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A picture is worth a thousand words: topological graph theory

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Abstract

Topological graph theory concerns geometric representations of graphs. In this paper we give a gentle introduction to the area and survey some of its results and problems.

1. Introduction

A *graph* is a (finite) set V of *vertices* together with a set E of *edges*, where each edge is an unordered pair of vertices. Graphs are sometimes described by an *adjacency matrix*: a $|V| \times |V|$ symmetric matrix whose rows and columns are indexed by elements of V, with a 1 in row i column j when $\{i, j\} \in E$ and a 0 otherwise. For example, Figure 1 gives a graph with 10 vertices and 15 edges (each edge gives rise to two 1's in the matrix).

Graphs are used to model symmetric relations, hence they have many applications. The adjacency matrix is useful for computer analyses of these relations: computers know 0's and 1's! But we are carbon-based, not silicon-based, life-forms. Humans can understand pictures much more easily. Figure 2 gives a picture of the graph represented by the adjacency matrix. In this picture the vertices are represented by points, and an edge is represented by a (possibly curved) line-segment joining its two incident vertices.

$$
\begin{array}{c}
\\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \\ j
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc} a & b & c & d & e & f & g & h & i & j \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}
\end{array}
$$

Figure 1: An adjacency matrix of a graph

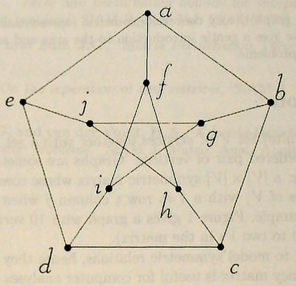

Figure 2: A drawing of a graph

Do you know this graph? It's one of the most famous in graph theory: the *Petersen graph.* It adorns the cover of the *Journal of Graph Theory.* The picture reveals some of its beautiful structure.

Topological graph theory deals with the pictorial representation of graphs. Historically graphs have been geometric objects, rather than abstract relations. The names vertices and edges come from considering the 0- and 1-dimensional structures of polyhedra, see Figure 4. For example, a tetrahedron has 4 vertices, and each pair of vertices are joined by an edge. More strongly, each triple of vertices represents a face of the tetrahedron.

Many applications of graphs represent this geometric flavor. For example, the vertices could represent airports, with the edges being flights between them. More recently, graphs have been used to represent computer processors and communication channels between them. Graphs may have started in the "slums of topology", but they have risen to heights unimagined by their originators.

In this article we give the reader a taste of problems and results in topological graph theory. In Section 2 we study which graphs can be drawn in the plane without edge crossings, and which graphs arise from convex polyhedra. In Section 3 we study how to draw graphs with the minimum number of crossings. In Section 4 we discuss the most famous result in topological graph theory: the Four-Color Theorem. In Section 5 we examine maps on other surfaces, like the torus. We close in Section 6 with some pointers to the literature for those interested in learning more.

2 Planar graphs

In our drawing of the Petersen graph each of the edges on the inside pentagram cross two other edges. These crossing points do not represent vertices of the graph: the edge that cross have no vertices in common. This can lead to confusion in reading off the vertices and edges. *Can the Petersen graph be drawn in the plane so that no two edges cross?* More generally, *Which graphs can be drawn in the plane without edge-crossings?* This is one of the most fundamental topics in topological graph theory, which we will now answer.

First we need some definitions. A *complete graph* on n vertices, K_n, has an edge between every pair of vertices. A *complete bipartite graph*, $K_{n,m}$, has its $n+m$ vertices divided into two sets of size n and m respectively, with an edge between every vertex in the first part and every vertex in the second part. For example, Figure 3 shows the complete graph K_5 and the complete bipartite graph $K_{3,3}$.

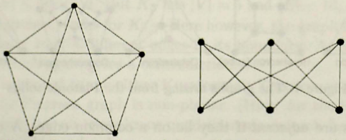

Figure 3: A complete and a complete bipartite graph

Try as one might, you cannot draw either of these two graphs in the plane without edge crossings (we also exclude an edge $\{a, b\}$ crossing through a third vertex c). We will prove this shortly after introducing Euler's Formula.

Consider the graph $S(K_5)$ formed from K_5 by deleting an edge $\{a, b\}$, adding a new vertex c, and adding new edges $\{a, c\}$ and $\{c, b\}$. You also cannot draw $S(K_5)$ in the plane without edge crossings: any such drawing could easily be modified to a planar drawing of K_5 (remember, edges need not be represented by straight-line segments). We call $S(K_5)$ an *elementary subdivision* of K_5. More generally, H is a *subdivision* of G if it is formed by a sequence of elementary subdivisions. Two graphs are *homeomorphic* if they have a common subdivision. Loosely speaking they are equivalent up to vertices of degree 2. More formally, they are homeomorphic as topological spaces.

