CUBO *Matemática Educacional*
Vol. 5, Nº 2, (159-183). JUNIO 2003.

An Invitation to Semilinear Elliptic Equations

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1 Introduction

In this work we present some problems from the Theory of Partial Differential Equations of semi-linear type. The topic here is not the generalization of results, but a presentation of different methods and techniques from Topology, Functional Analysis and the Theory of Critical Points. It is necessary to warn the reader about the danger he or she is exposed to: the glamour of the challenge and importance of the problems, the variety and beauty of the methods, and to find in this subject an important history, past and present, suggesting a fruitful future!

2 Elliptic Operators and semi-linear Equations

Let Ω be a domain contained in $\mathbb{R}^N$, that is, a conected open subset of $\mathbb{R}^N$. A partial differential operator $L(D)$ of order 2 acting on real functions defined on the closure of Ω, $\overline{\Omega}$, has the formula

$$L(D) = \sum_{i,j=1}^{N} a_{ij}(x) D_i D_j + \sum_{j=1}^{N} b_j(x) D_j + c(x) \qquad (1)$$

*Notes from a short course given at the Universidade Federal do Ceará, January 1996
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where the coefficients a_{ij} are real functions defined in $\overline{\Omega}$. This operator is called *elliptic* if

$$\sum_{i,j=1}^{N} a_{ij}(x)\psi_i\psi_j > 0, \qquad \forall x \in \overline{\Omega},\ \psi \in \mathbb{R}^N \backslash \{0\}. \tag{2}$$

The operator L is called *strongly elliptic* if there exists a constant $c > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\psi_i\psi_j \geq c|\psi|^2, \qquad \forall x \in \overline{\Omega},\ \psi \in \mathbb{R}^N. \tag{3}$$

We refer to Gilbarg and Trudinger [5] for more details. The Laplacian

$$\Delta = \sum_{j=1}^{N} D_j^2$$

is an example of a strongly elliptic operator.

Consider a function $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$. An equation of the form

$$L(D)u = f(x,u) \tag{4}$$

is called *semi-linear*. This kind of equations shows up in several problems of Mathematical Physics, Geometry, and others applied fields of Partial Differential Equations. The main problem related to the equation (4) is to find a function $u : \overline{\Omega} \to \mathbb{R}$ which satisfies the equation and some boundary condition. In case of applications such conditions have a specific physical meaning. The most common conditions are

- (i) Dirichlet condition: $u(x) = 0$ for $x \in \partial\Omega$,
- (ii) Neumann condition: $\frac{\partial u}{\partial n}(x) = 0$ for $x \in \partial\Omega$,
- (iii) Mixed condition: $\alpha u(x) + \beta\frac{\partial u}{\partial n}(x) = 0$ for $x \in \partial\Omega$ with α, β fixed constants.

All these conditions are homogeneous. There are also important nonhomogeneous conditions which the right hand side is replaced by a function $g(x)$ defined on $\partial\Omega$.

3 Model Problem

In this work we will only consider the problem below where the differential operator is the Laplacian, the boundary condition is Dirichlet, and Ω is a bounded domain. Moreover, this case already has several problems and difficulties with the theory. There is no way that the problem is trivial.

$$\begin{cases} -\Delta u = f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a bounded domain and f satisfies the condition

$$(F) \qquad f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \text{ is locally Lipschitz.}$$

Considering the different methods used to solve the problems of the Theory of Partial Differential Equations, for example in (5), there will be different concepts of solutions. To introduce them, we define some functional spaces: continuous functions spaces, Schauder spaces, $C^{k,\alpha}$, and Sobolev spaces $W^{m,p}$.

3.1 The $C^{k,\alpha}$ Spaces

Let A be a subset of $\mathbb{R}^N$. We denote by $C^k(A)$, k a nonnegative integer, the space of all functions $u : A \to \mathbb{R}$ with continuous derivatives of order up to k. In this work, we will consider two examples of sets A: the domain Ω and its closure $\overline{\Omega}$. When Ω is bounded, the space $C^k(\overline{\Omega})$ can be endowed with the structure of Banach space defined with the norm

$$\|u\|_{C^k} = \max\{|D^j u(x)| : j = 0, \ldots, k,\ x \in \overline{\Omega}\}. \tag{6}$$

For $0 < \alpha \leq 1$, we define the space $C^\alpha(\overline{\Omega})$ as the space of Hölder continuous functions on $\overline{\Omega}$. In this space the norm is defined by

$$\|u\|_{C^\alpha} = \sup\{\frac{|u(x) - u(y)|}{|x-y|^\alpha} : x, y \in \overline{\Omega}, x \neq y\}. \tag{7}$$

it Definition.

A function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ which satisfies (5) is called a *classical solution*.

3.2 Sobolev Spaces

Let $1 \leq p < \infty$ and $m = 1, 2, \ldots$. Define $W^{m,p}(\Omega)$ the set of all $L^p(\Omega)$ functions with derivatives (in the sense of distributions $D'(\Omega)$) up to the order m in $L^p(\Omega)$. What is the meaning of derivatives in the sense of distribution? To answer this question, we notice that each function $u \in L^1_{loc}(\Omega)$, in fact each function in $L^p(\Omega)$, is a distribution. In fact, every function $u \in L^1_{loc}(\Omega)$ can be identified with a distribution T_u defined by:

$$< T_u, \phi >= \int_\Omega u(x)\phi(x)dx, \qquad \forall \phi \in D(\Omega)$$

where $D(\Omega) = C_c^\infty(\Omega)$ is the space of functions with infinitely many derivatives and compact support in Ω. We define the derivative $D_i = \frac{\partial}{\partial x_i}$ of T_u as a distribution, which we denote $D_i T_u$, and it is described by

$$< D_i T_u, \phi >:= - < T_u, D_i \phi >, \qquad \forall \phi \in D(\Omega).$$

The space $W^{m,p}$ is a Banach space endowed with the norm

$$\|u\|_{W^{m,p}} = \Big(\sum_{|\alpha| \leq m} \int |D^\alpha u(x)|^p dx \Big)^{\frac{1}{p}}. \tag{8}$$

It is easy to see that $D(\Omega)$ is a vector subspace of $W^{m,p}$. The closure of $D(\Omega)$ with the norm (8) is denoted by $W_0^{m,p}(\Omega)$. Therefore, $u \in W_0^{m,p}(\Omega)$ satisfies, in a certain meaning (the notion of trace which we will not explain here), boundary conditions. For example, $u \in {W_0}^{m,p}(\Omega)$ satisfies the condition "$u = 0$" in $\partial\Omega$.

