CUBO *Matemática Educacional*
Vol. 5, Nº 2, (289-304). JUNIO 2003.

A new approach to congruences of Kummer type for Bernoulli numbers*

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ABSTRACT

By means of simple identities among rational functions of a particular type, we are able to produce identities among Bernoulli numbers and from them congruences of the form

$$(1-p^{m-1})\frac{B_m}{m} \equiv \frac{1}{mp^{N+1}} \sum_{\substack{1\le j<p^{N+1}\\ (j,p)=1}} j^m - \frac{1}{2} \sum_{\substack{1\le j<p^{N+1}\\ (j,p)=1}} j^{m-1} \pmod{p^{N+1}}$$

when the odd prime p has the property that $p-1$ is not a divisor of the positive even integer m. With such relations, we are able to produce new identities among Bernoulli numbers as well as reproving congruences of Kummer type such as

$$\sum_{l=0}^{r} \binom{r}{l} (-1)^{r-l} \frac{B_{m+\omega l}}{m+\omega l} \equiv 0 \pmod{(p^{er}, p^{m-1})}$$

when ω is a multiple of $(p-1)p^{e-1}, e \ge 1$.

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*1991 *Mathematics Subject Classification.* Primary 11M41

†This work was supported by the Department of Mathematics, National Chung Cheng University and by the National Science Fundation of Taiwan, Republic of China.

1 Zeta-functions associated with rational functions

In this section we outline the general theory to produce Bernoulli identities through zeta functions associated with rational functions initiated by the first author [1]. Let $m_1, m_2, \ldots, m_r$ be positive integers and $P(T)$ a polynomial function in T. Consider the rational function

$$F(T) = \frac{P(T)}{(1-T^{m_1})(1-T^{m_2})\ldots(1-T^{m_r})}.$$

For $|T| < 1, F(T)$ has a power series expansion

$$F(T) = \sum_{k=0}^{\infty} a(k) T^k.$$

The zeta function $Z_F(s)$ associated with $F(T)$ is defined as

$$Z_F(s) = \sum_{k=1}^{\infty} a(k) k^{-s}, \quad \text{Re} \quad s > r.$$

This zeta function is related to $F(T)$ via a Mellin transform

$$Z_F(s)\Gamma(s) = \int_0^{\infty} t^{s-1}[F(e^{-t}) - F(0)]dt$$

for Re $s > r$, where $\Gamma(s)$ is the gamma function defined by

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt.$$

For Re $s > r$, $Z_F(s)$ is an analytic function of s. It has an analytic continuation to the whole complex plane and its special value at the negative integer $s = -m(m = 1, 2, 3, \ldots)$ is given by

$$Z_F(-m) = (-1)^m m! \times [\text{the coefficient of } t^m \text{ in the asymptotic expansion at } t = 0 \text{ of } F(e^{-t})].$$

For example, if we consider $F(T) = \frac{1}{1-T}$, then the zeta function associated with $F(T)$ is the well-known Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \text{Re } s > 1$$

and

$$\zeta(-m) = \frac{(-1)^m B_{m+1}}{m+1},$$

where $B_m (m = 0, 1, 2, \ldots)$ are the Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!}, \quad |t| < 2\pi.$$

On the other hand, if we consider the rational function

$$F(T) = \frac{T^\alpha}{1 - T^k}, \quad \alpha > 0,$$

the zeta function associated with $F(T)$ is

$$\sum_{n=0}^{\infty} (\alpha + kn)^{-s},$$

which is the product of k^{-s} and the well-known Hurwitz zeta function

$$\zeta(s;\delta) = \sum_{n=0}^{\infty} (n + \delta)^{-s}, \quad \text{Re } s > 1,$$

with $\delta = \frac{\alpha}{k}$.

The value of the Hurwitz zeta function at the negative integer $-m$ is given by

$$\zeta(-m;\delta) = -\frac{B_{m+1}(\delta)}{m+1},$$

where $B_m(x) (m = 1, 2, 3, \ldots)$ are the Bernoulli polynomials defined by

$$B_m(x) = \sum_{j=0}^{m} \binom{m}{l} B_{m-l} x^l, \tag{1.1}$$

or equivalently

$$\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} \frac{B_m(x) t^m}{m!}, \quad |t| < 2\pi.$$

When we have more than one way to evaluate $Z_F(s)$ at negative integers in terms of Bernoulli numbers or Bernoulli polynomials, this often leads to identities among Bernoulli numbers and Bernoulli polynomials. Proposition 1 illustrates this approach.

