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Socratic Topology

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"For since the whole of nature is akin, and the soul has learned all things, there is nothing to prevent someone, upon being reminded of one single thing - which men call learning - from rediscovering all the rest, if he is courageous and faints not in the search". Socrates from the Dialogues of Plato

FORWARD.

To refer to this text as one on "Socratic Topology" is most likely to be an overstatement. Perhaps "SemiSocratic Topology" would provide more of a description. Exercises appear at first to be nonexistent. However, on more careful reading, one observes that many of the theorems come with no proof. The student reader of this text is expected to provide the proofs with some hints from time to time from the Professor. Proofs of the more complex results are included in the text. It is intended in part for the student to learn from these proofs some of the techniques required to handle some of the other results. In this author's opinion, this course will be of great value in preparing a student for research in mathematics.

The more rigid method of asking students to present proofs of theorems goes back to the late Professors Moore and Wall of the University of Texas. Thus the "Texas Method" was born and continued by their students and many of their mathematical descendants (which includes this author) . In more recent times, it appears to have lost some of its appeal. Perhaps, in this age of knowledge, it is difficult to find students that have not already retained enough information to make the Texas(or Socratic)Method practical.

These notes have been developed over a period of a few years and used in a number of graduate topology classes at Louisiana State University. The initial notes were inherited from the long chain of Socratic [Texas School] Professors and were already substantially different than those first developed by Professor Moore.

The material in this text is intended to be covered in two semesters. It is recommended that chapters 1-8 be covered in the first semester and the remaining chapters in the second semester with some selection. Chapters 9, 10, and 11 should be covered as preparation for algebraic topology. Chapters 12-19 can be selected according to the Professor's preference. If it is discovered that the first eight chapters cannot be covered in the first semester, then postpone some of this material to the second semester. It is important that the material be covered carefully and at a pace to accommodate the needs of the students, even if some later material goes uncovered.

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1 FUNDAMENTAL CONCEPTS

A **topological space** (X, τ) is a set X together with a collection τ of subsets of X such that: — (1) $\emptyset \in \tau$ and $X \in \tau$;

(2) if $A \in \tau$ and $B \in \tau$, then $A \cap B \in \tau$; and

(3) the union of a family of members of τ is a member of τ.

The members of τ are call **τ-open** (or simply **open**) subsets of X, and τ is called a **topology** on X.

A subcollection β of τ is called a **basis** for τ provided each member of τ is a union of members of β.

A subcollection σ of τ is called a **subbasis** for τ provided the collection of all finite intersections of members of σ is a basis for τ.

Theorem 1.1 *Let β be a collection of subsets of a set X. Then β is a basis for a unique topology on X if and only if:*

(1) X is a union of members of β; and

(2) if $A \in \beta$ and $B \in \beta$ and $p \in A \cap B$, then there exists $K \in \beta$ such that $p \in K \subseteq A \cap B$.

Theorem 1.2 *Let σ be a collection of subsets of a set X. Then σ is a subbasis for a unique topology on X.*

If X is a set and $\tau = \{X, \emptyset\}$, then (X, τ) is a topological space. The topology τ is called the **indiscrete topology** on X.

If X is a set and τ is the collection of all subsets of X, then (X, τ) is a topological space, and τ is called the **discrete topology** on X.

Example. Let X be a set and let $\tau = \{A\colon A \subseteq X \text{ and } X \backslash A \text{ is finite }\} \cup \{\emptyset\}$. Then (X, τ) is a topological space. The topology τ is called the **cofinite topology** on X.

Example. Let $\mathbb{R}$ be the set of real numbers and let $\beta = \{(a, b)\colon a < b \text{ in } \mathbb{R}\}$, where $(a, b) = \{x\colon a < x < b\}$. Then β is a basis for a unique topology $\mathcal{E}$ on $\mathbb{R}$ called the **Euclidean topology** or **usual topology**.

Example. Let $\mathbb{R}$ be the set of real numbers and let $\beta = \{[a, b)\colon a < b \text{ in } \mathbb{R}\}$, where $[a, b) = \{a \le x < b\}$. Then β is a basis for a unique topology τ on $\mathbb{R}$ called the **half open interval topology**.

Example. Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. For each $(a, b) \in \mathbb{R}^2$ and each $\epsilon > 0$ let $D((a, b), \epsilon) = \{(x, y) \in \mathbb{R}^2\colon (x - a)^2 + (y - b)^2 < \epsilon^2\}$ i.e., the open disk with center (a, b) and radius ϵ. Let $\beta = \{D((a, b), \epsilon)\colon (a, b) \in \mathbb{R}^2 \text{ and } \epsilon > 0\}$. Then β is a basis for a unique topology τ on $\mathbb{R}^2$. Note that τ is the product topology induced by the Euclidean topology on $\mathbb{R}$. Again, the topology τ on $\mathbb{R}^2$ is called the **Euclidean** (or **usual**) **topology**.

Example. Let $\mathbb{R}$ be the set of real numbers and let $\tau = \{A\colon A \subseteq \mathbb{R}$ and either $0 \in R\backslash A$ or $\mathbb{R}\backslash\{1,2\} \subseteq A\}$. Then $(\mathbb{R}, \tau)$ is a space.

Example. Let $\mathbb{N}$ be the set of positive integers and let $B_n = \{2n-1, 2n\}$ for each $n \in \mathbb{N}$. Then $\beta = \{B_n\colon n \in \mathbb{N}\}$ is a basis for a unique topology on $\mathbb{N}$.

Example. Let $H = \{(a,b) \in \mathbb{R}^2\colon 0 \leq b\}$ and let $L = \{(a,0) \in \mathbb{R}^2\}$. Let $\beta_1 = \{D((a,b),\epsilon) : D((a,b),\epsilon) \subseteq H\backslash L\}$, and let $\beta_2 = \{[D((a,0),\epsilon) \cap (H\backslash L)] \cup \{(a,0)\}\colon a \in \mathbb{R}, \epsilon > 0\}$. Then $\beta = \beta_1 \cup \beta_2$ is a basis for a unique topology on H.

Example. Let $\mathbb{N}$ denote the set of positive integers, and for each $n \in \mathbb{N}$ let $B_n = \{(\frac{1}{n}, y) \in \mathbb{R}^2\colon 0 \leq y \leq 1\}$. Let $B_0 = \{(0,y) \in \mathbb{R}^2\colon 0 \leq y \leq 1\}$ and let $A = \{(x,0) \in \mathbb{R}^2\colon 0 \leq x \leq 1$. Let $X = A \cup \bigcup_{n=0}^{\infty} B_n$ with the relative Euclidean topology of $\mathbb{R}^2$.

If (X,τ) is a topological space, then we frequently supress the mention of τ and simply refer to X as a space.

If X is a space and $p \in X$, then a subset N of X is called a **neighborhood** of p provided there exists an open set G such that $p \in G \subseteq N$.

If E is a subset of a space X and $p \in X$, then p is called a **limit point** of E provided that for each neighborhood N of p, $(N\backslash\{p\}) \cap E \neq \emptyset$. We use E' to denote the set of all limit points of E in X.

A subset E of a space X is said to be **closed** provided that $E' \subseteq E$.

1.3 **Theorem.** *Let X be a space. Then:*

(a) *$\emptyset$ and X are closed;*

(b) *the intersection of any family of closed sets is closed; and*

(c) *if A and B are closed sets, then $A \cup B$ is closed.*

If E is a subset of a space X, then the **closure** of E is defined $\overline{E} = E \cup E'$. A point $p \in X$ is called an **interior point** of E if E if there exists a neighborhood N of p such that $N \subseteq E$. Observe that p is an interior point of E provided E is a neighborhood of p. The set E° of all interior points of E in X is called the **interior** of E in X. A point $b \in X$ is called a **boundary point** of E provided that for each neighborhood M of b, we have that $M \cap E \neq \emptyset$ and $M \cap (X\backslash E) \neq \emptyset$. The set ∂E of all boundary points of E in X is called the **boundary** of E in X. Observe that $\partial E = \overline{E} \cap \overline{X\backslash E}$.

1.4 **Theorem.** *If E is a subset of a space X, then these are equivalent:*

(a) *E is open;*

(b) *$E = E^\circ$;*

(c) *$X \backslash E$ is closed; and*

(d) $E \cap \partial E = \emptyset$.

1.5 Theorem. *If E is a subset of a space X, then these are equivalent:*

(a) *E is closed;*

(b) $E = \overline{E}$;

(c) *$X \backslash E$ is open;*

(d) $\partial E \subseteq E$.

If E is a subset of a space X and $\tau = \{E \cap V : V \text{ is open in } X\}$, then observe that (E, τ) is a space. The topology τ is called the **relative topology** on E, and (E, τ) is called a **subspace** of X. Again, we refer to E as a subspace of X if the topology on E is the relative topology. The statement "A is open [closed] in E" means that $A \subseteq E$ and that A is open [closed] in the relative topology.

1.6 Theorem. *Let E be a subspace of a space X and let A be a subset of E. Then:*

(a) *A is open in E if and only if there exists an open set U in X such that $A = E \cap U$; and*

(b) *A is closed in E if and only if there exists a closed set F in X such that $A = E \cap F$.*

If E is a subspace of a space X and $A \subseteq E$, then:

(a) $\overline{A}^E$ denotes the closure of A in E;

(b) $\partial_E A$ denotes the boundary of A in E; and

(c) $A^{\circ E}$ donotes the interior of A in E.

1.7 Theorem. *Let E be a subspace of a space X, $A \subseteq E$, and let $p \in E$. Then p is a limit point of A in E if and only if p is a limit point of A in X.*

1.8 Problem. *Let E be a subspace of a space X and let $A \subseteq E$. Determine the relation (if any) between:*

(a) *$\overline{A}^E$ and $\overline{A}$;*

(b) *$\partial_E A$ and ∂A;*

(c) *$A^{\circ E}$ and A°.*

A space X is said to be **second countable** if the topology of X has a countable basis.

If X is a space and $p \in X$, the a **local basis at** p is a collection β of neighborhoods of p such that each neighborhood of p contains a member of β.

If X is a space and $p \in X$, then X is said to be **first countable at** p if X has a countable local basis at p. If X is first countable at each of its points, then X is said to be a **first countable space**.

A subset E of a space X is said to be **dense in** X if $\overline{E} = X$.

1.9 **Theorem**. *A subset E of a space X is dense in X if and only if each nonempty open subset of X contains a point of E.*

A space X is said to be **separable** if X contains a countable dense subset.

1.10 **Theorem**. *Each second countable space is a separable first countable space.*

A **cover** of a space X is a collection γ of subsets of X such that X is a union of members of γ. If the elements of γ are open [closed], then γ is called an **open** [**closed**] cover of X.

If γ is a cover of a space X and $\beta \subseteq \gamma$ is also a cover of X, then β is called a **subcover** of γ.

A space X is called a **Lindelöf space** if each open cover of X has a countable subcover.

1.11 **Theorem**. *Each second countable space is a Lindelöf space.*

2 FUNCTIONS, QUOTIENTS, AND PRODUCTS

If X and Y are spaces and $f\colon X \to Y$ is a function, then f is said to be **continuous at** $p \in X$ if for each neighborhood N of $f(p)$ in Y, there exists a neighborhood M of p in X such that $f(M) \subseteq N$. If f is continous at each point of X, then f is said to be **continuous**.

Note that $f\colon X \to Y$ is continuous at $p \in X$ if and only if for each open set U in Y with $f(p) \in U$, there exists an open set V in X such that $p \in V$ and $f(V) \subseteq U$.

2.1 **Theorem**. *Let $f\colon X \to Y$ be a function from a space X into a space Y. These are equivalent:*

(a) *f is continuous;*

(b) *If U is open in Y, then $f^{-1}(U)$ is open in X;*

(c) *If F is closed in Y, then $f^{-1}(F)$ is closed in X; and*

(d) *$f(\overline{A}) \subseteq \overline{f(A)}$ for each $A \subseteq X$.*

A function $f\colon X \to Y$ from a space X into a space Y is said to be **open** [**closed**] if the image of each open [closed] set in X is open [closed] in Y.

A function $f\colon X \to Y$ from a space X into a space Y is called a **homeomorphism** if f is bijective (one-one and onto), f is continuous, and f^{-1} is continuous. We say that X and Y are **homeomorphic** under f.

2.2 **Theorem**. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous functions. Then gf is continuous. Moreover, if f and g are homeomorphisms, then f^{-1}, g^{-1}, and gf are homeomorphisms.*

If $f\colon X \to Y$ is a function and $E \subseteq X$, then the function $g\colon E \to Y$ such that $g(x) = f(x)$ for each $x \in E$ is called the **restriction** of f to E and is denoted $g = f|E$.

If $g: E \to Y$ is a function and $E \subseteq X$, then $f: X \to Y$ is called an **extension** of g provided $g = f|E$.

A subspace E of a space X is called a **retract** of X if there is a continuous function $r: X \to E$ such that $r|E = 1_E$ (the identity map on E). The function r is called a **retraction** of X onto E.

2.3 **Theorem.** *A subspace E of a space X is a retract of X if and only if each continuous function $f: E \to Y$ has a continuous extension to X.*

If A is a subspace of a space X, $f: X \to Y$ a continuous function, and B is a subspace of Y such that $f(A) \subseteq B$, then the continuous function $g: A \to B$ defined by $g(x) = f(x)$ for each $x \in A$ is called the **function from A into B defined by f**.

An **embedding** of a space X into a space Y is an injective (one-one) continuous function $f: X \to Y$ which defines a homeomorphism $X \to f(X)$.

Let $\{X_\alpha : \alpha \in A\}$ be a collection of spaces, and let $\sigma = \{\prod_{\alpha \in A} U_\alpha : U_\beta$ is open in X_β for some $\beta \in A$ and $U_\lambda = X_\lambda$ for all $\lambda \neq \beta$ in $A\}$. Then σ is a subbasis for a unique topology τ on $P = \prod_{\alpha \in A} X_\alpha$. The space (P, τ) is called the **topological product** of the collection of spaces $\{X_\alpha : \alpha \in A\}$ and τ is called the **product topology**.

2.4 **Theorem.** *Let $\{X_\alpha : \alpha \in A\}$ be a collection of spaces. Then the projection*

$$\pi_\beta : \prod_{\alpha \in A} X_\alpha \to X_\beta$$

is an open continuous surjective (onto) function for each $\beta \in A$.

If X is a space and M is a set, then X^M denotes the topological product of M copies of X and $\Delta(X)$ denotes the diagonal of X^M.

2.5 **Theorem.** *Let $\{X_\alpha : \alpha \in A\}$ be a collection of spaces, Y a space, and $f: Y \to \prod_{\alpha \in A} X_\alpha$. Then f is continuous if and only if $\pi_\beta f$ is continuous for each $\beta \in A$.*

2.6 **Theorem.** *Let X be a space and let M be a set. Then the diagonal injection $\Delta: X \to X^M$ is an embedding and $\Delta(X)$ is a retract of X^M.*

2.7 **Theorem.** *Let $\{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in A}$ be a collection of continuous functions and let*

$$p_\beta : \prod_{\alpha \in A} X_\alpha \to X_\beta$$

and

$$\pi_\beta \colon \prod_{\alpha \in A} Y_\alpha \to Y_\beta$$

be projections for each $\beta \in A$. *Define*

$$f \colon \prod_{\alpha \in A} X_\alpha \to \prod_{\alpha \in A} Y_\alpha$$

such that $\pi_\beta f = f_\beta p_\beta$ *for each* $\beta \in A$. *Then* f *is continuous.*

The function $f = \prod_{\alpha \in A} f_\alpha$ in 2.7 is called the **product** of the functions $\{f_\alpha \colon \alpha \in A\}$.

A function $f \colon X \to Y$ from a space X to a space Y is called a **quotient map** provided f is surjective, and E is open in Y if and only if $f^{-1}(E)$ is open in X. Note that f is continuous.

2.8 **Theorem**. *Let* $f \colon X \to Y$ *be a quotient map and let* $E \subseteq Y$. *Then* E *is closed in* Y *if and only if* $f^{-1}(E)$ *is closed in* X.

2.9 **Theorem**. *If* $f \colon X \to Y$ *is an open [closed] surjective continuous function, then* f *is a quotient map.*

2.10 **Theorem**. *Let* $f \colon X \to Y$ *be a quotient map and let* $g \colon Y \to Z$ *be a function* (Z *is a space*). *Then* g *is continuous if and only if* gf *is continuous.*

2.11 **Theorem**. *Let* X *be a space,* Y *a set,* $f \colon X \to Y$ *a surjective function, and let* $\tau = \{V \colon V \subseteq Y$ *and* $f^{-1}(V)$ *is open in* $X\}$. *Then* (Y, τ) *is a space and* f *is a quotient map.*

The topology τ in 2.11 is called the **quotient topology** on Y induced by f.

If X is a space, R is an equivalence relation on X, $p \colon X \to X/R$ is the natural map, and τ the quotient topology on X/R inducted by p, then $(X/R, \tau)$ is called the **quotient space** of X mod R.

Let X be a space and A a subspace of X. Define $R = \{(a,b) \in X \times X : a = b$ or $a, b \in A\}$. Then R is an equivalence relation on X. The quotient space X/R is denoted X/A.

Let $f \colon X \to Y$ be a function and let $K(f) = \{(a,b) \in X \times X \colon f(a) = f(b)\}$. Then $K(f)$ is an equivalence relation on X. The space $X/K(f)$ is called the **decomposition space** of f, and the relation $K(f)$ is called the **kernel relation** of f. Let $p_f \colon X \to X/K(f)$ denote the natural map.

Let E be a space, $I = [0,1]$, $X = E \times I$, and let $A = E \times \{1\}$. The space $Cone(E) = X/A$ is called the **cone** over E. Let $q \colon X \to Cone(E)$ be the natural map. The point $q(A)$ is called the **vertex** of the cone and the quotient space $S(E) = Cone(E)/q(E \times \{0\})$ is called the **suspension** of

E. Let $p\colon Cone(E) \to S(E)$ be the natural map and let v be the vertex of $Cone(E)$. Then the points $p(v)$ and $p(q(E \times \{0\}))$ are called the **poles** of $S(E)$.

Let $\mathbb{R}$ denote the space of real numbers with the usual (open interval) topology, and let $n \in \mathbb{N}$. The space $\mathbb{R}^n$ is called ***n*-dimensional Euclidean space**. The **unit open *n*-cell** is $E^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n\colon x_1^2 + \cdots + x_n^2 < 1\}$. The **unit *n*-cell** is $B^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n\colon x_1^2 + \cdots + x_n^2 \leq 1\}$. The **unit *n*-sphere** is $S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}\colon x_0^2 + \cdots + x_n^2 = 1\}$. Note that S^1 is the boundary of B^2 in $\mathbb{R}^2$. The **unit *n*- cube** is I^n, where $I = [0, 1]$.

A space X is called an ***n*-cell** [***n*-sphere**] if X is homeomorphic to B^n [S^n].

2.12 Exercise. Show that I^2 is a 2-cell.

In fact, I^n is an n-cell for each n.

Let $X = R^{n+1}\backslash\{(0, \ldots, 0)\}$ and let $\rho = \{((a_0, \ldots, a_n), (b_0, \ldots, b_n)) \in X \times X\colon$ there exists $\lambda \in R\backslash\{0\}$ such that $b_i = \lambda a_i$ for $0 \leq i \leq n\}$. Then ρ is an equivalence relation on X. The quotient space $RP^n = X/\rho$ is called **real projective *n*-space**.

Two points $(x_0, \ldots, x_n)$ and $(y_0, \ldots, y_n)$ in S^n are said to be **antipodal** provided $x_i = -y_i$ for $i = 0, 1, \ldots, n$.

2.13 Theorem. (a) *The boundary of B^n in $\mathbb{R}^n$ is S^{n-1};*

(b) *The suspension of S^n is homeomorphic to S^{n+1};*

(c) *The quotient space obtained by identifying antipodal points on S^n is homeomorphic to RP^n.*

2.14 Exercise. Establish the table:

	CountableProducts	Subspaces	Continuous
First Countable	yes	yes	no
Second Countable	yes	yes	no
Separable	yes	if open	yes
Lindelöf	no	if closed	yes

A property of topological spaces which is preserved by homeomorphisms is called a **topological property**.

2.15 Theorem. *Each of the following is a topological property:*

(a) *First countable;*

(b) *Second countable;*

(c) *Separable; and*

(d) *Lindelöf.*

3 SEPARATION AXIOMS

A space X is called a **T_0-space** if for each pair a, b of distinct points of X, there exists an open set U such that either $a \in U$ and $b \in X \backslash U$ or $b \in U$ and $a \in X \backslash U$.

A space X is called a **T_1-space** if for each pair of distinct points a, b, there exists an open set U such that $a \in U$ and $b \in X \backslash U$.

A space X is called a **T_2-space** (or **Hausdorff space**) if for each pair a, b of distinct points of X, there exist open sets U and V such that $a \in U$, $b \in V$, and $U \cap V = \emptyset$.

If X is a space and $p \in X$, then X is said to be **regular at** p if each neighborhood of p contains a closed neighborhood of p. If X is regular at each of its points, then X is said to be a **regular space**.

3.1 **Theorem**. *If X is a space, then these are equivalent:*

(a) *X is regular;*

(b) *If A is closed and $p \in X \backslash A$, then there exist open sets U and V in X such that $A \subseteq U$, $p \in V$, and $U \cap V = \emptyset$; and*

(c) *If $p \in X$ and U is an open set containing p, then there exists an open set V such that $p \in V \subseteq \overline{V} \subseteq U$.*

If X is a space and $p \in X$, then X is said to be **completely regular at** p if for each neighborhood W of p, there exists a continuous function $f \colon X \to I = [0,1]$ (usual) such that $f(p) = 0$ and $f(X \backslash W) = 1$. If X is completely regular at each of its points, then X is said to be a **completely regular space**.

A space X is said to be a **normal space** if for each pair A, B of disjoint closed sets, there exist open sets U and V such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

3.2 **Theorem**. *A space X is normal if and only if for each closed set A and each open set U such that $A \subseteq U$, there exists an open set V such that $A \subseteq V \subseteq \overline{V} \subseteq U$.*

A space X is called a **T_3-space** if X is a regular T_1-space.

A space X is called a **T_4-space** if X is a normal T_1-space.

3.3 **Theorem**. *A space X is a T_1-space if and only if $\{p\}$ is closed for each $p \in X$.*

3.4 **Theorem**. *Each completely regular space is regular.*

3.5 **Theorem**. *A space X is a Hausdorff space if and only if $\Delta(X) = \{(x,x) \colon x \in X\}$ is closed in $X \times X$.*

3.6 **Theorem**. *If X is a T_i-space, then X is a T_{i-1}-space; for $i = 1, 2, 3, 4$.*

3.7 **Exercise**. Give an example of a T_{i-1}-space which is not a T_i-space for $i = 1, 2, 3, 4$.

A property of spaces is said to be **hereditary** if each subspace of a space having that property also has the property.

A property of spaces is said to be **productive** if the topological product of a family of spaces having this property also has the property.

3.8 **Theorem**. *T_i for $i = 0, 1, 2, 3, 4$, regular, completely regular, and normal are all topological properties.*

3.9 **Theorem**. *Each of T_i for $i = 0, 1, 2, 3$, regular, and completely regular are both hereditary and productive properties.*

Note that normal and T_4 are excluded in 3.9.

3.10 **Lemma**. *Let X be a space, D a dense subset of $I = [0, 1]$ and let $\{G_d : d \in D\}$ be an open cover of X such that $\overline{G}_a \subseteq G_b$ whenever $a < b$ in D. Define $f: X \to I$ by $f(x) = glb\{d \in D : x \in G_d\}$ for each $x \in X$. Then for $y \in I$:* (a) $\{x \in X : f(x) < y\} = \cup\{G_d : d \in D, d < y\}$; *and* (b) $\{x \in X : f(x) \leq y\} = \cap\{G_d : d \in D, y < d\}$. *Moreover, f is continuous.*

3.11 **Urysohn's Lemma**. *Let X be a normal space and A and B disjoint closed subsets of X. Then there exists a continuous function $f: X \to I$ such that $f(A) = 0$ and $f(B) = 1$.*

Proof. Let τ denote the topology of X, and let D denote the set of dyadic rational numbers in $[0, 1]$.

We will establish the existence of a function $g: D \to \tau$ such that $A \subseteq g(0)$, $g(1) = X$, $g(d) \subseteq X \backslash B$ if $d < 1$ in D, and $\overline{g(a)} \subseteq g(b)$ for $a < b$ in D. For this purpose, let $\mathbb{N}$ denote the set of all positive integers and define $D_n = \{k/2^n : k = 0, 1, \ldots, 2^n\}$ for each $n \in \mathbb{N}$.

Let $\mathcal{M} = \{(D_n, g_n): \quad g_n : D_n \to \tau,\ A \subseteq g(0),\ g_n(1) = X,\ \overline{g_n(k/2^n)} \subseteq g_n((k+1)/2^n)$ for $k = 0, 1, \ldots 2^n - 1$, and $g_n((2^n - 1)/2^n) \subseteq X \backslash B\}$.

Note that $\mathcal{M} \neq \emptyset$ by taking $n = 1$.

Define a partial order $\leq$ on $\mathcal{M}$ by $(D_n, g_n) \leq (D_m, g_m)$ if $n \leq m$ (and hence $D_n \subseteq D_m$) and $g_m | D_n = g_n$.

By HMP there exists a maximal chain $\mathcal{M}'$ in $\mathcal{M}$.

Let $\mathbb{N}' = \{n \in \mathbb{N} : (D_n, g_n) \in \mathcal{M}'\}$.