The spaces $W^{m,p}$, when m and/or p increases, are better spaces. This is translated in a precise way by the Sobolev embedding theorems.

- (i) If $mp < N$, $W^{m,p} \subset L^q$ for all $q \leq \frac{Np}{N-mp}$,
- (ii) If $mp > N$, $W^{m,p} \subset C^\alpha$, where $\alpha = m - \frac{N}{p}$.

The special case $mp = N$ is called the Trudinger case and the Orlicz spaces are called on stage.

When $p = 2$ $W^{m,2}(\Omega)$ are Hilbert spaces. The notation $H^m(\Omega)$ is also used to denote $W^{m,2}(\Omega)$, and $H_0^m(\Omega)$ to denote $W_0^{m,p}(\Omega)$, $m = 1, 2, \ldots$.

Theorem 3.1 Poincarè Inequality *(cf. [1])*
There exists a constant $c > 0$ (depending on Ω) such that

$$\int_\Omega u^2 \le c \int_\Omega |\nabla u|^2, \qquad u \in H_0^1(\Omega). \tag{9}$$

It follows from (9) that in $H_0^1(\Omega)$ the expression

$$\sqrt{\int_\Omega |\nabla u|^2} \tag{10}$$

is an equivalent norm with respect to its norm

$$\sqrt{\int_\Omega u^2 + \int_\Omega |\nabla u|^2}.$$

Consequently, now on we will assume that $H_0^1(\Omega)$ has its topology defined by the norm (10).

3.3 Generalized Solutions (also known as weak solutions)

Let us motivate its arrival Let u be a classical solution of (5). Multiplying this equation by a function $\phi \in D(\Omega)$ and integrating by parts, we obtain

$$\int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega f(x,u)\phi, \qquad \forall\, \phi \in D(\Omega). \tag{11}$$

(Integrate by parts is to use the Green's Theorem. Notice that there is not a boundary term because $u = 0$ on $\partial\Omega$).

The expression (11) has meaning for functions u which has only first derivative. This induces us to define generalized solution as a function $u \in W_0^{1,p}(\Omega)$ such that (11) is verified. Because the functions in $W_0^{1,p}(\Omega)$ are not neccesarily bounded, we must be careful about the integrability of the right hand side in (11). This is obtained requiring polynomial growth in f as a function of u, such that $f(x,u)$ belongs to $L^1_{loc}(\Omega)$. For example, if $p = 2$, we ask for

$$|f(x,s)| \le c|s|^{q-1} + c, \qquad \forall\, s \in \mathbb{R}, \tag{12}$$

where $q \le \frac{2N}{N-2} =: 2^*$. Check it! Find what growth must have f if $p \ne 2$.

3.4 The Variational Method for solving (5)

Suppose that f satisfies (12). Then, the functional Φ is well defined in $H_0^1(\Omega)$ by the expression

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x,u) \tag{13}$$

where

$$F(x,s) = \int_0^s f(x,t)dt.$$

Indeed, if $u \in H_0^1(\Omega)$, using the Sobolev embedding theorem we get $u \in L^{2^*}$, and by (12), $F(x,u) \in L^1$. On the other hand, the first term of $\Phi(u)$ is the square norm in $H_0^1(\Omega)$. The functional Φ is differentiable, and indeed, of C^1 class (cf. [3]). Its derivative $\Phi'(u)$ at a point $u \in H_0^1(\Omega)$ is given by the term

$$< \Phi'(u), \phi >= \int_\Omega \nabla u \cdot \nabla \phi - \int_\Omega f(x,u)\phi \tag{14}$$

for all $\phi \in H_0^1(\Omega)$. Here, $< \cdot, \cdot >$ denotes the duality pair between the dual $(H_0^1)^*$ and H_0^1, where $(H_0^1)^*$ is the space of all linear continuous functionals in H_0^1. Because H_0^1 is a Hilbert space, $(H_0^1)^*$ can be identified by the same H_0^1 (that is, the Riesz Representation Theorem). Also, we can see $< \cdot, \cdot >$ as an inner product in H_0^1. To obtain the expression (14) we can compute the derivative of a real function g with real variable t defined by

$$g(t) = \Phi(u + t\phi)$$

at the point $t = 0$ (that is, the Gâteaux derivative or the derivative in the direction ϕ).

Comparing now (11) and (14), we conclude that *the weak solutions of (5) are exactly the critical points of* Φ, that is, the points in $u \in H_0^1(\Omega)$ such that $\Phi'(u) = 0$.

4 The Spectrum of the Laplacian

The problem (5) can be seen as a perturbation of a linear problem which involves the Laplacian operator Δ. As we will see later, existence, nonexistence, and multiplicity of solutions depend on the interaction of the nonlinear part with the linear operator Δ. To understand this interaction, we need to know the spectrum of the operator, that is, the numbers $\lambda \in \mathbb{R}$ such that the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{16}$$

has a solution $u \neq 0$.

To comprise this study we need the following result which will be proved in Section 6.

Theorem 4.1 *For all $f \in L^2(\Omega)$, Ω a bounded domain, there exists a unique weak solution of the problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{17}$$

that is, $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega f\phi, \qquad \forall \phi \in H_0^1. \tag{18}$$

Thus, the Theorem 4.1 give us an operator

$$T : L^2 \to L^2$$

such that for each $f \in L^2$ corresponds $Tf = u$ defined by (18). It is clear that T is linear. We can also see that T is continuous. In fact, if in (18) we choose $\phi = u$ we obtain

$$\int |\nabla u|^2 = \int fu. \tag{19}$$

Using Poincarè's Inequality (inequality (9)) and the Cauchy-Schwarz inequality we obtain from (19)

$$c \int u^2 \leq \Big(\int f^2\Big)^{\frac{1}{2}} \Big(\int u^2\Big)^{\frac{1}{2}}$$

and then

$$\Big(\int |Tf|^2\Big)^{\frac{1}{2}} \leq c\Big(\int f^2\Big)^{\frac{1}{2}}.$$

Moreover, because the image of T is contained in $H_0^1(\Omega)$ and this space is compactly embedded in $L^2(\Omega)$, we conclude that T is a *compact operator*.