Proposition 1.1 *Let p be a prime number and m a positive even integer. Then for any non-negative integer N, one has*

$$(1-p^{m-1})\frac{B_m}{m} = \frac{p^{(N+1)(m-1)}}{m} \sum_{\substack{1\leq j<p^{N+1}\\(j,p)=1}} B_m\left(\frac{j}{p^{N+1}}\right).$$

Proof. Consider the rational function

$$F(T) = \frac{1}{1-T} - \frac{1}{1-T^p} = \frac{T+\ldots+T^{p-1}}{1-T^p}.$$

The zeta function associated with $F(T)$ is

$$Z_F(s) = \sum_{n=1}^{\infty} n^{-s} - \sum_{n=1}^{\infty} (np)^{-s} = (1-p^{-s})\zeta(s)$$

for Re $s > 1$. On the other hand, we also have

$$\begin{aligned} F(T) &= \frac{(T+\ldots+T^{p-1})(1+T^p+T^{2p}+\ldots+T^{p(p^N-1)})}{1-T^{p^{N+1}}} \\ &= \frac{1}{1-T^{p^{N+1}}} \sum_{\substack{1\leq j<p^{N+1}\\(j,p)=1}} T^j. \end{aligned}$$

Note that for each positive integer j, the zeta function associated with the rational function

$$\frac{T^j}{1-T^{p^{N+1}}}$$

is $p^{-(N+1)s}\zeta(s, \frac{j}{p^{N+1}})$. Consequently we have for Re $s > 1$,

$$(1-p^{-s})\zeta(s) = p^{-(N+1)s} \sum_{\substack{1\leq j<p^{N+1}\\(j,p)=1}} \zeta\left(s, \frac{j}{p^{N+1}}\right).$$

In the above identity, the zeta functions on both sides have analytic continuations. Setting $s = 1-m$ in this continuations we obtain the assertion of Proposition 1. ∎

In order to obtain a congruence for $\frac{B_m}{m}$ modulo a power of p, we need the following classical theorem concerning the denominators of Bernoulli numbers.

von Staudt-Claussen Theorem [2]. *Let p be a prime number and m a positive even integer. Then the following assertions hold.*

1. *If $p-1$ is not a divisor of m, then B_m is p-integral, i.e. p is not a divisor of the denominator of B_m.*

2. *If $p-1$ is a divisor of m, then pB_m is p-integral and*

$$pB_m \equiv -1 \pmod{p}.$$

Proposition 1.2 *Let p be an odd prime and m a positive even integer such that $p-1$ is not a divisor of m. Then for any non-negative integer N,*

$$(1-p^{m-1})\frac{B_m}{m} \equiv \frac{1}{mp^{N+1}} \sum_{\substack{1\le j<p^{N+1}\\(j,p)=1}} j^m - \frac{1}{2} \sum_{\substack{1\le j<p^{N+1}\\(j,p)=1}} j^{m-1} \pmod{p^{N+1}}.$$

Proof. By Proposition 1 and (1.1), we have

$$(1-p^{m-1})\frac{B_m}{m} = \sum_{l=0}^{m} C_l(m)$$

where

$$\begin{aligned} C_l &= \frac{1}{m}\binom{m}{l} B_l p^{(l-1)(N-1)} \sum_{\substack{1\le j<p^{N+1}\\(j,p)=1}} j^{m-l} \\ &= \frac{(m-1)\dots(m-l+1)}{l!} B_l p^{(l-1)(N-1)} \sum_{\substack{1\le j<p^{N+1}\\(j,p)=1}} j^{m-l}. \end{aligned}$$