We claim that $D = \bigcup\{D_n : n \in \mathbb{N}'\}$. Let $d \in D$ and suppose that $d \notin D_n$ for all $n \in \mathbb{N}'$. Let $d = k/2^m$. Since D_n contains those dyadic rational numbers with denominators smaller than 2^n, we have that $2^m > 2^n$ and hence $m > n$ for each $n \in \mathbb{N}'$. Thus $\mathbb{N}'$ is finite. Let $r = max\{n' : n' \in \mathbb{N}'\}$ and define $g_m : D_m \to \tau$ so that $(D_m, g_m) \in \mathcal{M}$ and $g_m | D_r = g_r$. Then $\mathcal{M}' \cup \{(D_m, g_m)\}$ is a chain in $\mathcal{M}$, contradicting the maximality of $\mathcal{M}'$. We conclude that $D = \bigcup\{D_n : n \in \mathbb{N}'\}$.

Now define $g: D \to \tau$ so that $g|D_n = g_n$ for each $n \in \mathbb{N}'$, and let $G_d = g(d)$ for each $d \in D$, and apply the Lemma 3.10. ■

3.12 **Theorem**. *Each T_4-space is completely regular.*

3.13 **Theorem**. *Each regular Lindelöf space is normal.*

Proof. Let X be a regular Lindelöf space and let A and B be disjoint closed subsets of X. For each $p \in A$, let W_p be an open set containing p such that $\overline{W_p} \subseteq X \backslash B$. Now A is Lindelöf, so there exists a countable subcover $\{W_n : n \in \mathbb{N}\}$ of the W_p's ,i.e., $A \subseteq \bigcup_{n\in\mathbb{N}} W_n$ and $\overline{W_n} \subseteq X\backslash B$ for each n. Similarly, there exists a countable collection $\{H_n : n \in \mathbb{N}\}$ of open sets such that $B \subseteq \bigcup_{n\in\mathbb{N}} H_n$ and $\overline{H_n} \subseteq X\backslash A$ for each n. For each $n \in \mathbb{N}$, let $U_n = W_n \backslash \bigcup_{i=1}^{n} \overline{H_i}$ and let $V_n = H_n \backslash \bigcup_{i=1}^{n} \overline{W_i}$. Then $U_n = W_n \cap (X \backslash \bigcup_{i=1}^{n} \overline{H_i})$ and $V_n = H_n \cap (X\backslash \bigcup_{i=1}^{n} \overline{W_i})$, so that each U_n and each V_n is open. Let $U = \bigcup_{n\in\mathbb{N}} U_n$ and let $V = \bigcup_{n\in\mathbb{N}} V_n$. Then U and V are open sets.

We claim that $A \subseteq U$. Let $p \in A$. Then $p \in W_n$ for some n and $p \notin \overline{H_m}$ for all m, so that $p \in W_n \backslash \bigcup_{i=1}^{n} \overline{H_i} = U_n$. Thus $p \in U$ and $A \subseteq U$.

A similar argument shows that $B \subseteq V$.

It remains to show that $U \cap V = \emptyset$.

Suppose that $U \cap V \neq \emptyset$ and let $p \in U \cap V$. Then $p \in U_n$ for some n and $p \in V_m$ for some m. We can assume that $m \leq n$. Then $p \in W_n \backslash \bigcup_{i=1}^{n} \overline{H_i}$ so that $p \notin H_m$, since $m \leq n$. But $p \in V_m$ implies that $p \in H_m$. This contradiction proves that $U \cap V = \emptyset$, and hence X is normal. ■

In 3.14 we will use $\mathbb{R}$ to denoted the real numbers with the usual topology.

3.14 **Lemma**. *Let X be a space and let $f_n : X \to \mathbb{R}$ a sequence of continuous functions. If $\sum_{n=1}^{\infty} M_n$ is a convergent series of positive numbers such that $|f_n(x)| \leq M_n$ for each $x \in X$ and each n, then for each $x \in X$, the series $\sum_{n=1}^{\infty} f_n(x)$ converges, and the function $f: X \to \mathbb{R}$ defined by $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is continuous.*

Proof. (i). Let $x \in X$, and let $S_n = \sum_{j=1}^{n} f_j(x)$. We will show that S_n is a Cauchy sequence. Let $K_n = \sum_{j=1}^{n} M_j$. Then K_n is convergent and hence Cauchy. Let $\epsilon' > 0$. Then there exists a positive integer N such that $|K_n - K_m| <$

ϵ' when $m,n > N$. Let $n \geq m > N$, then $|S_n - S_m| = |\sum_{j=m+1}^{n} f_j(x)| \leq \sum_{j=m+1}^{n} |f_j(x)| \leq \sum_{j=m+1}^{n} M_j = K_n - K_m = |K_n - K_m| < \epsilon'$. Thus S_n is Cauchy, and hence convergent, so that $\sum_{j=1}^{\infty} f_j(x)$ is convergent.

(ii). Define $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for each $x \in X$. Let $K = \sum_{j=1}^{\infty} M_j$ and let $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $|K - \sum_{j=1}^{k} M_j| < \epsilon/3$ when $k \geq n_0$; and for $k \geq n_0$ and $z \in X$, $|\sum_{j=k+1}^{\infty} f_j(z)| \leq \sum_{j=k+1}^{\infty} |f_j(z)| \leq \sum_{j=k+1}^{\infty} M_j = K - \sum_{j=1}^{k} M_j < \epsilon/3$.

(iii). We claim that f is continuous. Let $x \in X$ and $\epsilon > 0$. We will show that f is continuous at x. Let n_0 be as in (ii). For each $j = 1, 2, \ldots, n_0$ there exists an open set U_j containing x such that $|f_j(x) - f_j(z)| < \epsilon/(3n_0)$ when $z \in U_j$, since each f_j is continuous. Let $U = \bigcap_{j=1}^{n_0} U_j$. Then $x \in U$ and U is open.

Let $y \in U$. Then $|\sum_{j=1}^{n_0}(f_j(x) - f_j(y))| \leq \sum_{j=1}^{n_0} |f_j(x) - f_j(y)| < \sum_{j=1}^{n_0}(\epsilon/(3n_0))$ $= \epsilon/3$, so that $|f(x) - f(y)| = |\sum_{j=1}^{n_0}(f_j(x) - f_j(y)) + \sum_{j=n_0+1}^{\infty} f_j(x) - \sum_{j=n_0+1}^{\infty} f_j(y)|$ $\leq |\sum_{j=1}^{n_0}(f_j(x) - f_j(y))| + |\sum_{j=n_0+1}^{\infty} f_j(x)| + |\sum_{j=n_0+1}^{\infty} f_j(y)| < (\epsilon/3) + (\epsilon/3) + (\epsilon/3)$ $= \epsilon$, and we conclude that f is continuous. ■

3.15 **The Tietze Extension Theorem.** *Let X be a normal space, C a closed subspace of X, and let $f\colon C \to I$ a continuous function. Then there exists a continuous function $g\colon X \to I$ such that $g|C = f$.*

Proof. We represent $I = [-1, 1]$. Let $H_1 = \{x \in C\colon f(x) \geq 1/3\}$ and let $K_1 = \{x \in C\colon f(x) \leq -1/3\}$. Then $H_1 = f^{-1}[1/3, 1]$ and $K_1 = f^{-1}[-1, -1/3]$, so that H_1 and K_1 are disjoint closed subsets of X. By Urysohn's Lemma, there exists a continuous function $f_1\colon X \to [-1/3, 1/3]$ such that $f_1(H_1) = 1/3$ and $f_1(K_1) = -1/3$. Note that $|f(x) - f_1(x)| \leq 2/3$ for all $x \in C$. Let $H_2 = \{x \in C\colon f(x) - f_1(x) \geq 2/3\}$ and let $K_2 = \{x \in C\colon f(x) - f_1(x) \leq -2/9\}$. Then H_2 and K_2 are disjoint closed subsets of X. Again, by Urysohn's Lemma, there exists a continuous function $f_2\colon X \to [-2/9, 2/9]$ such that $f_2(H_2) = 2/9$ and $f_2(K_2) = -2/9$. We have $|f(x) - f_1(x) - f_2(x)| \leq 4/9$ for all $x \in C$.

Continuing this construction, we obtain a sequence of continuous functions

$$f_n\colon X \to [-2^{n-1}/3^n, 2^{n-1}/3^n]$$

such that $|f(x) - f_1(x) - f_2(x) - \cdots - f_n(x)| \le (2/3)^n$ for each $x \in C$. Let $M_n = 2^{n-1}/3^n$. Then $\sum_{n=1}^{\infty} M_n = 1$. Applying 3.14, we define $g(x) = \sum_{n=1}^{\infty} f_n(x)$, so that g is continuous. For $x \in C$, we have $|f(x) - \sum_{j=1}^{n} f_j(x)| < (2/3)^n$, and hence $|f(x) - g(x)| = 0$ and $f(x) = g(x)$. We conclude that $g|C = f$. ∎

4 COMPACTNESS

A space X is said to be **compact** if each open cover of X has a finite subcover.

4.1 **Theorem.** *Let E be a subspace of a space X. Then E is compact if and only if each open cover of E by open sets in X has a finite subcover.*

A family α of sets has the **finite intersection property** if the intersection of any finite subfamily of α in nonempty.

4.2 **Theorem.** *A space X is compact if and only if each family of closed sets in X which has the finite intersection property has a nonempty intersection.*

4.3 **Theorem.** *A closed subspace of a compact space is compact.*

4.4 **Theorem.** *The continuous image of a compact space is compact.*

4.5 **Theorem.** *A compact subspace of a Hausdorff space is closed.*

4.6 **Theorem.** *Let X be a compact space, Y a Hausdorff space, and $f\colon X \to Y$ a bijective continuous function. Then f is a homeomorphism.*

4.7 **Theorem.** *A compact Hausdorff space is normal.*

4.8 **Theorem.** *A compact regular space is normal.*

4.9 **Theorem.** *Let X and Y be spaces with Y compact. Then the first projection $\pi\colon X \times Y \to X$ is a closed map.*

Proof. Let C be a closed subset of $X \times Y$ and let $p \in X\backslash\pi(C)$. Then $(p \times Y) \cap C = \emptyset$. For each $y \in Y$ let U_y be open in X and let V_y be open in Y such that $(p, y) \in U_y \times V_y$ and $(U_y \times V_y) \cap C = \emptyset$. Now $\{V_y\colon y \in Y\}$ is an open cover of Y. Let $V_1, \ldots, V_n$ be a finite subcover with $U_1, U_2, \ldots, U_n$ corresponding. Let $U = \bigcap_{j=1}^{n} U_j$. Then $p \in U$ and U is open in X.

Suppose $U \cap \pi(C) \neq \emptyset$ and let $q \in U \cap \pi(C)$. Then there exists $y_0 \in Y$ such that $(q, y_0) \in C$. Now $y_0 \in V_{j_0}$ for some j_0 and $q \in U_{j_0}$. Hence $(q, y_0) \in (U_{j_0} \times V_{j_0}) \cap C$. This contradiction proves that $U \cap \pi(C) = \emptyset$ and $\pi(C)$ is closed. ∎

4.10 **Wallace's Lemma.** *Let X, Y, and Z be spaces, A a compact subset of X, B a compact subset of Y, $f: X \times Y \to Z$ a continuous function, and let W be an open subset of Z such that $f(A \times B) \subseteq W$. Then there exists an open set U in X and an open set V in Y such that $A \subseteq U$, $B \subseteq V$, and $f(U \times V) \subseteq W$.*

4.11 **Lemma.** *Let X be a space and let γ be a collection of subsets of X with the finite intersection property. Then γ is maximal with respect to having the finite intersection property if and only if $A \cap H \neq \emptyset$ for all $A \in \gamma$ implies that $H \in \gamma$.*

Proof. Suppose that γ is maximal with respect to having the finite intersection property. Then $A_1, A_2, \ldots, A_n \in \gamma$ implies that $A_1 \cap A_2 \cap \cdots \cap A_n \in \gamma$. Suppose that $H \cap A \neq \emptyset$ for all $A \in \gamma$. Then $\gamma \cup \{H\}$ has the finite intersection property and hence $H \in \gamma$.

Suppose that γ has the finite intersection property and that $H \cap A \neq \emptyset$ for all $A \in \gamma$ implies that $H \in \gamma$. Let $\gamma \subseteq \gamma'$, where γ' has the finite intersection property. Let $H \in \gamma'$. Then $H \cap A \neq \emptyset$ for all $A \in \gamma$, so that $H \in \gamma$, $\gamma' \subseteq \gamma$, $\gamma' = \gamma$, and γ is maximal with respect to the finite intersection property. ∎

4.12 **Lemma.** *A space X is compact if and only if $\cap\{\overline{A}: A \in \gamma\} \neq \emptyset$ for each collection γ of subsets of X which is maximal with respect to having the finite intersection property.*

Proof. Suppose that X is compact and that γ is a collection of subsets of X which is maximal with respect to having the finite intersection property. Then $\{\overline{A}: A \in \gamma\}$ is a collection of closed sets with the finite intersection property, and hence $\bigcap\{\overline{A}: A \in \gamma\} \neq \emptyset$ by 4.2.

Suppose, on the other hand, that the condition holds. Let σ be a collection of closed subsets of X with the finite intersection property. In view of 4.2, we need only show that $\bigcap\{A: A \in \sigma\} \neq \emptyset$. Let $\mathcal{A} = \{\alpha: \alpha$ is a collection of subsets of X with fip and $\sigma \subseteq \alpha\}$. Then $\mathcal{A} \neq \emptyset$, since $\sigma \in \mathcal{A}$. Now $\mathcal{A}$ is partially ordered by $\subseteq$. By HMP, there exists a maximal chain $\mathcal{A}'$ in $\mathcal{A}$. Let $\gamma = \bigcup\{\alpha: \alpha \in \mathcal{A}'\}$ $(= \{A: A \in \alpha$ for some $\alpha \in \mathcal{A}'\})$.

We will show that γ is maximal with respect to fip.

Let $A_1, A_2, \ldots, A_n \in \gamma$. Then there exist $\gamma_1, \gamma_2, \ldots, \gamma_n \in \mathcal{A}'$ such that $A_i \in \gamma_i$ for $i = 1, 2, \ldots, n$. Since $\mathcal{A}'$ is a chain, there exists m with $1 \leq m \leq n$, such that $\bigcup_{i=1}^{n} \gamma_i \subseteq \gamma_m$. Thus $A_1, A_2, \ldots, A_n \in \gamma_m$, and since γ_m has fip, we have $A_1 \cap A_2 \cap \cdots \cap A_n \neq \emptyset$. This implies that γ has fip.

To see that γ is maximal with respect to fip, suppose that $\gamma \subseteq \gamma'$ and that γ' has fip. Now $\sigma \subseteq \gamma$, and hence $\gamma' \in \mathcal{A}$. Thus $\mathcal{A}' \cup \{\gamma'\}$ is a chain in $\mathcal{A}$, and since $\mathcal{A}'$ is maximal, we have $\gamma' \in \mathcal{A}'$, $\gamma' \subseteq \gamma$, and hence $\gamma = \gamma'$. It follows that γ is maximal with respect to fip.

Now $\emptyset \neq \bigcap_{A \in \gamma} \overline{A} \subseteq \bigcap_{A \in \sigma} \overline{A} = \bigcap_{A \in \sigma} A$, so that X is compact. ∎

4.13 **The Tychonoff Theorem.** *Let $\{X_\alpha \colon \alpha \in A\}$ be a collection of nonempty spaces. Then $\prod_{\alpha \in A} X_\alpha$ is compact if and only if X_α is compact for each $\alpha \in A$.*

Proof. Let $X = \prod_{\alpha \in A} X_\alpha$.

Suppose that X is compact and let $\alpha \in A$. Then $\pi_\alpha \colon X \to X_\alpha$ is a continuous surjection, so that X_α is compact for each $\alpha \in A$.

Suppose that X_α is compact for each $\alpha \in A$. If V_α is an open set in X_α for $\alpha \in A$, denote by $\tilde{V}_\alpha = \prod_{\beta \in A} T_\beta$, where $T_\beta = V_\alpha$ if $\alpha = \beta$, and $T_\beta = X_\beta$ if $\alpha \neq \beta$. Let γ be a collection of subsets of X which is maximal with respect to the finite intersection property. Then for each $\alpha \in A$, $\{\pi_\alpha(B) \colon B \in \gamma\}$ has fip in X_α. Let $p_\alpha \in \bigcap\{\overline{\pi_\alpha(B)} \colon B \in \gamma\}$ for each $\alpha \in A$, and let $p \in X$ such that $\pi_\alpha(p) = p_\alpha$ for each $\alpha \in A$.

We will show that $p \in \bigcap\{\overline{B} \colon B \in \gamma\}$, so that, in view of 4.12, we will have that X is compact.

Let W be a basic open set in X containing p. Then $W = \prod_{\alpha \in A} V_\alpha$, where V_α is open in X_α for $\alpha \in F$ (some finite subset of A), and $V_\alpha = X_\alpha$ for $\alpha \in A \backslash F$. Then, for each $\alpha \in A$, we have $p_\alpha \in V_\alpha \bigcap \{\overline{\pi_\alpha(B)} \colon B \in \gamma\}$, so that $V_\alpha \cap \pi_\alpha(B) \neq \emptyset$ for each $B \in \gamma$, since V_α is open. Thus, for each $\alpha \in A$, $\tilde{V}_\alpha \cap B \neq \emptyset$ for each $B \in \gamma$, so that $\tilde{V}_\alpha \in \gamma$ by 4.11. Now $W = \bigcap\{\tilde{V}_\alpha \colon \alpha \in F\}$, so that $W \cap B \neq \emptyset$ for each $B \in \gamma$. We obtain that $p \in \overline{B}$ for each $B \in \gamma$, $p \in \bigcap\{\overline{B} \colon B \in \gamma\}$, and X is compact. ∎

4.14 **Theorem.** *Let X be a Hausdorff space, γ a tower of compact subsets of X, and G an open subset of X such that $\cap\{A \colon A \in \gamma\} \subseteq G$. Then $B \subseteq G$ for some $B \in \gamma$.*

If X is a space and $p \in X$, then X is said to be **locally compact at** p if p has a compact neighborhood. If X is locally compact at each of its points, then X is said to be a **locally compact space**.

4.15 **Theorem.** *Let X be a locally compact Hausdorff space and let $p \in X$. Then the family of compact neighborhoods of p is a local basis at p.*

4.16 **Theorem.** *Let X be a locally compact space and let $f \colon X \to Y$ be and open continuous function. Then $f(X)$ is locally compact.*

Note that, as a consequence of 4.4 and 4.15, both compactness and local compactness are topological properties.

4.17 **Theorem.** *Let E be a subspace of a locally compact Hausdorff space*

X. Then E is locally compact if and only if there exists an open set G in X and a closed set F in X such that $E = G \cap F$.

Proof. Suppose that E is locally compact. Then for each $a \in E$, there exists an open set G_a such that $a \in G_a$ and $\overline{G_a} \cap E$ is compact. Since X is Hausdorff, $\overline{G_a} \cap E$ is closed for each $a \in E$. Let $G = \bigcup\{G_a : a \in E\}$. Then G is open and $E \subseteq G$. We will show that E is closed in G. Now for each $a \in E$, $E \cap G_a = G_a \cap (\overline{G_a} \cap E)$, so that $E \cap G_a$ is closed in G_a. Thus $G_a \backslash (E \cap G_a)$ is open in G_a and hence open in G. We obtain that $E = G \backslash \bigcup_{a \in E} [G_a \backslash (E \cap G_a)]$ is closed in G, so that $E = G \cap F$ for some closed subset F of X.

Suppose, on the other hand, that $E = G \cap F$ for some open set G and some closed set F in X. Let $a \in E$. Then there exists an open set U such that $a \in U \subseteq \overline{U} \subseteq G$, and $\overline{U}$ is compact. Then $a \in U \cap E \subseteq \overline{U} \cap E = \overline{U} \cap F \cap G = \overline{U} \cap F = \overline{U}$. Since $\overline{U} \cap F$ is closed and $\overline{U}$ is compact, we have that $\overline{U} \cap E$ is a compact neighborhood of a in E. We conclude that E is locally compact. ■

4.18 **Theorem.** *A locally compact Hausdorff space is completely regular.*

Proof. Let X be a locally compact Hausdorff space. Let $p \in X$ and A a closed subset of X such that $p \in X \backslash A$. Then there exist open sets U and V such that $\overline{U}$ and $\overline{V}$ are compact and $p \in U \subseteq \overline{U} \subseteq V \subseteq \overline{V} \subseteq X \backslash A$. Now $\overline{V}$ is a compact Hausdorff space and hence is normal. By Urysohn's Lemma, there exists a continuous function $f : \overline{V} \to I$ such that $f(p) = 0$ and $f(\overline{V} \backslash U) = 1$. Define $F: X \to I$ by $F|\overline{V} = f$ and $F(X \backslash \overline{V}) = 1$. Now $X = \overline{V} \cup (X \backslash U)$ is a closed cover of X, and $\overline{V} \cap (X \backslash U) = \overline{V} \backslash U$ and F is 1 on $\overline{V} \backslash U$, so that F is continuous, and X is completely regular. ■

4.19 **Theorem.** *Let X be a Hausdorff space and let E be a dense locally compact subspace of X. Then E is open in X.*

Proof. For each $e \in E$, let G_e be an open set such that $\overline{G_e} \cap E$ is compact (and hence closed in X). For each $e \in E$, $E \cap G_e = G_e \cap (\overline{G_e} \cap E)$ is closed in G_e, so that $G_e \backslash (E \cap G_e)$ is open in G_e. Let $G = \bigcup_{e \in E} G_e$. Then G is open, so that $G \backslash \bigcup_{e \in E} [G_e \backslash (E \cap G_e)] = E$ is closed in G. Thus $E = \overline{E} \cap G = X \cap G = G$ and G is open, so that E is open. ■

A subset E of a space X is said to be **nowhere dense in** X provided $\overline{E}^{\circ} = \emptyset$.

4.20 **Lemma.** *Let $\{X_\alpha : \alpha \in A\}$ be a collection of spaces such that an infinite number of the X_α's are not compact, and let E be a closed compact subset of $\prod_{\alpha \in A} X_\alpha$. Then E is nowhere dense in $\prod_{\alpha \in A} X_\alpha$.*

Proof. Suppose that E fails to be nowhere dense. Then $E^{\circ} = (\overline{E})^{\circ} \neq \emptyset$.

Let W be a basic open subset of $\prod_{\alpha \in A} X_\alpha$ contained in E. Then there exists a finite subset F of A such that $\pi_\alpha(W) = X_\alpha$ for $\alpha \in A \backslash F$. Thus $\pi_\alpha(E) = X_\alpha$ is compact for $\alpha \in A \backslash F$, so that all but a finite number of the X_α 's are compact. This contradiction proves that E is nowhere dense. ■

4.21 **Theorem**. *Let $\{X_\alpha : \alpha \in A\}$ be a collection of spaces. Then $\prod_{\alpha \in A} X_\alpha$ is locally compact if and only if each X_α is locally compact and all but a finite number of the X_α's are compact.*

Proof. Suppose that each X_α is locally compact and F is a finite subset of A such that X_α is compact for $\alpha \in A \backslash F$. Let $p \in \prod_{\alpha \in A} X_\alpha$. For each $\alpha \in F$, let N_α be a compact neighborhood of $\pi_\alpha(p)$, and let $N_\alpha = X_\alpha$ for $\alpha \in A \backslash F$. Then $\prod_{\alpha \in A} N_\alpha$ is a compact neighborhood of p, so that $\prod_{\alpha \in A} X_\alpha$ is locally compact.

Suppose that $X = \prod_{\alpha \in A} X_\alpha$ is locally compact. Let $\alpha \in A$ and let $p \in X_\alpha$. For $\beta \in A \backslash \{\alpha\}$, let $q_\beta \in X_\beta$ and let $q \in X$ such that $\pi_\beta(q) = q_\beta$ for $\beta \neq \alpha$ and $\pi_\alpha(q) = p$. Then q has a compact neighborhood N in X. Since π_α is open and continuous, $\pi_\alpha(N)$ is a compact neighborhood of p in X_α, so that X_α is locally compact. Thus X_α is locally compact for each $\alpha \in A$.

Let $p \in X$ and let N be a compact neighborhood of p. Let W be a basic open subset of X such that $p \in W \subseteq N$. Then there exists a finite subset F of A such that $\pi_\alpha(W) = X_\alpha$ for $\alpha \in A \backslash F$ and so $\pi_\alpha(N) = X_\alpha$ is compact for $\alpha \in A \backslash F$. Thus all but a finite number of the X_α 's are compact. ■

A family α of subsets of a space X is said to be **locally finite** if each point of X has a neighborhood which meets at most a finite number of members of α.

If α and β are covers of a space X such that each member of α is contained in some member of β, then α is said to be a **refinement** of β.

A space X is said to be **paracompact** if each open cover of X has a locally finite open refinement.

4.22 **Theorem**. *Each regular Lindelöf space is paracompact.*

Proof. Let X be a regular Lindelöf space and let $\mathcal{V}$ be an open cover of X. For each $p \in X$, let $V_p \in \mathcal{V}$ such that $p \in V_p$, and let U_p be an open set such that $p \in U_p \subseteq \overline{U_p} \subseteq V_p$. Let $\{U_{p_n} : n \in \mathbb{N}\}$ be a subcover of $\{U_p : p \in X\}$, and let $V_{p_n} \in \mathcal{V}$ such that $\overline{U_{p_n}} \subseteq V_{p_n}$ for each $n \in \mathbb{N}$. Note that $\{V_{p_n} : n \in \mathbb{N}\}$ covers X. For each $n \in \mathbb{N}$, let $W_n = V_{p_n} \backslash \bigcup_{j=1}^{n-1} \overline{U_{p_j}}$. Then W_n is open for each $n \in \mathbb{N}$. Let $\mathcal{W} = \{W_n : n \in \mathbb{N}\}$.

