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Fractional calculus and its applications*

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Abstract

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained importance and popularity during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse fields of science and engineering. Indeed it provides several potentially useful tools for solving differential, integral, and integro-differential equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables. The main purpose of this expository article is to provide a rather brief introduction to the theory and applications of fractional calculus.

1. Fractional Calculus: A Brief Historical Introduction

The concept of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) seems to have stemmed from a question raised in the year 1695 by Marquis de l'Hôpital (1661-1704) to Gottfried Wilhelm Leibniz (1646-1716), which sought the meaning of Leibniz's (currently popular) notation

$$\frac{d^n y}{dx^n}$$

for the derivative of order $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ when $n = \frac{1}{2}$ (What if $n = \frac{1}{2}$?). In his reply, dated 30 September 1695, Leibniz wrote to l'Hôpital as follows:

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"... *This is an apparent paradox from which, one day, useful consequences will be drawn.* ..."

Subsequent mention of fractional derivatives was made, in some context or the other, by (for example) Euler in 1730, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Liouville in 1832, Riemann in 1847, Greer in 1859, Holmgren in 1865, Grünwald in 1867, Letnikov in 1868, Laurent in 1884, Nekrassov in 1888, Krug in 1890, and Weyl in 1917. In fact, in his 700-page textbook, entitled "*Traité du Calcul Différentiel et du Calcul Intégral*" (Second edition; Courcier, Paris, 1819), S.F. Lacroix devoted two pages (pp. 409-410) to fractional calculus, showing eventually that

$$\frac{d^{\frac{1}{2}}}{dv^{\frac{1}{2}}}v = \frac{2\sqrt{v}}{\sqrt{\pi}}.$$

In addition, of course, to the theories of differential, integral, and integro-differential equations, and special functions of mathematical physics as well as their extensions and generalizations in one and more variables, some of the areas of present-day applications of fractional calculus include

1. Fluid Flow
2. Rheology
3. Dynamical Processes in Self-Similar and Porous Structures
4. Diffusive Transport Akin to Diffusion
5. Electrical Networks
6. Probability and Statistics
7. Control Theory of Dynamical Systems
8. Viscoelasticity
9. Electrochemistry of Corrosion
10. Chemical Physics

and so on (see, for details, [4] and [17]).

The first work, devoted exclusively to the subject of fractional calculus, is the book by Oldham and Spanier [16]. One of the most recent works on the subject of fractional calculus is the book by Podlubny [17]. The latest (but certainly not the last) work on the subject of fractional calculus is the volume edited by Hilfer [4]. Indeed, in the meantime, numerous other works (books, edited volumes, and conference proceedings) have also appeared (see, *e.g.*, [5], [6], [8], [9], [10], [11], [12], [13], [18], [19], [20], [21], and [27]). And today there exist at least two international journals which are devoted almost entirely to the subject of fractional calculus: (i) *Journal of Fractional Calculus* and (ii) *Fractional Calculus and Applied Analysis*.

Here, in this expository article, we aim at presenting an introductory overview of the theory of fractional calculus and of some of its applications.

2. The Riemann-Liouville and Weyl Operators of Fractional Calculus

We begin by defining the linear integral operators $\mathcal{I}$ and $\mathcal{K}$ by

$$(\mathcal{I}f)(x) := \int_0^x f(t)\,dt \tag{2.1}$$

and

$$(\mathcal{K}f)(x) := \int_x^\infty f(t)\,dt, \tag{2.2}$$

respectively. Then it is easily seen by iteration (and mathematical induction) that

$$(\mathcal{I}^n f)(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)\,dt \quad (n \in \mathbb{N}) \tag{2.3}$$

and

$$(\mathcal{K}^n f)(x) = \frac{1}{(n-1)!}\int_x^\infty (t-x)^{n-1} f(t)\,dt \quad (n \in \mathbb{N}), \tag{2.4}$$

where, and elsewhere in this presentation,

$$\mathbb{N} := \{1,2,3,\ldots\} = \mathbb{N}_0 \setminus \{0\}.$$

With a view to interpolating $(n-1)!$ between the positive integer values of n, one can set