For sufficiently large l, we have $C_l(m) \equiv 0 \pmod{p^{N+1}}$. We now estimate how large l should be. Note that the exponent of p in $l!$ is no greater than

$$\left[\frac{l}{p}\right] + \left[\frac{l}{p^2}\right] + \dots + \left[\frac{l}{p^k}\right] + \dots \le \sum_{k=1}^{\infty} \frac{l}{p^k} = \frac{l}{p-1} \le \frac{l}{2}.$$

Also pB_l is p-integral by the von-Staudt-Claussen theorem. Thus, $C_l(m)$ is p-integral and

$$C_l(m) \equiv 0 \pmod{p^{N+1}}$$

provided that

$$(N+1)(l-1)-1-\frac{l}{2}\geq N+1.$$

The above inequality is true for $l \geq 6$. Hence we have

$$(1-p^{m-1})\frac{B_m}{m}\equiv\sum_{l=0}^{4}C_l(m)\pmod{p^{N+1}}.$$

Thus, to obtain our assertion, we have to show that $C_4(m)$ and $C_2(m)$ are divisible by p^{N+1}. Note that

$$C_4(m)=-\frac{(m-1)(m-2)(m-3)}{2^3\cdot 3^2\cdot 5}p^{3(N+1)}\sum_{\substack{1\leq j<p^{N+1}\\(j,p)=1}} j^{m-4},$$

so that $C_4(m)\equiv 0 \pmod{p^{N+1}}$ for any odd prime p. Also

$$C_2(m)=\frac{m-1}{2^2\cdot 3}p^{N+1}\sum_{\substack{1\leq j<p^{N+1}\\(j,p)=1}} j^{m-2},$$

and hence, $C_2(m)\equiv 0 \pmod{p^{N+1}}$ for any prime p except $p=3$. However $p=3$ is excluded under the assumption that $p-1$ is not a divisor of m. ■

2 Congruences of kummer type

The classical Kummer congruence for Bernoulli numbers asserts that

$$\frac{B_m}{m}\equiv\frac{B_{m+p-1}}{m+p-1}\pmod{p},$$

if $p-1$ is not a divisor of the positive even integer m. See page 385 of [2] for the details. This was generalized to

$$(1-p^{m-1})\frac{B_m}{m}\equiv(1-p^{n-1})\frac{B_n}{n}\pmod{p^e}$$

if $m\equiv n \pmod{(p-1)p^{e-1}}$ and $p-1$ is not a divisor of m [5]. Here we shall use Proposition 2 to prove a further generalization.

Proposition 2.1 [5]. *Let p be an odd prime and m a positive even integer such that $p-1$ is not a divisor of m. Suppose that ω is a multiple of $(p-1)p^{e-1}, e\geq 1$ and r is a positive integer. Then*

$$\sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}(1-p^{m+\omega l-1})\frac{B_{m+\omega l}}{m+\omega l}\equiv 0\pmod{p^{er}.}$$

To prove Proposition 3, we need the following lemma.

Lemma 1. *Let r, m and ω be positive integers. Then*

$$\sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}\frac{1}{x^{m+\omega l}-1}=\frac{(x^{\omega}-1)^{r}P_{r}(x^{m},x^{\omega})}{(x^{m}-1)(x^{m+\omega}-1)\dots(x^{m+r\omega}-1)},$$

where $P_r(X,Y)$ is a polynomial in X, Y with integral coefficients.

Proof. We shall prove the assertion by induction on r. For $r = 1$, we have

$$\frac{1}{x^{m}-1}-\frac{1}{x^{m+\omega}-1}=\frac{(x^{\omega}-1)x^{m}}{(x^{m}-1)(x^{m+\omega}-1)},$$

so the assertion is true for $r = 1$.