Finally, we observe (we leave the verification of this fact to the reader) that T is a symmetric operator in L^2, that is,

$$\int Tf \cdot g = \int f \cdot Tg.$$

We have to use the spectral theory of symmetric compact operators, cf. [1], which can be applied to our operator T. First, we see that the eigenvalues of T are positive. Indeed, suppose that $Tf = \mu f$ for $\mu \in \mathbb{R}$ and $f \in L^2$, $f \neq 0$. Let $u = Tf$, then from (18) we get

$$\int \nabla u \cdot \nabla \phi = \frac{1}{\mu} \int u\phi, \qquad \phi \in H_0^1. \tag{20}$$

Choosing $\phi = u$ in (20) we obtain

$$\int |\nabla u|^2 = \frac{1}{\mu} \int u^2$$

which implies that $\mu > 0$. To obtain (20) we assumed that $\mu \neq 0$, what it is a consequence of (18).

Thus, we can conclude from the theory of symmetric compact operators that there exists a sequence of eigenvalues $\{\mu_n\}$ of T, all positive, and such that $\mu_n \to 0$; we denote by ϕ_n a normalized eigenfunction, that is,

$$\int |\phi_n|^2 = 1.$$

Consequently, we have

$$T\phi_n = \mu_n \phi_n, \tag{21}$$

what shows, in particular, that $\phi_n \in H_0^1$. Using (18) with $f = \phi_n$, $u = T\phi_n$, we obtain

$$\mu_n \int \nabla\phi_n \nabla v = \int \phi_n v, \qquad \forall\, v \in H_0^1,$$

where it follows that ϕ_n is a weak solution of

$$\begin{cases} -\Delta\phi_n = \mu_n\phi_n & \text{in } \Omega \\ \phi_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{22}$$

Thus, we obtain the spectrum of the Laplacian (with Dirichlet boundary conditions), $\lambda_n = \frac{1}{\mu_n}$:

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \to +\infty.$$

But more Mathematics tells us that $\lambda_1 < \lambda_2$, that is, λ_1 is a single eigenvalue. It can also be proved that it is possible to obtain an eigenfunction corresponding to λ_1, such that $\phi_1(x) > 0$ for $x \in \Omega$, and moreover, $\frac{\partial\phi_1}{\partial\nu}(x) < 0$ for $x \in \partial\Omega$, if Ω is regular.

4.1 The regularity of the eigenfunctions

Now, we will use the following result called L^p regularity of the elliptic equations (cf. [5]). This result generalizes the Theorem 4.1.

Theorem 4.2 *For all $f \in L^p(\Omega)$, $p > 1$, Ω a bounded domain, there exists a unique solution of the problem (17) $u \in W_0^{1,p} \cap W^{2,p}$.*

We also need a result about the C^α regularity of the solutions of the problem (17).

Theorem 4.3 *If $f \in C^\alpha(\overline{\Omega})$, $0 < \alpha \leq 1$, then the solutions of (17) belongs to $C^{2,\alpha}(\overline{\Omega})$. Moreover, if $f \in C^{k,\alpha}(\overline{\Omega})$ then $u \in C^{k+2,\alpha}(\overline{\Omega})$.*

Now, we will use these two results to show that the eigenfunctions ϕ_n introduced above are, in fact, C^∞. We use a process known as "bootstrap". Because $\phi_n \in H_0^1$, it follows from the Sobolev embedding that $\phi_n \in L^{2^*}$, and from the L^p regularity obtained from (22) it follows that $\phi_n \in W^{2,2^*}$. Again by the Sobolev embeddings we obtain that $\phi_n \in L^q$ where $\frac{1}{q} = \frac{1}{2^*} - \frac{2}{N}$. We follow this process until we obtain that $\phi_n \in W^{2,s}$ where $\frac{1}{s} < \frac{2}{N}$. (We leave to the reader to check that this will happen after a suitable number of iterations of this process). Once at this stage, we obtain that $\phi_n \in C^\alpha$ and the C^α regularity give us $\phi_n \in C^{2,\alpha}$. From here, it is only to continue the process.

5 A First Example of Nonexistence

Using some examples we see that the problem (5) not always has a solution. In this section we see an example which the nonexistence comes from the location of the nonlinear part with respect to the spectrum of the Laplacian. Let Ω be a bounded domain in $\mathbb{R}^N$. Which values $\lambda \in \mathbb{R}$ the problem

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{23}$$

has a solution? What we will prove is the following assertion: there exists $\lambda_* > 0$ such that if $\lambda \geq \lambda_*$ then (23) has no classical solutions. To prove this fact, we obtain a necessary condition to solve (23). Integrating by parts twice and using the fact that $-\Delta\phi_1 = \lambda_1\phi_1$, we obtain

$$\lambda_1 \int u\phi_1 = \lambda \int e^u \phi_1. \tag{24}$$

Because $e^s \geq es$ is valid for all $s \in \mathbb{R}$, it follows from (24)

$$\lambda_1 \int u\phi_1 \geq e\lambda \int u\phi_1. \tag{25}$$

At this point, we need the Maximum Principle (cf. [5]) in the following form which says "If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is such that $-\Delta u \geq 0$ in Ω and $u = 0$ on $\partial\Omega$, then $u \geq 0$".

We conclude that if $\lambda > 0$, then a possible solution of (23) is nonnegative. Because $\phi_1 > 0$ in Ω, we obtain from (25) that

$$\lambda_1 \geq e\lambda$$

which is a necessary condition to obtain a solution of (23) for $\lambda > 0$. Consequently, if $\lambda > \frac{\lambda_1}{e}$, the problem (23) has no solution.