We claim that $\mathcal{W}$ covers X. Let $p \in X$. Then $p \in U_{p_n}$ for some $n \in \mathbb{N}$, and $\overline{U_{p_n}} \subseteq V_{p_n}$. Let $k = min\{n : p \in V_{p_n}\}$. Then $p \in V_{p_k}$ and $p \notin V_{p_n}$ for $n < k$, so that $p \notin \overline{U_{p_n}}$ for $n < k$. It follows that $p \in W_k$, and $\mathcal{W}$ covers X.

Since $W_n \subseteq V_{p_n} \in \mathcal{V}$ for each $n \in \mathbb{N}$, we have that $\mathcal{W}$ is a refinement of $\mathcal{V}$.

To see that $\mathcal{W}$ is locally finite, let $p \in X$, and again let $k = min\{n : p \in U_{p_n}\}$. Then $p \in U_{p_k}$ and $U_{p_k} \cap W_n = \emptyset$ for $n > k$, since $W_n = V_{p_n} \setminus \bigcup_{j=1}^{n-1} \overline{U_{p_j}}$. Thus U_{p_k} is a neighborhood of p which meets at most a finite number of the elements of $\mathcal{W}$.

We conclude that X is paracompact. ■

Note that each compact space is paracompact.

4.23 **Theorem**. *Each closed subspace of a paracompact space is paracompact.*

4.24 **Lemma**. *Let $\{M_\alpha : \alpha \in A\}$ be a locally finite collection of subsets of a space X. Then:*

(a) *$\{\overline{M_\alpha} : \alpha \in A\}$ is locally finite; and*

(b) *If $B \subseteq A$, then $\cup\{\overline{M_\beta} : \beta \in B\}$ is closed.*

4.25 **Lemma**. *Let X be a paracompact space and let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of X. Then $\mathcal{U}$ has a locally finite open refinement $\{V_\alpha : \alpha \in A\}$ (indexed by A) such that $V_\alpha \subseteq U_\alpha$ for each $\alpha \in A$.*

Proof. Let $\mathcal{W}$ be a locally finite open refinement of $\mathcal{U}$. For each $W \in \mathcal{W}$, let $H_W \in \mathcal{U}$ such that $W \subseteq H_W$. For each $\alpha \in A$, let $V_\alpha = \bigcup\{W \in \mathcal{W} : H_W = U_\alpha\}$. The V_α is open and $V_\alpha \subseteq U_\alpha$ for each $\alpha \in A$.

Let $p \in X$. Then $p \in W$ for some $W \in \mathcal{W}$, and $W \subseteq H_W = U_\alpha$ for some $\alpha \in A$, so that $p \in V_\alpha$. Thus $\{V_\alpha : \alpha \in A\}$ is an open refinement of $\mathcal{U}$.

We claim that $\{V_\alpha : \alpha \in A\}$ is locally finite. Let $p \in X$. Then there exists a neighborhood N of p such that N meets $W_1, W_2, \dots, W_n \in \mathcal{W}$ and $N \cap W \neq \emptyset$, $W \in \mathcal{W}$ implies that $W = W_j$ for some $j = 1, 2, \dots, n$. Let $\alpha_1, \alpha_2, \dots, \alpha_n \in A$ such that $H_{W_j} = U_{\alpha_j}$; $j = 1, 2, \dots, n$. Suppose $N \cap V_\alpha \neq \emptyset$. Then $N \cap W \neq \emptyset$ for some $W \in \mathcal{W}$ such that $H_W = U_\alpha$, so that $W = W_j$ for some $j = 1, 2, \dots, n$. We obtain that $H_{W_j} = U_{\alpha_j}$, and hence $\alpha = \alpha_j$ for some $j = 1, 2, \dots, n$. We conclude that $\{V_\alpha : \alpha \in A\}$ is locally finite. ■

4.26 **Theorem**. *A paracompact Hausdorff space is normal.*

Proof. We first show that X is regular. For this purpose let A be a closed subset of X and let $p \in X \setminus A$. For each $a \in A$, there exists an open set U_a such that $p \notin \overline{U_a}$ and $a \in U_a$. Now $\{U_a : a \in A\} \cup \{X \setminus A\}$ is an open cover of X. Let $\{V_a : a \in A\} \cup \{G\}$ be a locally finite open refinement such that $V_a \subseteq U_a$ for each $a \in A$ and $G \subseteq X \setminus A$ (from 4.25). Then $W = \bigcup\{V_a : a \in A\}$ is an open set containing A. By 4.24 (b), $\overline{W} = \bigcup\{\overline{V_a} : a \in A\}$. Now $p \notin \overline{V_a}$ for each $a \in A$, since $V_a \subseteq U_a$, so that $p \notin \overline{W}$. Then $p \in X \setminus \overline{W}$ and $A \subseteq W$, so that X is regular.

To see that X is normal, let A and B be disjoint closed subsets of X. Then for each $a \in A$, there exists an open set U_a such that $B \cap \overline{U_a} = \emptyset$ and $a \in U_a$. Now $\{U_a \colon a \in A\} \cup \{X \backslash A\}$ is an open cover of X. Let $\{V_a \colon a \in A\} \cup \{G\}$ be a locally finite open refinement such that $V_a \subseteq U_a$ for each $a \in A$ and $G \subseteq X \backslash A$ (from 4.25). Then $W = \bigcup\{V_a \colon a \in A\}$ is an open set containing A. By 4.25 (b), $\overline{W} = \bigcup\{\overline{V_a} \colon a \in A\}$. Now $B \cap \overline{V_a} = \emptyset$ for each $a \in A$, since $V_a \subseteq U_a$, so that $\overline{W} \cap B = \emptyset$. We conclude that $B \subseteq X \backslash \overline{W}$ and $A \subseteq W$, so that X is normal. ∎

If X is a space and $f \colon X \to I = [0,1]$ is a function, then the closed set $S(f) = \overline{\{x \colon f(x) \neq 0\}}$ is called the **support** of f. Note that $x \in X \backslash S(f)$ if and only if there exists a neighborhood U of x such that $f(U) = 0$.

If X is a space, then a family $\{f_\alpha \colon X \to I, \alpha \in A\}$ of continuous functions is called a **partition of unity** provided:

(a) $\{S(f_\alpha) \colon \alpha \in A\}$ is a locally finite closed cover of X; and (b) $\sum_{\alpha \in A} f_\alpha(x) = 1$ for each $x \in X$.

If X is a space and $\mathcal{U} = \{U_\alpha \colon \alpha \in A\}$ is an open cover of X, then a partition of unity $\{f_\alpha \colon \alpha \in A\}$ is said to be **subordinate** to $\mathcal{U}$ provided that $S(f_\alpha) \subseteq U_\alpha$ for each $\alpha \in A$.

4.27 **Lemma**. *Let X be a T_4-space and let $\{U_\alpha \colon \alpha \in A\}$ be a locally finite open cover of X. Then there exists an open cover $\{V_\alpha \colon \alpha \in A\}$ of X such that $\overline{V}_\alpha \subseteq U_\alpha$ for each $\alpha \in A$, and $V_\alpha \neq \emptyset$ whenever $U_\alpha \neq \emptyset$ for each $\alpha \in A$.*

Proof. Let τ denote the topology on X and let $\mathcal{A} = \{(B, f_B) \colon B \subseteq A, f_B \colon B \to \tau$ such that $\overline{f_B(\alpha)} \subseteq U_\alpha$, $f_B(\alpha) \neq \emptyset$ if $U_\alpha \neq \emptyset$, and $\bigcup_{\alpha \in B} f(\alpha) \cup \bigcup_{\alpha \in A \backslash B} U_\alpha = X\}$.

We first claim that $\mathcal{A} \neq \emptyset$. Let $\alpha \in A$ such that $U_\alpha \neq \emptyset$, and let $B = \{\alpha\}$. We consider two cases:

Case 1. $U_\alpha \subseteq \bigcup\{U_\beta \colon \beta \in A \backslash B\}$.

Let $p \in U_\alpha$ and let $f_B(\alpha)$ be an open set such that $p \in f(\alpha)$ be an open set such that $p \in f(\alpha) \subseteq \overline{f(\alpha)} \subseteq U_\alpha$. Then $(B, f_B) \in \mathcal{A}$.

Case 2. $U_\alpha \not\subseteq \bigcup\{U_\beta \colon \beta \in A \backslash B\}$.

Then $\bigcap\{X \backslash U_\beta \colon \beta \in A \backslash B\} = N$ is a nonempty closed subset of U_α. Let $f_B(\alpha)$ be an open subset of U_α such that $N \subseteq f_B(\alpha) \subseteq \overline{f_B(\alpha)} \subseteq U_\alpha$. Then $(B, f_B) \in \mathcal{A}$.

We conclude that $\mathcal{A} \neq \emptyset$. Define a partial order $\leq$ on $\mathcal{A}$ by $(B, f_B) \leq (C, f_C)$ provided $B \subseteq C$ and $f_C | B = f_B$. By HMP, there exists a maximal chain $\mathcal{C}$ in $\mathcal{A}$. Let $H = \bigcup\{B \colon (B, f_B) \in \mathcal{C}\}$ and define $f_H \colon H \to \tau$ by

$f_H|B = f_B$ for $(B, f_B) \in \mathcal{C}$.

We claim that $X = \bigcup_{\alpha\in H} f_H(\alpha) \cup \bigcup_{\alpha\in A\backslash H} U_\alpha$. Let $p \in X$. Then there exists a finite subset F of A such that $p \in U_\beta$ if $\beta \in F$ and $p \notin U_\beta$ if $\beta \in A\backslash F$. Again, we consider two cases:

Case 1. $F \cap (A\backslash H) \neq \emptyset$.

Let $\beta \in F \cap (A\backslash H)$. Then $p \in U_\beta$ and $\beta \in A\backslash H$, so that $p \in \bigcup_{\alpha\in A\backslash H} U_\alpha$.

Case 2. $F \cap (A\backslash H) = \emptyset$.

In this case, $F \subseteq H$, and there exists $(B, f_B) \in \mathcal{C}$ such that $F \subseteq B$. We have that $p \notin \bigcup_{\alpha\in A\backslash B} U_\alpha$, $p \in \bigcup_{\alpha\in B} f_B(\alpha)$, and hence $p \in \bigcup_{\alpha\in H} f_H(\alpha)$.

The two cases above prove that $X = \bigcup_{\alpha\in H} f_H(\alpha) \cup \bigcup_{\alpha\in A\backslash H} U_\alpha$.

We claim that $A = H$. Suppose that $A \neq H$ and let $p \in A\backslash H$. Now $X = \bigcup_{\alpha\in H} f_H(\alpha) \cup U_\beta \cup \bigcup_{\alpha\in A\backslash(H\cup\beta)} U_\alpha$, so that $X\backslash[\bigcup_{\alpha\in H} f_H(\alpha) \cup \bigcup_{\alpha\in A\backslash(H\cup\beta)} U_\alpha] \subseteq U_\beta$. There exists an open set V ($\neq \emptyset$ if $U_\beta \neq \emptyset$) such that $X\backslash[\bigcup_{\alpha\in H} f_H(\alpha) \cup \bigcup_{\alpha\in A\backslash(H\cup\beta)} U_\alpha] \subseteq V \subseteq \overline{V} \subseteq U_\beta$. Define $F\colon H \cup \{\beta\} \to \tau$ by $F|H = f_H$ and $f(\beta) = V$. Then $(H \cup \beta, F) \in \mathcal{C}$; contradicting the maximality of $\mathcal{C}$. We conclude that $A = H$. Define $V_\alpha = f_A(\alpha)$ for each α. ■

4.28 **Partition of Unity Theorem**. *Let X be a paracompact Hausdorff space and let γ be an open cover of X. Then there exists a partition of unity which is subordinate to γ.*

Proof. Since X is paracompact, γ has a locally finite open refinement which covers X. We can assume that γ is locally finite. Let $\gamma = \{G_\alpha\colon \alpha \in A\}$ and $\mathcal{U} = \{U_\alpha\colon \alpha \in A\}$ a locally finite open refinement of γ such that $\emptyset \neq U_\alpha \subseteq \overline{U_\alpha} \subseteq G_\alpha$ (we assume that $G_\alpha \neq \emptyset$ for each $\alpha \in A$). Let $\mathcal{V} = \{V_\alpha\colon \alpha \in A\}$ be a locally finite open refinement of $\mathcal{U}$ such that $\emptyset \neq V_\alpha \subseteq \overline{V_\alpha} \subseteq U_\alpha$ for each $\alpha \in A$. Let $g_\alpha\colon X \to [0,1]$ be a continuous function such that $g_\alpha(\overline{V_\alpha}) = 1$ and $g_\alpha(X\backslash U_\alpha) = 0$ for each $\alpha \in A$. Define $\psi\colon X \to \mathbb{R}$ by $\psi(x) = \sum_{\beta\in A} g_\beta(x)$. Note that for each $x \in X$, there exists a finite subset $F \subseteq A$ such that $x \in U_\alpha$ for $\alpha \in F$ and $x \in X\backslash U_\alpha$ for $\alpha \in A\backslash F$, since $\mathcal{U}$ is locally finite, so that $g_\alpha(x) = 0$, if $\alpha \in A\backslash F$, and $g_\beta(x) = 1$ for some $\beta \in A$, since $x \in \overline{V_\beta}$ for some $\beta \in A$. Thus ψ is well-defined and $\psi(x) > 0$ for each $x \in X$.

We claim that ψ is continuous. For each $p \in X$, let N_p be an open set containing p such that N_p meets only a finite number of members of $\mathcal{U}$. Then $\mathcal{N} = \{N_p\colon p \in X\}$ is an open cover of X. We will show that $\psi|N_p$ is continuous for each $p \in X$. Let $F = \{\alpha_1, \ldots, \alpha_n\} \subseteq A$ such that $N_p \cap U_{\alpha_j} \neq \emptyset$ for $j = 1, \ldots, n$; and $N_p \cap U_\alpha = \emptyset$ for $\alpha \in A \backslash F$. The function $+\colon \mathbb{R}^n \to \mathbb{R}$ defined by $+(x_1, \ldots, x_n) = x_1 + \cdots + x_n$ is continuous, and the function $k\colon N_p \to \mathbb{R}^n$ defined by $k(x) = (g_{\alpha_1}(x), \ldots, g_{\alpha_n}(x))$ is continuous. Now $\psi | N_p = + \circ k$, so that $\psi | N_p$ is continuous for each p, and hence ψ is continuous.

The function $\delta\colon [0,1] \times (0, \infty) \to \mathbb{R}$ defined by $\delta(r, s) = r/s$ is continuous. Define $f_\alpha\colon X \to [0,1]$ by $f_\alpha(x) = g_\alpha(x)/\psi(x)$ for $x \in X$. Let $m_\alpha\colon X \to [0,1] \times (0, \infty)$ be defined by $m_\alpha(x) = (g_\alpha(x), \psi(x))$, so that $f_\alpha = \delta \circ m_\alpha$ for each $\alpha \in A$ is continuous.

Let $\alpha \in A$. Then $\{x\colon f_\alpha(x) \neq 0\} \subseteq U_\alpha$, so that $S(f_\alpha) \subseteq \overline{U_\alpha} \subseteq G_\alpha$.

Let $x \in X$. Then $\sum_{\alpha \in A} f_\alpha(x) = \sum_{\alpha \in A} g_\alpha(x)/\psi(x) = 1$. By 4.24, $\{\overline{U_\alpha}\colon \alpha \in A\}$ is locally finite, so that $S(f_\alpha) \subseteq \overline{U_\alpha}$ yields that $\{S(f_\alpha)\colon \alpha \in A$ is locally finite. Since for each $x \in X$, $x \in \overline{V_\alpha}$ for some $\alpha \in A$, $g_\alpha(x) = 1$, so that $x \in S(f_\alpha)$, and hence $\{S(f_\alpha)\colon \alpha \in A\}$ is a cover of X. ∎

A space X is said to be **σ-compact** if X is a union of a countable family of compact subspaces.

4.29 **Lemma.** *Let X be a locally compact σ-compact Hausdorff space. Then there exists a countable open cover $\{U_n\colon n = 1, 2, \ldots\}$ of X such that $\overline{U}_n$ is compact and $\overline{U}_n \subseteq U_{n+1}$ for each n.*

Proof. Since X is σ-compact, there exists a collection $\{C_n\colon n \in \mathbb{N}\}$ of compact subsets of X such that $X = \bigcup_{n \in \mathbb{N}} C_n$. For each $n \in \mathbb{N}$ define $F_n = \bigcup_{j=1}^{n}$. Then each F_n is compact, $F_n \subseteq F_{n+1}$ for each n, and $X = \bigcup_{n \in \mathbb{N}} F_n$.

Let τ denote the topology on X, and let $\mathcal{K} = \{(K, g_K)\colon K \subseteq \mathbb{N}$, $1 \in K$, if $m \in K$, and $1 < n < m$, then $n \in K$; $g_K\colon K \to \tau$, $F_n \subseteq g_K(n)$, and $\overline{g_K(n)}$ is compact for each $n \in K$, and $\overline{g_K(n)} \subseteq g_K(n+1)$, whenever $n + 1 \in K\}$.

We claim that $\mathcal{K} \neq \emptyset$. Let $K = \{1\}$. Let $\mathcal{W}$ be an open cover of F_1 such that $\overline{W}$ is compact for each $W \in \mathcal{W}$ (using that X is locally compact Hausdorff), and let $W_1, W_2, \ldots, W_n$ be a finite subcover of F_1. Define $g_K(1) = \bigcup_{j=1}^{n} W_j$. Then $F_1 \subseteq g_K(1) \subseteq \overline{g_K(1)} = \bigcup_{j=1}^{n} \overline{W_j}$ (compact). We see that $(K, g_K) \in \mathcal{K}$ and hence $\mathcal{K} \neq \emptyset$.

Let $\mathcal{C}$ be a maximal chain in $\mathcal{K}$, and let $H = \bigcup\{K\colon (K, g_K) \in \mathcal{C}\}$. Define $g_H\colon H \to \tau$ so that $g_H | K = g_K$ for each $(K, g_K) \in \mathcal{C}$. Then $(H, g_H) \in \mathcal{K}$ and hence in $\mathcal{C}$ by maximality.

We claim that $H = \mathbb{N}$. Suppose that $H \neq \mathbb{N}$. Then H has a largest member h. Now $\overline{g_H(h)} \cup F_{h+1}$ is compact. Let V be an open set containing $\overline{g_K(h)} \cup F_{h+1}$ such that $\overline{V}$ is compact. Define $g_{H \cup \{h+1\}}(h+1) = V$ and

$g_{H\cup\{h+1\}}(n) = g_H(n)$ if $n \in H$. Then $(H \cup \{h+1\}, g_{H\cup\{h+1\}}) \in \mathcal{K}$ and in $\mathcal{C}$; contradicting the maximality of $\mathcal{C}$. It follows that $H = \mathbb{N}$.

Let $U_n = g_{\mathbb{N}}(n)$ for each $n \in \mathbb{N}$. Then each U_n is open, $\overline{U_n}$ is compact, and $\overline{U_n} \subseteq U_{n+1}$. Since $F_n \subseteq U_n$, and $X = \bigcup_{n\in\mathbb{N}} F_n$, $\{U_n : n \in \mathbb{N}\}$ covers X. ∎

4.30 **Theorem.** *Each locally compact σ-compact Hausdorff space is paracompact.*

Proof. Let X be a locally compact σ-compact Hausdorff space.

Since X is locally compact Hausdorff, X is regular.

Since X is σ-compact, X is Lindelöf.

The conclusion now follows from 4.22. ∎

A **cube** is a topological product space I^M for some set M, where $I = [0,1]$.

4.31 **The Tychonoff Embedding Theorem.** *Each completely regular Hausdorff space can be embedded into a cube.*

Proof. Let X be a completely regular Hausdorff space, and let $M = \{(p, A) : A$ is a closed subset of X and $p \in X\backslash A\}$. For each $(p, A) \in M$, there exists a continuous function $f_{(p,A)} : X \to I$ such that $f_{(p,A)}(p) = 0$ and $f_{(p,A)}(A) = 1$. Define $\phi : X \to I^M$ so that $\pi_{(p,A)}\phi = f_{(p,A)}$ for each $(p, A) \in M$. Then ϕ is continuous.

To see that ϕ is injective, let x and y be distinct points of X. Since X is Hausdorff, $\{y\}$ is closed, so that $(x, \{y\}) \in M$. Note that $\pi_{(x,\{y\})}\phi(x) = 0$ and $\pi_{(x,\{y\})}\phi(y) = 1$, and hence $\phi(x) \neq \phi(y)$ and ϕ is injective.

To complete the proof that ϕ is an embedding, let U be an open subset of X. We will show that $\phi(U)$ is open in $\phi(X)$. Let $q \in \phi(U)$ and let $u \in U$ such that $\phi(u) = q$. Then $(x, X\backslash U) \in M$. Let $W = \prod_{(p,A)\in M} W_{(p,A)}$ such that $W_{(p,A)} = I$ if $(p, A) \neq (u, X\backslash U)$ and $W_{(u,X\backslash U)} = [0,1)$. Then W is a basic open set in I^M, so that $W \cap \phi(X)$ is open in $\phi(X)$. Now $q = \phi(u) \in W \cap \phi(X)$, since $f_{(u,X\backslash U)}(u) = 0$. We claim that $W \cap \phi(X) \subseteq \phi(U)$. Let $b \in W \cap \phi(X)$ and let $a \in X$ such that $\phi(a) = b$. Then $\pi_{(u,X\backslash U)}(b) = \pi_{(u,X\backslash U)}\phi(a) = f_{(u,X\backslash U)}(a) \neq 1$, since $b \in W$, and hence $a \notin X\backslash U$, and $a \in U$. Thus $b \in \phi(U)$, and $W \cap \phi(X) \subseteq \phi(U)$, and $\phi(U)$ is open in $\phi(X)$. ∎

If M is a countable set, the cube I^M is called the **Hilbert cube** and is denoted I^∞.

4.32 **Theorem.** *A second countable T_3-space is normal.*

4.33 **Theorem.** *A second countable T_3-space can be embedded into I^∞.*

Proof. Let X be a second countable T_3-space, and let β be a countable basis for the topology of X. Let $M = \{(A, B) : A, B \in \beta$ and $\overline{A} \subseteq B\}$. Then M is countable. For each $(A, B) \in M$, let $f_{(A,B)} : X \to I$ be a continuous function such that $f_{(A,B)}(A) = 0$ and $f_{(A,B)}(X\backslash B) = 1$ (X is normal from 4.32). Define $\phi\colon X \to I^M$ so that $\pi_{(A,B)}\phi = f_{(A,B)}$ for each $(A, B) \in M$. Then ϕ is the desired embedding. ■

A function $f\colon X \to Y$ from a space X into a space Y is said to be **dense** if $f(X)$ is dense in Y.

A **compactification** of a space X is a dense embedding $f\colon X \to Y$ of X into a compact space Y.

4.34 **Theorem.** *Each locally compact Hausdorff space has a Hausdorff compactification.*

4.35 **Theorem.** *Let X be a non-compact space and ∞ a point not in X. Let $X_\infty = X \cup \{\infty\}$ and let $\beta = \{U\colon U \subseteq X_\infty$ and either U is open in X or $X\backslash U$ is a closed compact subset of $X\}$. Then β is a basis for a unique topology τ on X_∞ such that (X_∞, τ) is compact. Moreover, the inclusion $i\colon X \to X_\infty$ is a compactification of X.*

The compactification $i\colon X \to X_\infty$ in 4.35 is called the **one point compactification** of X.

4.36 **Theorem.** *Let X be a non-compact space. Then X_∞ is Hausdorff if and only if X is a locally compact Hausdorff space.*

4.37 **Exercise.** *The one point compactification of $\mathbb{R}^n$ is homeomorphic to S^n for each $n \in \mathbb{N}$.*

A **Stone-Čech compactification** of a space X is a compactification $\beta\colon X \to B$ such that if $f\colon X \to Y$ is a continuous function from X into a compact Hausdorff space Y, then there is a unique continuous function $g\colon B \to Y$ such that $f = g\beta$.

4.38 **Lemma.** *Let X and Y be Hausdorff spaces, D a dense subset of X, and $f, g\colon X \to Y$ continuous functions such that $f|D = g|D$. Then $f = g$.*

4.39 **Theorem.** *Each completely regular Hausdorff space has a unique (Hausdorff) Stone-Čech compactification.*

5 CONNECTEDNESS

If X is a space and U and V are disjoint nonempty open subsets of X such that $X = U \cup V$, then the pair U, V is called a **separation** of X. We write $X = U|V$ to denoted that U, V is a separation of X.

A space X is said to be **connected** if X has no separation.

5.1 **Theorem.** *Let C be a connected subspace of a space X and let E be a subspace of X such that $C \subseteq E \subseteq \overline{C}$. Then E is connected. In particular, $\overline{C}$ is connected.*

5.2 **Theorem.** *A space X is connected if and only if X contains no proper subset which is both open and closed.*

5.3 **Theorem.** *The continuous image of a connected space is connected.*

If a and b are points of a space X, then a and b are said to be **connected in** X if there exists a connected subset of X containing both a and b.

5.4 **Clover Leaf Theorem.** *If $\{C_\alpha : \alpha \in A\}$ is a family of connected subsets of X such that $\bigcap_{\alpha \in A} C_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in A} C_\alpha$ is connected.*

A **component** of a space X is a maximal connected subset of X.

5.5 **Theorem.** *Let X be a space and $p \in X$. Then p belongs to exactly one component of X.*

If X is a space and $p \in X$, then the component of X containing p is denoted by $C(p)$.

5.6 **Theorem.** *Each component of a space is closed.*

5.7 **Theorem.** *Let X be a space and let $R = \{(x, y) \in X \times X : x$ and y are connected in $X\}$. Then:*

(a) *R is an equivalence relation on X; and*

(b) *If $p \in X$, then $C(p)$ is the R-class of p.*

The relation R in 5.7 is called the **component equivalence** on X.