Suppose that it is true for $r = k$. Then for $r = k + 1$, we have

$$\sum_{l=0}^{k+1}\binom{k+1}{l}(-1)^{k+1-l}\frac{1}{x^{m+\omega l}-1}$$

$$=\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\frac{1}{x^{m+\omega+\omega l}-1}-\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\frac{1}{x^{m+\omega l}-1}$$

$$=\frac{(x^{\omega}-1)^{k}P_{k}(x^{m+\omega},x^{\omega})}{(x^{m+\omega}-1)(x^{m+2\omega}-1)\dots(x^{m+(k+1)\omega}-1)}$$

$$-\frac{(x^{\omega}-1)^{k}P_{k}(x^{m},x^{\omega})}{(x^{m}-1)(x^{m+\omega}-1)\dots(x^{m+k\omega}-1)}$$

$$=\frac{(x^{\omega}-1)^{k}[P_{k}(x^{m+\omega},x^{\omega})(x^{m}-1)-P_{k}(x^{m},x^{\omega})(x^{m+(k+1)\omega}-1)]}{(x^{m}-1)(x^{m+\omega}-1)\dots(x^{m+(k+1)\omega}-1)}$$

Note that

$$Q(x^{m},x^{\omega})=P_{k}(x^{m+\omega},x^{\omega})(x^{m}-1)-P_{k}(x^{m},x^{\omega})(x^{m+(k+1)\omega}-1)$$

is a polynomial function in variables $X = x^m$ and $Y = x^\omega$ with integral coefficients, which is zero if $x^\omega = 1$. This implies that

$$Q(x^{m},x^{\omega})=P_{k+1}(x^{m},x^{\omega})(x^{\omega}-1)$$

for some polynomial $P_{k+1}(X, Y)$ with integral coefficients. This completes our proof. ∎

Proof of Proposition 3. By Proposition 2, we have for $0 \le l \le r$,

$$(1-p^{m+\omega l-1})\frac{B_{m+\omega l}}{m+\omega l} \equiv \frac{p^{-er}}{m+\omega l}\sum_{\substack{1\le j<p^{er}\\(j,p)=1}} j^{m+\omega l} - \frac{1}{2}\sum_{\substack{1\le j<p^{er}\\(j,p)=1}} j^{m+\omega l-1} \pmod{p^{er}}.$$

Multiplying both sides of this congruence by $\binom{r}{l}(-1)^{r-l}$ and summing over $l = 0, 1, \ldots, r$, we obtain

$$\sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}(1-p^{m+\omega l-1})\frac{B_{m+\omega l}}{m+\omega l}$$
$$\equiv \sum_{l=0}^{r}\binom{r}{l}\frac{(-1)^{r-l}p^{-er}}{m+\omega l}\sum_{\substack{1\le j<p^{er}\\(j,p)=1}} j^{m+\omega l} - \frac{1}{2}\sum_{\substack{1\le j<p^{er}\\(j,p)=1}} j^{m-1}(j^{\omega}-l)^{r} \pmod{p^{er}}.$$

For $(j,p)=1$ we have

$$(j^{\omega}-1)^{r} \equiv 0 \pmod{p^{er}}$$

so that we can drop the second terms.

Let g be a generator of the cyclic group $(\mathbb{Z}/p^{er}\mathbb{Z})^{*}$, the multiplicative group of the ring $\mathbb{Z}/p^{er}\mathbb{Z}$. Then

$$\sum_{l=0}^{r}\binom{r}{l}\frac{(-1)^{r-l}p^{-er}}{m+\omega l}\sum_{\substack{1\le j<p^{er}\\(j,p)=1}} j^{m+\omega l}$$
$$\equiv \sum_{l=0}^{r}\binom{r}{l}\frac{(-1)^{r-l}p^{-er}}{m+\omega l}\,\frac{g^{(m+\omega l)(p-1)p^{er-1}}-1}{g^{m+\omega l}-1} \pmod{p^{er}}.$$

Now set

$$g^{(p-1)p^{er-1}} = 1 + \alpha p^{er},$$

so that

$$\frac{p^{-er}}{m+\omega l}\left(g^{(m+\omega l)(p-1)p^{er-1}}-1\right) \equiv \alpha \pmod{p^{er}}.$$

Thus it suffices to prove

$$\sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}\frac{\alpha}{g^{m+\omega l}-1} \equiv 0 \pmod{p^{er}}.$$

But it follows from Lemma 1 that

$$\sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}\frac{1}{g^{m+\omega l}-1} = \frac{(g^{\omega}-1)^{r}P_r(g^{m},g^{\omega})}{(g^{m}-1)(g^{m+\omega}-1)\ldots(g^{m+r\omega}-1)}.$$ ∎