6 The Pohozaev's Identity and a Second Example of Nonexistence

Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a classical solution of the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{26}$$

where Ω is a regular bounded domain in $\mathbb{R}^N$, $N \geq 3$. Then, u satisfies the following identity

$$2N \int_\Omega F(u) - (N-2) \int_\Omega uf(u) = \oint_{\partial\Omega} (x \cdot \nu)|\nabla u|^2 \tag{27}$$

where $F(s) = \int_0^s f(t)dt$ and ν is the exterior unit normal vector on $\partial\Omega$ at a point x.

The proof of (27) consists in multiplying the equation in (26) by $x \cdot \nabla u$ and then integrating by parts. We leave to the reader this homework.

Now let us consider the problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{28}$$

which has a solution $u = 0$. We ask if there is values of p for what the unique solution is the trivial solution. The answer comes from the application of the identity (27) with $f(t) = |t|^{p-1}t$. In this case $F(t) = \frac{1}{p}|t|^p$ and (27) is reduced to

$$\left(\frac{2N}{p} - (N-2)\right) \int_\Omega |u|^p = \oint_{\partial\Omega} (x \cdot \nu)|\nabla u|^2. \tag{29}$$

From here we obtain a necessary condition for the existence of a non-trivial solution when Ω is a star-shaped domain. (Without loss of generality we can assume that a star-shaped domain is star-shaped with respect to the origin, that is, $x \cdot \nu > 0$). Thus, from (29) we have the condition $\frac{2N}{p} - (N-2) > 0$. If $p \geq \frac{N+2}{N-2}$, the problem (28) has no non-trivial solution.

7 Existence of a solution for Problem (5)

With our goal of illustrating several methods of solving equations of the kind (5), we will concentrate in problems asymptotically non-linear. Other examples can be seen in [K], [R], [S], [MN]. The equation (5) is *asymptotically linear* if the following limits exist and there are $L^\infty(\Omega)$ functions:

$$\lim_{s\to+\infty} \frac{f(x,s)}{s} = a(x), \qquad \text{and} \qquad \lim_{s\to-\infty} \frac{f(x,s)}{s} = b(x). \tag{30}$$

For the asymptotic linear problems we have the following results

Theorem 7.1 *Asumme that f satisfies the condition (F) in Section 2. Assume also that*

$$a(x), b(x) \leq \mu < \lambda_1, \qquad \forall\, x \in \Omega \text{ a.e.}, \tag{31}$$

where μ is a constant. Then, the problem (5) has a classical solution.

Theorem 7.2 *Assume that f satisfies the condition (F) and*

$$\lambda_j < c_j \leq a(x), b(x) \leq c_{j+1} < \lambda_{j+1}, \qquad \forall\, x \in \Omega \text{ a.e.}, \tag{32}$$

where c_j and c_{j+1} are constants. Then, the problem (5) has a classical solution.

The proofs for the two theorems above are made using two methods: topological, via degree theory, and variational, minimization for Theorem 7.1 and Saddle Point Theorem for Theorem 6.2. In any of the methods we need to prove a compactness result, apriori bounds and Palais Smale condition which we will show next.

8 A priori bounds and the Fučik Spectrum

In this section we will prove the following result:

Theorem 8.1 *Assume that the problem (5) is asymptotically linear with* $a, b \in L_1^\infty(\Omega)$. *Let us suppose also that the unique solution* $v \in H_0^1(\Omega)$ *of*

$$\begin{cases} -\Delta v = av^+ - bv^- & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{33}$$

is the trivial solution. Then, there exists $K > 0$ *such that*

$$\|u\|_{H'} \leq K$$

for all the solutions of (5).

Remark 8.1 *In the statement of this theorem we have used the notation*

$$v^+ = \max\{v(x), 0\} \qquad v^- = v^+(x) - v(x).$$

The problem (33) can be studied as the linearization of (5) at infinity. This reflects very well the symmetry of the problem.

Proof. *Theorem 8.1*

By contradiction, assume that there exists a sequence $u_n \in H_0^1(\Omega)$ of solutions of (5) such that $\|u_n\|_{H'} \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|_{H'}}$. Because $\|v_n\|_{H'} = 1$, using the weak compactness of the unit ball in a Hilbert space and the Sobolev immersion theorem, we see that there exists a subsequence of (v_n), which we will also call (v_n) , and a $v_0 \in H_0^1$ such that

$$v_n \rightharpoonup v_0 \qquad \text{weak in } H_0^1 \tag{34}$$
$$v_n \to v_0 \qquad \text{on norm of } L^2. \tag{35}$$

In addition, this subsequence can be chosen in such a way that $v_n(x) \to v_0(x)$ a.e. and $|v_n(x)| \leq h(x)$, where $h \in L^2$. These last assertions can be proved using the Riesz-Fisher theorem about the completeness of L^2. Then, from the fact that u_n is a generalized solution of (5) we have

$$\int \nabla v_n \cdot \nabla \phi = \int \frac{f(x, u_n)}{\|u_n\|_{H'}}\phi, \qquad \forall \phi \in H_0^1(\Omega). \tag{36}$$

It is possible to prove that

$$\frac{f(x,u_n)}{\|u_n\|_{H'}} \to a(x)v_0^+ - b(x)v_0^-, \qquad \text{weak in } L^2. \tag{37}$$

Taking limits in (36) we obtain

$$\int \nabla v_0 \cdot \nabla \phi = \int (a(x)v_0^+ - b(x)v_0^-)\phi$$

from we obtain that $v_0 = 0$ because of the hypothesis of the theorem. On the other hand, making $\phi = v_n$ in (36) we obtain

$$1 = \int |\nabla v_n|^2 = \int \frac{f(x,u_n)}{\|u_n\|} v_n.$$

Taking limits and using the fact that $v_n \to v_0 = 0$ (strong convergence in norm), we conclude that this is a contradiction.

