}} + \sum_{\substack{m,n>0\\ m+n=\text{odd}}}.$$

Then we have

$$\sum_{\substack{m,n\geq 0\\ m+n=\text{odd}}} (-1)^{\frac{m-n+1}{2}} u^{\frac{m+n-1}{2}} v^m q^{\frac{mn}{2}}$$

$$= \frac{1}{1+u} - \frac{v}{1+uv^2} + \sum_{\substack{m,n>0 \\ m+n=\mathrm{odd}}} (-1)^{\frac{m-n+1}{2}} u^{\frac{m+n-1}{2}} v^m q^{\frac{mn}{2}}$$
$$= \frac{(1-v)(1-uv)}{(1+u)(1+uv^2)} + \sum_{\substack{m,n>0 \\ m+n=\mathrm{odd}}} (-1)^{\frac{m-n+1}{2}} u^{\frac{m+n-1}{2}} v^m q^{\frac{mn}{2}},$$

and so the right side of (39) becomes as follows :

$$\begin{aligned}
\text{right side of (39)} \; &= \frac{(1-v)(1-uv)}{(1+u)(1+uv^2)} \\
&+ \sum_{\substack{m,n>0 \\ m+n=\mathrm{odd}}} \left\{ (-1)^{\frac{m-n+1}{2}} u^{\frac{m+n-1}{2}} v^m - (-1)^{\frac{-m+n+1}{2}} u^{\frac{-m-n-1}{2}} v^{-m} \right\} q^{\frac{mn}{2}} \\
&= \frac{(1-v)(1-uv)}{(1+u)(1+uv^2)} + \left\{ \sum_{\substack{m\in\mathbb{N}_{\mathrm{even}} \\ n\in\mathbb{N}_{\mathrm{odd}}}} + \sum_{\substack{n\in\mathbb{N}_{\mathrm{even}} \\ m\in\mathbb{N}_{\mathrm{odd}}}} \right\} (-1)^{\frac{m-n+1}{2}} \cdot \frac{u^{m+n} v^{2m} + 1}{u^{\frac{m+n+1}{2}} v^m} \cdot q^{\frac{mn}{2}} \\
&= \frac{(1-v)(1-uv)}{(1+u)(1+uv^2)} + \sum_{\substack{m\in\mathbb{N}_{\mathrm{even}} \\ n\in\mathbb{N}_{\mathrm{odd}}}} (-1)^{\frac{m-n+1}{2}} \left\{ \frac{u^{m+n} v^{2m} + 1}{u^{\frac{m+n+1}{2}} v^m} - \frac{u^{m+n} v^{2n} + 1}{u^{\frac{m+n+1}{2}} v^n} \right\} q^{\frac{mn}{2}} \\
&= \frac{(1-v)(1-uv)}{(1+u)(1+uv^2)} + \sum_{\substack{m\in\mathbb{N}_{\mathrm{even}} \\ n\in\mathbb{N}_{\mathrm{odd}}}} (-1)^{\frac{m-n+1}{2}} \cdot \frac{(v^n - v^m)(1-(uv)^{m+n})}{u^{\frac{m+n+1}{2}} v^{m+n}} \cdot q^{\frac{mn}{2}}. \qquad (40)
\end{aligned}$$

By (40), the $\widehat{osp}(3|2)$-identity (39) is rewritten as follows :

$$\prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-vq^n)(1-v^{-1}q^n)(1-(uv)^2q^n)(1-(uv)^{-2}q^n)}{(1+uq^n)(1+u^{-1}q^n)(1+uvq^n)(1+(uv)^{-1}q^n)(1+uv^2q^n)(1+u^{-1}v^{-2}q^n)}$$
$$= 1 + \sum_{\substack{m\in\mathbb{N}_{\mathrm{even}} \\ n\in\mathbb{N}_{\mathrm{odd}}}} (-1)^{\frac{m-n+1}{2}} \cdot \frac{(1+u)(1+uv^2)(v^n-v^m)(1-(uv)^{m+n})}{u^{\frac{m+n+1}{2}} v^{m+n}(1-v)(1-uv)} \cdot q^{\frac{mn}{2}}. \qquad (41)$$