By the preceeding comments, any graph homeomorphic to K_5 or to $K_{3,3}$ cannot be planar. But we can say more. Call H a *subgraph* of G if you can form H from G by deleting some vertices and edges, subject to the restriction that when you delete a vertex you must also delete all edges on that vertex. If G has a planar drawing without crossings, then so does any subgraph H: we merely delete the vertices and edges from the drawing.

Combining these ideas, any graph that contains a subgraph homeomorphic to K_5 or to $K_{3,3}$ has no planar drawing. The following remarkable theorem [10] shows that this necessary condition is also sufficient.

Theorem 2.1 (Kuratowski) *A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or to $K_{3,3}$.*

Popular examples of graphs come from the vertices and edges of polyhedra, see Figure 4. We ask *"What other examples of planar graphs come from convex polyhedra?"* These graphs must be planar: any graph that embeds on the sphere also embeds on the sphere with a point deleted, and this is topologically equivalent to the plane. We need some additional definitions to state another necessary condition.

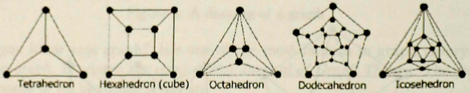

Figure 4: The graphs arising from the Platonic solids

Two vertices are *adjacent* if they lie on a common edge. A *component* of a graph is the transitive closure of the relation on vertices induced by adjacency.

Thinking geometrically, a component is all parts of the graph that can be reached by walking along the edges from a fixed vertex. The graph is *connected* if it has only one component. A *cut-set* in a graph G is a collection of vertices S such that the subgraph $G - S$ formed by deleting the vertices in S has more components than G does. Finally, a graph is *k-connected* if it has at least $k+1$ vertices, is connected, and has no cut-sets of size strictly less than k.

When G arises from the vertices and edges of a convex polyhedron, it can be shown that G is 3-connected. Steinitz showed in (1922) that these necessary conditions are sufficient.

Theorem 2.2 (Steinitz) *A graph is the vertices and edges of a convex polyhedron if and only if it is planar and 3-connected.*

Polyhedra are an important part of classical mathematics and have been studied for thousands of years. Surprisingly, it wasn't until 1750 that a remarkable numerical relationship for polyhedra was discovered by Euler. By Theorem 2.2 this is also a relationship for planar graphs. Let $|V|$ and $|E|$ denote the number of vertices and edges in a planar graph. The graph divides the plane into different regions called the *faces* of the embedding (we include the outside region as a face). If the graph is 3-connected, these correspond to the faces of the polyhedron. Let $|F|$ denote the number of faces.

Theorem 2.3 (Euler) *If G is connected and planar, then*

$$|V| - |E| + |F| = 2.$$

We now give the promised proof that the "Kuratowski graphs" are non-planar.

Theorem 2.4 *K_5 and $K_{3,3}$ are non-planar.*

Proof: Suppose by way of contradiction that K_5 was planar. We will count the number of pairs (e, f) such that the edge e lies in the boundary of the face f. First, each edge lies in the boundary of 2 separate faces, so the number of pairs is $2|E|$. Second, each face has at least 3 different edges in its boundary, so the number of pairs is at least $3|F|$. Hence, $2|E| \geq 3|F|$. Combining this with Euler's formula gives $|E| \leq 3|V| - 6$. But K_5 has $|V| = 5$ and $|E| = 10$, a contradiction.

A similar argument works for $K_{3,3}$. Here however, the graph has no triangles, so each face has at least 4 different edges in its boundary. As above, this gives $|E| \leq 2|V| - 4$ which again gives a contradiction. ■

The reader is invited to use either Kuratowski's Theorem or Euler's Formula to prove that the Petersen graph is non-planar. (Hint: for the latter, note that the Petersen graph has no triangles or quadrilaterals, so each face has at least 5 different edges in its boundary).

3 Drawings and crossing numbers

We have discovered that not all graphs are planar. Suppose that you did want to connect 5 processors on a circuit board so that there was a communication line between every pair of processors. You couldn't do it without these lines crossing. You would look, then, for the layout that had the fewest number of crossings.

More formally, a *drawing* of a graph in the plane is a representation of the vertices by distinct points, and the edges by curves that only contain the vertices at their ends, with the requirement that curves representing two distinct edges intersect in a finite number of points. For convenience, we also require that no 3 distinct edges cross at a single point. The minimum number of crossings over all possible drawings of G is called the *crossing number*, $cr(G)$. The requirement that no 3 edges cross at a point can be modified by counting this point as 3 separate crossings.