5.8 **Theorem.** *Let $\{E_\alpha : \alpha \in A\}$ be a tower of compact connected subsets of a Hausdorff space X. Then $\bigcap_{\alpha \in A} E_\alpha$ is connected.*

5.9 **Theorem.** *The product of a family of connected spaces is connected.*

Proof. We first prove that the product of a finite number of connected spaces is connected. It is sufficient to show that the product of two connected spaces is connected.

Let M and N be connected spaces and let (p, q) and (a, b) be in $M \times N$. Then $(a, q) \in M \times N$, and $M \times \{q\}$ is a connected subspace containing (a, q) and (p, q), so that (a, q) and (p, q) are connected in $M \times N$. Also (a, b) and (a, q) are connected in $m \times N$ by $\{a\} \times N$, so that (a, b) and (p, q) are connected in $M \times N$ (5.7). It follows that $M \times N$ is connected.

Let $\{X_\alpha : \alpha \in A\}$ be a family of connected spaces and let $X = \prod_{\alpha \in A} X_\alpha$. Suppose that $X = U|V$ is not connected. Let $p \in U$ and $q \in V$, and let W be a basic open set containing q such that $W \subseteq V$. Then there exists a finite subset F of A such that $W = \prod_{\alpha \in A} W_\alpha$, W_α is open in X_α for each $\alpha \in F$ and $W_\alpha = X_\alpha$ for $\alpha \in A \backslash F$. Let $T = \prod_{\alpha \in A} T_\alpha$, where $T_\alpha = X_\alpha$ if $\alpha \in F$ and $T_\alpha = \{\pi_\alpha(p)\}$ if $\alpha \in A \backslash F$. Then T is connected and $p \in T$, so that $T \subseteq U$. Let $z \in X$ such that $\pi_\alpha(z) = \pi_\alpha(q)$ if $\alpha \in F$ and $\pi_\alpha(z) = \pi_\alpha(p)$ if $\alpha \in A \backslash F$. Then

$z \in T \cap W$. But $W \subseteq V$ and $T \subseteq U$, so that $U \cap V \neq \emptyset$. This contradition yields that X is connected. ■

5.10 **Exercise.** Let $n \in \mathbb{N}$ and prove that $\mathbb{R}^n, I^n$, and S^n are connected.

5.11 **Exercise.** Show that if X is connected, then $Cone(X)$ is connected.

5.12 **Theorem.** *Let $f: X \to Y$ be a continuous function and let A be a connected subset of X. Then $f(A)$ is contained in exactly one component of Y.*

A space X is said to be **totally disconnected** if each component of X is degenerate (a single point).

A function $f: X \to Y$ from a space X into a space Y is said to be **monotone** if $f^{-1}(p)$ is connected for each $p \in Y$.

5.13 **Theorem.** *If $f: X \to Y$ is a monotone quotient map and K is a connected open [closed] subset of Y, then $f^{-1}(K)$ is connected.*

Proof. We prove 5.13 in the case that K is a closed subset of Y.

Suppose that $f^{-1}(K) = A|B$, where A and B are closed in $f^{-1}(K)$. Since $f^{-1}(K)$ is closed, A and B are closed. Let $p \in A$. Then $f^{-1}f(p)$ is a connected subset of $f^{-1}(K)$ containing p, so that $f^{-1}f(p) \subseteq A$. We obtain that $f^{-1}f(A) = A$ and similarly, $f^{-1}f(B) = B$. It follows that $f(A)$ and $f(B)$ are closed, since f is quotient. Now $K = f(A) \cup f(B) = f(A)|f(B)$; contradicting that K is connected. Thus $f^{-1}(K)$ is connected. ■

5.14 **Theorem.** *If R is the component equivalence on a space X, then X/R is totally disconnected.*

A space X is said to be **locally connected** at $p \in X$ if for each neighborhood N of p, there exists a neighborhood M of p such that $M \subseteq N$ and each pair of points in M are connected in N. If X is locally connected at each of its points, then we say that X is a **locally connected space**.

5.15 **Theorem.** *Let X be a space and let $p \in X$. Then X is locally connected at p if and only if each neighborhood of p contains a connected neighborhood of p.*

5.16 **Theorem.** *If a space X is locally connected at $p \in X$, then $p \in C(p)^\circ$.*

5.17 **Theorem.** *An open subspace of a locally connected space is locally connected.*

5.18 **Theorem.** *Let X be a locally connected space and let $f: X \to Y$ be a continuous closed surjective function. Then Y is locally connected.*

Proof. Let $p \in Y$ and V be an open subset of Y containing p. Then $f^{-1}(p) \subseteq f^{-1}(V)$ and $f^{-1}(V)$ is open. Since X is locally connected, for each $q \in f^{-1}(p)$, there exists a connected neighborhood T_q of q such that $T_q \subseteq f^{-1}(V)$. For each $q \in f^{-1}(p)$, let M_q be an open set such that $q \in M_q \subseteq T_q$. Let $M = \bigcup\{M_q : q \in f^{-1}(p)\}$ and let $T = \bigcup\{T_q : q \in f^{-1}(p)\}$.

Then M is open and $f^{-1}(p) \subseteq M \subseteq T$, so that $p \in f(M) \subseteq f(T) \subseteq V$. Now $f(T) = \bigcup\{f(T_q): q \in f^{-1}(p)\}$, and each $f(T_q)$ is connected, since f is continuous and T_q is connected. Since $p \in \bigcap\{f(T_q): q \in f^{-1}(p)\}$, we have that $f(T)$ is connected by the cloverleaf theorem.

We will show that $f(T)$ is a neighborhood of p. Since M is open, $X \setminus M$ is closed, and hence $f(X \setminus M)$ is closed, since f is closed. Thus $H = \{Y \setminus f(X \setminus M)\} \cap V$ is open. Now $p \notin f(X \setminus M)$, since $f^{-1}(p) \subseteq M$, so that $p \in H$. We claim that $H \subseteq f(M)$. Let $t \in H$. Then $t \notin f(X \setminus M)$ and $t \in V$. Thus $t \in f(M)$, since f is surjective.

We conclude that $p \in H \subseteq f(M) \subseteq f(T) \subseteq V$, H is open, and $f(T)$ is connected. It follows that $f(T)$ is a connected neighborhood of p contained in V, and so Y is locally connected. ■

From 5.3 and 5.18, we see that connectedness and local connectedness are topological properties.

5.19 **Theorem**. *Let X be a space. These are equivalent:*

(a) *X is locally connected*; (b) *The components of each open subspace of X are open*; *and* (c) *The connected open subsets of X form a basis for the topology of X.*

5.20 **Theorem**. *Let X be a Hausdorff space and let $f: I \to X$ be a continuous function (where $I = [0,1]$ with the usual topology). Then $f(I)$ is locally connected.*

5.21 **Theorem**. *Let X be a connected space and let $f: X \to R$ (reals with the ususal topology) be a continuous function such that $f(p) < 0$ and $f(q) > 0$ for some $p, q \in X$. Then $f(a) = 0$ for some $a \in X$.*

6 METRIC SPACES

A **metric** on a set X is a function $d: X \times X \to [0, \infty)$ such that:

(a) $d(a,b) = 0$ if and only if $a = b$ in X;

(b) $d(a,b) = d(b,a)$ for each $a, b \in X$; and

(c) $d(a,c) \leq d(a,b) + d(b,c)$ for each $a, b, c \in X$.

If d is a metric on a set X, $r > 0$ is a real number, and $p \in X$, then the r-**sphere** in X with **center** p is defined $N_r(p) = \{x \in X: d(p,x) < r\}$.

6.1 **Theorem**. *Let d be a metric on a set X and let $\beta = \{N_r(p): p \in X, 0 < r\}$. Then β is a basis for a unique topology on X.*

The topology on X generated by β in 6.1 is called the **topology defined by the metric** d.

A space (X, τ) is said to be **metrizable** if there exists a metric d on X such that τ is the topology defined by d.

6.2 **Theorem.** *A subspace of a metrizable space is metrizable.*

Two metrics d and δ on a set X are said to be **equivalent** if they define the same topology on X.

6.3 **Lemma.** *Let d be a metric on a set X. Then there exists and equivalent metric δ on X such that $0 \leq \delta(a,b) \leq 1$ for each $a,b \in X$.*

Proof. Define $\delta(p,q) = \min\{1, d(p,q)\}$ for each $p,q \in X$. To see that δ is a metric on X, let $a,b,c \in X$. Then

(1) $\delta(a,b) = 0 \Leftrightarrow d(a,b) = 0 \Leftrightarrow a = b$;

(2) $\delta(a,b) = \delta(b,a)$ is clear; and

(3) $\delta(a,b) + \delta(b,c) = \min\{1, d(a,b)\} + \min\{1, d(b,c)\} =$

$$\begin{cases} 1 + 1 > \delta(a,c); & or \\ 1 + d(b,c) \geq \delta(a,c); & or \\ d(a,b) + 1 \geq \delta(a,c); & or \\ d(a,b) + d(b,c) \geq d(a,c) \geq \delta(a,c) \end{cases}$$

Thus δ is a metric on X with $0 \leq \delta(a,b) \leq 1$ for $a,b \in X$.

To see that d and δ are equivalent, let $M_r(x)$ denote neighborhoods in the δ-topology and let $N_r(x)$ neighborhoods in the d-topology.

Let $x \in X$ and $r > 0$. We will show that $M_r(x)$ is open in the d-topology. Let $p \in M_r(x)$. Then $\delta(p,x) < r$. Let $\epsilon = r - \delta(p,x)$. We claim that $N_\epsilon(p) \subseteq M_r(x)$. Let $t \in N_\epsilon(p)$. Then $d(p,t) < \epsilon$, so that $d(p,t) < r - \delta(p,x)$. We then have $\delta(t,x) \leq \delta(p,x) + \delta(p,t) \leq \delta(p,x) + d(p,t) < r$, and hence $t \in M_r(x)$, $N_\epsilon(p) \subseteq M_r(x)$, and $M_r(x)$ is open.

Let $x \in X$ and $r > 0$. We will show that $N_r(x)$ is δ-open. Let $p \in N_r(x)$. Then $d(p,x) < r$. Let $\epsilon = \min\{r - d(p,x), 1\}$. We claim that $M_\epsilon(p) \subseteq N_r(x)$. Let $t \in M_\epsilon(p)$. Then $\delta(p,t) < \epsilon \leq 1$, so that $\delta(p,t) = d(p,t)$. We have $d(p,t) < \epsilon \leq r - d(p,x)$ and so $d(t,x) \leq d(p,t) + d(p,x) < r$. We conclude that $t \in N_r(x)$, $M_\epsilon(p) \subseteq N_r(x)$ and $N_r(x)$ is δ-open. ∎

6.4 **Theorem.** *If $\{X_i : i = 1,2,\ldots\}$ is a countable collection of metrizable spaces, then $\prod_{i=1}^{\infty} X_i$ is metrizable.*

Proof. For each $i \in \mathbb{N}$, let d_i denote the metric on X_i which defines the topolgy on X_i and is such that $0 \leq d_i(x,y) \leq 1$ for each $x,y \in X_i$ (6.3). Let $X = \prod_{i \in \mathbb{N}} X_i$. For each $x,y \in X$, define $d(x,y) = \sum_{i=1}^{\infty} 2^{-i} d_i(x_i, y_i)$, where $x_i = \pi_i(x)$ and $y_i = \pi_i(y)$ for each $i \in \mathbb{N}$. Then d is a metric on X. It remains to show that d defines the product topology on X.

Let W be a basic open set in X in the product topology. We will show that W is d-open. Now $W = \prod_{i=1}^{\infty} W_i$ where W_i is open for $i \in F$ (a finite subset of $\mathbb{N}$) and $W_i = X_i$ for $i \in \mathbb{N} \setminus F$. We can assume that $F = \{1, 2, \dots, m\}$. Let $p \in W$, and let $p_i = \pi_i(p)$ for each $i \in \mathbb{N}$. Now for each $i \in F$, there exists $r_i > 0$ such that $N_{r_i}(p_i) \subseteq W_i$ (in the d_i metric). Let $r = 2^{-m} \cdot \min\{r_i : i \in F\}$. Then $r > 0$ and $N_r(p) \subseteq W$ (in the d-metric), and it follows that W is d-open.

Let G be open in the d-metric topology on X and let $x \in G$. Then there exists $m \in \mathbb{N}$ such that $N_{2^{-m+1}}(x) \subseteq G$ (in the d-metric). Let $F = \{1, 2, \dots, m\}$, $W_i = N_{2^{-m+1}}(x_i)$ for $i \in F$ and $W_i = X_i$ for $i \in \mathbb{N} \setminus F$. Then $x \in W = \prod_{i \in \mathbb{N}} W_i$, W is a basic open set in the product topology on X and $W \subseteq N_{2^{-m+1}}(x) \subseteq G$. It follows that G is open in the product topology on X. ■

6.5 **Theorem.** *Each of the spaces $\mathbb{R}^n$, I^n, S^n, and I^∞ is metrizable.*

6.6 **Theorem.** *Metrizability is a topological property.*

6.7 **Metrization Theorem.** *Each second countable T_3-space is metrizable.*

A **metric space** (X, d) is a space X together with a metric d which defines the topology of X.

6.8 **Theorem.** *If (X, d) is a metric space, then $d \colon X \times X \to [0, \infty)$ is continuous, where $[0, \infty)$ has the relative usual topology of the reals.*

Proof. Let $r < x$ in $[0, \infty)$. We claim that $d^{-1}(r, s)$ is open in $X \times X$. Let $(x, y) \in d^{-1}(r, s)$. Then $r < d(x, y) < s$. Let $\epsilon = \min\left\{\frac{s - d(x,y)}{2}, \frac{d(x,y) - r}{2}\right\}$. Then $(x, y) \in N_\epsilon(x) \times N_\epsilon(y)$. A straightforward argument shows that $N_\epsilon(x) \times N_\epsilon(y) \subseteq d^{-1}(r, s)$. A similar argument shows that if $s \in (0, \infty)$, then $d^{-1}[0, s)$ is open. It follows that d is continuous. ■

If (X, d) is a metric space, and A and B are subsets of X, then the **distance between** A **and** B is defined $d(A, B) = glb\{d(x, y) : x \in A, y \in B\}$. If $A = \{a\}$, then this distance is denoted $d(a, B)$.

6.9 **Theorem.** *Let (X, d) be a metric space, $E \subseteq X$, and $f \colon X \to [0, \infty)$ the function defined by $f(x) = d(x, E)$ for each $x \in X$. Then f is continuous.*

Proof. Let $r < s$ in $[0, \infty)$. We will show that $f^{-1}(r, s)$ is open in X. Let $x \in f^{-1}(r, s)$. Then $r < f(x) < s$, i.e., $r < d(x, E) < s$. Let $\epsilon = \min\{s - d(x, E), (d(x, E) - r)/2\}$. Then $N_\epsilon(x) \subseteq f^{-1}(r, s)$. It follows that $f^{-1}(r, s)$ is open and f is continuous. ■

6.10 **Lemma.** *Let (X, d) be a metric space, let A be a closed subset of X, and let $p \in X$. Then $p \in A$ if and only if $d(p, A) = 0$.*

6.11 **Theorem.** *Each metric space is a first countable T_4-space.*

Proof. Let (X, d) be a metric space. That (X, d) is a first countable T_2-space is straightforward. To see that this space is normal, let A and B be disjoint closed subsets of X. Define $h \colon X \to \mathbb{R}$ by $h(x) = d(x, A) - d(x, B)$.

Then h is continuous from 6.9. Let $U = h^{-1}(-\infty, 0)$ and $V = h^{-1}(0, \infty)$. Then U and V are disjoint open subsets of X containing A and B, respectively. ∎

6.12 **Theorem.** *Let (X, d) be a metric space. These are equivalent:*

(a) *X is second countable;*

(b) *X is separable; and*

(c) *X is Lindelöf.*

6.13 **Theorem.** *Let (X, d) be a compact metric space, Y a Hausdorff space, and let $f: X \to Y$ be a continuous function. Then $f(X)$ is metrizable.*

Proof. Now $f(X)$ is a compact Hausdorff space, and hence is T_3. We will show that $f(X)$ is second countable. Now X is second countable from 6.12. Let β be a countable basis for the topology of X, and let $\sigma = \{U : U \subseteq X$ and U is a finite union of members of $\beta\}$. Then σ is countable, and $\{f(X) \backslash f(X \backslash V) : V \in \sigma\}$ is a basis for the topology of $f(X)$. ∎

A space X is said to be **countably compact** if each countable open cover of X has a finite subcover.

6.14 **Theorem.** *Let X be a T_1-space. Then X is countable compact if and only if each sequence in X clusters to a point of X.*

Proof. Suppose that each sequence in X clusters. Let $\{U_n : n \in \mathbb{N}\}$ be a countable open cover of X. Suppose, for the purpose of proof by contradiction, that there is no finite subcover. Then $X \backslash \bigcup_{j=1}^{n} U_j \neq \emptyset$ for each $n \in \mathbb{N}$. Let $x_n \in X \backslash \bigcup_{j=1}^{n} U_j$. Then $x_n \xrightarrow{f} p$ for some $p \in X$. Now $p \in U_m$ for some $m \in \mathbb{N}$; but $x_n \in X \backslash U_m$ for $n > m$, i.e., $x_n \in^e X \backslash U_m$; and this contradicts $x_n \xrightarrow{f} p$. We conclude that $\{U_n : n \in \mathbb{N}\}$ has a finite subcover, and hence X is countably compact.

Suppose, on the other hand, that X is countably compact, and let $\{x_n\}$ be a sequence in X. Suppose, for the purpose of proof by contradiction, that x_n does not cluster in X. Then for each $p \in X$, there exists an open set V_p containing p such that $x_n \in^e X \backslash V_p$, i.e., for each $p \in X$, there exists $n_p \in \mathbb{N}$ such that $x_n \in X \backslash V_p$ when $n > n_p$. For each $p \in X$ such that $p \notin \{x_n\}$, there exist open sets $W_1, W_2, \ldots, W_{n_p}$ containing p such that $x_n \notin W_n$ for $n \leq n_p$; and for $p \in \{x_n\}$, there exist open sets $W_1, W_2, \ldots, W_{n_p}$ containing p such that $x_n \notin W_n$ for $n \leq n_p$ when $p \neq x_n$. Let $U_p = V_p \cap W_1 \cap W_2 \cap \cdots \cap W_{n_p}$ for each $p \in X$. Then if $p \neq x_n$, $\{x_n\} \cap U_p = \emptyset$ and if $p = x_n$, $\{x_n\} \cap U_p = \{x_n\}$. Let $U = \bigcup\{U_p : p \notin \{x_n\}\}$. Then $\{U, U_{x_1}, U_{x_2}, \ldots\}$ is a countable open cover of X. Let $\{U, U_{x_1}, U_{x_2}, \ldots, U_{x_k}\}$ be a finite subcover. But $x_{k+1} \in X \backslash [U \cup \bigcup_{j=1}^{k} U_{x_j}]$; which is a contradiction. ∎

6.15 **Theorem.** *Let X be a first countable space and let $p \in X$. Then each sequence which clusters to p has a subsequence which converges to p.*

Proof. Let $x_n \xrightarrow{f} p$ and let $\{U_n : n \in \mathbb{N}\}$ be a local basis at p. For each $n \in \mathbb{N}$ let $V_n = U_1 \cap U_2 \cap \cdots \cap U_n$. Then $\{V_n : n \in \mathbb{N}\}$ is a local basis at p with $V_{n+1} \subseteq V_n$ for each $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $n_k \in \mathbb{N}$ such that $x_{n_k} \in V_k$, with $n_1 < n_2 < n_3, \cdots$. Then $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. We claim that $x_{n_k} \xrightarrow{e} p$. Let W be a neighborhood of p. Then $V_m \subseteq W$ for some $m \in \mathbb{N}$. Let $k > m$. Then $n_k > n_m$ and $x_{n_k} \in V_k \subseteq V_m \subseteq W$. Thus for $k > m$, $x_{n_k} \in W$, i.e., $x_{n_k} \in^e W$, and hence $x_{n_k} \xrightarrow{e} p$. ■

6.16 **Theorem.** *A metric space X is compact if and only if X is countably compact.*

Proof. If X is compact, then clearly X is countably compact.

Suppose that X is countably compact. Let $\epsilon > 0$. We first claim that there is a finite ϵ-sphere cover of X. Suppose not. Define a sequence $\{p_n\}$ as follows: let $p_1 \in X$, and let $p_n \in X \setminus \bigcup_{j=1}^{n-1} N_\epsilon(p_j)$; recursively. Then, by 6.14, $p_n \xrightarrow{f} p$ for some $p \in X$. There exists $p_n \in N_{\frac{\epsilon}{2}}(p)$, and hence $d(p_n, p_m) < \epsilon$, so that $p_m \in N_\epsilon(p_n)$. This is a contradiction, and hence there is a finite ϵ-sphere cover of X for each $\epsilon > 0$.

We claim that X is separable. Let Q denote the set of all positive rational numbers. For each $r \in Q$, let $\mathcal{U}_r$ be a finite cover of X by r-spheres. Select an element from each member of the finite set $\mathcal{U}_r$ and denote the resulting finite set as D_r. Let $D = \bigcup\{D_r : r \in Q\}$. Then D is a countable dense subset of X. Thus X is seperable and hence Lindelöf (6.12). Clearly, a countably compact Lindelöf space is compact. ■

If (X, d) is a metric space, E is a subset of X, and $\epsilon > 0$, then a subset M of X is called an **ϵ-net for** E if for each $x \in E$, there exists $y \in M$ such that $d(x, y) < \epsilon$.

A subset E of a metric space X is said to be **totally bounded** if for each $\epsilon > 0$, there exists a finite ϵ-net for E.

6.17 **Theorem.** *Let E be a subset of a metric space X. Then E is totally bounded if and only if for each $\epsilon > 0$, There exists $x_1, \ldots, x_n$ in X such that $E \subseteq \bigcup_{j=1}^{n} N_\epsilon(x_j)$.*

A subset E of a metric space X is said to be **bounded** if there exists $p \in E$ and $0 < r$ such that $E \subseteq N_r(p)$.

Note that a totally bounded set is bounded.

6.18 **Theorem.** *If E is a totally bounded subset of a metric space X, then*

$\overline{E}$ *is totally bounded.*

6.19 **Exercise**. Show that a subset of R^n is bounded if and only if it is totally bounded.

A sequence x_n in a metric space (X, d) is called a **Cauchy sequence** if for each $\epsilon > 0$, there exists a positive integer k such that $d(x_n, x_m) < \epsilon$ whenever $m, n > k$.

A subset E of a metric space X is said to be **complete** if each Cauchy sequence in E converges to a point of E.

6.20 **Theorem**. *A closed subset of a complete metric space is complete.*

6.21 **Theorem**. *A complete subset of a metric space is closed.*

6.22 **Theorem**. *Let E be a closed subset of a metric space X. Then E is compact if and only if E is complete and totally bounded.*

6.23 **Exercise**. Show that a subset E of $\mathbb{R}^n$ is compact if and only if E is closed and bounded.

6.24 **Lemma**. *Let (X, d) be a metric space, A and B closed subsets of X, and let $p \in X$. Then $d(A, B) \leq d(A, p) + d(p, B)$.*

Proof. Suppose that $d(A, B) > d(A, p) + d(p, B)$. Then $d(A, B) - d(p, B) > d(A, p) = \text{glb}\,\{d(x, p) \colon x \in A\}$, so that for some $a \in A$, $d(A, B) - d(p, B) > d(a, p)$. Now $d(A, B) - d(a, p) > d(p, B)$ and for some $b \in B$, $d(A, B) - d(a, p) > d(p, b)$. Thus $d(A, B) > d(a, p) + d(p, b) \geq d(a, b)$. We have that $d(A, B) > d(a, b)$, $a \in A$, $b \in B$. But $d(A, B) = \text{glb}\,\{d(x, y) \colon x \in A, y \in B\}$. This contradiction yields that $d(A, B) \leq d(A, p) + d(p, B)$. ∎

6.25 **Lemma**. *Let (X, d) be a metric space and A and B subsets of X. Then $d(A, B) = glb\{d(x, B) \colon x \in A\}$,*

6.26 **Lemma**. *Let (X, d) be a metric space and for each positive integer n and each subset E of X define: $S_n(E) = \{x \in X \colon d(x, E) < 2^{-n}\}$ and define $C_n(E) = \{x \in X \colon S_n \subseteq E\}$. Then:*

(a) *$S_n(E)$ is open;*

(b) *$E \subseteq S_n(E)$;*

(c) *$C_n(E)$ is closed;*

(d) *$C_n(E) \subseteq E$; and*

(e) *$S_n(C_n(E)) \subseteq E$.*

Proof. (a) follows from 6.9.

(b) If $e \in E$, then $d(x, e) = 0 < 2^{-n}$, so that $e \in S_n(E)$.

(c) $C_n(E) = X \backslash S_n(X \backslash E)$ and (c) follows from (a).

(d) Let $x \in C_n(E)$. Then $S_n(x) \subseteq E$, and $x \in S_n(x)$, so that $x \in E$ and hence $C_n(E) \subseteq E$.

(e) Let $x \in S_n(C_n(E))$. Then $d(x, C_n(E)) < 2^{-n}$, and hence $d(x, p) < 2^{-n}$ for some $p \in C_n(E)$. We have that $S_n(p) \subseteq E$ and $x \in S_n(p)$, so that $x \in E$.