■

8.0.1 The Fučik spectrum

If a and b are constants, we will give the following definition: the pair (a,b) belongs to the *Fučik spectrum* if the equation (33) has a non-trivial solution. Notice that if $a = b$ we have the usual spectrum studied in Section 3. It is easy to see that the pairs (λ_1, b) for any $b \in \mathbb{R}$, and (a, λ_1) for all $a \in \mathbb{R}$ also belong to the Fučik spectrum. It is still unknow the whole characterization of the Fučik spectrum. There are several works about it: Gallouet-Kavian, Ruf, d'Aujourd'hui, Micheletti, Magalhães, Gossez and the author. For more details, see [F] and [K]. If $\Omega = (0,1)$, we deal with an ordinary differential equation (ODE) and it is possible to describe the spectrum completely. In this case, we are helped by the theorem of existence and uniqueness which ensure us that the non-trivial solutions have only simple zeros. From here, we obtain the fact that where a solution v of (33) has definite sign, it satisfies an ODE. For example, on an interval where $v > 0$, it satisfies the equation $-v'' = av$, which is the same equation for each interval. Then, if l_+ is the length of those intervals, we have

$$a = \frac{\pi^2}{l_+^2}. \tag{38}$$

Thus, all intervals where $v > 0$ have the same length. In the same way, for the intervals where $v < 0$ we have

$$b = \frac{\pi^2}{l_-^2}. \tag{39}$$

Finally, using (38) and (39) we obtain the following equations:

- If v has k intervals where $v > 0$ and k intervals where $v < 0$:

$$\frac{k\pi}{\sqrt{a}} + \frac{k\pi}{\sqrt{b}} = 1.$$

- If v has $k+1$ intervals where $v > 0$ and k intervals where $v < 0$:

$$\frac{(k+1)\pi}{\sqrt{a}} + \frac{k\pi}{\sqrt{b}} = 1.$$

- If v has k intervals where $v > 0$ and $k+1$ intervals where $v < 0$:

$$\frac{k\pi}{\sqrt{a}} + \frac{(k+1)\pi}{\sqrt{b}} = 1.$$

We invite the reader to draw the pictures corresponding to these curves on the (a, b) -plane.

Theorem 8.2 *If $a(x)$ and $b(x)$ satisfy the condition (31) or the condition (32), we obtain that the problem (33) possesses only the trivial solution.*

Proof. We will do it under condition (32). The other case can be proved in a similar way. Let

$$\lambda = \frac{\lambda_j + \lambda_{j+1}}{2} \qquad \text{and} \qquad \mu = \lambda_{j+1} - \lambda_j.$$

Let us write (33) in the form

$$\begin{cases} -\Delta v - \lambda v = (a-\lambda)v^+ - (b-\lambda)v^- & \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{40}$$

Let $T : L^2 \to L^2$ the linear operator defined in the following way: for each $f \in L^2$, Tf is the solution w (unique) to the problem

$$-\Delta w - \lambda w = f \quad \text{in } \Omega \tag{41}$$

$$w = 0 \quad \text{on } \partial\Omega. \tag{42}$$

It can be proved that (see Section 9) the norm of T is

$$\|T\| = \frac{2}{\mu}. \tag{43}$$

Consequently (40) is equivalent to

$$v = T[(a-\lambda)v^+ - (b-\lambda)v^-] =: Sv.$$

Then, a solution of (33) is a fixed point of S and vice-versa. Because $v = 0$ is a solution of (33), it is enought to prove that S is a contraction to conclude the theorem. Let us denote by $W : L^2 \to L^2$ the operator (non-linear) defined by

$$Wv = (a-\lambda)v^+ - (b-\lambda)v^-.$$

Observe from the hypoteses (32):

$$\|a(x)-\lambda\|_{L^\infty}, \|b(x)-\lambda\|_{L^\infty} \le \max\{|\lambda - c_j|, |\lambda - c_{j+1}|\} =: k \qquad (44)$$

and $k < \frac{\mu}{2}$. Denoting $f(x,s) = (a-\lambda)s^+ - (b-\lambda)s^-$, we observe that for each $x \in \Omega$ fixed, we have

$$|f(x,s_1) - f(x,s_2)| \le k|s_1 - s_2|, \qquad \forall s_1, s_2 \in \mathbb{R}.$$

Then

$$\int_\Omega |Wv_1(x) - Wv_2(x)|^2 \le k^2 \int_\Omega |v_1(x) - v_2(x)|^2.$$

From here we obtain

$$\|Sv_1 - Sv_2\| \le \frac{2}{\mu}\|Wv_1 - Wv_2\|_{L^2} \le \frac{2k}{\mu}\|v_1 - v_2\|_{L^2},$$

which shows that S is a contraction. ■

9 The Problem (37) and the Fredholm Alternative

We will show that (37) has a unique solution.

Theorem 9.1 *If $\lambda \in \mathbb{R}$ is not in the spectrum of $-\Delta$ under Dirichlet condition, then the problem (37) has a unique solution.*

Proof.
Uniqueness: if u and v are soultiuons of (37), then $w = u - v$ is solution of $-\Delta w = \lambda w$. Because $\lambda \neq \lambda_j$, we conclude that $w = 0$.

Existence: We use the fact that the eigenfunctions ϕ_j corresponding to λ_j normalized by $\int \phi_j^2 = 1$ form an complete orthonormal system in L^2. Thus we will solve problem (37) using Fourier Series. Let $f \in L^2$, we can write it as $f = \sum_{j=0}^{\infty} c_j \phi_j$. Denote by $w \in L^2$ the solution of (37) we are looking for, and let us write $w = \sum_{j=0}^{\infty} a_j \phi_j$. Then, (37) can be written as

$$\sum_{j=0}^{\infty} a_j(\lambda_j - \lambda)\phi_j = \sum_{j=0}^{\infty} c_j \phi_j. \quad (45)$$

Then $a_j = \frac{c_j}{\lambda_j - \lambda)}$. The convergence of the serie comes from the fact that

$$|a_j| \leq |c_j| \frac{1}{d} \quad (46)$$

where d is the distance of λ to the spectrum, that is, $d = \min\{|\lambda - \lambda_j| : j = 1, 2, \ldots\}$. From (46) immediately it follows that

$$\|T\| \leq \frac{1}{d}. \quad (47)$$

We notice that $\|u\|_{L^2} = \sum a_j^2$. In (47) we obtain the equality if we solve the problem

$$\begin{cases} -\Delta w - \lambda w = \phi_j & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \quad (48)$$

where ϕ_j is an eigenfunction corresponding to the nearest eigenvalue λ_j to λ.