$$(n-1)! = \Gamma(n) \tag{2.5}$$

in terms of the familiar Gamma function. Thus, in general, Equations (2.3) and (2.4) would lead eventually to the Riemann-Liouville operator $\mathcal{R}^\mu$ and the Weyl operator $\mathcal{W}^\mu$ of fractional integral of order $\mu\ (\mu \in \mathbb{C})$, defined by (*cf.*, *e.g.*, Erdélyi *et al.* [1, Chapter 13])

$$(\mathcal{R}^{\mu} f)(x) := \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t)\, dt \quad (\Re(\mu) > 0) \tag{2.6}$$

and

$$(\mathcal{W}^{\mu} f)(x) := \frac{1}{\Gamma(\mu)} \int_x^{\infty} (t-x)^{\mu-1} f(t)\, dt \quad (\Re(\mu) > 0), \tag{2.7}$$

respectively, it being assumed that the function $f(t)$ is so constrained that the integrals in (2.6) and (2.7) exist.

There are operators of fractional derivatives $\mathcal{D}^{\mu}_{x;0}$ and $\mathcal{D}^{\mu}_{x;\infty}$ of order μ ($\mu \in \mathbb{C}$), which correspond to the fractional integral operators $\mathcal{R}^{\mu}$ and $\mathcal{W}^{\mu}$, respectively, and we have

$$\left(\mathcal{D}^{\mu}_{x;0} f\right)(x) := \frac{d^m}{dx^m} (\mathcal{R}^{m-\mu} f)(x)$$

$$(m - 1 \leqq \Re(\mu) < m; \quad m \in \mathbb{N}), \tag{2.8}$$

and

$$(\mathcal{D}^{\mu}_{x;\infty} f)(x) := \frac{d^m}{dx^m} (\mathcal{W}^{m-\mu} f)(x)$$

$$(m - 1 \leqq \Re(\mu) < m; \quad m \in \mathbb{N}), \tag{2.9}$$

There also exist, in the considerably vast literature on fractional calculus, numerous further extensions and generalizations of the operators $\mathcal{R}^{\mu}$, $\mathcal{W}^{\mu}$, $\mathcal{D}^{\mu}_{x;0}$, and $\mathcal{D}^{\mu}_{x;\infty}$, each of which we have chosen to introduce here for the sake of the non-specialists in this subject.

3. Unified Investigations of Initial-Value Problems by Using Fractional Calculus

Define, as usual, the Laplace transform operator $\mathcal{L}$ by

$$\mathcal{L}\{f(t) : s\} := \int_0^{\infty} e^{-st} f(t)\, dt =: F(s), \tag{3.1}$$

provided that the integral exists. Then, for the Riemann-Liouville fractional derivative operator $\mathcal{D}^{\mu}_{t;0}$ of order μ, we have

$$\mathcal{L}\left\{\left(\mathcal{D}_{t;0}^{\mu} f\right)(t) : s\right\} = s^{\mu} F(s) - \sum_{k=0}^{n-1} s^{k} \left(\mathcal{D}_{t;0}^{\mu-k-1}\right) f(t)\Big|_{t=0}$$

$$(n-1 \leqq \Re(\mu) < n;\quad n \in \mathbb{N}). \tag{3.2}$$

Such initial values as those occurring in (3.2) are usually *not* interpretable physically in a given initial-value problem. This situation is overcome at least partially by making use of the so-called *Caputo fractional derivative* which arose in several important works, dated 1969 onwards, by M. Caputo.

In many recent works, especially in the theory of viscoelasticity and in hereditary solid mechanics, the following (*Caputo's*) definition is adopted for the fractional derivative of order $\alpha > 0$ of a *causal* function $f(t)$ (*i.e.*, $f(t) = 0$ for $t < 0$):

$$\frac{d^{\alpha}}{dt^{\alpha}} f(t) := \begin{cases} f^{(n)}(t) & \text{if } \alpha = n \in \mathbb{N}_0, \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau & \text{if } n-1 < \alpha < n \quad (n \in \mathbb{N}), \end{cases} \tag{3.3}$$

where $f^{(n)}(t)$ denotes the usual (ordinary) derivative of order n and Γ is the Gamma function occurring already in (2.5), (2.6), and (2.7). We apply the above notion in order to generalize some basic topics of classical mathematical physics, which are treated by simple, linear, ordinary or partial, differential equations.