Corollary. *Let r be a positive integer, p an odd prime number and m a positive even integer such that $p-1$ is not a divisor of m. Then for any positive integer e and $\omega = (p-1)p^{e-1}$, one has*

$$\sum_{l=0}^{r} \binom{r}{l} (-1)^{r-l} \frac{B_{m+\omega l}}{m+\omega l} \equiv 0 \pmod{(p^{er}, p^{m-1})}.$$

As shown in [3], p-adic integration on p-adic spaces can be used to prove congruences of Kummer type. Here we shall give another proof of Proposition 3 via p-adic integration.

Let p be a prime number. $\mathbb{Z}_p$ and $\mathbb{Q}_p$ are the rings of p-adic integers and the field of p-adic numbers, respectively. ν_p is the p-adic valuation on $\mathbb{Q}_p$. Ω_p is the algebraic completion of $\mathbb{Q}_p$.

Fix a k-th root of unity $\epsilon (\epsilon \neq 1)$ with k relatively prime to p. $\mathbb{Z}_p^*$ is the set of invertible elements in $\mathbb{Z}_p$ and $a + p^N\mathbb{Z}_p$ is the set of x in $\mathbb{Z}_p$ which maps to a in $\mathbb{Z}/p^N\mathbb{Z}$ under the natural projection from $\mathbb{Z}_p$ to $\mathbb{Z}/p^N\mathbb{Z}$. Define

$$\mu_\epsilon(a + p^N\mathbb{Z}_p) = \frac{\epsilon^a}{1-\epsilon^{p^N}},$$

and

$$\mu(a + p^N\mathbb{Z}_p) = \sum_{\epsilon^k=1, \epsilon\neq 1} \mu_\epsilon(a + p^N\mathbb{Z}_p).$$

Also for any continuous function $f : \mathbb{Z}_p \to \Omega_p$, we define

$$\int_{\mathbb{Z}_p} f(x)d\mu(x) = \lim_{N\to\infty} \sum_{0\leq a<p^N} f(a)\mu(a + p^N\mathbb{Z}_p).$$

Proposition 2.2 *For any positive integer m, one has*

$$\int_{\mathbb{Z}_p} x^{m-1}d\mu(x) = (1-k^m)\frac{B_m}{m}.$$

Proof. For each t in Ω_p with $\nu_p(t) > 1/(p-1)$, the exponential function e^{tx} defined by the power series

$$e^{tx} = \sum_{i=0}^{\infty} \frac{t^i x^i}{i!}$$

is a continuous function on $\mathbb{Z}_p$.

Hence we have

$$\begin{aligned}\int_{\mathbb{Z}_p} e^{tx} d\mu_\epsilon(x) &= \lim_{N\to\infty} \frac{1}{1-\epsilon^{p^N}} \sum_{0\le a<p^N} e^{at}\epsilon^a \\ &= \lim_{N\to\infty} \frac{1}{1-\epsilon^{p^N}} \frac{1-\epsilon^{p^N} e^{p^N t}}{1-\epsilon e^t} \\ &= \frac{1}{1-\epsilon e^t},\end{aligned}$$

since $e^{p^N t} \to 1$ as $N \to \infty$. It follows that

$$\int_{\mathbb{Z}_p} e^{tx} d\mu(x) = \sum_{\epsilon^k=1,\epsilon\ne1} \frac{1}{1-\epsilon e^t} = \frac{k}{1-e^{kt}} - \frac{1}{1-e^t}.$$

Comparing the coefficients of t^{m-1}, we get

$$\int_{\mathbb{Z}_p} x^{m-1} d\mu(x) = (1-k^m)\frac{B_m}{m}.$$

■

Note that

$$\int_{\mathbb{Z}_p^*} x^{m-1} d\mu(x) = \int_{\mathbb{Z}_p} x^{m-1} d\mu(x) - \int_{p\mathbb{Z}_p} x^{m-1} d\mu(x)$$

and

$$\int_{p\mathbb{Z}_p} x^{m-1} d\mu(x) = p^{m-1}(1-k^m)\frac{B_m}{m}.$$

by a similar calculation as in the proof of the previous proposition. Thus we obtain the following.