Now letting $u, v \to 1$ in (41), we obtain Jacobi's formula [3] :

$$\left(\frac{\varphi(q)^2}{\varphi(q^2)}\right)^6 = 1 + 4 \sum_{\substack{m\in\mathbb{N}_{\mathrm{even}} \\ n\in\mathbb{N}_{\mathrm{odd}}}} (-1)^{\frac{m-n+1}{2}} (m^2-n^2) q^{\frac{mn}{2}},$$

which gives

$$\square_6(N) = 4 \sum_{\substack{m\in\mathbb{N}_{\mathrm{even}} \\ n\in\mathbb{N}_{\mathrm{odd}} \\ mn=2N}} (-1)^{\frac{(m-1)(n+1)}{2}} (n^2-m^2) \qquad \text{for } N \in \mathbb{N}.$$

Furthermore putting $u=-xq^{\frac{1}{2}}$ and $v=y$ in (39), we get

$$\prod_{n=1}^{\infty}\frac{(1-q^n)^2(1-yq^{n-1})(1-y^{-1}q^n)(1-(xy)^2q^{n-1})(1-(xy)^{-2}q^n)}{(1-xq^{n-\frac{1}{2}})(1-x^{-1}q^{n-\frac{1}{2}})(1-xyq^{n-\frac{1}{2}})(1-(xy)^{-1}q^{n-\frac{1}{2}})}$$
$$\times\prod_{n=1}^{\infty}\frac{1}{(1-xy^2q^{n-\frac{1}{2}})(1-x^{-1}y^{-2}q^{n-\frac{1}{2}})}$$
$$=\Bigg\{\sum_{\substack{m,n\geq 0\\ m\equiv n+1 \mod 2}}-\sum_{\substack{m,n<0\\ m\equiv n+1 \mod 2}}\Bigg\}(-1)^{m+1}x^{\frac{m+n+3}{2}}y^{m+2}q^{\frac{2mn+m+n-1}{4}}. \tag{42}$$

Putting $m=-m'-1$ and $n=-n'-1$, the second term in this right side is written as

$$\sum_{\substack{m,n<0\\ m\equiv n+1 \mod 2}}(-1)^{m+1}x^{\frac{m+n+3}{2}}y^{m+2}q^{\frac{2mn+m+n-1}{4}}$$
$$=\sum_{\substack{m',n'\geq 0\\ m'\equiv n'+1 \mod 2}}(-1)^{m'}x^{\frac{-m'-n'+1}{2}}y^{-m'+1}q^{\frac{2m'n'+m'+n'-1}{4}}.$$

So the right side of (42) is rewritten as follows :

$$\text{right side of (42)}$$
$$=\sum_{\substack{m,n\geq 0\\ m\equiv n+1 \mod 2}}(-1)^{m+1}\left(x^{\frac{m+n+3}{2}}y^{m+2}+\frac{1}{x^{\frac{m+n-1}{2}}y^{m-1}}\right)q^{\frac{2mn+m+n-1}{4}}$$
$$=\sum_{\substack{m,n\geq 0\\ m\equiv n+1 \mod 2}}(-1)^{m+1}\cdot\frac{x^{m+n+1}y^{2m+1}+1}{x^{\frac{m+n-1}{2}}y^{m-1}}\cdot q^{\frac{2mn+m+n-1}{4}}. \tag{43}$$