First of all, it follows easily from (3.1) and (3.3) that [*cf.* Equation (3.2) *and* Definition (3.3)]

$$\mathcal{L}\left\{\frac{d^{\alpha}}{dt^{\alpha}} f(t) : s\right\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0+), \tag{3.4}$$

which obviously is *more* suited for initial-value problems than (3.2).

The basic processes of relaxation, diffusion, oscillations, and wave propagation have been generalized by several authors by introducing fractional derivatives in the governing (ordinary or partial) differential equations. This leads to *superslow* or *intermediate processes* that, in mathematical physics, we may refer to as *fractional phenomena*. Our analysis of these phenomena, carried out by means of fractional calculus and Laplace transforms, leads to certain special functions in one variable of Mittag-Leffler and Wright types. These useful special functions are investigated systematically as relevant cases of the general class of functions which are popularly known as Fox's H-function after Charles Fox (1897-1977) who initiated a detailed study of these functions as symmetrical Fourier kernels (see, for details, Srivastava *et al.* [24]).

We choose to summarize below some recent investigations by Gorenflo *et al.* [2] who did indeed make references to numerous earlier related works on this subject.

I. The Fractional (Relaxation-Oscillation) Ordinary Differential Equation

$$\frac{d^\alpha u}{dt^\alpha} + c^\alpha \, u(t;\alpha) = 0$$
$$(c > 0;\ 0 < \alpha \leqq 2), \tag{3.5}$$

Case I.1: Fractional Relaxation $(0 < \alpha \leqq 1)$

Initial Condition: $u(0+;\alpha) = u_0$

Case I.2: Fractional Oscillation $(1 < \alpha \leqq 2)$

Initial Conditions: $u(0+;\alpha) = u_0$, $\dot{u}(0+;\alpha) = v_0$

with $v_0 \equiv 0$ for continuous dependence of the solution on the parameter α also in the transition from $\alpha = 1-$ to $\alpha = 1+$.

Explicit Solution (in *both* cases):

$$u(t;\alpha) = u_0 \, E_\alpha(-(ct)^\alpha),$$

where $E_\alpha(z)$ denotes the familiar Mittag-Leffler function defined by (*cf.*, *e.g.*, Srivastava and Kashyap [25, p. 42, Equation II.5 (23)])

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} \frac{\zeta^{\alpha-1} e^{\zeta}}{\zeta^\alpha - z} \, d\zeta$$

$$(\alpha > 0;\ z \in \mathbb{C}).$$

II. The Fractional (Diffusion-Wave) Partial Differential Equation

$$\frac{\partial^{2\beta} u}{\partial t^{2\beta}} = k \frac{\partial^2 u}{\partial x^2}$$
$$(k > 0;\ -\infty < x < \infty;\ 0 < \beta \leqq 1), \tag{3.6}$$

where $u = u(x, t; \beta)$ is assumed to be a *causal* function of time $(t > 0)$ *with*

$$u(\mp\infty, t; \beta) = 0.$$

Case II.1: Fractional Diffusion $\quad \left(0 < \beta \leqq \frac{1}{2}\right)$

$$\text{Initial Condition:} \quad u(x, 0+; \beta) = f(x)$$

Case II.2: Fractional Wave $\quad \left(\frac{1}{2} < \beta \leqq 1\right)$

$$\text{Initial Conditions:} \quad \begin{array}{l} u(x, 0+; \beta) = f(x) \\ \dot{u}(x, 0+; \beta) = g(x) \end{array}$$

with $g(x) \equiv 0$ for continuous dependence of the solution on the parameter β also in the transition from $\beta = \frac{1}{2}-$ to $\beta = \frac{1}{2}+$.