The Fredholm Alternative follows from the coupling of Theorem 9.1 and knowledge about the spectrum. It is the following

Given the problem

$$\begin{cases} -\Delta u - \lambda u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (49)$$

- (i) or the problem has a unique solution
- (ii) or the problem

$$\begin{cases} -\Delta u - \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (50)$$

has a nontrivial solution, that is, λ is an eigenvalue. In this case, (50) has a solution if and only if $\int f\phi$ for all eigenfunctions corresponding to the eigenvalue λ. (See [B]). It is well known from the Theory of symmetric compact operators that those eigenfucntions form a finite dimensional subespace. ∎

10 Proof of Theorem 6.2 using Degree Theory

We consider the family of problems

$$\begin{cases} -\Delta u = \theta\lambda u + (1-\theta)f)(x,u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{51}$$

where $\theta \in [0,1]$ and $\lambda = (\lambda_j + \lambda_{j+1})/2$. Let

$$a_\theta = \lim_{s\to+\infty} \frac{\theta\lambda s + (1-\theta)f(x,s)}{s},$$

$$b_\theta = \lim_{s\to-\infty} \frac{\theta\lambda s + (1-\theta)f(x,s)}{s}.$$

We can see that there exist constants c_1 and c_2 such that

$$\lambda_j < c_1 \leq a_\theta(x), b_\theta(x) \leq c_2 < \lambda_{j+1}.$$

Then, from a result in Section 7, all the possible solutions of problem (51) are bounded by a constant K. (There is a small reason to show that K does not depend of θ). We continue by defining the operators $S : L^2 \to L^2$ as the inverse operator of $-\Delta$ under Dirichlet boundary condition, and by $T_\theta : L^2 \to L^2$ such that

$$T_\theta = S(\theta\lambda u + (1-\theta)f(x,u)).$$

We see that T_θ is because S is compact and the nonlinear operator involved is continuous and bounded in L^2 (See [deF]). Thus, (51) is equivalent to

$$(I - T_\theta)u = 0.$$

Because of the apriori estimate, for all $\|u\|_{L^2} = K+1$, $(I - T_\theta)u \neq 0$. Then

$$deg(I - T_1, B_r(0), 0) \neq 0$$

for all $r > 0$. Then,

$$deg(I - T_0, B_{K+1}(0), 0) \neq 0,$$

and from this fact we get a $u \in B_{K+1}(0)$ such that $(I - T_0)u = 0$. But this is equivalent to say that u is a solution of the equation (5).

11 The Palais-Smale Condition

Let X be a Banach space and $\Phi : X \to \mathbb{R}$ a C^1 functional. We say that Φ satisfies the *Palais-Smale condition* at the level c (denoted by $(PS)_c$) if all sequences $\{u_n\} \in X$ such that

$$\Phi(u_n) \to c, \qquad \text{and} \qquad \Phi'(u_n) \to 0 \tag{52}$$

have a convergent subsequence (in the X norm). If Φ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$, we simply say that Φ satisfies the (PS) condition.

Theorem 11.1 *Suppose that the condition (F) and also hypothesis that (33) only possesses the solution $u = 0$. Then the functional Φ defined in (13) satisfies the Palais-Smale condition.*

Proof. Suppose that there exists a sequence satisfying (52) (such sequences are called *Palais-Smale sequences*) and such that

$$\|u_n\|_{H'} \to \infty.$$

We proceed exactly as in the proof of Theorem 7.1 and we obtain a contradiction. We leave to the reader to complete the proof.

■

12 Proof of Theorem 6.1 by Minimization

It follows from (31) that there exists $\overline{\mu}$ such that $\mu \le \overline{\mu} < \lambda_1$, and that

$$f(x,s) \le \overline{\mu}|s| + c, \qquad \forall s \in \mathbb{R}, \forall x \in \Omega$$

and then

$$F(x,s) \le \frac{\overline{\mu}}{2}|s|^2 + C, \qquad \forall s \in \mathbb{R}, \forall x \in \Omega.$$

Consequently, we can obtain the following estimates

$$\Phi(u) \ge \frac{1}{2}\int |\nabla u|^2 - \frac{\overline{\mu}}{2}\int u^2 - C$$

and then

$$\Phi(u) \ge \frac{1}{2}\left(1 - \frac{\overline{\mu}}{\lambda_1}\right)\int |\nabla u|^2 - C \tag{53}$$

where we have used the Poincaré inequality. The last inequality tells us that $\Phi : H_0^1(\Omega) \to \mathbb{R}$ is bounded below. Moreover, it tells us that Φ is *coercive*, that is, if $\|u\| \to +\infty$, then $\phi(u) \to +\infty$.

Thus, we can prove Theorem 7.1 showing that the infimum of Φ is attained. There is two ways to prove this fact. The first one uses the following result from General Topology.

Theorem 12.1 *Let X be a compact topological space, and $\Phi :\to \mathbb{R}$ a lower semicontinuous function. Then Φ is bounded below and there exists $u \in X$ such that*

$$\Phi(u) = \inf_X \Phi.$$

The second way to prove that our infimum of Φ (the one defined at (13)) is attained is using the Ekeland's Variational Principle.

Theorem 12.2 *Let X be a complete metric space, and $\Phi : X \to \mathbb{R}$ a function bounded below and lower semicontinuous. Then, given $\epsilon > 0$ there exists $u_\epsilon \in X$ such that*

$$\Phi(u_\epsilon) \leq \inf_X \Phi + \epsilon, \qquad \Phi(u_\epsilon) \leq \Phi(u) + \epsilon d(u, u_\epsilon), \quad \forall x \in X.$$