A graph is planar if and only if it has crossing number 0. Using Kuratowski's Theorem, a graph has crossing number at least 1 if and only if it contains one of the two Kuratowski graphs. Is there a similar theorem for graphs with crossing number at least k? Equivalently, can you find all graphs with crossing number at least k, but every proper subgraph has crossing number less that k? The answer is unknown even for $k = 2$. It is known that for every k there are infinitely many such graphs.

Another natural question would be to find the crossing numbers of a nice class of graphs. *What is the crossing number of K_n?* Surprisingly, this too is not known.

Conjeture 3.1 *The crossing number of K_n is*

$$cr(K_n) = \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor.$$

The "greatest integer" (floor) function is included only to unify the separate cases of n even or odd. The conjecture is known to be true for $n \leq 12$. It is also known that the quantity above is an upper bound on the crossing number. This is shown by demonstrating a drawing with that number of crossings.

The *rectilinear crossing number*, $\bar{cr}(G)$, is the minimum number of crossings over all drawings with the added requirement that the edges are represented by straight-line segments. It is known that $\bar{cr}(G) > cr(G)$ for $n = 8$ and $n \geq 10$. For example, the rectilinear crossing number of K_{10} was just shown to be 62, versus the known value of 60 for the crossing number.

The next most natural class of graphs are the complete bipartite graphs. Determining their crossing number is sometimes known as "Turan's brickyard problem": there are n kilns each to be connected to m warehouses by railroad tracks, and the number of track crossings is to be minimized to prevent derailings.

Once more, the exact bound is not known except for some small cases.

Conjeture 3.2 *The crossing number of $K_{n,m}$, $n \leq m$, is*

$$cr(K_{n,m}) = \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor.$$

The bound is exact for $n \leq 6$, and for $n = 7$, $m \leq 10$. In contrast to the complete graph, the rectilinear crossing of $K_{n,m}$ is conjectured to be equal to its crossing number.

The perverse graph drawer may wish to have many crossings instead of few. *"What is the maximum possible number of crossings in a drawing?"* To avoid trivial crossings, we require that two edges have at most one crossing (so the number of crossings is bounded) and that two edges on a common vertex do not cross. This maximum crossing number is known for complete and complete bipartite graphs.

The extremal case would be a drawing where every pair of edges that do not share a vertex cross exactly once. Conway calls such a drawing a *thrackle*; a word he heard refers to a tangled fishing line. A quadrilateral cannot be thrackled. It is unknown which graphs can be thrackled. See [13] for a recent survey on thrackles.

4 Coloring maps

No article on topological graph theory is complete without the 4-Color Problem. Let G be a planar graph so that every edge lies on the boundary of two different faces. The goal is to color the faces of the graph with 4 colors so that anytime two faces share an edge, they get different colors. *Can this always be done regardless of the map?* This restriction is common when, say, you are coloring the countries on a political map. The different colors help to distinguish the boundaries between the countries.

The 4-Color Problem was first posed by a student, Francis Guthrie, in 1852 while coloring a map of England. The number of colors needed for such a coloring depends on the map. It is not clear at first why there should a fixed number of colors that work for all maps. Let's give an argument that 6 colors suffice.

Theorem 4.1 *We can color the faces of any planar map with 6 colors such that any two faces that share an edge receive different colors.*

Proof: First we show that, without loss of generality, each vertex lies on at least

3 edges. We assume that no vertex lies on exactly 1 edge, because that edge lies twice on the boundary of the same face and adds nothing to the difficulty of coloring. If there are vertices on exactly 2 edges, then there is a graph homeomorphic to the given one without degree 2 vertices. Any coloring of this homeomorph corresponds to a coloring of the original map.

Second, we count the number of pairs (v, e) such that the vertex v lies on the edge e. First, each edge has exactly 2 vertices, so the number of pairs is $2|E|$. Second, each vertex lies in at least 3 edges, so the number of pairs is at least $3|V|$. Hence, $3|V| \leq 2|E|$.

Third, we show that there must be a face that shares an edge with at most 5 neighboring faces. If not, then each face is at least a hexagon. Counting pairs (e, f) with e on the boundary of f as before we get. $6|F| \leq 2|E|$. Now we substitute into Euler's Formula $|V|-|E|+|F| = 2$ to get $2|E|/3-|E|+|E|/3 \geq 2$. Simplifying gives $0 \geq 2$, our desired contradiction.

Finally, we use induction on the number of faces colored. The start of the induction is easy: if there are 6 or fewer faces, color each of them with their own color. For the inductive step, let f be a face with 5 or fewer neighbors. Delete an edge between f and one of its neighbors g. The resulting map has fewer faces, so by induction it can be 6-colored. Now replace the edge and remove the color on f. Since f has at most 5 neighbors, we can always color it with a 6^{th} color unused by any neighbor. ■

A more subtle argument shows that in fact five colors suffice. One way a map could require five colors would be that there exist five different faces, each of which shares an edge with another. These five faces would need five different colors. However, such a map is impossible by an argument similar to that which shows that K_5 does not embed in the plane. This fact is sometimes confused with the statement that five colors are never necessary, however, there could possibly be more subtle reasons why they would be.