Proposition 2.3 *For any positive integer m, one has*

$$\int_{\mathbb{Z}_p^*} x^{m-1} d\mu(x) = (1-p^{m-1})(1-k^m)\frac{B_m}{m}.$$

Now the application of Proposition 5 to congruences of Kummer type is clear. By Proposition 5, we have the identity

$$\int_{\mathbb{Z}_p^*}(x^w-1)^r x^{m-1}d\mu(x) = \int_{\mathbb{Z}_p^*}\sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}x^{m+\omega l-1}d\mu(x)$$
$$= \sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}(1-p^{m+\omega l-1})\frac{B_{m+\omega l}}{m+\omega l}(1-k^{m+\omega l}).$$

Clearly

$$(x^\omega-1)^r \equiv 0 \pmod{p^{er}}$$

it x is relatively prime to p. This implies

$$\int_{\mathbb{Z}_p^*}(x^\omega-1)^r x^{m-1}d\mu(x) \equiv 0 \pmod{p^{er}}.$$

It follows that

$$\sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}(1-p^{m+\omega l-1})\frac{B_{m+\omega l}}{m+\omega l}(1-k^{m+\omega l}) \equiv 0 \pmod{p^{er}}.$$

On the other hand, Lemma 1 implies that

$$\sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}\frac{1}{1-k^{m+\omega l}} \equiv 0 \pmod{p^{er}}.$$

Therefore,

$$\sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}(1-p^{m+\omega l})\frac{B_{m+\omega l}}{m+\omega l} \equiv 0 \pmod{p^{er}},$$

by applying Theorem 1 in [5], which we restate as follows.

Proposition 2.4 Theorem 1 of [5]). *Let p be a fixed prime and let $\{a_m\}$, $\{b_m\}$ be two sequences of rational numbers that are integral* $\pmod p$. *Suppose that*

$$\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}a_{m+s(p-1)}a_p^{r-s} \equiv 0 \pmod{p^{er}}. \tag{2.1}$$

and

$$\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}b_{m+s(p-1)}b_p^{r-s} \equiv 0 \pmod{p^{er}}. \tag{2.2}$$

for all $m \geq r \geq 1$. Then the same type of congruence is true for $\{c_m\} = \{a_m b_m\}$.

3 Congruences of Kummer type for Bernoulli polynomials

Congruences of Kummer type for Bernoulli polynomials were first considered by the authors [3] in 1997. Again we begin with a simple rational function

$$F(T) = \frac{T^a}{1-T^k}.$$

For a prime number p relatively prime to k, there exists an integer j such that

$$a + kj = p\beta, \quad 0 \le j < p.$$

Thus it follows that

$$\begin{aligned}\frac{T^\alpha}{1-T^k} - \frac{T^{p\beta}}{1-T^{kp}} &= \frac{1}{1-T^{kp}} \sum_{\substack{j=\alpha+kl \\ 0\le l<p^{N+1} \\ (j,p)=1}} T^j \\ &= \frac{1}{1-T^{kp^{N+1}}} \sum_{\substack{j=\alpha+kl \\ 0\le l<p^{N+1} \\ (j,p)=1}} T^j.\end{aligned}$$

From the above with a similar argument as in Proposition 2, we have the following

Proposition 3.1 *Let p be a prime number and m a positive integer such that $p-1$ is not a divisor of m. Suppose that k is a positive integer relatively prime to k and α, β are non-negative numbers such that $\alpha + kj = p\beta$, $0 \le j < p$. Then for any nonnegative integer N,*

$$\begin{aligned}&= \frac{1}{m}\left\{B_m\left(\frac{\alpha}{k}\right) - p^{m-1}B_m\left(\frac{\beta}{k}\right)\right\} \\ &= \frac{p^{(m-1)(N+1)}}{m} \sum_{\substack{j=\alpha+kl \\ 0\le l<p^{N+1} \\ (j,p)=1}} B_m\left(\frac{j}{kp^{N+1}}\right) \\ &\equiv \frac{1}{mk^m p^{N+1}} \sum_{\substack{j=\alpha+kl \\ 0\le l<p^{N+1} \\ (j,p)=1}} j^m - \frac{1}{2k^{m-1}} \sum_{\substack{j=\alpha+kl \\ 0\le l<p^{N+1} \\ (j,p)=1}} j^{m-1} \pmod{p^{N+1}}.\end{aligned}$$