Explicit Solution(in *both* cases):

$$u(x, t; \beta) = \int_{-\infty}^{\infty} \mathcal{G}_c(\xi, t; \beta) \; f(x - \xi) \, d\xi, \tag{3.7}$$

where the Green function $\mathcal{G}_c(x, t; \beta)$ is given by

$$|x| \, \mathcal{G}_c(x, t; \beta) = \frac{z}{2} \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(1 - \beta - \beta n)} \quad \left(z = \frac{|x|}{\sqrt{k t^{\beta}}}; \; 0 < \beta < 1 \right), \tag{3.8}$$

which can readily be expressed in terms of Wright's (generalized Bessel) function $J_{\nu}^{\mu}(z)$ defined by (*cf.*, *e.g.*, Srivastava and Kashyap [25, p. 42, Equation II.5(22)])

$$J_{\nu}^{\mu}(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(1 + \nu + \mu n)}. \tag{3.9}$$

4. Operators of Fractional Calculus Based Upon the Cauchy-Goursat Integral Formula

Operators of fractional integrals and fractional derivatives, which are based essentially upon the familiar Cauchy-Goursat integral formula, were considered by (among others) Sonin in 1869, Letnikov in 1868 onwards, and Laurent in 1884. In recent years, many authors have demonstrated the usefulness of fractional calculus operators (based upon the Cauchy-Goursat integral formula) in obtaining particular solutions of numerous families of homogeneous (as well as nonhomogeneous) linear ordinary and partial differential equations which are associated, for example, with many of the following *celebrated* equations:

I. The Gauss Equation:

$$z(1-z)\frac{d^2w}{dz^2}+[\gamma-(\alpha+\beta+1)z]\frac{dw}{dz}-\alpha\beta w=0 \tag{4.1}$$

II. The Kummer Equation:

$$z\frac{d^2w}{dz^2}+(\gamma-z)\frac{dw}{dz}-\alpha w=0 \tag{4.2}$$

III. The Euler Equation:

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}-\rho^2 w=0 \tag{4.3}$$

IV. The Coulomb Equation:

$$z\frac{d^2w}{dz^2}+(2\lambda-z)\frac{dw}{dz}+(\mu-\lambda)w=0 \tag{4.4}$$

V. The Laguerre-Sonin Equation:

$$z\frac{d^2w}{dz^2}+(\alpha+1-z)\frac{dw}{dz}+\lambda w=0 \tag{4.5}$$

VI. The Chebyshev Equation:

$$\left(1-z^2\right)\frac{d^2w}{dz^2}-z\frac{dw}{dz}+\lambda^2 w=0 \tag{4.6}$$

VII. The Weber-Hermite Equation:

$$\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+(\lambda-1)w=0 \tag{4.7}$$

Numerous earlier contributions on fractional calculus along the aforementioned lines are reproduced, with proper credits, in the works of Nishimoto (*cf.* [11] and [12]). Moreover, a rather systematic analysis (including interconnections) of many of the results involving (homogeneous or nonhomogeneous) linear differential equations associated with (for example) the Gauss hypergeometric equation (4.1) can be found in the works of Nishimoto *et al.* ([14] and [15]).

In the cases of (ordinary as well as partial) differential equations of *higher* orders, which have stemmed naturally from the Gauss hypergeometric equation (4.1) and its many relatives and extensions, including some of the above-listed linear differential equations (4.2) to (4.7), there have been several seemingly independent attempts to present a remarkably large number of scattered results in a unified manner. We choose to furnish here the generalizations (and unifications) proposed in one of the latest works on this subject by Tu *et al.* [30] in which references to many earlier related works can be found. We find it to be convenient to begin by recalling the following definition of a *fractional differintegral* (that is, *fractional derivative* and *fractional integral*) of $f(z)$ of order $\nu \in \mathbb{R}$.

Definition (*cf.* [11], [12], and [29]). If the function $f(z)$ is analytic and has no branch point inside and on $\mathcal{C}$, where

$$\mathcal{C} := \{\mathcal{C}^-, \mathcal{C}^+\}, \tag{4.8}$$

$\mathcal{C}^-$ is a contour along the cut joining the points z and $-\infty + i\Im(z)$, which starts from the point at $-\infty$, encircles the point z once counter-clockwise, and returns to the point at $-\infty$, $\mathcal{C}^+$ is a contour along the cut joining the points z and $\infty + i\Im(z)$, which starts from the point at ∞, encircles the point z once counter-clockwise, and returns to the point at ∞,

$$f_\nu(z) = cf_\nu(z) := \frac{\Gamma(\nu+1)}{2\pi i} \int_{\mathcal{C}} \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{\nu+1}}$$