Saaty and Kainen [16] write, "One of the many surprising aspects of the four-color conjecture is that a number of the most important contributions to the subject were originally made with the belief that they were solutions." These include work by Kempe and Tait. Indeed, the 4-Color Problem motivated many of the now central problems in graph theory. For details see [4].

The problem was finally solved (yes, 4 colors do suffice!) in 1977 by Appel and Haken [1, 2]. Their proof was at first controversial, in part because of their use of long computer computations. Since then these calculations have been done independently, including a very careful check by Robertson, Sanders, Seymour, and Thomas [15].

We close this section with a variation on the 4-Color Problem. Suppose that we have two maps, say one on the earth and the other on the moon. Each country on the earth has exactly one colony on the moon. Our goal is to color the regions on both bodies so that every country receives the same color as its colony, and whenever two countries or two colonies share an edge they receive different colors.

It is not unexpected that the number of colors required goes up. There is a calculation, again based on Euler's Formula, that shows 12 colors always suffice. However, the highest number of colors known to be necessary is 9. Hutchinson [8] has an excellent article on this "Earth-Moon Coloring Problem", including an unexpected application to computer science.

5 Maps on other surfaces

Our work to date has focused on graphs in the plane or sphere. What are some other possible drawing boards? We focus on ***surfaces***, compact connected 2-manifolds. The simplest example of such a surface is the ***torus*** (see the left half of Figure 5). Informally, it resembles the boundary of a donut. More formally, consider a rectangle in the plane. Bend the rectangle so that the top edge aligns with the bottom edge, making a cylinder. Next bend the cylinder so that one rim aligns with the other rim.

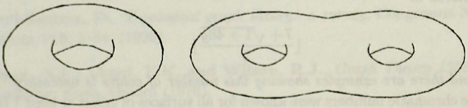

Figure 5: The torus and a genus 2 surface

Brahana has classified the surfaces [5]. There are two infinite classes, orientable and nonorientable; we won't explain the terms and consider only the former. Every orientable surface resembles a series of tori glued together. The number of torii is called the *genus* of the surface. The right half of Figure 5 shows a surface of genus 2.

Many of the questions we have considered for graphs in the plane extend to graphs on these surfaces. For example, *"How many colors suffice to color the faces of every possible map on a surface of genus g?"* For the torus, the answer is 7. That 7 colors suffice uses a generalization of Euler's formula. An example showing that 7 colors are necessary is shown in Figure 6. The torus is formed by identifying the top and bottom of the rectangle as well as the left and right sides as described above. When making these identifications the four corners combine into the single face labeled "3". Each of the 7 faces share an edge with each other, so they must all get distinct colors.

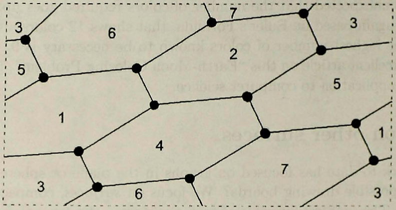

Figure 6: A map on the torus requiring seven colors

The *chromatic number* of these surfaces is known [14].

Theorem 5.1 (Ringel et al) *The faces of any map on a surface of genus $g \geq 1$ can be colored in*

$$\left\lfloor \frac{7+\sqrt{1+48g}}{2} \right\rfloor$$

colors, and there are examples showing this number of colors is necessary.

These chromatic numbers were known for all surfaces of genus at least 1 before the 4-Color-Theorem was proven. In this case, the simplest surface turned out to be the hardest!

6 Conclusion

We have hardly scratched the surface of topological graph theory. For a more extensive survey article the reader should see [3] (this also contains references for many of the results stated in this paper).

For those interested in colorings, the excellent book by Jensen and Toft [9] cannot be recommended highly enough. For more about the history and solution of the Four-Color Problem see [6] and [16]. Biggs, Lloyd, and Wilson [4] give an excellent history of the first 200 years of graph theory, including its topological origins.

There are several in-depth books for the researcher in the area. We mention in particular those of White [18], Gross and Tucker [7], and Mohar and Thomassen [11]. For the reader seeking more background, we recommend the introductory graph theory text by West [17].

For an on-going list of open problems in topological graph theory we refer the reader to ⟨http://www.emba.uvm.edu/~archdeac/problems/⟩. Some open problems about crossing numbers are also given in [12].

Topological graph theory is an exciting and growing area of research. We hope that the reader has enjoyed this small taste of its delights.

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