Proof. (of Theorem 7.1 using Theorem 12.1)
Let $R > 0$ be such that

$$\Phi(u) \geq 1, \qquad \forall u \in H_0^1, \|u\| \geq R,$$

which follows from (53). Now consider the functional Φ restricted to the closed ball $B_R(O)$ of radius R and center at O in H_0^1. Such ball is weakly compact. On the other hand, the functional Φ is weakly lower semicontinuous. In fact, its first part is a norm (it is a fact that the norm is weakly lower semicontinuous) and the second part is more than that, indeed is continuous, which is a consequence of the compact embeeding from H_0^1 in L^2. Then, we can apply the Theorem 12.1 and conclude that there exists a $u_0 \in B_R(O)$ such that

$$\Phi(u_0) = \inf_{B_R(O)} \Phi.$$

Because $\Phi(O) = 0$, it follows that $\Phi(u_0) \leq 0$. Then, $\|u_0\|_{H'} < R$, due to the boundary of the ball we have $\Phi(u) \geq 1$. Consequently, there exists $\epsilon > 0$ such that $\|u_0 + \frac{1}{\epsilon} w\|_{H'} < R$ for all $0 < t \leq \epsilon$ and all $\|w\|_{H'} = 1$. Thus,

$$\Phi(u_0) \leq \Phi(u_0 + tw)$$

and then

$$\frac{\Phi(u_0 + tw) - \Phi(u_0)}{t} \geq 0.$$

Taking limits we obtain

$$< \Phi'(u_0), w > \geq 0, \qquad \forall w \in H_0^1, \ \|w\| = 1.$$

Then $< \Phi'(u_0), w >= 0$ for all $w \in H_0^1$, which implies that $\Phi'(u_0) = 0$. That is, u_0 is a critical point of Φ and then it is a solution of (5). ∎

Proof. (of Theorem 7.1 using Theorem 12.2)

It follows from (53) that the functional Φ is bounded below. We also have that Φ is continuous in the H_0^1-norm. Then, we can apply the Ekeland's Variational Principle. Thus, given $\epsilon = 1/n$, there exists $u_n \in H_0^1$ such that

$$\Phi(u_n) \leq \inf_{H_0^1} \Phi + \frac{1}{n} \tag{54}$$

and

$$\Phi(u_n) \leq \Phi(u) + \frac{1}{n} d(u, u_n), \qquad \forall u \in H_0^1. \tag{55}$$

Taking $u = u_n + tw$ in the last equation, where $t > 0$ and $\|w\|_{H^1} = 1$, we obtain

$$\Phi(u_n) \leq \Phi(u_n + tw) + \frac{1}{n} t,$$

that is

$$\frac{\Phi(u_n) - \Phi(u_n + tw)}{t} \leq \frac{1}{n}.$$

Taking limits we obtain

$$- < \Phi'(u_n), w > \leq \frac{1}{n},$$

and then

$$| < \Phi'(u_n), w > | \leq \frac{1}{n}, \qquad \forall \|w\|_{H^1} = 1. \tag{56}$$

Thus

$$\|\Phi'(u_n)\| \leq \frac{1}{n}. \tag{57}$$

Consequently $\{u_n\}$ is a Palais-Smale sequence. Because Φ satisfies (PS), we obtain a subsequence $\{u_{n_j}\}$ and $u_0 \in H_0^1$ such that $\{u_{n_j}\} \to u_0$. Because Φ is a C^1 functional, we follow from (54) and (57) that

$$\Phi(u_0) = \inf_{H_0^1} \Phi \qquad \text{and} \qquad \Phi'(u_0) = 0.$$

∎

13 Proof of Theorem 7.2 using Variational Methods

Initially we observe that the following inequalities come from the variational characterizationof the eigenvalues of $-\Delta$ under Dirichlet boundary conditions. Given $\lambda_j < \lambda_{j+1}$ two consecutives eigenvalues. Denote by H_j is the subspace of H_0^1 generated by the eigenfunctions of λ_j with $i \leq j$. Then, $H_j^\perp$ the orthogonal complement of H_j which is generated by the eigenfunctions corresponding to the eigenvalues λ_i with $i \geq j+1$. We observe that H_j has finite dimension. The inequalities we are refering to are the following

$$\int |\nabla v|^2 \leq \lambda_j \int v^2, \qquad \forall v \in H_j, \tag{58}$$

$$\int |\nabla w|^2 \geq \lambda_{j+1} \int w^2, \qquad \forall w \in H_j^\perp. \tag{59}$$

The idea is to apply the following result, The Saddle Point Theorem, due to Rabinowitz.

Theorem 13.1 *Let X ba a Banach space and V a finite dimensional subspace. Let W be the topological orthogonal complement of V, that is, a subspace of X such that $X = V \oplus W$. Let $\Phi : V \to \mathbb{R}$ a C^1 that satisfies (PS). Suppose there exist constants a and b, and a real number $R > 0$ such that*

$$\Phi|_{\partial B_R(O)\cap V} \leq a \tag{60}$$

$$\Phi|_W \geq b \tag{61}$$

$$a < b \tag{62}$$

where $B_R(O)$ is the ball of radius R centered at O in the space X. Then Φ has a critical point ta the level c definde by

$$c = \inf_{\gamma\in\Gamma} \max_{u\in \overline{B_R(O)\cap V}} \Phi(\gamma(u))$$

where $\Gamma = \{\gamma : \overline{B_R(O)} \cap V \to X; \gamma \text{ continuous}, \gamma(u) = u \ \forall u \in \partial B_R(O) \cap V\}$.

Let us return to the functional Φ defined in (13) and let us see that satisfies the hypothesis of the Theorem 13.1 because of the hypothesis of Theorem 7.2. We obtain from (F) and (32) the estimates

$$\hat{c}_1 \leq \frac{f(x,s)}{s} \leq \hat{c}_2 \qquad \forall x \in \Omega \text{ and } |s| > s_0$$

where $\lambda_j < \hat{c}_1 < \hat{c}_2 < \lambda_{j+1}$. Integrating we obtain the constants c_1, c_2, d_1 and d_2 such that $\lambda_j < c_1 < c_2 < \lambda_{j+1}$, $d_1, d_2 > 0$ and

$$\frac{1}{2}c_1 s^2 + d_1 \leq F(x,s) \leq \frac{1}{2}c_2 s^2 + d_2. \tag{63}$$

Using (63) we obtain for $v \in H_j$

$$\Phi(v) \leq \frac{1}{2}\int |\nabla v|^2 - \frac{1}{2}c_1 \int v^2 + c$$

and using (58) we obtain

$$\Phi(v) \leq \frac{1}{2}\int |\nabla v|^2 - \frac{1}{2}\frac{c_1}{\lambda_j}\int |\nabla v|^2 + c.$$

Because $c_1 > \lambda_j$, it follows that $\Phi(v) \to -\infty$ when $\|v\| \to \infty$.