In summing

$$\frac{1}{mp^{N+1}} \sum_{\substack{j=\alpha+kl \\ 0\le l<p^{N+1} \\ (j,p)=1}} j^m,$$

we note that in general j does not range over a set of representatives of $(\mathbb{Z}/p^{N+1}\mathbb{Z})^*$. Suppose that

$$j_1 = j_2 + \epsilon(j_2)p^{N+1}$$

with $0 \le j_2 < p^{N+1}$ and $\epsilon(j_2) \in \mathbb{Z}$, then

$$\frac{1}{mp^{N+1}}(j_1^m - j_2^m) \equiv \epsilon(j_2)j_2^{m-1} \pmod{p^{N+1}}.$$

So if we let j range over a set of representatives of $(\mathbb{Z}/p^{N+1}\mathbb{Z})^*$ in the summation, it will cause a perturbation in the term

$$\sum_{\substack{j=\alpha+kl \\ 0\le l<p^{N+1} \\ (j,p)=1}} j^{m-1}.$$

If we proceed as in the proof of Proposition 3, we obtain

Proposition 3.2 *Let p be an odd prime and m be a positive integer such that $p-1$ is not a divisor of m. Suppose that k is a positive integer relative prime to p and α, β are non-negative integers such that $\alpha + kj = p\beta, 0 \le j < p$. Then for any positive integer r and ω, a multiple of $(p-1)p^{e-1}$,*

$$\sum_{l=0}^{r} \binom{r}{l}(-1)^{r-l}\left[\frac{B_{m+\omega l}(\frac{\alpha}{k})}{m+\omega l} - \frac{p^{m+\omega l-1}B_{m+\omega l}(\frac{\beta}{k})}{m+\omega l}\right] \equiv 0 \pmod{p^{er}}.$$

4 Congruences of Kummer type for generalized Bernoulli polynomials

Let f be a positive integer and χ a primitive character of conductor f. The generalized Bernoulli polynomials B_χ^n are defined by

$$\sum_{j=1}^{f} \chi(j)\frac{te^{jt}}{e^{ft}-1} = \sum_{n=0}^{\infty} B_\chi^n \frac{t^n}{n!}, \quad |t| \le \frac{2\pi}{f}.$$

In terms of Bernoulli polynomials, we have

$$B_\chi^n = f^{n-1}\sum_{j=1}^{f}\chi(j)B_n(\frac{j}{f}).$$

In particular, if χ is a nontrivial character, then

$$B_\chi^1 = \frac{1}{f}\sum_{j=1}^{f} j\chi(j).$$

Generalized Bernoulli polynomials are used to give the values at negative integers of the L-series defined by

$$L(s,\chi) = \sum_{n=1}^{\infty}\chi(n)n^{-s}, \quad \mathrm{Re}\ s \geq 1.$$

Indeed, $L(s,\chi)$ has a meromorphic continuation in the whole complex plane and for each positive integer n,

$$L(1-n,\chi) = -\frac{B_\chi^n}{n}.$$

Congruences of Kummer type for generalized Bernoulli numbers can be obtained as a simple application of those for Bernoulli polynomials, see Ernvall [5], Eie and Ong [3].

Theorem 1. *Suppose that χ is a nontrivial character with conductor $f \geq 1$. Let m, n be positive integers, and p be an odd prime such that $p-1$ is not a divisor of m and p is not a divisor of f. Then we have*

$$(1-\chi(p)p^{m-1})\frac{B_\chi^m}{m} \equiv (1-\chi(p)p^{n-1})\frac{B_\chi^n}{n} \pmod{p^{N+1}}$$

if $m \equiv n \pmod{(p-1)p^N}$.