$$\left(\nu \in \mathbb{R} \setminus \mathbb{Z}^-;\ \mathbb{Z}^- := \{-1, -2, -3, \ldots\}\right) \tag{4.9}$$

and

$$f_{-n}(z) := \lim_{\nu \to -n} \{f_\nu(z)\} \qquad (n \in \mathbb{N}), \tag{4.10}$$

where $\zeta \neq z$,

$$-\pi \leqq \arg(\zeta - z) \leqq \pi \qquad \text{for} \qquad \mathcal{C}^-, \tag{4.11}$$

and

$$0 \leqq \arg(\zeta - z) \leqq 2\pi \qquad \text{for} \qquad \mathcal{C}^+, \tag{4.12}$$

then $f_\nu(z)$ $(\nu > 0)$ is said to be the *fractional derivative of* $f(z)$ *of order* ν and $f_\nu(z)$ $(\nu < 0)$ is said to be the *fractional integral of* $f(z)$ *of order* $-\nu$, provided that

$$|f_\nu(z)| < \infty \qquad (\nu \in \mathbb{R}). \tag{4.13}$$

Throughout the remainder of this section, we shall simply write f_ν for $f_\nu(z)$ whenever the argument of the differintegrated function f is clearly understood by the surrounding context. Moreover, in case f is a many-valued function, we shall tacitly consider the *principal value* of f in this investigation.

Each of the following general results is capable of yielding particular solutions of many simpler families of linear ordinary fractional differintegral equations (*cf.* Tu *et al.* [30]) including (for example) the classical differential equations listed above [*cf.* Equations (4.1) to (4.7)].

Theorem 1. *Let $P(z;p)$ and $Q(z;q)$ be polynomials in z of degrees p and q, respectively, defined by*

$$P(z;p) := \sum_{k=0}^{p} a_k\, z^{p-k} \tag{4.14}$$

$$= a_0 \prod_{j=1}^{p} (z - z_j) \qquad (a_0 \neq 0;\ p \in \mathbb{N})$$

and

$$Q(z;q) := \sum_{k=0}^{q} b_k\, z^{q-k} \qquad (b_0 \neq 0;\ q \in \mathbb{N}). \tag{4.15}$$

Suppose also that $f_{-\nu}\,(\neq 0)$ exists for a given function f.

Then the nonhomogeneous linear ordinary fractional differintegral equation:

$$P(z;p)\,\phi_\mu(z) + \left[\sum_{k=1}^{p} \binom{\nu}{k} P_k(z;p) + \sum_{k=1}^{q} \binom{\nu}{k-1} Q_{k-1}(z;q)\right] \phi_{\mu-k}(z)$$

$$+ \binom{\nu}{q} q!\; b_0\; \phi_{\mu-q-1}(z) = f(z)$$

$$(\mu, \nu \in \mathbb{R};\ p, q \in \mathbb{N}), \tag{4.16}$$

has a particular solution of the form:

$$\phi(z) = \left(\left(\frac{f_{-\nu}(z)}{P(z;p)} e^{H(z;p,q)} \right)_{-1} e^{-H(z;p,q)} \right)_{\nu-\mu+1}$$

$$(z \in \mathbb{C} \setminus \{z_1, \dots, z_p\}), \tag{4.17}$$

where, for convenience,

$$H(z;p,q) := \int^{z} \frac{Q(\zeta;q)}{P(\zeta;p)} d\zeta \quad (z \in \mathbb{C} \setminus \{z_1, \dots, z_p\}), \tag{4.18}$$

provided that the second member of (4.17) *exists.*

Theorem 2. *Under the various relevant hypotheses of Theorem* 1, *the homogeneous linear ordinary fractional differintegral equation:*

$$P(z;p)\,\phi_\mu(z) + \left[\sum_{k=1}^{p} \binom{\nu}{k} P_k(z;p) + \sum_{k=1}^{q} \binom{\nu}{k-1} Q_{k-1}(z;q) \right] \phi_{\mu-k}(z)$$

$$+ \binom{\nu}{q} q!\, b_0\, \phi_{\mu-q-1}(z) = 0 \tag{4.19}$$

$$(\mu, \nu \in \mathbb{R};\ p, q \in \mathbb{N})$$

has solutions of the form:

$$\phi(z) = K \left(e^{-H(z;p,q)} \right)_{\nu-\mu+1}, \tag{4.20}$$

where K is an arbitrary constant and $H(z;p,q)$ is given by (4.18), *it being provided that the second member of* (4.20) *exists.*