It follows that $S_n(C_n(E)) \subseteq E$. ■

6.27 **Lemma**. *Let (X,d) be a metric space and A and B subsets of X. For each positive integer n and each subset E of X define $S_n(E) = \{x \in X: d(x,E) < 2^{-n}\}$.*

(a) *If $C = \overline{S_n(A)}$, then $d(x,A) \leq 2^{-n}$ for each $x \in C$, and hence $d(C,A) \leq 2^{-n}$;*

(b) *If $S_n(A) \cap B = \emptyset$, then $d(A,B) \geq 2^{-n}$;*

(c) *If $G = \overline{S_{n+3}(A)}$, $H = \overline{S_{n+3}(B)}$, and $S_n(A) \cap B = \emptyset$, then $d(G,H) \geq 2^{-(n+1)}$; and*

(d) *If $G = S_{n+2}(A)$, $H = S_{n+2}(B)$, and $d(A,B) \geq 2^{-n}$, then $d(G,H) \geq 2^{-(n+1)}$.*

Proof. (a) Let $x \in C$ and let $\epsilon > 0$. Then $N_\epsilon(x) \cap S_n(A) \neq \emptyset$. Let $p \in N_\epsilon(x) \cap S_n(A)$. Then $d(p,x) < \epsilon$ and $d(p,A) < 2^{-n}$. By 6.24, $d(x,A) < \epsilon + 2^{-n}$ for each $\epsilon > 0$. Thus $d(x,A) \leq 2^{-n}$, so that $d(C,A) \leq 2^{-n}$.

(b) Suppose $d(A,B) < 2^{-n}$. By 6.25, $d(A,B) = \text{glb}\,\{d(A,y): y \in B\}$. Thus there exists $b \in B$ such that $d(a,b) < 2^{-n}$, and hence $b \in S_n(A)$. This contradicts $S_n(A) \cap B = \emptyset$, so that $d(A,B) \geq 2^{-n}$.

(c) Suppose $d(G,H) < 2^{-(n+1)}$. Then there exists $g \in G$ and $h \in H$ such that $d(g,h) < 2^{-(n+1)}$. By (a), $d(g,A) \leq 2^{-(n+3)}$ and $d(h,B) \leq 2^{-(n+3)}$. By 6.24, $d(A,B) \leq d(A,g) + d(g,h) + d(h,B) \leq 2^{-(n+3)} + d(g,h) + 2^{-(n+3)} = 2^{-(n+2)} + d(g,h) < 2^{-(n+2)} + 2^{-(n+1)} = \frac{3}{2} \cdot 2^{-(n+1)}$. By (b), we have $2^{-n} < \frac{3}{2} \cdot 2^{-(n+1)}$, so that $1 < \frac{3}{4}$. This contradiction yields that $d(G,H) \geq 2^{-(n+1)}$.

(d) Suppose $d(G,H) < 2^{-(n+1)}$. Then $d(g,h) < 2^{-(n+1)}$ for some $g \in G$ and $h \in H$. By (a), $d(g,A) \leq 2^{-(n+2)}$ and $d(h,B) \leq 2^{-(n+2)}$. Thus, by 6.24, $d(A,B) \leq d(A,g) + d(g,h) + d(h,B) \leq 2^{-(n+2)} + d(g,h) + 2^{-(n+2)} = 2^{-(n+1)} + d(g,h) < 2^{-(n+1)} + 2^{-(n+1)} = 2^{-n}$, which contradicts our assumption. ■

6.28 **Lemma**. *Let (X,d) be a metric space, $\epsilon > 0$, and let $\{F_\alpha: \alpha \in A\}$, be a family of closed subsets of X such that $d(F_\alpha, F_\beta) \geq \epsilon$ for $\alpha \neq \beta$ in A. Then $F = \cup\{F_\alpha: \alpha \in A\}$ is closed.*

Proof. Let $x \in \overline{F}$. Then $N_{\frac{\epsilon}{2}}(x) \cap F_\alpha \neq \emptyset$ and $N_{\frac{\epsilon}{2}}(x) \cap F_\beta \neq \emptyset$. Let $p \in N_{\frac{\epsilon}{2}}(x) \cap F_\alpha$ and let $q \in N_{\frac{\epsilon}{2}}(x) \cap F_\beta$. Then $d(x,p) < \frac{\epsilon}{2}$, $p \in F_\alpha$ and $d(x,q) < \frac{\epsilon}{2}$, $q \in F_\beta$. Thus $d(p,q) \leq d(p,x) + d(q,x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, so that $d(p,q) < \epsilon$, $p \in F_\alpha$, and $q \in F_\beta$. It follows that $d(F_\alpha, F_\beta) < \epsilon$ and hence $\alpha = \beta$. We conclude that $N_{\frac{\epsilon}{2}}(x) \cap F_{\alpha_0} \neq \emptyset$ for exactly one $\alpha_0 \in A$, and hence each neighborhood of x meets F_{α_0}. Since F_{α_0} is closed, $x \in F_{\alpha_0}$, and $x \in F$; and F is closed. ■

6.29 **Theorem**. *Each metric space is paracompact.*

Proof. Let (X,d) be a metric space and let $\{U_\alpha: \alpha \in A\}$ be an open cover of X. Let $<$ be a well-ordering of A. For $E \subseteq X$ and $n \in \mathbb{N}$, let $S_n(E) = \{x \in X: d(x, E) < 2^{-n}\}$ and let $C_n(E) = \{x \in X: S_n(x) \subseteq E\}$. Then by 6.26:

$S_n(E)$ is open;
$E \subseteq S_n(E)$;
$C_n(E)$ is closed;
$C_n(E) \subseteq E$; and
$S_n(C_n(E)) \subseteq E$.

Let $n \in \mathbb{N}$. We claim that there exists a collection $\{D^n_\alpha: \alpha \in A\}$ of subsets of X such that $D^n_\alpha = C_n(U_\alpha \setminus \bigcup_{\gamma<\alpha} D^n_\gamma)$ for each $\alpha \in A$. To establish this, let $\mathcal{A}$ be the collection of all pairs (σ, B) satisfying:

(a) $B \subseteq A$;
(b) $\alpha_0 \in B$, where α_0 is the least element of A;
(c) If $\gamma \in A$, $\alpha \in B$, and $\gamma < \alpha$, then $\gamma \in B$; and
(d) $\sigma = \{M_\alpha: \alpha \in B\}$ is a collection of subsets of X such that $M_\alpha = C_n(U_\alpha \setminus \bigcup\{M_\gamma: \gamma < \alpha, \gamma \in B\})$ for each $\alpha \in B$.

Now $(\{C_n(U_{\alpha_0})\}, \{\alpha_0\}) \in \mathcal{A}$, so that $\mathcal{A} \neq \emptyset$. Define $(\sigma_1, B_1) \leq (\sigma_2, B_2)$ on $\mathcal{A}$ provided $\sigma_1 \subseteq \sigma_2$ and $B_1 \subseteq B_2$. Then $\leq$ is a partial order on $\mathcal{A}$. Let $\mathcal{A}^*$ be a maximal chain in $\mathcal{A}$, and let $H = \bigcup\{B: (\sigma, B) \in \mathcal{A}^*\}$. We will show that $H = A$.

Suppose that $H \neq A$, and let α_1 be the least element of $A \setminus H$. Then $\alpha_1 \neq \alpha_0$, since $\alpha_0 \in H$. Now from (c), $\gamma < \alpha_1$, implies $\gamma \in H$, and $\alpha_1 < \beta$ implies $\beta \in A \setminus H$. Define $M_{\alpha_1} = C_n(U_{\alpha_1} \setminus \bigcup_{\gamma \in H} M_\gamma)$ and let $\sigma_1 = \bigcup\{\sigma: (\sigma, B) \in \mathcal{A}^*\}$. Then $(\sigma_1 \cup \{M_{\alpha_1}\}, H \cup \{\alpha_1\}) \in \mathcal{A}$ and is above each member of $\mathcal{A}^*$; contradicting the maximality of $\mathcal{A}^*$. It follows that $H = A$ and that σ_1 is the desired collection.

We claim that $\{D^n_\alpha: \alpha \in A, n \in \mathbb{N}\}$ is a cover of X. Let $x \in X$, and let λ be the least element of A such that $x \in U_\lambda$. Let $n \in \mathbb{N}$ such that $N_{\frac{1}{2^n}}(x) = S_n(x) \subseteq U_\lambda$. Suppose that $x \notin D^n_\lambda$. Then $S_n(x) \not\subseteq U_\lambda \setminus \bigcup_{\gamma<\lambda} D^n_\gamma$, by definition of D^n_λ. Thus $S_n(x) \cap \bigcup_{\gamma<\lambda} D^n_\gamma \neq \emptyset$, and hence $S_n(x) \cap D^n_\beta \neq \emptyset$ for some $\beta < \lambda$. We obtain that $x \in S_n(D^n_\beta) = S_n(C_n(U_\beta \setminus \bigcup_{\gamma<\beta} D^n_\gamma)) \subseteq$ [from 6.26(e)] $U_\beta \setminus \bigcup_{\gamma<\beta} D^n_\gamma \subseteq U_\beta$. Thus $x \in U_\beta$ and $\beta < \lambda$; contradicting the fact that λ is the least such member of A. We conclude that $x \in D^n_\lambda$ and $\{D^n_\alpha: \alpha \in A, n \in \mathbb{N}\}$ covers X.

For each $n \in \mathbb{N}$ and $\alpha \in A$ define:

$F_\alpha^n = \overline{S_{n+3}(D_\alpha^n)}$; and
$G_\alpha^n = S_{n+2}(D_\alpha^n)$.
Note that G_α^n is open from 6.26(a).

Let $x \in F_\alpha^n$. Then, by 6.27(a), $d(x, D_\alpha^n) \leq 2^{-(n+1)}$, so that $d(x, D_\alpha^n) < 2^{-(n+2)}$ and hence $x \in G_\alpha^n$. It follows that $F_\alpha^n \subseteq G_\alpha^n$.

Suppose that $\alpha < \beta$ in A. Then $S_n(D_\beta^n) = S_n(C_n(U_\beta \setminus \bigcup_{\gamma<\beta} D_\gamma^n)) \subseteq U_\beta \setminus \bigcup_{\gamma<\beta} D_\gamma^n \subseteq X \setminus D_\alpha^n$, so that $S_n(D_\beta^n) \cap D_\alpha^n = \emptyset$. By 6.27(b), $d(D_\beta^n, D_\alpha^n) \geq 2^{-n}$. Thus for $\alpha \neq \beta$ in A and $n \in \mathbb{N}$, $d(D_\beta^n, D_\alpha^n) \geq 2^{-n}$. Also from 6.27(c), $d(F_\alpha^n, F_\beta^n) \geq 2^{-(n+1)}$ for $\alpha \neq \beta$ in A and $n \in \mathbb{N}$.

Let $F^n = \bigcup_{\alpha \in A} F_\alpha^n$ for each $n \in \mathbb{N}$. Then, by 6.28, F^n is closed for each $n \in \mathbb{N}$.

Define $U_\alpha^n = G_\alpha^n \setminus \bigcup_{i<n} F^i$ for $n \in \mathbb{N}$, $\alpha \in A$. Then U_α^n is open.

We claim that $\{U_\alpha^n : \alpha \in A, n \in \mathbb{N}\}$ is a locally finite refinement of $\{U_\alpha : \alpha \in A\}$.

To see that $\{U_\alpha^n : \alpha \in A, n \in \mathbb{N}\}$ covers X, let $x \in X$. Now, since $\{D_\alpha^n : \alpha \in A, n \in \mathbb{N}\}$ is a cover of X, so is $\{F_\alpha^n : \alpha \in A, n \in \mathbb{N}\}$. Let m be the least positive integer such that $x \in F_\beta^m$ for some $\beta \in A$. Then $x \in F_\beta^m \setminus \bigcup\{F_\alpha^i : \alpha \in A, i < m\} = F_\beta^m \setminus \bigcup\{F^i : i < m\} \subseteq G_\beta^m \setminus \bigcup\{F^i : i < m\} = U_\beta^m$.

To see that $\{U_\alpha^n : \alpha \in A, n \in \mathbb{N}\}$ is a refinement of $\{U_\alpha : \alpha \in A\}$, let $\alpha \in A$ and $n \in \mathbb{N}$. Then $U_\alpha^n \subseteq G_\alpha^n = S_{n+2}(D_\alpha^n) \subseteq S_n(D_\alpha^n) = S_n(C_n(U_\alpha \setminus \bigcup_{\gamma<\alpha} D_\gamma^n)) \subseteq U_\alpha \setminus \bigcup_{\gamma<\alpha} D_\gamma^n \subseteq U_\alpha$.

It remains to show that $\{U_\alpha^n : \alpha \in A, n \in \mathbb{N}\}$ is locally finite. Let $x \in X$. Then $x \in D_\beta^m$ for some $m \in \mathbb{N}$ and $\beta \in A$. Now $S_{m+3}(x)$ is an open set containing x. We will show that $S_{m+3}(x)$ meets at most m members of $\{U_\alpha^n : \alpha \in A, n \in \mathbb{N}\}$. Now $S_{m+3} \subseteq S_{m+3}(D_\beta^m) \subseteq F_\beta^m \subseteq F^m$, so that by definition of U_α^n, we have that $S_{m+3}(x) \cap U_\alpha^i = \emptyset$ for $\alpha \in A$ and $i > m$. Suppose that $i \leq m$ and $\alpha \neq \gamma$ in A. Then $d(D_\alpha^i, D_\gamma^i) \geq 2^{-i}$, so that, by 6.27(d), $d(G_\alpha^i, G_\gamma^i) \geq 2^{-(i+1)} \geq 2^{-(m+1)}$. Suppose that $S_{m+3}(x)$ meets both G_α^i and G_γ^i. Let $p \in S_{m+3}(x) \cap G_\alpha^i$ and let $q \in S_{m+3}(x) \cap G_\gamma^i$, so that $d(p, q) < 2^{-(m+2)}$, and hence $d(G_\alpha^i, G_\gamma^i) < 2^{-(m+2)}$; contradicting $d(G_\alpha^i, G_\gamma^i) \geq 2^{-(m+1)}$. Thus $S_{m+3}(x)$ meets at most one G_α^i for each $i \leq m$. It follows that $S_{m+3}(x)$ meets at most m members of $\{G_\alpha^n : \alpha \in A, n \in \mathbb{N}\}$, and hence at most m members of $\{U_\alpha^n : \alpha \in A, n \in \mathbb{N}\}$. ■

If (X, d) and (Y, e) are metric spaces, and $f : X \to Y$ is a function, then f is said to be **uniformly continuous** if for each $\epsilon > 0$, there exists $\delta > 0$ such that $e(f(a), f(b)) < \epsilon$ whenever $a, b \in X$ and $d(a, b) < \delta$.

6.30 **Theorem.** *Let X be a compact metric space, Y a metric space, and $f: X \to Y$ a continuous function. Then f is uniformly continuous.*

Proof. Let $\epsilon > 0$. Since f is continuous, and X is compact, $f(X)$ is compact. Let $y_1, y_2, \ldots, y_n$ be a finite $\frac{\epsilon}{2}$ -net for $f(X)$. Then $f(X) \subseteq \bigcup_{j=1}^{n} N_{\frac{\epsilon}{2}}(y_j)$ and $\{f^{-1}(N_{\frac{\epsilon}{2}}(y_j)): 1 \leq j \leq n\}$ is an open cover of X. For each $p \in X$, let $\delta_p > 0$ such that $N_{2\delta_p}(p) \subseteq f^{-1}(N_{\frac{\epsilon}{2}}(y_j))$ for some j. Now $\{N_{\delta_p}(p): p \in X\}$ is an open cover of X. Let $N_{\delta_{p_1}}(p_1), \ldots, N_{\delta_{p_k}}(p_k)$ be a finite subcover, and let $\delta = \min\{\delta_{p_1}, \ldots, \delta_{p_k}\}$.

Suppose that $a, b \in X$ and $d(a, b) < \delta$. Now $a \in N_{\delta_{p_i}}$ for some i, so that $d(a, p_i) < \delta_{p_i}$ and $d(a, b) < \delta_{p_i}$. We obtain that $d(b, p_i) < 2\delta_{p_i}$ and $a, b \in N_{2\delta_{p_i}}(p_i)$. Thus $f(a), f(b) \in N_{\frac{\epsilon}{2}}(y_j)$ for some j, and hence $d(f(a), f(b)) < \epsilon$. ∎

7 BAIRE CATEGORY THEORY

Recall that a subset E of a space X is nowhere dense in X provided $\overline{E}^{\circ} = \emptyset$.

7.1 **Theorem.** *Let E be a subset of a space X.* (a) *E is nowhere dense in X if and only if for each nonempty open subset U of X, there exists a nonempty open subset V of X such that $V \subseteq U$ and $V \cap E = \emptyset$.* (b) *If E is nowhere dense in X and $B \subseteq E$, then B is nowhere dense in X.* (c) *If $B \subseteq E$ and B is nowhere dense in E, then B is nowhere dense in X.* (d) *If E is nowhere dense in X, then $X \backslash \overline{E}$ is dense in X.* (e) *If E is open, then $\overline{E} \backslash E$ is nowhere dense in $\overline{E}$.*

A subset E of a space X is said to be **first category in** X if there exists a countable collection E_i for $i \in \mathbb{N}$ of subsets of X each of which is nowhere dense in X such that $E = \bigcup_{i \in \mathbb{N}} E_i$. If E is not first category in X, then E is said to be **second category in** X. If X is first [second] category in itself, then X is said to be a **first** [**second**] **category space**.

7.2 **Lemma.** *Let E be a subset of a space X and let $B \subseteq E$. Then:*

(a) *If E is first category in X, then B is first category in X.*

(b) *If B is second category in X, the E is second category in X.*

(c) *If B is first category in E, then B is first category in X.*

(d) *If B is second category in X, then B is second category in E.*

(e) *If E is open and B is second category in E, then B is second category in X.*

7.3 **Theorem.** *Let X be a locally compact Hausdorff [or complete metric]* *space, and let G_n for $n \in \mathbb{N}$ be a countable collection of open subsets of X each of which is dense in X. Then $\bigcap_{i \in \mathbb{N}} G_i$ is dense in X.*

Proof. Suppose that A and B are open dense subsets of X, and let U be an open subset of X. Then $U \cap A$ is nonempty and open in X, so that $(U \cap A) \cap B \neq \emptyset$. We obtain that $A \cap B$ is dense in X.

Case 1. Suppose that X is a locally compact Hausdorff space. Let U be an open subset of X, and let V_0 be an open subset of X such that $\overline{V}_0$ is compact and $V_0 \subseteq \overline{V}_0 \subseteq U$. Let $V_1 = V_0 \cap G_1$. There exists an open set V_2 such that $\overline{V}_2$ is compact and $V_2 \subseteq \overline{V}_2 \subseteq V_1 \cap G_2$. Define recursively a sequence $\{V_n\}$ of open subsets of X such that for each $n \in \mathbb{N}$, $\overline{V}_n$ is compact and $V_{n+1} \subseteq \overline{V}_{n+1} \subseteq V_n \cap G_{n+1}$. Now $\overline{V}_1 \supseteq \overline{V}_2 \supseteq \cdots$ is a tower of compact subsets of X and hence $\bigcap_{n \in \mathbb{N}} \overline{V}_n \neq \emptyset$. Let $p \in \bigcap_{n \in \mathbb{N}} \overline{V}_n$. We claim that $p \in U \cap \bigcap_{n \in \mathbb{N}} G_n$.

Let $n \in \mathbb{N}$. Then $p \in \overline{V}_{n+1} \subseteq V_n \cap G_{n+1} \subseteq \overline{V}_n \cap G_{n+1} \subseteq V_{n-1} \cap G_n \cap G_{n+1} \subseteq \cdots \subseteq V_0 \cap G_1 \cap G_2 \cap \cdots \cap G_{n+1} \subseteq U \cap G_1 \cap \cdots \cap G_{n+1}$, so that $p \in U$ and $p \in G_n$. We obtain that $p \in U \cap \bigcap_{n \in \mathbb{N}} G_n$, and hence $\bigcap_{n \in \mathbb{N}} G_n$ is dense in X.

Case 2. Suppose that (X, d) is a complete metric space. For $n \in \mathbb{N}$ and $p \in X$, let $S_n(p) = \{x \in X : d(x,p) < \frac{1}{n}\}$ and let $S_n^\sharp(p) = \{x \in X : d(x,p) \leq \frac{1}{n}\}$.

Let U be an open subset of X. We want to show that $U \cap \bigcap_{n \in \mathbb{N}} G_n \neq \emptyset$. There exists $n_1 \in \mathbb{N}$ and $p_1 \in X$ such that $S_{n_1}^\sharp(p_1) \subseteq U$. Let $V_1 = S_{n_1}(p_1) \cap G_1$ ($\neq \emptyset$, since G_1 is dense). There exists $n_2 > n_1$ and $p_2 \in X$ such that $S_{n_2}^\sharp(p_2) \subseteq V_1$. Let $V_2 = S_{n_2}(p_2) \cap G_2$ and $n_3 > n_2$ and $p_3 \in X$ such that $S_{n_3}^\sharp \subseteq V_2$. We obtain recursively, sequences $\{p_n\}$ in X, $n_1 < n_2 < n_3 < \cdots$, $\{S_{n_j}(p_j)\}$ and $\{V_n\}$ such that $V_j = S_{n_j}(p_j) \cap G_j$, $S_{n_{j+1}}(p_j) \subseteq S_{n_{j+1}}^\sharp(p_{j+1}) \subseteq V_j$, and $S_{n_{j+1}}(p_{j+1}) \subseteq S_{n_{j+1}}^\sharp \subseteq S_{n_j}(p_j) \subseteq S_{n_j}^\sharp(p_j) \subseteq \cdots \subseteq U$.

7.4 **Baire Category Theorem.** *Let X be a locally compact Hausdorff [or complete metric] space. Then X is second category.*

Proof. Suppose that X is first category. Then $X = \bigcup_{i \in \mathbb{N}} A_i$, where $\overline{A}_i^\circ = \emptyset$. Let $B_i = \overline{A}_i$ for each $i \in \mathbb{N}$. Then $X = \bigcup_{i \in \mathbb{N}} B_i$, B_i is closed, and $B_i^\circ = \emptyset$. For each $i \in \mathbb{N}$, $X \backslash B_i$ is open and dense in X from 7.1(d). Thus from 7.3, $\bigcap_{i \in \mathbb{N}} (X \backslash B_i)$ is dense in X. But $\bigcap_{i \in \mathbb{N}} (X \backslash B_i) = X \backslash \bigcup_{i \in \mathbb{N}} B_i = \emptyset$. This contradiction proves that X is second category. ■

7.5 **Theorem.** *Let X be a locally compact Hausdorff [or complete metric] space. Then:*

(a) *Each nonempty open subset of X is second category in X.*

(b) *If $E \subset X$ and E is first category in X, then $X \backslash E$ is second category in X and dense in X.*

Throughout the remainder of this section we let $\mathbb{R}$ denote the space of real numbers with the Euclidean (usual) topology, and [0,1] the unit interval with the relative topology. Note that the topology of $\mathbb{R}$ is induced by the metric defined by $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$ and that [0,1] is compact. Also note that $\mathbb{R}$ is complete.

Let $C[0,1]$ denote the continuous functions $f\colon [0,1] \to \mathbb{R}$ with a metric defined by $d(f,g) = sup_{x\in[0,1]} \; |f(x) - g(x)|$. This metric is called the **sup metric** on $C[0,1]$.

7.6 **Lemma.** *The set $C[0,1]$ with the sup metric is a metric space.*

7.7 **Lemma.** *Let f_n converge to f in $C[0,1]$ and let x_n converge to x in $[0,1]$. Then $f_n(x_n)$ converges to $f(x)$ in $\mathbb{R}$.*

7.8 **Lemma.** *If f_n is a Cauchy sequence in $C[0,1]$ and $x \in [0,1]$, then $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$.*

7.9 **Theorem.** *The space $C[0,1]$ is a complete metric space.*

If $n \in \mathbb{N}$, let P_n denoted the set of all $f \in C[0,1]$ such that there exists $p \in [0, 1 - \frac{1}{n}]$ such that $|\frac{f(p+h)-f(p)}{h}| \leq n$ for all $h \in (0, \frac{1}{n})$.

7.10 **Lemma.** *For each $n \in \mathbb{N}$, P_n is closed and nowhere dense in $C[0,1]$.*

7.11 **Lemma.** The space $C[0,1]$ is not equal to $\bigcup_{n\in\mathbb{N}} P_n$.

7.12 **Theorem.** *There exists a continuous function $f\colon [0,1] \to \mathbb{R}$ which is nowhere differentiable on $[0,1)$.*

Proof. Let $f \in C[0,1]$ have a derivative at $p \in [0,1)$ and let $a = |f'(p)|$. Then there exists $\epsilon > 0$ such that $|\frac{f(p+h)-f(p)}{h}| \leq a + 1$ for $|h| < \epsilon$. Let $n \in \mathbb{N}$ such that $a + 1 < n$, $\frac{1}{n} < \epsilon$, and $p < 1 - \frac{1}{n}$. Then $f \in P_n$. Thus $\bigcup_{n\in\mathbb{N}} P_n$ contains all functions which are differentiable at some point of $[0,1)$. ∎

8 NETS

A **directed set** $(D, \leq)$ is a set D together with a relation $\leq$ on D such that:

(a) $a \leq a$ for each $a \in D$ (reflexive);

(b) If $a \leq b$ and $b \leq c$ in D, then $a \leq c$ (transitive); and

(c) If $a, b \in D$, then there exists $c \in D$ such that $a \leq c$ and $b \leq c$ (directed property).

A **net** in a space X is a function $\phi\colon D \to X$ from a directed set D into X.

If $\phi\colon D \to X$ is a net and $E \subseteq X$, then ϕ is **eventually** in E if there exists $a \in D$ such that $\phi(d) \in E$ for each $d \in D$ such that $a \leq d$. This is denoted $\phi \in^e E$.