We continue calculation dividing the sum in this right side into two parts where $(m,n)=(\text{even},\text{odd})$ and $(m,n)=(\text{odd},\text{even})$. Then

$$\text{right side of (43)}=-\sum_{\substack{m=\text{even}\geq 0\\ n=\text{odd}>0}}\frac{x^{m+n+1}y^{2m+1}+1}{x^{\frac{m+n-1}{2}}y^{m-1}}\cdot q^{\frac{2mn+m+n-1}{4}}$$
$$+\sum_{\substack{m=\text{odd}>0\\ n=\text{even}\geq 0}}\frac{x^{m+n+1}y^{2m+1}+1}{x^{\frac{m+n-1}{2}}y^{m-1}}\cdot q^{\frac{2mn+m+n-1}{4}}.$$

Exchanging $m\leftrightarrow n$ in this second term, the right side of (43) is written as follows:

$$\text{right side of (43)}=-\sum_{\substack{m=\text{even}\geq 0\\ n=\text{odd}>0}}\frac{x^{m+n+1}y^{2m+1}+1}{x^{\frac{m+n-1}{2}}y^{m-1}}\cdot q^{\frac{2mn+m+n-1}{4}}$$

$$+ \sum_{\substack{m=\text{even}\geq 0 \\ n=\text{odd}>0}} \frac{x^{m+n+1}y^{2n+1}+1}{x^{\frac{m+n-1}{2}}\, y^{n-1}} \cdot q^{\frac{2mn+m+n-1}{4}}$$

$$= \sum_{\substack{m=\text{even}\geq 0 \\ n=\text{odd}>0}} \frac{(y^n - y^m)((xy)^{m+n+1}-1)}{x^{\frac{m+n-1}{2}}\, y^{m+n-1}} \cdot q^{\frac{2mn+m+n-1}{4}}. \tag{44}$$

Now putting $m = 2s$ and $n = 2r+1$ in (44), formulas (42) and (43) and (44) give the following :

$$\prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-yq^{n-1})(1-y^{-1}q^n)(1-(xy)^2q^{n-1})(1-(xy)^{-2}q^n)}{(1-xq^{n-\frac{1}{2}})(1-x^{-1}q^{n-\frac{1}{2}})(1-xyq^{n-\frac{1}{2}})(1-(xy)^{-1}q^{n-\frac{1}{2}})}$$

$$\times \prod_{n=1}^{\infty} \frac{1}{(1-xy^2q^{n-\frac{1}{2}})(1-x^{-1}y^{-2}q^{n-\frac{1}{2}})}$$

$$= \sum_{r,s\geq 0} \frac{(y^{2r+1}-y^{2s})((xy)^{2(r+s+1)}-1)}{x^{r+s}y^{2(r+s)}} \cdot q^{2rs+\frac{r+3s}{2}}. \tag{45}$$

Dividing both sides of (45) by $(1-y)(1-(xy)^2)$ and letting $x, y \to 1$ and then replacing q by q^2, we obtain

$$\left(\frac{\varphi(q^2)^2}{\varphi(q)}\right)^6 = \sum_{r,s\geq 0} (r+s+1)(2r-2s+1)q^{\frac{(4r+3)(4s+1)-3}{4}}$$

$$= \frac{1}{8} \sum_{\substack{j,k\in\mathbb{N}_{\text{odd}} \\ j\equiv 3 \mod 4 \\ k\equiv 1 \mod 4}} (j^2-k^2)q^{\frac{jk-3}{4}},$$

which gives

$$\triangle_6(n) = \frac{1}{8} \sum_{\substack{j,k\in\mathbb{N}_{\text{odd}} \\ j\equiv 3 \mod 4 \\ k\equiv 1 \mod 4 \\ jk=4n+3}} (j^2-k^2) \qquad \text{for } n \in \mathbb{Z}_{\geq 0}.$$

In this note we have shown some calculations on the denominator identity of the simplest affine superalgebras $\widehat{\mathfrak{sl}}(2|1)$ and $\widehat{\mathfrak{osp}}(3|2)$. Similar analysis using other affine superalgebras gives more formulas on $\Box_k(n)$ and $\triangle_k(n)$ (cf. [7]), and functions appearing in this context related to affine Lie superalgebras have deep connections with modular functions (cf. [12]).