Proof. For each j with $1 \leq j \leq f$, there exists an unique $\bar{j}$ with $1 \leq \bar{j} \leq f$ and

$$j + \mu_j f = p\bar{j}$$

for some μ_j with $0 \leq \mu_j < p-1$. Hence, by the Kummer type congruence for Bernoulli polynomials, we have

$$\frac{1}{m}\left\{B_m(\frac{j}{f}) - p^{m-1}B_m(\frac{\bar{j}}{f})\right\} \equiv \frac{1}{n}\left\{B_n(\frac{j}{f}) - p^{n-1}B_n(\frac{\bar{j}}{f})\right\} \pmod{p^{N+1}}$$

if $m \equiv n \pmod{(p-1)p^N}$.

Consequently,

$$\frac{f^{m-1}}{m}\left\{\sum_{j=1}^{f}\chi(j)B_m(\frac{j}{f}) - p^{m-1}\sum_{j=1}^{f}\chi(j)B_m(\frac{\bar{j}}{f})\right\}$$
$$\equiv \frac{f^{n-1}}{n}\left\{\sum_{j=1}^{f}\chi(k)B_n(\frac{j}{p}) - p^{n-1}\sum_{j=1}^{f}\chi(j)B_n(\frac{\bar{j}}{f})\right\} \pmod{p^{N+1}}$$

if $m \equiv n \pmod{(p-1)p^N}$.

Note that for any positive integer k,

$$\begin{aligned}\sum_{j=1}^{f}\chi(j)B_k(\frac{\bar{j}}{f}) &= \sum_{j=1}^{f}\chi(j+\mu_j f)B_k(\frac{\bar{j}}{f}) \\ &= \sum_{\bar{j}=1}^{f}\chi(p\bar{j})B_k(\frac{\bar{j}}{f}) \\ &= \chi(p)\sum_{j=1}^{f}\chi(j)B_k(\frac{j}{f}).\end{aligned}$$

Finally we have

$$(1-\chi(p)p^{m-1})\frac{B_\chi^m}{m} \equiv (1-\chi(p)p^{n-1})\frac{B_\chi^n}{n} \pmod{p^{N+1}}$$

if $m \equiv n \pmod{(p-1)p^N}$. ■

As a simple application of Proposition 8, we have the following extension of Theorem 1.

Proposition 4.1 *Suppose that χ is a non-trivial character with conductor $f \geq 1$. Let m be a positive even integer, and p be an odd prime number such that $p-1$ is not a divisor of m and p is not a divisor of f. One has for any positive integer r and ω, a multiple of $(p-1)p^{e-1}$,*

$$\sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}(1-\chi(p)p^{m+\omega l-1})\frac{B_\chi^{m+\omega l}}{m+\omega l} \equiv 0 \pmod{p^{er}}.$$

As a final application of our general procedure to produce congruences of Kummer type, we shall mention here a new identity resulting from the observation that

$$a^{(p-1)/2} \equiv \left(\frac{a}{p}\right) \pmod{p} \text{ so that } \left[a^{(p-1)/2} - \left(\frac{a}{p}\right)\right]^r \equiv 0 \pmod{p^r},$$

where p is an odd prime number, a is an integer relatively by prime to p and $(\frac{a}{p})$ is the Legendre symbol defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if a is a quadratic residue modulo } p, \\ -1, & \text{otherwise.} \end{cases}$$

Theorem 2. *Let p be an odd prime number and χ the character defined by*

$$\chi(a) = \begin{cases} (\frac{a}{p}), & \text{if } a \text{ is relatively prime to } p, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $\omega = (p-1)/2$ and m is a positive even integer such that $(p-1)$ is not a divisor of m. Then for any positive even integer r,

$$\sum_{\substack{0 \le l \le r \\ l \equiv r \pmod 2}} \binom{r}{l}(1 - p^{m+\omega l-1})\frac{B_{m+\omega l}}{m+\omega l} \equiv \sum_{\substack{0 \le l < r-1 \\ l \equiv r-1 \pmod 2}} \binom{r}{l}\frac{B_{\chi}^{m+\omega l}}{m+\omega l} \pmod{p^r}.$$

The authors would like to thank an unknown referee for suggestions improving the presentation of the results in this paper.

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