If $\phi\colon D \to X$ is a net and $E \subseteq X$, then ϕ is **frequently** in E if for each $d \in D$, there exists $e \in D$ such that $d \leq e$ and $\phi(e) \in E$. This is denoted $\phi \in^f E$.

If $\phi\colon D \to X$ is a net in a space X and $p \in X$, then ϕ **converges** [**clusters**] to p provided ϕ is eventually [frequently] in each neighborhood of p. This is denoted $\phi \xrightarrow{e} p$ [$\phi \xrightarrow{f} p$].

8.1 **Theorem**. *A space X is Hausdorff if and only if each net in X has at most one point of convergence.*

A **sequence** is a net whose domain is the set of positive integers with the usual ordering.

8.2 **Theorem**. *Let E be a subset of a space X. These are equivalent:*

(a) E is closed;

(b) If ϕ is a net in E and $\phi \xrightarrow{e} p$, then $p \in E$; and

(c) If ϕ is a net in E and $\phi \xrightarrow{f} p$, then $p \in E$.

If E and D are directed sets, then a function $\lambda\colon E \to D$ is **cofinal** in D provided that for each $d \in D$, there exists $e \in E$ such that $d \leq \lambda(x)$ in D whenever $e \leq x$ in E.

A **subnet** of a net $\phi\colon D \to X$ is a net $\psi\colon E \to X$ such that there exists a cofinal function $\lambda\colon E \to D$ with $\psi = \phi\lambda$.

8.3 **Theorem**. *Let X be a space, $\phi\colon D \to X$ a net, and let $p \in X$. Then $\phi \xrightarrow{f} p$ if and only if there exists a subnet $\psi\colon E \to X$ such that $\psi \xrightarrow{e} p$.*

Proof. Suppose $\phi \xrightarrow{f} p$, and let $\mathcal{U}$ be the collection of all neighborhoods of p. Let $E = \{(d,U) \in D \times \mathcal{U}\colon \phi(d) \in U\}$ and define $(d_1,U_1) \leq (d_2,U_2)$ provided $d_1 \leq d_2$ and $U_2 \subseteq U_1$. Then E is directed set. Define $\psi\colon E \to X$ by $\psi(d,U) = \phi(d)$ and $\lambda\colon E \to D$ by $\lambda(d,U) = d$. Then, for $(d,U) \in E$, $\phi\lambda(d,U) = \phi(d) = \psi(d,U)$, so that $\psi = \phi \circ \lambda$. To see that λ is cofinal in D, let $d \in D$, and let $U \in \mathcal{U}$ such that $\phi(d) \in U$. Then $(d,U) \in E$. Suppose that $(d,U) \leq (d',U')$ in E. Then $d \leq d'$ and $U' \subseteq U$, so that $d \leq d' = \lambda(d',U')$ and hence λ is cofinal in D. We obtain that ψ is a subnet of ϕ.

To see that $\psi \xrightarrow{e} p$, let U be a neighborhood of p. Then $\phi(d) \in U$ for some $d \in D$. Suppose that $(d,U) \leq (d',U')$ in E. Then $d \leq d'$ and $U' \subseteq U$. Now $\psi(d',U') = \phi(d') \in U' \subseteq U$, so that $\psi(d',U') \in U$, $\psi \in^e U$, and $\psi \xrightarrow{e} p$.

Suppose there is a subnet $\psi\colon E \to X$ of $\phi\colon D \to X$ such that $\psi \xrightarrow{e} p$. Let $\lambda\colon E \to D$ be a cofinal function such that $\psi = \phi \circ \lambda$. Let U be a neighborhood of p, and let $d' \in D$. Now there exists $e' \in E$ such that $e' \leq e$ implies that

$\psi(e) \in U$, and there exists $e'' \in E$ such that $e'' \leq x$ implies that $d' \leq \lambda(x)$. Let $\tilde{e} \in E$ such that $e' \leq \tilde{e}$ and $e'' \leq \tilde{e}$. Then $\psi(\tilde{e}) \in U$ and $d' \leq \lambda(\tilde{e})$, so that $\psi(\tilde{e}) = \phi(\lambda(\tilde{e})) \in U$. Let $d = \lambda(\tilde{e})$. Then $d' \leq d$ and $\phi(d) \in U$, so that $\phi \xrightarrow{f} p$. ∎

8.4 **Theorem.** *Let X be a space, $p \in X$, and $\phi\colon D \to X$ a net in X such that $\phi \xrightarrow{e} p$. If $\psi\colon E \to X$ is a subnet of ϕ, then $\psi \xrightarrow{e} p$.*

8.5 **Theorem.** *A space X is compact if and only if each net in X has a subnet which converges to a point of X.*

Proof. Suppose that X is compact and that $\phi\colon D \to X$ is a net. Suppose that for each $p \in X$, the net ϕ does not cluster to p. Then for each $p \in X$, there exists an open set U_p containing p such that $\phi \in^e X \backslash U_p$. Now $\{U_p\colon p \in X\}$ is an open cover of X. Since X is compact, there exists a finite subcover $\{U_1, U_2, \ldots, U_n\}$. For each U_j, there exists $d_j \in D$ such that $d_j \leq d$ in D implies that $\phi(d) \in X \backslash U_j$. Let $d \in D$ such that $d_1, d_2, \ldots, d_n \leq d$. Then $\phi(d) \in \bigcap_{j=1}^{n} (X \backslash U_j) = X \backslash \bigcup j = 1^n U_j = \emptyset$. This contradiction yields that $\phi \xrightarrow{f} p$ for some $p \in X$, and hence by 8.3, there is a subnet converging to p.

Suppose that each net in X has a subnet which converges to a point of X. Then, by 8.3, each net in X clusters to a point in X. Let $\mathcal{A}'$ be a collection of closed subsets of X with the finite intersection property and let $\mathcal{A}$ be the collection of all finite intersections of memember of $\mathcal{A}'$. For $A \in \mathcal{A}$, let $p_A \in A$. Define $\phi\colon \mathcal{A} \to X$ by $\phi(A) = p_A$. Now $\mathcal{A}$ is ordered by reverse inclusion and is thus a directed set, so that ϕ is a net in X. We have that $\phi \xrightarrow{f} p$ for some $p \in X$. Suppose that $p \in X \backslash A$ for some $A \in \mathcal{A}$. Then there exists $B \subseteq A$, $B \in \mathcal{A}$, such that $\phi(B) \in X \backslash A$, since $X \backslash A$ is open and $\phi \xrightarrow{f} p$. Now $\phi(B) \in B$ and hence $B \cap (X \backslash A) \neq \emptyset$. This contradicts $B \subseteq A$, so that $p \in A$ for each $A \in \mathcal{A}$, $\bigcap \mathcal{A}' = \bigcap \mathcal{A} \neq \emptyset$. From 4.2, X is compact. ∎

8.6 **Theorem.** *Let $f\colon X \to Y$ be a function and let $p \in X$. Then these are equivalent:*

(a) f is continuous at p;

(b) If $\phi \xrightarrow{e} p$, then $f\phi \xrightarrow{e} f(p)$; and

(c) If $\phi \xrightarrow{f} p$, then $f\phi \xrightarrow{f} f(p)$.

A net $\phi\colon D \to X$ is said to be a **universal net** if for each $A \subseteq X$, either $\phi \in^e A$ or $\phi \in^e X \backslash A$.

Note that if $\phi\colon D \to A$ is a universal net, $A \subseteq X$, and $\phi \in^f A$, then $\phi \in^e A$.

8.7 **Theorem.** *If $\phi\colon D \to X$ is a universal net in a space X and $p \in X$ such that $\phi \xrightarrow{f} p$, then $\phi \xrightarrow{e} p$.*

8.8 **Theorem.** *Each subnet of a universal net is a universal net.*

8.9 **Theorem**. *If $\phi\colon D \to X$ is a universal net in a space X, and $f\colon X \to Y$ is a function, then $f\phi\colon D \to Y$ is a universal net in Y.*

8.10 **Lemma**. *Let $\phi\colon D \to X$ be a net in a space X. Then there exists a family $\mathcal{A}$ of subsets of X such that:*

(a) $\phi \in^f A$ for all $A \in \mathcal{A}$;

(b) If $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$; and

(c) If $S \subseteq X$, then either $S \in \mathcal{A}$ or $X \backslash S \in \mathcal{A}$.

Proof. Let $\mathcal{C} = \{\widetilde{\mathcal{A}}\colon \widetilde{\mathcal{A}}$ is a family of subsets of X satisfying (a) and (b) $\}$. Then $\mathcal{C}$ is nonempty, since $\{X\} \in \mathcal{C}$, and $\mathcal{C}$ is partially ordered by inclusion. Let $\mathcal{C}'$ be a maximal chain in $\mathcal{C}$, and let $\mathcal{A} = \{A\colon A \in \widetilde{\mathcal{A}}$ for some $\widetilde{\mathcal{A}} \in \mathcal{C}'\}$. Then $\mathcal{A}$ is a maximal member of $\mathcal{C}$. Let $S \subseteq X$. We consider two cases.

Case 1. $\phi \notin^f S$. In this case $\phi \in^e X \backslash S$, so that $\phi \in^f (X \backslash S) \cap A$ for each $A \in \mathcal{A}$. Thus $\mathcal{A} \cup \{A \cap (X \backslash S)\colon A \in \mathcal{A}\} \cup \{X \backslash S\}$ is a member of $\mathcal{C}$ containing $\mathcal{A}$. The maximality of $\mathcal{A}$ yields that $X \backslash S \in \mathcal{A}$.

Case 2. $\phi \in^f S$. We consider two subcases.

Subcase 1. $\phi \in^f S \cap A$ for each $A \in \mathcal{A}$. In this case we obtain that $S \in \mathcal{A}$.

Subcase 2. $\phi \notin^f S \cap A$ for some $A \in \mathcal{A}$. Then $\phi \in^e X \backslash (S \cap A) = (X \backslash S) \cup (X \backslash A)$, so that $(X \backslash S) \cup (X \backslash A) \in \mathcal{A}$ (as in case 1). But $A \cap [(X \backslash S) \cup (X \backslash A)] = A \cap (X \backslash S) \in \mathcal{A}$. Let $C \in \mathcal{A}$. Now $A \cap C \in \mathcal{A}$ implies that $(A \cap C) \cap [(X \backslash S) \cup (X \backslash A)] = (A \cap C) \cap (X \backslash S) \in \mathcal{A}$. Then $\phi \in^f C \cap (X \backslash S)$ and $\mathcal{A} \cup \{X \backslash S\}$ satisfies (a) and (b), so that $X \backslash S \in \mathcal{A}$. ■

8.11 **Theorem**. *Each net in a space X has a universal subnet.*

Proof. Let $\phi\colon D \to X$ be a net in X, and let $\mathcal{A}$ be a family of subsets of X satisfying 8.10. Let $E = \{(d, A) \in D \times \mathcal{A}\colon \phi(d) \in A\}$ and define $(d_1, A_1) \leq (d_2, A_2)$ on E if $d_1 \leq d_2$ in D and $A_2 \subseteq A_1$. Then $(E, \leq)$ is a directed set. Define $\lambda\colon E \to D$ by $\lambda(d, A) = d$ and let $\psi = \phi \circ \lambda$. Then ψ is a subnet of ϕ.

Let $A \in \mathcal{A}$. Then $\phi(d) \in A$ for some $d \in D$. Suppose $(d, A) \leq (e, B)$ in E. Then $\psi(e, B) = \phi\lambda(e, B) = \phi(e) \in B \subseteq A$, so that $\psi \in^e A$ for each $A \in \mathcal{A}$. For $S \subseteq A$ either $S \in \mathcal{A}$ or $X \backslash S \in \mathcal{A}$, so that either $\psi \in^e S$ or $\psi \in^e X \backslash S$, and hence ψ is universal. ■

8.12 **Theorem**. *A space X is compact if and only if each universal net in X converges.*

8.13 **Exercise**. Construct a proof of the Tychonoff Theorem using universal nets.

9 HOMOTOPY

If X and Y are spaces and $f\colon X \to Y$ and $g\colon X \to Y$ are continuous functions, then f and g are said to be **homotopic** (denoted $f \sim g$) provided there exists a continuous function $H\colon X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for each $x \in X$, where $I = [0,1]$ is the unit interval with the relative topology of the reals. The continuous function H is called a **homotopy**.

9.1 **Theorem.** *Let X be a space. Then any pair $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ of continuous functions from X into $\mathbb{R}^n$ are homotopic.*

9.2 **Theorem.** *Let X be a space and $f\colon X \to S^n$ a continuous function. Then either f is surjective or f is homotopic to a constant map.*

If X and Y are spaces, then $C(X,Y)$ denotes the set of all continuous functions from X into Y.

9.3 **Theorem.** *Let X and Y be spaces. Then $\sim$ is an equivalence relation on $C(X,Y)$.*

The equivalence classes of $\sim$ on $C(X,Y)$ are called **homotopy classes**.

A subspace E of a space X is said to be **contractible in** X if the inclusion map $i\colon E \hookrightarrow X$ is homotopic to a constant map $E \to X$.

A space X is said to be **contractible** if the identity map $1_X\colon X \to X$ is homotopic to a constant map $X \to X$, i.e. X is contractible in itself.

9.4 **Theorem.** *Each proper subspace of S^n is contractible in S^n.*

9.5 **Theorem.** *A product of contractible spaces is contractible.*

If X and Y are spaces and $f\colon X \to Y$ is a continuous function, then f is said to be a **homotopy equivalence** if there exists a continuous function $g\colon Y \to X$ such that $g \circ f \sim 1_X$ and $f \circ g \sim 1_Y$. We say, in this case, that X and Y are **homotopically equivalent** and denote this fact by writing $X \sim Y$.

9.6 **Theorem.** *Homotopy equivalence is an equivalence relation on any set of spaces.*

A property of spaces is called a **homotopy property** if it is preserved by every homotopy equivalence.

9.7 **Theorem.** *Contractibility is a homotopy property.*

9.8 **Theorem.** *Every homotopy property is a topological property.*

9.9 **Theorem.** *Let R be the component equivalence on a space X and let T be the component equivalence on a space Y. If $f\colon X \to Y$ is a continuous function, then there exists a continuous function $\tilde{f}$ such that the diagram:*

$$\begin{array}{ccc} X/R & \xrightarrow{\tilde{f}} & Y/T \\ \uparrow & & \uparrow \\ X & \xrightarrow{f} & Y \end{array}$$

commutes, where the vertical maps are projections.

If $f: X \to Y$ as in 9.9, then $\widetilde{f}$ denotes the induced map as in 9.9.

9.10 **Theorem.** *If two continuous functions $f, g: X \to Y$ are homotopic, then $\widetilde{f} = \widetilde{g}$.*

Proof. Let R and T be the component equivalences on X and Y respectively, and let $H: X \times I \to Y$ be continuous such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for each $x \in X$. Then each of the diagrams:

$$\begin{array}{ccc} X/R & \xrightarrow{\widetilde{f}} & Y/T \\ \alpha \uparrow & & \uparrow \beta \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} X/R & \xrightarrow{\widetilde{g}} & Y/T \\ \alpha \uparrow & & \uparrow \beta \\ X & \xrightarrow{g} & Y \end{array}$$

commute, where α and β are natural projections. Let $p \in X/R$, and let A be a component of X such that $\alpha(A) = p$. Then $A \times I$ is a component of $X \times I$ so that $H(A \times I)$ is contained in some component B of Y. Thus $f(A) = H(A \times \{0\}) \subseteq B$ and $g(A) = H(A \times \{1\}) \subseteq B$. We have that $\widetilde{f}(p) = \widetilde{f}\alpha(A) = \beta(B) = \beta g(A) = \widetilde{g}\alpha(A) = \widetilde{g}(p)$ and $\widetilde{f} = \widetilde{g}$. ∎

9.11 **Theorem.** *If $f: X \to Y$ is a homotopy equivalence, then $\widetilde{f}$ is bijective.*

Proof. Let $g: Y \to X$ be a continuous function such that $gf \sim 1_X$ and $fg \sim 1_Y$. Let R and T be the component equivalences on X and Y, respectively. Then each of the diagrams:

$$\begin{array}{ccc} X/R & \xrightarrow{\widetilde{f}} & Y/T \\ \alpha \uparrow & & \uparrow \beta \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} Y/T & \xrightarrow{\widetilde{g}} & X/R \\ \beta \uparrow & & \uparrow \alpha \\ Y & \xrightarrow{g} & X \end{array}$$

commutes, where α and β are natural projections. Observe that $\widetilde{1}_X$ and $\widetilde{1}_Y$ are the identity maps on X/R and Y/T, respectively. By 9.10, we have $\widetilde{gf} = \widetilde{1}_X$ and $\widetilde{fg} = \widetilde{1}_Y$. We will show that $\widetilde{gf} = \widetilde{g}\widetilde{f}$. Consider the diagrams:

$$\begin{array}{ccccc} X/R & \xrightarrow{\tilde{f}} & Y/T & \xrightarrow{\tilde{g}} & X/R \\ \alpha\uparrow & & \beta\uparrow & & \uparrow\alpha \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & X \end{array}$$

and

$$\begin{array}{ccc} X/R & \xrightarrow{\widetilde{gf}} & X/R \\ \alpha\uparrow & & \uparrow\alpha \\ X & \xrightarrow{gf} & X \end{array}$$

and both diagrams commute, so that $\widetilde{gf} = \tilde{g}\tilde{f}$. Similarly, $\widetilde{fg} = \tilde{f}\tilde{g}$, and we have that $\tilde{g}\tilde{f} = \tilde{1}_X$ and $\tilde{f}\tilde{g} = \tilde{1}_Y$. It follows that $\tilde{f}$ is bijective. ∎

9.12 **Theorem**. *Connectedness is a homotopy property.*

Proof. Suppose that $X \sim Y$ and that X is connected. Let $f: X \to Y$ be a homotopy equivalence, and let R and T be the component equivalences of X and Y, respectively. Then the diagram:

$$\begin{array}{ccc} X/R & \xrightarrow{\tilde{f}} & Y/T \\ \uparrow & & \uparrow \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Since X is connected, X/R is degenerate, and since $\tilde{f}$ is surjective (9.11), Y/T is degenerate. It follows that Y is connected. ∎

9.13 **Theorem**. *Let X be a space. Then X is contractible if and only if for each space Y and each pair of continuous functions $f, g: Y \to X$, $f \sim g$.*

Proof. Suppose that $f, g: Y \to X$ implies that $f \sim g$. Let $Y = X$, $f = 1_X$, and $g: X \to X$ a constant map. Then $f \sim g$, so that X is contractible.

Suppose, on the other hand, that X is contractible. Let Y be a space and let $f, g: Y \to X$ be continuous functions. Let $c \in X$. Then $1_X \sim c$. Let $H: X \times I \to X$ denote this homotopy with $H(x, 0) = 1_X(x) = x$ and $H(x, 1) = c$.

Define $F: Y \times I \to X$ by

$$F(y,t) = \begin{cases} H(f(y), 2t) & \text{for } 0 \le t \le \frac{1}{2} \\ H(g(y), 2-2t) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

Then F is continuous, and $F(y,0) = H(f(y),0) = f(y)$ and $F(y,1) = H(g(y),0)$ $= g(y)$ so that $F\colon f \sim g$.

10 PATHS

A **path** in a space X is a continuous function $\sigma\colon I \to X$. We call $\sigma(0)$ the **initial point** and $\sigma(1)$ the **final point** of the path.

10.1 **Theorem.** *Let X be a space and $R = \{(a,b) \in X \times X\colon$ there exists a path $\sigma\colon I \to X$ such that $\sigma(0) = a$ and $\sigma(1) = b\}$. Then R is an equivalence relation on X.*

A equivalence class of R in 10.1 is called a **path component** of X and R is called the **path component equivalence** on X. If X has exactly one path component, then X is said to be **pathwise connected**.

10.2 **Theorem.** *Each pathwise connected space is connected.*

10.3 **Theorem.** *The continuous image of a pathwise connected space is pathwise connected.*

10.4 **Theorem.** *A contractible space is pathwise connected.*

10.5 **Theorem.** *The product of a family of pathwise connected spaces is pathwise connected.*

10.6 **Theorem.** *Let $f\colon X \to Y$ be a continuous function and let K be a path component of X. Then $f(K)$ is contained in exactly one path component of Y.*

10.7 **Theorem.** *Let R be the path component equivalence on a space X and let T be the path component equivalence on a space Y. If $f\colon X \to Y$ is a continuous function, then there exists a continuous function $f^\natural\colon X/R \to Y/T$ such that the diagram:*

$$\begin{array}{ccc} X/R & \xrightarrow{f^\natural} & Y/T \\ \uparrow & & \uparrow \\ X & \xrightarrow{f} & Y \end{array}$$

commutes, where the vertical maps are projections.

If $f\colon X \to Y$ as in 10.7, then $f^\natural$ denotes the induced map.

10.8 **Theorem.** *If $f, g\colon X \to Y$ are homotopic continuous functions, then $f^\sharp = g^\sharp$.*

10.9 **Theorem.** *If $f\colon X \to Y$ is a homotopy equivalence, then $f^\sharp$ is bijective.*

If X is a space and $p \in X$, then X is said to be **locally pathwise connected at** p provided each neighborhood of p contains a pathwise connected neighborhood of p. If X is locally pathwise connected at each of its points, then X is said to be **locally pathwise connected.**

10.10 **Theorem.** *Let X be a space and $p \in X$ such that X is locally pathwise connected at p. Then p is an interior point of its path component in X.*

10.11 **Theorem.** *Pathwise connectedness and local pathwise connectedness are topological properties.*

10.12 **Theorem.** *Let X be a space and $p \in X$ such that X is locally pathwise connected at p. Then X is locally connected at p.*

10.13 **Theorem.** *Let X be a locally pathwise connected space, $p \in X$, and let K denote the path component of p in X. Then K is both open and closed in X and $K = C(p)$.*

10.14 **Theorem.** *If a space X is connected and locally pathwise connected, then X is pathwise connected.*

11 THE FUNDAMENTAL GROUP

A **loop** in a space X is a continuous function $\sigma\colon I \to X$ (where $I = [0,1]$ with the usual topology) such that $\sigma(0) = \sigma(1)$. The point $b = \sigma(0) = \sigma(1)$ is called the **base point** of the loop σ. The collection of all loops in X with basepoint b is denoted $L(X, b)$.

If X is a space, then two loops $\sigma, \tau \in L(X, b)$ are said to be **equivalent** provided there exists a homotopy $H\colon I \times I \to X$ such that $H(s, 0) = \sigma(s)$ and $H(0, s) = \tau(s)$ for each $s \in I$, and $H(0, t) = H(1, t) = b$ for each $t \in I$. We use $\sigma \simeq \tau$ to denote that σ and τ are equivalent.

11.1 **Theorem.** *If X is a space and $b \in X$, then $\simeq$ is an equivalence relation on $L(X, b)$.*

If X is a space, $b \in X$, and $\sigma \in L(X, b)$, then $[\sigma]$ denoted the $\simeq$ equivalence class of σ, and $\pi_1(X, b) = L(X, b)/\simeq$.

11.2 **Theorem.** *Let X be a space, $b \in X$, and $\sigma, \tau \in L(X, b)$. Define $\sigma\tau\colon I \to X$ by*

$$\sigma\tau(t) = \begin{cases} \sigma(2t) & 0 \le t \le \frac{1}{2} \\ \tau(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Then $\sigma\tau \in L(X, b)$.

11.3 Theorem. *Let X be a space,* $b \in X$, *and* $\sigma \in L(X, b)$. *Define* $\sigma^{-1}: I \to X$ *by* $\sigma^{-1}(t) = \sigma(1-t)$ *for* $t \in I$. *Then* $\sigma^{-1} \in L(X, b)$.

11.4 Theorem. *Let X be a space and* $b \in X$. *Then* $\pi_1(X, b)$ *is a group under* $[\sigma][\tau] = [\sigma\tau]$.

Proof. We first show that multiplication is well defined. Suppose that $\sigma \simeq \sigma'$ and $\tau \simeq \tau'$ in $L(X, b)$. We claim that $\sigma\tau \simeq \sigma'\tau'$. Let $F: I \times I \to X$ be a homotopy with $F(s, 0) = \sigma(s)$, $F(s, 1) = \sigma'(s)$, and $F(0, t) = F(1, t) = b$. Let $G: I \times I \to X$ be a homotopy with $G(s, 0) = \tau(s)$, $G(s, 1) = \tau'(s)$, and $G(0, t) = G(1, t) = b$. Define $H: I \times I \to X$ by

$$H(s,t) = \begin{cases} F(2s, t) & \text{for } 0 \le s \le \frac{1}{2} \text{ and all } t \\ G(2s-1, t) & \text{for } \frac{1}{2} \le s \le 1 \text{ and all } t \end{cases}$$

Since for $s = \frac{1}{2}$, $F(1, t) = b = G(1, t)$; H is continuous. Now

$$H(s,0) = \begin{cases} F(2s, 0) = \sigma(2s) & \text{for } 0 \le s \le \frac{1}{2} \\ G(2s-1, 0) = \tau(2s-1) & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$

$= \sigma\tau(s)$, and

$$H(s,1) = \begin{cases} F(2s, 1) = \sigma'(2s) & \text{for } 0 \le s \le \frac{1}{2} \\ G(2s-1, 1) = \tau'(2s-1) & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$

$= \sigma'\tau'(s)$, $H(0, t) = F(0, t) = b$, and $H(1, t) = G(1, t) = b$. Thus H is a homotopy between $\sigma\tau$ and $\sigma'\tau'$, so that multiplication is well defined.