In concluding this brief note, we explain the relation of these denominator identities with the Ramanujan's mock theta functions. First look at the formula (17.6) in p.34 of [1] which is equivalent to the Ramanujan's famous ${}_1\psi_1$-summation formula (17.1) in p.32 of [1]. This formula reads as follows :

$$\begin{aligned}&\prod_{n=1}^{\infty}\frac{(1-azq^{n-1})(1-a^{-1}z^{-1}q^n)(1-q^n)(1-a^{-1}bq^{n-1})}{(1-zq^{n-1})(1-(az)^{-1}bq^{n-1})(1-bq^{n-1})(1-a^{-1}q^n)}\\ &= 1+\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}\frac{1-aq^{k-1}}{1-bq^{k-1}}\right)z^n+\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}\frac{1-bq^{-k}}{1-aq^{-k}}\right)z^{-n},\end{aligned} \tag{46}$$

since the q-binomial symbol in (17.6) is

$$(a)_n := (a;q)_n := \begin{cases}\displaystyle\prod_{k=0}^{n-1}(1-aq^k) & \text{if } n\geq 0,\\ \displaystyle\prod_{n\leq k<0}\frac{1}{1-aq^k} & \text{if } n<0.\end{cases}$$

In this formula, the expansion is taken in the domain $|\frac{b}{a}|<|z|<1$ and $|q|<|a|<1$. Letting $a=v$, $b=vq$ and $z=u$, the products in the second and third summands in the right side of (46) become as follows :

$$\prod_{k=1}^{n}\frac{1-aq^{k-1}}{1-bq^{k-1}} = \prod_{k=1}^{n}\frac{1-vq^{k-1}}{1-vq^k} = \frac{1-v}{1-vq^n},$$
$$\prod_{k=1}^{n}\frac{1-bq^{-k}}{1-aq^{-k}} = \prod_{k=1}^{n}\frac{1-vq^{-k+1}}{1-vq^{-k}} = \frac{1-v}{1-vq^{-n}}.$$

So the formula (46) gives

$$\prod_{n=1}^{\infty}\frac{(1-uvq^{n-1})(1-u^{-1}v^{-1}q^n)(1-q^n)^2}{(1-uq^{n-1})(1-u^{-1}q^n)(1-vq^n)(1-v^{-1}q^n)}=\sum_{n\in\mathbb{Z}}\frac{1-v}{1-vq^n}u^n,$$

namely

$$\prod_{n=1}^{\infty}\frac{(1-q^n)^2(1-uvq^{n-1})(1-u^{-1}v^{-1}q^n)}{(1-uq^{n-1})(1-u^{-1}q^n)(1-vq^{n-1})(1-v^{-1}q^n)}=\sum_{n\in\mathbb{Z}}\frac{u^n}{1-vq^n}. \tag{47}$$

Expanding this right side by using (19) just gives the formula (23).

Now looking at the Hickerson's paper [2] on mock theta identities, one sees that the formula (1.29) in Theorem 1.5 of [2] is just the $\widehat{s\ell}(2|1)$-identity (23) and also, via an easy calculation, that both formulas (1.30) and (1.32) in p.646 - 647 of [2], which are also deduced from the ${}_1\psi_1$-summation formula, are equivalent to each other and exactly the same with the denominator identity (39) of $\widehat{\mathfrak{osp}}(3|2)$. Thus the denominator identities for the simplest affine Lie superalgebras $\widehat{\mathfrak{sl}}(2|1)$ and $\widehat{\mathfrak{osp}}(3|2)$ are Ramanujan's mock theta functions. In this sense, denominator identities of affine Lie superalgebras provide a general class of mock theta functions.

The denominator identity is the special case of the character formula, namely is the character formula for the *trivial* representation and, in this note, we explained one of its related topics. The representation theory of Lie superalgebras has quite different aspects from that of Lie algebras, and includes a lot of problems to be investigated.

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