In the same way from (63) and (59) it follows for $w \in H_j^\perp$ that

$$\Phi(w) \geq \frac{1}{2}\int |\nabla w|^2 - \frac{1}{2}\frac{c_2}{\lambda_{j+1}}\int |\nabla w|^2 - c.$$

But $c_2 < \lambda_{j+1}$, so we conclude that

$$\inf\{\Phi(w) : w \in H_j^\perp\} = b > -\infty.$$

Let us choose now $R > 0$ in such a way that

$$\sup\{\Phi(v) : v \in H_j, \|v\| = R\} = a < b.$$

Finally, because Φ satisfies (PS), cf. Section 11, we can apply Theorem 13.1 and conclude the existence of a critical point for our functional Φ and consequently a solution of (5), proving the Theorem 7.2.

14 Proof of Theorem 13.1

The proof of Theorem 13.1 depends on the next two lemmata. The first one is a weak version of the Deformation Lemma. This is enough for our purposes.

Lemma 14.1 *Let X be a Banach space and $\Phi : X \to \mathbb{R}$ a C^1 functional satisfying the (PS) condition. Suppose that there is no critical point of Φ at the level c. Then, there exists $\epsilon_0 > 0$ such that satisfies the following property: given $0 < \epsilon < \bar{\epsilon} < \epsilon_0$ and a continuous function $\eta : [0,1] \times X \to X$ such that*

- $\eta(t,x) = x$, $\forall t \in [0,1]$ *and* x *such that* $|\Phi(x) - c| \geq \bar{\epsilon}$,
- $\Phi(\eta(1,x)) < c - \epsilon$ *if* $\Phi(x) \leq c + \epsilon$.

The reader can see a more general proof in the reference [R].

The next lemma says that there is a 'linking' between the sets W and $\partial B_R(O) \cap V$, which appeared in Theorem 13.1

Lemma 14.2 *Let* X, V, W *and* Γ *as in Theorem 13.1. Then, given* $\gamma \in \Gamma$ *there exists* $u_0 \in B_R(O) \cap V$ *such that* $\gamma(u_0) \in W$.

Proof. Let $P : X \to X$ a projection of X on V through W. This is a continuous linear operator defined in the following way: given $u \in X$, u can be written in a unique way as $u = v + w$, and P is defined by $Pu = w$. The continuity of P is included in the assertion $X = V \oplus W$, and that such decomposition is possible because V is finite dimensional. (Observe that if X is a banach space, in general, there exist closed subspaces F_1 which have no topological complement, that is, it is not possible to write $X = F_1 \oplus F_2$ for some F_2).

We define the following mapping

$$S : \overline{B_R(O)} \cap V \to V$$

by

$$Sv = P(\gamma(v)).$$

Observe that $Sv = v$ if $v \in \partial B_R(O) \cap V$. Consider the homotopy

$$H(t, \cdot) = (1-t)I + tS, \qquad 0 \leq t \leq 1.$$

Because $H(t,v) = v \neq 0$ for all $t \in [0,1]$, $v \in \partial B_R(O) \cap V$, we conclude that

$$deg(H(t,\cdot), B_R(O), O) = const.$$

Because $H(0,\cdot) = I$, that constant must be equal to 1. Then, because $H(1,\cdot) = S$, we have that there exists $u_0 \in B_R(O) \cap V$ such that $Su_0 = 0$. That is, we have $\gamma(u_0) \in W$. ■

Proof. (Theorem 13.1)

Suppose by contradiction that c is not a critical value. Then, there exists $\epsilon_0 > 0$ such that Φ has no critical points between levels $c - \epsilon_0$ and $c + \epsilon_0$. In fact, if there is no such ϵ_1 we would have a sequence $\{u_n\} \subset X$ such that

$$\Phi(u_n) \to c \qquad \text{and} \qquad \Phi'(u_n) = 0. \tag{64}$$

Using the $(PS)_c$ condition, it follows that there exists a subsequence and $u_0 \in X$ such that $u_{n_j} \to u_0$. Because Φ is C^1, we obtain

$$\Phi(u_0) = c \qquad \text{and} \qquad \Phi'(u_0) = 0$$

which is a contradiction. (This is the ϵ_0 which appears in Lemma 14.1). Now, let us take $\bar{\epsilon} < c - a$ and $\bar{\epsilon} < \epsilon_0$. Observe that, by Lemma 14.2, $c \geq b$. Now take $\epsilon < \bar{\epsilon}$, and let $\gamma \in \Gamma$ such that

$$\max_{v \in B_R(O) \cap V} \Phi(\gamma(v)) < c + \epsilon. \tag{65}$$

We affirm that $\tilde{\gamma}(v) = \eta(1, \gamma(v))$ belongs to Γ. Because the continuity is clear, it is enough to show that $\tilde{\gamma}(v) = v$ for $v \in \partial B_R(O) \cap V$. That is, it follows from the fact that $\gamma(v) = v$ and $\Phi(v) \leq a < c - \bar{\epsilon}$. Then, the conclusion of the first statement in Lemma 14.1 tells us that $\eta(1, v) = v$. Now, it follows from the second statement in Lemma 14.1 that

$$\max_{v \in B_R(O) \cap V} \Phi(\tilde{\gamma}(v)) < c - \epsilon,$$

contradicting the fact of being infimum. ■

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Remark 14.1 *The previous bibliography only refers to books. In those books the reader will find the references to the corresponding articles. Obviously, the interested reader in knowing more about this area or doing research must read the articles published in technical journals. As a reference we will mentioned the most used journals in this area:*

Advances in Differential Equations
Annales Institute Henri Poincaré
Annali della Scuola Normale Superiore di Pisa
Archive Rational Mechanics and Analysis
Communications in Mathematical Physics
Communications in Partial Differential Equations
Communications in Pure and Applied Mathematics
Dynamic Systems and Applications
Journal of Differential Equations
Journal of Mathematical Analysis and its Applications
Nonlinear Analysis - Theory, Methods and Applications
Proccedings of the Royal Society of Edinburg
Transactions of the American Mathematical Society