To see that multiplication is associative, let $\alpha, \beta, \gamma \in L(X, b)$. We claim that $\alpha(\beta\gamma) \simeq (\alpha\beta)\gamma$. Note that

$$(\alpha\beta)\gamma(s) = \begin{cases} \alpha(4s) & \text{for } 0 \le s \le \frac{1}{4} \\ \beta(4s-1) & \text{for } \frac{1}{4} \le s \le \frac{1}{2} \\ \gamma(2s-1) & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$

and

$$\alpha(\beta\gamma)(s) = \begin{cases} \alpha(2s) & \text{for } 0 \le s \le \frac{1}{2} \\ \beta(4s-2) & \text{for } \frac{1}{2} \le s \le \frac{3}{4} \\ \gamma(4s-3) & \text{for } \frac{1}{4} \le s \le 1 \end{cases}$$

Define $F: I \times I \to X$ by

$$F(s,t) = \begin{cases} \alpha\left(\frac{4s}{t+1}\right) & for 0 \le 4s \le t+1 \\ \beta(4s-t-1) & \text{for } t+1 \le 4s \le t+2 \\ \gamma\left(\frac{4s-t-2}{2-t}\right) & \text{for } t+2 \le 4s \le 4 \end{cases}$$

Then F is a homotopy between $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$, and hence multiplication is associative.

To see that $\pi_1(X,b)$ has identity, define $e\colon I \to X$ be $e(t) = b$ for all $t \in I$. Now let $\sigma \in L(X,b)$ and define $F\colon I \times I \to X$ by

$$F(s,t) = \begin{cases} \sigma(2s) & \text{for } 0 \le 2s \le t \\ \sigma(\frac{2s+t}{2}) & \text{for } t \le 2s \le 2-t \\ b & \text{for } 2-t \le 2s \le 2 \end{cases}$$

Then F is a homotopy between σ and σe, i.e., $[\sigma][e] = [\sigma e] = [\sigma]$, and $[e]$ is a right identity.

To see that right inverses exist, let $\sigma \in L(X,b)$ and define $\sigma^{-1} \in L(X,b)$ as in 11.3. Then

$$\sigma\sigma^{-1}(s) = \begin{cases} \sigma(2s) & \text{for } 0 \le s \le \frac{1}{2} \\ \sigma(2-2s) & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$

Define $F\colon I \times I \to X$ by

$$F(s,t) = \begin{cases} \sigma(2s) & \text{for } 0 \le 2s \le t \\ \sigma(t) & \text{for } t \le 2s \le 2-t \\ \sigma(2-2s) & \text{for } 2-t \le 2s \le 2 \end{cases}$$

Then F is a homotopy between $\sigma\sigma^{-1}$ and e. We conclude that $\pi_1(X,b)$ is a group.

The notation $f\colon (X,x) \to (Y,y)$ means that $f\colon X \to Y$ is a continuous funtion from the space X into the space Y, $x \in X$, $y \in Y$, and $f(x) = y$.

11.5 **Theorem**. *If $f\colon (X,x) \to (Y,y)$, then the function $\pi_1(f)\colon \pi_1(X,x) \to \pi_1(Y,y)$ defined by $\pi_1(f)([\sigma]) = [f\sigma]$ is a homomorphism.*

11.6 **Theorem**. *If $f\colon (X,x) \to (Y,y)$ is a homeomorphism, then $\pi_1(f)\colon \pi_1(X,x) \to \pi_1(Y,y)$ is an isomorphism.*

11.7 **Theorem**. *If X is a space and $b \in X$, then $\pi_1(1_X)$ is the identity homomophism on $\pi_1(X,b)$.*

11.8 **Theorem**. *If $f\colon (X,x) \to (Y,y)$ and $g\colon (Y,y) \to (Z,z)$, then $\pi_1(gf) = \pi_1(g)\pi_1(f)$.*

11.9 **Theorem**. *Let X and Y be spaces, $x \in X$, and $y \in Y$. Then $\pi_1(X \times Y,(x,y))$ is isomorphic to the direct sum $\pi_1(X,x) \oplus \pi_1(Y,y)$.*

11.10 **Theorem**. *Let X be a pathwise connected space and let $a, b \in X$. Then $\pi_1(X,a)$ is isomorphic to $\pi_1(X,b)$.*

Proof. Let $\alpha\colon I \to X$ be a path such that $\alpha(0) = a$ and $\alpha(1) = b$. Define $\alpha^{-1}(t) = \alpha(1-t)$ and

$$\alpha\alpha^{-1}(t) = \begin{cases} \alpha(2t) & \text{for } 0 \le t \le \frac{1}{2} \\ \alpha^{-1}(2t-1) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

Then $\alpha\alpha^{-1} \in L(X,a)$ and $\alpha\alpha^{-1} \sim a$. Also $\alpha^{-1}\alpha \in L(X,b)$ and $\alpha^{-1}\alpha \sim b$. For $\sigma \in L(X,a)$ define

$$(\alpha^{-1}\sigma\alpha)(t) = \begin{cases} \alpha^{-1}(3t) & \text{for } 0 \le t \le \frac{1}{3} \\ \sigma(3t-1) & \text{for } \frac{1}{3} \le t \le \frac{2}{3} \\ \alpha(3t-2) & \text{for } \frac{2}{3} \le t \le 1 \end{cases}$$

(1) We claim that if $\sigma \simeq \tau$ in $L(X,a)$, then $\alpha^{-1}\sigma\alpha \simeq \alpha^{-1}\tau\alpha$ in $L(X,b)$. Let $F\colon I \times I \to X$ be a homotopy such that $F(s,0) = \sigma(s)$, $F(s,1) = \tau(s)$, and $F(0,t) = F(1,t) = a$. Define $H\colon I \times I \to X$ by

$$H(s,t) = \begin{cases} \alpha^{-1}(3s) & \text{for } 0 \le s \le \frac{1}{3} \\ F(3s-1,t) & \text{for } \frac{1}{3} \le s \le \frac{2}{3} \\ \alpha(3s-2) & \text{for } \frac{2}{3} \le s \le 1 \end{cases}$$

Then H is a homotopy between $\alpha^{-1}\sigma\alpha$ and $\alpha^{-1}\tau\alpha$.

For $\sigma' \in L(X,b)$ define

$$\alpha\sigma'\alpha^{-1} = \begin{cases} \alpha(3t) & \text{for } 0 \le t \le \frac{1}{3} \\ \sigma'(3t-1) & \text{for } \frac{1}{3} \le t \le \frac{2}{3} \\ \alpha^{-1}(3t-2) & \text{for } \frac{2}{3} \le t \le 1 \end{cases}$$

Then $\alpha\sigma'\alpha^{-1} \in L(X,a)$ and

(2) If $\sigma' \simeq \tau'$ in $L(X,b)$, then $\alpha\sigma'\alpha^{-1} \simeq \alpha\tau'\alpha^{-1}$ in $L(X,a)$. The proof of (2) is similar to the proof of (1).

(3) We claim that if $\sigma \in L(X,a)$, then $\alpha(\alpha^{-1}\sigma\alpha)\alpha^{-1} \simeq (\alpha\alpha^{-1})\sigma(\alpha\alpha^{-1})$ in $L(X,a)$. The desired homotopy is $G\colon I \times I \to X$ defined by:

$$G(s,t) = \begin{cases} \alpha\left(\frac{6s}{t+1}\right) & \text{for } 0 \le 18s \le 3t+2 \\ \alpha^{-1}\left(\frac{18s-3t-3}{3-t}\right) & \text{for } 3t+3 \le 18s \le 2t+6 \\ \sigma\left(\frac{9s-t-3}{3-2t}\right) & \text{for } 2t+6 \le 18s \le 12-2t \\ \alpha\left(\frac{18s+2t-12}{3-t}\right) & \text{for } 12-2t \le 18s \le 15-3t \\ \alpha^{-1}\left(\frac{6s+t-5}{t+1}\right) & \text{for } 15-3t \le 18s \le 18 \end{cases}$$

(4) $\alpha(\alpha^{-1}\sigma\alpha)\alpha^{-1} \simeq \sigma$ for $\sigma \in L(X,a)$

(5) $\alpha^{-1}(\alpha\sigma'\alpha^{-1})\alpha \simeq \sigma'$ for each $\sigma' \in L(X,b)$

Now define $\psi\colon \pi_1(X,a) \to \pi_1(X,b)$ by $\psi([\sigma]) = [\alpha^{-1}\sigma\alpha]$. Then ψ is well-defined by (1).

To see that ψ is surjective, let $\sigma' \in L(X,b)$. Then $\psi([\alpha\sigma'\alpha^{-1}]) = [\alpha^{-1}(\alpha\sigma'\alpha^{-1})\alpha] = [\sigma']$, by (5), so that ψ is surjective.

To see that ψ is injective, suppose that $\psi([\sigma_1]) = \psi([\sigma_2])$. Then $\alpha^{-1}\sigma_1\alpha \simeq \alpha^{-1}\sigma_2\alpha$ in $L(X,b)$, and so $\alpha(\alpha^{-1}\sigma_1\alpha)\alpha^{-1} \simeq \alpha(\alpha^{-1}\sigma_2\alpha)\alpha^{-1}$ by (2), and hence by (4) and the transitivity of $\simeq$, we have have $\sigma_1 \simeq \sigma_2$ and ψ is injective.

Finally, to see that ψ is a homomorphism, observe that $\psi([\sigma_1][\sigma_2]) = \psi([\sigma_1\sigma_2]) = [\alpha^{-1}\sigma_1\sigma_2\alpha] = [\alpha^{-1}\sigma\alpha\alpha^{-1}\sigma_2\alpha] = [\alpha^{-1}\sigma\alpha] = [\alpha^{-1}\sigma\alpha][\alpha^{-1}\sigma_2\alpha] = \psi([\sigma_1])\psi([\sigma_2])$. ■

If X is a pathwise connected space and $b \in X$, then $\pi_1(X,b)$ is denoted $\pi_1(X)$ and is called **the fundamental group** of X.

A space X is said to be **simply connected** provided X is pathwise connected and $\pi_1(X)$ is trivial.

11.11 **Theorem**. *Each contractible space is simply connected.*

Proof. Let X be a contractible space.

To see that X is pathwise connected, let $a, b \in X$. In view of 9.13, we have that $1_X \sim a$ and $1_X \sim b$. Let $H\colon X \times I \to X$ be a homotopy with $H(x,0) = a$ and $H(x,1) = x$, and let $F\colon X \times I \to X$ be a homotopy with $F(x,0) = x$ and $F(x,1) = b$. Define $\alpha\colon I \to X$ by

$$\alpha(t) = \begin{cases} H(a,2t) & \text{for } 0 \le t \le \frac{1}{2} \\ F(a,2t-1) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

Since $H(a,1) = a = F(a,0)$, α is continuous, and $\alpha(0) = H(a,0) = a$, $\alpha(1) = F(a,1) = b$. Thus X is pathwise connected.

To see that $\pi_1(X)$ is trivial, let $b \in X$, $\sigma \in L(X,b)$, and let $e\colon I \to X$ be $e(t) = b$. Then $\sigma\colon I \to X$ is continuous with $\sigma(0) = \sigma(1) = b$. We want to show that $\sigma \simeq e$. Now $\sigma \sim e$ from 9.13. Let $H\colon I \times I \to X$ be a homotopy with $H(s,0) = \sigma(s)$ and $H(s,1) = e(s) = b$. Define $F\colon I \times I \to X$ by:

$$F(s,t) = \begin{cases} H(s,2s) & \text{for } 0 \le 2s \le t \\ H(s,t) & \text{for } t \le 2s \le 2-t \\ H(s,2-2s) & \text{for } 2-t \le 2s \le 2 \end{cases}$$

Then F is continuous with $F(s,0) = \sigma(s)$, $F(s,1) = b$, $F(0,t) = b = F(1,t)$ and hence $\sigma \simeq e$. ■

11.12 **Theorem**. *Simple connectivity is a topological property.*

Let S^1 be endowed with the multiplication it inherits as a subset of the complex plane. Then S^1 is a group. The reals $\mathbb{R}$ under addition is also a group. The **exponential map** $\phi\colon \mathbb{R} \to S^1$ is defined by $\phi(t) = exp(2\pi i t)$ for $t \in \mathbb{R}$, and is a continuous surjective homomorphism. We use $\mathbb{Z}$ to denote the additive subgroup of $\mathbb{R}$ consisting of the integers.

11.13 **Theorem.** *The kernel of ϕ is $\mathbb{Z}$, i.e., $\phi^{-1}(1) = \mathbb{Z}$.*

11.14 **Theorem.** $\phi|(\frac{-1}{2}, \frac{1}{2})$ *is a homeomorphism onto* $S^1\backslash\{-1\}$.

11.15 **Lifting Lemma.** *Let $\sigma\colon I \to S^1$ be a path such that $\sigma(0) = 1$. Then there exists a path $\sigma^\star\colon I \to \mathbb{R}$ such that $\sigma^\star(0) = 0$ and $\phi\sigma^\star = \sigma$, i.e., the diagram:*

$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{\sigma^\star}{\nearrow} & \Big\downarrow \phi \\ I & \xrightarrow{\ \sigma\ } & S^1 \end{array}$$

commutes.

Proof. Let $\psi = \left(\phi|(\frac{-1}{2}, \frac{1}{2})\right)^{-1}$. Now there exists a real number $\delta > 0$ such that if $|t - s| < \delta$, then $\|\sigma(t) - \sigma(s)\| < 1$, since σ is uniformly continuous from 6.30. Thus if $|t - s| < \delta$, then $\sigma(t) \neq -\sigma(s)$ and hence $\sigma(t)/\sigma(s) \neq -1$. Let $n \in \mathbb{N}$ such that $|t| < nt$ for all $t \in I$, and define:

$$\sigma^\star(t) = \sum_{j=1}^{n} \psi\left(\frac{\sigma(\frac{j}{n}t)}{\sigma(\frac{j-1}{n}t)}\right)$$

Then $\sigma^\star\colon I \to \mathbb{R}$ is the desired path. ∎

11.16 **Covering Homotopy Lemma.** *Let $\sigma\colon I \to S^1$ and $\tau\colon I \to S^1$ be paths and $F\colon \sigma \sim \tau$ a homotopy such that $F(0,t) = \sigma(0) = \tau(0) = 1$ and $F(1,t) = \sigma(1) = \tau(1)$ for all $t \in I$. Then there exists a unique homotopy $F^\star\colon \sigma^\star \sim \tau^\star$ such that $F^\star(0,t) = \sigma^\star(0) = \tau^\star(0) = 0$ and $F^\star(1,t) = \sigma^\star(1) = \tau^\star(1)$ for all $t \in I$, and $\phi F^\star = F$.*

Proof. Now $F(s,0) = \sigma(s)$ and $F(s,1) = \tau(s)$ for all $s \in I$. Let $\psi = \left(\phi|(-\frac{1}{2}, \frac{1}{2})\right)^{-1}$. Now, from 9.13, $F\colon I \times I \to S^1$ is uniformly continuous, and hence there exists $\delta > 0$ such that if $\|(s,t) - (s',t')\| < \delta$, then $\|F(s,t) - F(s',t')\| < 1$. Let $n \in \mathbb{N}$ such that $\|(s,t)\| < n\delta$ for all $(s,t) \in I \times I$, and define:

$$F^\star(s,t) = \sum_{j=1}^{n} \psi\left(\frac{F(\frac{j}{n}(s,t))}{F(\frac{j-1}{n}(s,t))}\right)$$

for $(s,t) \in I \times I$. Then $F^\star\colon I \times I \to \mathbb{R}$ is continuous, and $\phi F^\star = F$.

It is simple to establish that $F^\star(s,0) = \sigma^\star(s)$.

To see that $F^\star(s,1) = \tau^\star(s)$, observe that $\phi F^\star(s,1) = F(s,1) = \tau(s)$ and $F^\star(0,1) = 0$, so that $F^\star(s,1) = \tau^\star(s)$.

Now $\phi F^\star(0\times I) = F(0\times I) = 1$, so that $F^\star(0\times I) \subseteq \mathbb{Z}$, and hence $F^\star(0\times I)$ is constant. Now $F^\star(0,0) = 0$, so that $F^\star(0 \times I) = 0 = \sigma^\star(0) = \tau^\star(0)$.

Since $F^\star(s,1) = \tau^\star(s)$, we have that $F^\star(1,1) = \tau^\star(1)$. Now $\phi F^\star(1 \times I) = F(1 \times I) = \sigma(1) = \tau(1)$, and hence $F^\star(1 \times I) \subseteq \phi^{-1}\sigma(1)$ (discrete), so that $F^\star(1 \times I) = \sigma^\star(1)$, and we obtain that $F^\star(1 \times I) = \sigma^\star(1) = \tau^\star(1)$.

To establish uniqueness, suppose that $\phi F' = F$ and $F'(0,t) = 0$, $F'(1,t) = \sigma^\star(1) = \tau^\star(1)$. Then $\phi(F^\star - F')(I \times I) = 0$ and hence $F^\star - F'$ is constant. Since $(F^\star - F')(0,t) = 0$, we have $F^\star - F' = 0$ and hence $F^\star = F'$. ∎

If $\sigma \in L(S^1, 1)$, then the **degree** of σ is defined $\deg \sigma = \sigma^\star(1)$.

11.17 **Lemma**. *If σ and τ are equivalent loops in $L(S^1,1)$, then* $\deg \sigma = \deg \tau$.

Define $\deg\colon \pi_1(S^1,1) \to \mathbb{Z}$ by $\deg([\sigma]) = \deg \sigma$.

11.18 **Theorem**. $\deg\colon \pi_1(S^1,1) \to \mathbb{Z}$ *is an isomorphism.*

Proof. For $\sigma \in L(S^1,1)$, $\phi\sigma^\star(1) = \sigma(1) = 1$, so that $\sigma^\star(1) \in \mathbb{Z}$ and hence the codomain of deg is $\mathbb{Z}$. Now deg is well-defined by 11.17.

To see that deg is surjective, let $n \in \mathbb{Z}$. Define $\tau\colon I \to \mathbb{R}$ by $\tau(t) = nt$. Define $\sigma\colon I \to S^1$ by $\sigma = \phi\tau$. Then $\sigma(0) = \phi\tau(0) = \phi(0) = 1$ and $\sigma(1) = \phi\tau(1) = \phi(n) = 1$, so that $\sigma \in L(S^1,1)$, and $\tau = \sigma^\star$. Thus $\sigma^\star(1) = \tau(1) = n$ and $\deg[\sigma] = n$.

To see that deg is injective, suppose that $\deg \sigma = \deg \tau$. Then $\sigma^\star(1) = \tau^\star(1)$. Define $H\colon I \times I \to \mathbb{R}$ by $H(s,t) = t\sigma^\star(s) + (1-t)\tau^\star(s)$ and define $F\colon I \times I \to S^1$ by $F(s,t) = \phi(H(s,t))$. Then $F(0,t) = \phi H(0,t) = \phi(t\sigma^\star(0) + (1-t)\tau^\star(0)) = \phi(0) = 1$, $F(1,t) = \phi H(1,t) = \phi(t\sigma^\star(1) + (1-t)\tau^\star(1)) = \phi\tau^\star(1) = 1$, $F(s,0) = \phi\tau^\star(s) = \tau(s)$, and $F(s,1) = \phi\sigma^\star(s) = \sigma(s)$, so that $\tau \simeq \sigma$ and $[\sigma] = [\tau]$. We conclude that deg is injective.

To see that deg is a homomorphism, first observe that $\deg[\sigma_1][\sigma_2] = \deg[\sigma_1\sigma_2] = (\sigma_1\sigma_2)^\star(1)$ and $\deg[\sigma_1] + \deg[\sigma_2] = \sigma_1^\star(1) + \sigma_2^\star(1)$. Define $\tau\colon I \to \mathbb{R}$ by $\tau(t) = \sigma_1^\star(t) + \sigma_2^\star(t)$, and define $\sigma\colon I \to S^1$ by $\sigma = \phi\tau$. Then $\tau(0) = 0$, and hence $\tau = \sigma^\star$. Define $H\colon I \times I \to \mathbb{R}$ by $H(s,t) = t(\sigma_1\sigma_2)^\star(s) + (1-t)\tau(s)$ and define $F\colon I \times I \to S^1$ by $F = \phi H$. Then we have $F(0,t) = \phi(0) = t = (\sigma_1\sigma_2)(0) = \sigma(0)$, $F(1,t) = \phi(0) = 1 = (\sigma_1\sigma_2)(1) = \sigma(1)$, $F(s,0) = \phi\tau(s) = \sigma(s)$, and $F(s,1) = \phi(\sigma_1\sigma_2)^\star(s) = (\sigma_1\sigma_2)(s)$. It follows that $(\sigma_1\sigma_2)^\star(1) = \sigma^\star(1) = \tau(1) = \sigma_1^\star(1) + \sigma_2^\star(1)$. ∎

11.19 **Theorem**. *The fundamental group of S^1 is $\mathbb{Z}$.*

If $f\colon X \to X$ is a function and $p \in X$, then p is called a **fixed point** of f provided $f(p) = p$.

A space X is said to have the **fixed point property** if each continuous function $f: X \to X$ has a fixed point.

11.20 **Theorem.** *The fixed point property is a topological property.*

11.21 **Lemma.** *The space S^1 is not a retract of B^2.*

11.22 **Theorem.** *The space B^2 has the fixed point property.*

12 THE CANTOR SET

A space X is said to be **perfect** if each point of X is a limit point of X.

A point p in a space X is called an **isolated point** provided p is not a limit point of X.

Note that a space is perfect if it contains no isolated points.

12.1 **Theorem.** *No locally compact Hausdorff space is both countable and perfect.*

Proof. Let X be a locally compact Hausdorff space. Suppose that X is both countable and perfect. Let $p \in X$. Then $\{p\}$ is not open, since p is a limit point of $X\backslash\{p\}$. Hence $X\backslash\{p\}$ is not closed and $\{p\}^\circ = \emptyset$. Thus X is a countable union of nowhere dense sets. This contradicts the Baire Category Theorem. ■

Let $X_n = \{0,1\}$ with the discrete topology for each $n \in \mathbb{N}$. The **Cantor Set** is the space $\prod_{n\in\mathbb{N}} X_n$.

Let $K = \{x \in [0,1]: x = \sum_{n=1}^{\infty} \alpha_n \left(\frac{1}{3}\right)^n : \alpha_n = 0 \text{ or } \alpha_n = 2\}$ with the relative topology of $[0,1]$.

12.2 **Theorem.** *The space K is homeomorphic to the Cantor set.*

Proof. Now the Cantor set is $X = \prod_{n\in\mathbb{N}} X_n$, where $X_n = \{0,1\}$ discrete. Define $\phi: X \to [0,\infty)$ by $\phi(x) = \sum_{i=1}^{\infty} 2x_i\left(\frac{1}{3}\right)^i$, where $\pi_i(x) = x_i$ for each $i \in \mathbb{N}$. Note that $K = \phi(X)$. Since X is compact and K is Hausdorff, we need only show that ϕ is continuous and injective. It is clear that ϕ is injective, so we show that ϕ is continuous.

Let $x \in \phi^{-1}(a,\infty)$. Then $a < \sum_{i=1}^{\infty} 2x_i\left(\frac{1}{3}\right)^i = \phi(x)$. There exists $n \in \mathbb{N}$ such that $a < \sum_{i=1}^{n} 2x_i\left(\frac{1}{3}\right)^i$. Let $W = \{x_1\} \times \{x_2\} \times \cdots \times \{x_n\} \times \prod_{j=n+1}^{\infty} X_j$. Then $x \in W$, and W is open in X. We claim that $W \subseteq \phi^{-1}(a,\infty)$. Let $t \in W$. Then $\phi(t) = \sum_{i=1}^{\infty} 2t_i\left(\frac{1}{3}\right)^i = \sum_{i=1}^{n} 2t_i\left(\frac{1}{3}\right)^i + \sum_{i=n+1}^{\infty} 2t_i\left(\frac{1}{3}\right)^i = \sum_{i=1}^{\infty} 2x_i\left(\frac{1}{3}\right)^i + \sum_{i=n+1}^{\infty} 2t_i\left(\frac{1}{3}\right)^i >$ a, and hence $\phi(t) \in (a,\infty)$, and $W \subseteq \phi^{-1}(a,\infty)$.

Let $x \in \phi^{-1}[0,b)$. Then $\sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^i = \phi(x) < b$. Let $0 < \epsilon$ such that $\phi(x) < \epsilon < b$. Then there exists $m \in \mathbb{N}$ such that $\sum_{i=m}^{\infty} 2\left(\frac{1}{3}\right)^i < b - \epsilon$. Let $W = \{x_1\} \times \cdots \times \{x_{m-1}\} \times \prod_{i=m}^{\infty} X_i$. Then $x \in W$ and W is open in X. We claim that $W \subseteq \phi^{-1}[0,b)$. Let $t \in W$. Then $\phi(t) = \sum_{i=1}^{m-1} 2x_i\left(\frac{1}{3}\right)^i + \sum_{i=m}^{\infty} 2t_i\left(\frac{1}{3}\right)^i < \epsilon + (b-\epsilon) = b$, and hence $\phi(t) \in [0,b)$, and $W \subseteq \phi^{-1}[0,b)$. We have that $\phi^{-1}(a,\infty)$ and $\phi^{-1}[0,b)$ are both open, and hence ϕ is continuous. ■

12.3 **Theorem.** *The set K is nowhere dense in* $[0,1]$.

Proof. Let $a, b \in K$ with $a < b$. We show that there exists $r \in [0,1]$ such that $a < r < b$ and $r \notin K$. Let $a = \sum_{i=1}^{\infty} a_i\left(\frac{1}{3}\right)^i$ and $b = \sum_{i=1}^{\infty} b_i\left(\frac{1}{3}\right)^i$, where $a_i, b_i = 0$ or 2.

Note that $\sum_{i=1}^{n}\left(\frac{1}{3}\right)^i = \frac{1}{2}\left[1-\left(\frac{1}{3}\right)^n\right]$, $\sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^i = \frac{1}{2}$, and $\sum_{i=n+1}^{\infty}\left(\frac{1}{3}\right)^i = \frac{1}{2}\left(\frac{1}{3}\right)^n$.