Next, for a function $u = u(z,t)$ of two *independent* variables z and t, we find it to be convenient to use the notation:

$$\frac{\partial^{\mu+\nu} u}{\partial z^\mu \partial t^\nu}$$

to abbreviate the partial fractional differintegral of $u(z,t)$ of order μ with respect to z *and* of order ν with respect to t $(\mu, \nu \in \mathbb{R})$. And we now state the following general result (*cf.* Tu *et al.* [30]):

Theorem 3. *Let the polynomials $P(z;p)$ and $Q(z;q)$ be defined by* (4.14)*and* (4.15), *respectively. Suppose also that the function $H(z;p,q)$ is given by* (4.18).

Then the partial fractional differintegral equation:

$$P(z;p)\frac{\partial^{\mu}u}{\partial z^{\mu}}+\left[\sum_{k=1}^{p-1}\binom{\nu}{k}P_k(z;p)+\sum_{k=1}^{q}\binom{\nu}{k-1}Q_{k-1}(z;q-1)\right]\frac{\partial^{\mu-k}u}{\partial z^{\mu-k}}$$

$$+\gamma\frac{\partial^{\mu-p}u}{\partial z^{\mu-p}}=\alpha\frac{\partial^{\mu-p+2}u}{\partial z^{\mu-p}\partial t^2}+\beta\frac{\partial^{\mu-p+1}u}{\partial z^{\mu-p}\partial t} \tag{4.21}$$

$$(\mu,\nu\in\mathbb{R};\ p,q\in\mathbb{N})$$

has solutions of the form:

$$u(z,t)=\begin{cases} K_1\left(e^{-H(z;p,q-1)}\right)_{\nu-\mu+1}e^{\xi t} & (\alpha\neq 0)\\ K_2\left(e^{-H(z;p,q-1)}\right)_{\nu-\mu+1}e^{\eta t} & (\alpha=0;\ \beta\neq 0),\end{cases} \tag{4.22}$$

where K_1 and K_2 are arbitrary constants, α, β, and γ are given constants, and (for convenience)

$$\xi:=\frac{-\beta\pm\sqrt{\beta^2+4(\gamma-\delta)\alpha}}{2\alpha}\quad(\alpha\neq 0)\quad\text{and}\quad\eta:=\frac{\gamma-\delta}{\beta}\quad(\alpha=0;\ \beta\neq 0), \tag{4.23}$$

with

$$\delta:=\binom{\nu}{p}p!\,a_0, \tag{4.24}$$

provided that the second member of (4.22) *exists in each case.*

We conclude this section by remarking further that either or both of the polynomials $P(z;p)$ and $Q(z;q)$, involved in Theorems 1 to 3, can be of degree 0 as well. Thus, in the definitions (4.14) and (4.15) (as also in Theorems 1 to 3), $\mathbb{N}$ may easily be replaced (if and where needed) by $\mathbb{N}_0$. Furthermore, it is fairly straightforward to see how each of these general theorems can be suitably specialized to yield numerous simpler results scattered throughout the ever-growing literature on fractional calculus.

5. Miscellaneous Further Applications

For the purpose of those readers who are interested in pursuing investigations on the subject of fractional calculus, we give here references to *some* of the other applications of fractional calculus operators in the mathematical sciences, which are *not* mentioned in the preceding sections.

(i) Theory of Generating Functions of Orthogonal Polynomials and Special Functions (*cf.* [26]);

(ii) Geometric Function Theory (especially the Theory of Analytic, Univalent, and Multivalent Functions) (*cf.* [27] and [28]);

(iii) Integral Equations (*cf.* [3], [22], and [23]);

(iv) Integral Transforms (*cf.* [6] and [8]);

(v) Generalized Functions (*cf.* [8]);

(vi) Theory of Potentials (*cf.* [19]).

A remarkably significant number of publications are emerging regularly in many of these *additional* areas of applications of fractional calculus *as well*;

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