Let n be the least positive integer such that $a_n = 0$ and $b_n = 2$ (n exists, since $a < b$) and $a_i = b_i$ for $i < n$.

Let $r = \sum_{i=1}^{n-1} b_i\left(\frac{1}{3}\right)^i + \frac{3}{2}\left(\frac{1}{3}\right)^n$. Then $a = \sum_{i=1}^{n-1} a_i\left(\frac{1}{3}\right)^i + \sum_{i=n+1}^{\infty} a_i\left(\frac{1}{3}\right)^i \leq \sum_{i=1}^{n-1} b_i\left(\frac{1}{3}\right)^i + \left(\frac{1}{3}\right)^n < r$, and $b = \sum_{i=1}^{n-1} b_i\left(\frac{1}{3}\right)^i + 2\left(\frac{1}{3}\right)^n + \sum_{i=n+1}^{\infty} b_i\left(\frac{1}{3}\right)^i > r$.

To show that $r \notin K$, we show that $r = \sum_{i=1}^{\infty} r_i\left(\frac{1}{3}\right)^i$, where $r_i = 0$ or 2 is not possible.

We first claim that $\frac{1}{2} = \sum_{i=1}^{\infty} p_i\left(\frac{1}{3}\right)^i$, where $p_i = 0$ or 2 is not possible. If $p_1 = 0$, then $\frac{1}{2} = \sum_{i=2}^{\infty} p_i\left(\frac{1}{3}\right)^i \leq \sum_{i=2}^{\infty} 2\left(\frac{1}{3}\right)^i = \frac{1}{3}$; a contradiction. If $p_1 = 2$, then $\frac{1}{2} = \frac{2}{3} + \sum_{i=2}^{\infty} p_i\left(\frac{1}{3}\right)^i$, and again we have a contradiction.

Suppose that $r = b_1\left(\frac{1}{3}\right) + \cdots + b_{n-1}\left(\frac{1}{3}\right)^{n-1} + \frac{3}{2}\left(\frac{1}{3}\right)^n = p_1\left(\frac{1}{3}\right) + p_2\left(\frac{1}{3}\right)^2 + \cdots$ $(n > 1)$. We will first show that $b_i = p_i$ for $i = 1, 2, \ldots, n-1$ (by induction on n), where $b_j, p_j = 0$ or 2 for all j.

We first claim that $b_1 = p_1$ for all n. Suppose that $b_1 = 0$ and $p_1 = 2$. Then $p_1\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \cdots \geq \frac{2}{3}$. On the other hand, $r = b_2\left(\frac{1}{3}\right)^2 + \cdots + b_{n-1}\left(\frac{1}{3}\right)^{n-1} + \left(\frac{3}{2}\right)\left(\frac{1}{3}\right)^n \leq 2\left(\frac{1}{3}\right)^2 + \cdots + 2\left(\frac{1}{3}\right)^{n-1} + \left(\frac{3}{2}\right)\left(\frac{1}{3}\right)^n = 2\left(\frac{1}{3}\right)^2 + \cdots + 2\left(\frac{1}{3}\right)^{n-1} + 2\left(\frac{1}{3}\right)^n - \frac{1}{2}\left(\frac{1}{3}\right)^n = \frac{1}{3} - \frac{3}{2}\left(\frac{1}{3}\right)^n < \frac{1}{3}$, which is a contradiction. Suppose that $b_1 = 2$ and

$p_1 = 0$. Then $b_1 = 2$ yields that $r \geq \frac{2}{3} + (\frac{3}{2})(\frac{1}{3})^n$ and hence $r > \frac{2}{3}$. Since $p_1 = 0$, $\sum_{i=1}^{\infty} p_i(\frac{1}{3})^i \leq \frac{2}{3}$, so that we have a contradiction and conclude that $b_1 = p_1$ in any case.

Suppose that $\sum_{i=1}^{n-1} c_i(\frac{1}{3})^i + (\frac{3}{2})^n = \sum_{i=1}^{\infty} d_i(\frac{1}{3})^i$ implies that $c_i = d_i$ for $i = 1, 2, \ldots, n-1$; where $c_i, d_i = 0$ or 2. Assume that $\sum_{i=1}^{n} b_i(\frac{1}{3})^i + (\frac{3}{2})(\frac{1}{3})^{n+1} = \sum_{i=1}^{\infty} p_i(\frac{1}{3})^i$. Then, as we have observed, $b_1 = p_1$, and hence $\sum_{i=2}^{n} b_i(\frac{1}{3})^i + (\frac{3}{2})(\frac{1}{3})^{n+1} = \sum_{i=2}^{\infty} p_i(\frac{1}{3})^i$. We obtain that $\frac{1}{3}\left[\sum_{i=1}^{n-1} b_{i+1}(\frac{1}{3})^i + (\frac{3}{2})(\frac{1}{3})^n\right] = \frac{1}{3}\sum_{i=1}^{\infty} p_{i+1}(\frac{1}{3})^i$, so that $\sum_{i=1}^{n-1} b_{i+1}(\frac{1}{3})^i + (\frac{3}{2})(\frac{1}{3})^n = \sum_{i=1}^{\infty} p_{i+1}(\frac{1}{3})^i$. Thus $b_{i+1} = p_{i+1}$ for $i = 1, 2, \ldots, n-1$, i.e., $b_i = p_i$ for $i = 2, \ldots, n$. Since we already have that $b_1 = p_1$, we have established that $b_i = p_i$ for $i = 1, 2, \ldots, n$.

Now $r = \sum_{i=1}^{n-1} b_i(\frac{1}{3})^i + (\frac{3}{2})(\frac{1}{3})^n = \sum_{i=1}^{n-1} p_i(\frac{1}{3})^i + \sum_{i=n}^{\infty} p_i(\frac{1}{3})^i$, we have that $(\frac{3}{2})(\frac{1}{3})^n = \sum_{i=n}^{\infty} p_i(\frac{1}{3})^i = \sum_{i=0}^{\infty} p_{i+n}(\frac{1}{3})^{i+n}$, so that $(\frac{1}{2})(\frac{1}{3})^n = (\frac{1}{3})\sum_{i=1}^{\infty} p_{i+n}(\frac{1}{3})^i$, and $\frac{1}{2} = \sum_{i=1}^{\infty} p_{i+n}(\frac{1}{3})^i$, which , as we first observed, is not possible. We conclude that $r \notin K$. ■

12.4 **Theorem.** *The Cantor set is compact totally disconnected perfect metric space.*

If x and y are points of a space X, then X is said to be **separated between** x and y provided $X = A|B$, where $x \in A$ and $y \in B$.

12.5 **Theorem.** *Let X be a compact Hausdorff space, $x, y \in X$, and $\{H_\alpha : \alpha \in A\}$ a tower of closed subsets of X such that $x, y \in H_\alpha$ and H_α is not separated between x and y for each $\alpha \in A$. Then $\bigcap_{\alpha \in A} H_\alpha$ is not separated between x and y.*

12.6 **Theorem.** *Let X be a compact Hausdorff space and let $x, y \in X$. Then these are equivalent:*

(1) X is not separated between x and y;

(2) x and y are connected in X; and

(3) There exists a compact connected subset of X containing both x and y.

12.7 **Theorem.** *Let X be a compact totally disconnected Hausdorff space, $p \in X$, and let U be an open neighborhood of p. Then there exists an open and closed set H such that $p \in H \subseteq U$.*

Proof. By 12.6, for each $t \in X\backslash U$, there exist open and closed sets A_t and B_t such that $p \in A_t$ and $t \in B_t$ with $X = A_t|B_t$. Now $\{B_t : t \in X\backslash U\}$ is an open cover of $X\backslash U$. Let $B_{t_1}, \ldots, B_{t_n}$ be a finite subcover, and let $H = \bigcap_{j=1}^{n} A_{t_j}$. Then H is open and closed, and $p \in H \subseteq U$. ■

If (X, d) is a metric space and E is bounded subset of X, then the **diameter** of E is defined $\operatorname{diam} E = \sup\{d(x,y) : x, y \in E\}$.

12.8 **Theorem.** *Let X be a compact metric space and $\{F_n\}$ a sequence of non-empty closed subsets of X such that $F_{n+1} \subseteq F_n$ for each $n \in \mathbb{N}$ and* $\operatorname{diam} F_n \xrightarrow{e} 0$. *Then $\bigcap_{n=1}^{\infty} F_n$ is degenerate.*

12.9 **Lemma.** *Let X be a compact metric space, α an open cover of X and let $0 < \epsilon$. Then there exists a finite refinement β of α such that* $\operatorname{diam} G < \epsilon$ *for each $G \in \beta$.*

12.10 **Lemma.** *Let X be a compact totally disconnected metric space. Then there exists a sequence $\{\alpha_n\}$ of finite open covers of X such that each α_n is a collection of disjoint open and closed sets with diameter less than $\frac{1}{n}$, and α_{n+1} is a refinement of α_n for each $n \in \mathbb{N}$.*

Proof. We first show that if α is an open cover of X, then there exists a finite refinement β of α consisting of disjoint open sets. If view of 12.7, for each $p \in X$, there exists an open and closed set H_p such that $p \in H_p \subseteq A$ for some $A \in \alpha$. Now $\{H_p : p \in X\}$ is an open cover of X. Let $H_{p_1}, H_{p_2}, \ldots, H_{p_n}$ be a finite subcover. Let $K_1 = H_{p_1}$ and $K_j = H_{p_j} \backslash \bigcup_{i=1}^{j-1} H_{p_i}$ for $2 \leq j \leq n$. Then $\beta = \{K_1, K_1, \ldots, K_n\}$ is the desired refinement.

Let β_1 be a finite open cover of X such that $\operatorname{diam} B < 1$ for each $B \in \beta_1$ (12.9). Let α_1 be a finite refinement of β_1 consisting of disjoint open and closed sets. Clearly, $\operatorname{diam} A < 1$ for each $A \in \alpha_1$. Let β_2 be a finite open refinement of α_1 such that $\operatorname{diam} B < \frac{1}{2}$ for each $B \in \beta_2$. Let α_2 be a finite refinement of β_2 consisting of disjoint open and closed sets. Again, $\operatorname{diam} A < \frac{1}{2}$ for each $A \in \alpha_2$, and α_2 is a refinement of α_1. Continuing recursively, we obtain the desired sequence $\{\alpha_n\}$. ■

12.11 **Lemma.** *Let X be a compact totally disconnected perfect Hausdorff space, U a non-empty open and closed subset of X, and let $n \in \mathbb{N}$. Then U is a union of n disjoint non-empty open and closed sets.*

Proof. We can assume that $2 \leq n$. Since X is perfect, U contains at least n distinct points $x_1, x_2, \ldots, x_n$. In view of 12.7 and the fact that X is Hausdorff, there exist disjoint open and closed sets $M_1, M_2, \ldots, M_{n-1}$ with $x_j \in M_j \subseteq U$

for $j = 1, 2, \ldots, n-1$ and $x_n \notin \bigcup_{j=1}^{n-1} M_j$. Let $M_n = U \setminus \bigcup_{j=1}^{n-1} M_j$. Then $x_n \in M_n$ and M_n is open and closed, so that $U = \bigcup_{j=1}^{n} M_j$ is the desired union. ■

12.12 **Characterization of the Cantor Set**. *Each compact totally disconnected perfect metric space is homeomorphic to the Cantor set.*

Proof. Let $\{\alpha_n\}$ be a sequence of open covers of X such that each α_n is a collection of disjoint open and closed sets with diameter less than $\frac{1}{n}$ and α_{n+1} is a refinement of α_n for each $n \in \mathbb{N}$ (12.10). Let m_1 be a sufficiently large positive integer so that α_1 consists of 2^{m_1} disjoint non-empty open and closed sets $\{U(x_1, x_2, \ldots, x_{m_1}): x_j = 0 \text{ or } 1\}$ (12.11). Let $m_1 < m_2$ be a sufficiently large positive integer so that $2 < m_2$ and α_2 consists consists of 2^{m_2} disjoint non-empty open and closed sets $\{U(x_1, x_2, \ldots, x_{m_1}, x_{m_1+1}, \ldots, x_{m_2}): x_j = 0 \text{ or } 1\}$ labeled so that $U(x_1, \ldots, x_{m_2}) \subset U(x_1, \ldots, x_{m_1})$. Continuing recursively, we obtain an increasing sequence $\{m_n\}$ of positive integers so that $n < m_n$ and α_n consists of 2^{m_n} disjoint non-empty open and closed sets $\{U(x_1, x_2, \ldots, x_{m_n}): x_j = 0 \text{ or } 1\}$ with $U(x_1, \ldots, x_{m_{n+1}}) \subset U(x_1, \ldots, x_{m_n})$.

Define $\phi\colon \prod_{n\in\mathbb{N}} \{0,1\}_n \to X$ by $\phi(x_1, x_2, \ldots) = \bigcap_{n\in\mathbb{N}} U(x_1, x_2, \ldots, x_{m_n})$, where $(x_1, x_2, \ldots, x_{m_n})$ is the first m_n terms of $(x_1, x_2, \ldots)$. Now $\phi(x_1, x_2, \ldots)$ is a point in X (12.8). To see that ϕ is continuous, let V be an open subset of X such that $\phi(x_1, x_2, \ldots) \in V$. Now $\{U(x_1, x_2, \ldots, x_{m_n})\}$ is a tower of compact sets whose intersection is $\phi(x_1, x_2, \ldots)$, so that $U(x_1, x_2, \ldots, x_{m_k}) \subseteq V$ for some $k \in \mathbb{N}$. Thus $\phi(\{x_1\} \times \cdots \times \{x_{m_k}\} \times \{0,1\} \times \cdots) \subseteq U(x_1, \ldots, x_k) \subseteq V$, so that ϕ is continuous.

To see that ϕ is injective, let $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ be distinct points of $\prod_{n\in\mathbb{N}} \{0,1\}_n$. Then $x_n \neq y_n$ for some $n \in \mathbb{N}$. Thus since $n < m_n$, $U(x_1, \ldots, x_{m_n})$ and $U(y_1, \ldots, y_{m_n})$ are disjoint, $\phi(x_1, x_2, \ldots) \in U(x_1, \ldots, x_{m_n})$ and $\phi(y_1, y_2, \ldots) \in U(y_1, \ldots, y_{m_n})$, so that $\phi(x_1, x_2, \ldots) \neq \phi(y_1, y_2, \ldots)$, and ϕ is injective.

To see that ϕ is surjective, let $p \in X$. Then p is in some member of α_n for each $n \in \mathbb{N}$, and hence in $\bigcap_{n\in\mathbb{N}} U(x_1, x_2, \ldots, x_{m_n})$ for some $(x_1, x_2, \ldots) \in \prod_{n\in\mathbb{N}} \{0,1\}_n$, so that $\phi(x_1, x_2, \ldots) = p$.

Since ϕ is a continuous bijection from a compact space onto a Hausdorff space, ϕ is a homeomorphism. ■

12.13 **Theorem**. *Each compact totally disconnected metric space is homeomorphic to a subspace of the Cantor set.*

12.14 **Theorem.** *Each compact metric space is a continuous image of the Cantor set.*

Proof. Let X be a compact metric space and let $\{U_n : n \in \mathbb{N}\}$ be a countable base for the topology of X. For each $n \in \mathbb{N}$ define $F_n : \{0,1\} \to 2^X$ by $F_n(0) = \overline{U}_n$ and $F_n(1) = X \backslash U_n$. Let $\mathbb{C}$ denote the Cantor set and let $H = \{(p_1, p_2, \ldots) \in \mathbb{C} : \bigcap_{n \in \mathbb{N}} F_n(p_n) \neq \emptyset\}$.

To see that $H \neq \emptyset$, let $x \in X$, and select $m_1 < m_2 < \cdots$ such that $x \in U_{m_j}$ and $x \in X \backslash U_m$ for $m \neq m_j$. Let $p = (p_i) \in \mathbb{C}$ such that $p_i = 0$ if $i = m_k$ for some k, and $p_i = 1$ otherwise. Then $x \in F_n(p_n)$ for each $n \in \mathbb{N}$, and hence $\bigcap_{n \in \mathbb{N}} F_n(p_n) \neq \emptyset$, so that $p \in H$ and $H \neq \emptyset$.

We next claim that H is a closed subset of $\mathbb{C}$. Let $q \in \mathbb{C} \backslash H$, with $q = (q_i)$. Then $\bigcap_{n \in \mathbb{N}} F_n(q_n) = \emptyset$. Now each $F_n(q_n)$ is compact and hence $\{F_n(q_n) : n \in \mathbb{N}\}$ does not have the finite intersection property. Thus there exists $m \in \mathbb{N}$ such that $\bigcap_{n=1}^{m} F_n(q_n) = \emptyset$. Note that q is in the open set $\{q_1\} \times \cdots \times \{q_m\} \times \{0,1\} \times \cdots$ and this set does not meet H. It follows that $\mathbb{C} \backslash H$ is open, H is closed, and hence H is a compact totally disconnected metric space.

Next we claim that if $\bigcap_{n \in \mathbb{N}} F_n(p_n) \neq \emptyset$, then it is degenerate; for $p = (p_n) \in \mathbb{C}$. Let $x \in \bigcap_{n \in \mathbb{N}} F_n(p_n)$. Now there exists a sequence U_{n_j} of basic open sets containing x such that $\operatorname{diam} \overline{U}_{n_j} < \frac{1}{j}$ and $\overline{U}_{n_{j+1}} \subseteq \overline{U}_{n_j}$ for each j. Thus $p_{n_j} = 0$ for each $j \in \mathbb{N}$, and $\bigcap_{n \in \mathbb{N}} F_n(p_n) \subseteq \bigcap_{n \in \mathbb{N}} \overline{U}_{n_j}$. Since $\operatorname{diam} \overline{U}_{n_j} \to 0$, $\bigcap_{j \in \mathbb{N}} \overline{U}_{n_j}$ is degenerate and hence $\bigcap_{n \in \mathbb{N}} F_n(p_n)$ is degenerate.

Now define $\phi : H \to X$ by $\phi(p) = \bigcap_{n \in \mathbb{N}} F_n(p_n)$, where $p = (p_n)$. The argument that $H \neq \emptyset$ shows that ϕ is surjective.

We claim that ϕ is continuous. Let $p = (p_n) \in H$ and W and open subset of X such that $\phi(p) \in W$. In view of 4.14, there exists $m \in \mathbb{N}$ such that $\phi(p) \in \bigcap_{n=1}^{m} F_n(p_n) \subseteq W$. Let $V = \{p_1\} \times \cdots \times \{p_m\} \times \{0,1\} \times \cdots$. Then $p \in V$ and $\phi(V) \subseteq \bigcap_{n=1}^{m} F_n(p_n) \subseteq W$. It follows that ϕ is continuous.

Since H is a compact totally disconnected metric space, $H \times \mathbb{C}$ is homomorphic to $\mathbb{C}$ (12.12). Then $H \times \mathbb{C} \xrightarrow{\pi_1} H \xrightarrow{\phi} X$ is the desired surjection. ∎

13 QUASICOMPONENTS

If X is a space and $p \in X$, then the intersection of all open and closed subsets of X containing p is called the **quasicompontent** of X containing p and is denoted Q_p.

13.1 **Theorem.** *If Q is a quasicomponent of a space X and A is a compact subset of X such that $Q \cap A = \emptyset$, then there exists an open and closed subset N of X such that $Q \subseteq N \subseteq X \backslash A$.*

Proof. Let $Q = Q_p$ for $p \in X$, and let $\mathcal{K}$ be the collection of all open and closed subsets of X containing p. Then $Q = \bigcap\{K : K \in \mathcal{K}\}$.

Suppose that $K \cap A \neq \emptyset$ for each $K \in \mathcal{K}$. Since $\mathcal{K}$ is closed under finite intersections, the collection $\{K \cap A : K \in \mathcal{K}\}$ is a collection of closed subsets of the compact space A with the finite intersection property, and hence $\bigcap_{K \in \mathcal{K}} (K \cap A) \neq \emptyset$ and $\bigcap_{K \in \mathcal{K}} K \cap A = Q \cap A \neq \emptyset$. Thus $K \subseteq X \backslash A$ for some $K \in \mathcal{K}$ and $Q \subset K$. ∎

13.2 **Lemma.** *Let X be a locally compact Hausdorff space and A and B disjoint compact subsets of X. Then there exist disjoint open sets U and V such that $A \subseteq U$, $B \subseteq V$, $\overline{U}$ is compact, $\overline{V}$ is compact, and $\overline{U} \cap \overline{V} = \emptyset$.*

Recall that if X is a space and $U \subseteq X$, then $\partial U = \overline{U} \cap \overline{X \backslash U}$ is the **boundary of U in X**.

13.3 **Theorem.** *Let X be a locally compact Hausdorff space and Q a compact quasicomponent of X. Then Q is a component.*

Proof. Let Q be the quasicomponent of $p \in X$. We first show that Q is connected. Suppose that $Q = A|B$ with $p \in A$. Since Q is compact and A and B are closed in Q, A and B are disjoint compact subsets of X. Using 13.2, there exist open sets U and V in X such that $A \subseteq U$, $B \subseteq V$, $\overline{U}$ and $\overline{V}$ are compact, and $\overline{U} \cap \overline{V} = \emptyset$. Let $W = U \cup V$. Then $Q \subseteq W$ and ∂W is compact. By 13.1, there exists an open and closed set N in X such that $Q \subseteq N \subseteq X \backslash \partial W$. Thus $N \cap U$ is an open and closed set containing p. Since $N \cap U \cap B = \emptyset$, we have that $B = \emptyset$, and hence Q is connected.

Let C_p denoted the component of p. Then, since Q is a connected set containing p, we have $Q \subseteq C_p$. Suppose $C_p \backslash Q \neq \emptyset$, and let $x \in C_p \backslash Q$. Since $\{x\}$ is compact, there exists an open and closed set M such that $Q \subseteq M \subseteq X \backslash \{x\}$ (13.1). We see that $M \cap C_p$ is a proper open and closed subset of C_p, and this contradicts that C_p is connected. We conclude that $C_p \backslash Q = \emptyset$, and $Q = C_p$. ∎

13.4 **Boundary Bumping Theorem.** *Let X be a locally compact connected Hausdorff space, U a proper open subset of X such that $\overline{U}$ is compact, and let C be a component of U. Then $\overline{C} \cap \partial U \neq \emptyset$.*

Proof. Suppose that $\overline{C} \cap \partial U = \emptyset$.

Suppose that $\overline{C} \cap (X \backslash U) \neq \emptyset$, and let $p \in \overline{C} \cap (X \backslash U)$. Then $p \in \overline{C} \cap \overline{(X \backslash U)}$ and hence $p \notin \overline{U}$, since $\partial U = \overline{(X \backslash U)} \cap \overline{U}$. But $C \subseteq U$ implies that $\overline{C} \subseteq \overline{U}$, and $p \in \overline{U}$. This contradiction yields that $\overline{C} \cap (X \backslash U) = \emptyset$ and it follows that $\overline{C} \subseteq U$.

Since C is a component of U, C is closed in U, and hence $C = \overline{C}$, since $\overline{C} \subseteq U$. Also $C = \overline{C} \subseteq \overline{U}$ and $\overline{U}$ is compact, so that C is compact, and hence C is closed in X.

Now since $\overline{U}$ is compact, ∂U is compact. Since X is regular, C is closed and $C \cap \partial U = \emptyset$, there exist open sets G and V such that $C \subseteq G$, $\partial U \subseteq V$, and $G \cap V = \emptyset$.

Let $W = G \cap U$. Then W is open, $C \subseteq W$, $W \cap V = \emptyset$, $\partial U \subseteq V$, and V is open. Since $W \subseteq U$, we have $\overline{W} \subseteq \overline{U}$, and hence $\overline{W}$ is compact.

Let $p \in C$, and let Q_p be the quasicomponent of p in $\overline{W}$. Since $\overline{W}$ is compact and Q_p is an intersection of closed (and open) subsets of $\overline{W}$, we have that Q_p is compact. Note that $C \subseteq W \subseteq U$ and C is a component of $\overline{W}$, since $\overline{W} \subseteq \overline{U} = U \cup \partial U$ and $\overline{W} \cap \partial U = \emptyset$, we have $\overline{W} \subseteq U$. Thus $Q_p = C$ (13.3).

Since $\partial \overline{W}$ is compact, there exists M which is open and closed in $\overline{W}$ such that $C \subseteq M \subseteq \overline{W} \backslash \partial W = W$ (13.1). Since $\overline{W}$ is compact, we have that M is compact, and hence M is closed in X. Since M is open in $\overline{W}$, there exists H open in X such that $M = \overline{W} \cap H$. Since $M \subseteq W$, $M = W \cap H$ and hence M is open in X. We now have that M is open and closed in X; and since $M \subseteq U$ and U is a proper subset of X, M is a proper subset of X. This contradicts that X is connected. ∎

13.5 **Theorem.** *No locally compact connected Hausdorff space is the countable union of pairwise disjoint compact sets.*

Proof. Let X be a locally compact connected Hausdorff space and assume that $X = \bigcup_{n \in \mathbb{N}} A_n$, where each A_n is compact and $A_j \cap A_i = \emptyset$ if $i \neq j$.

Let U_1 be an open set such that $A_1 \subseteq U_1$ and $\overline{U}_1$ is compact. Let C_1 be a component of U_1. Then $\overline{C}_1 \cap \partial U_1 \neq \emptyset$ (13.4). Let $p_1 \in \overline{C}_1 \cap \partial U_1$. Then $p_1 \notin A_1$. We can assume that $p_1 \in A_2$. Let U_2 be an open set such that $A_2 \subseteq U_2 \subseteq \overline{U}_2 \subseteq X \backslash A_1$ and $\overline{U}_2$ is compact. Note that $p_1 \in \partial U_1 \cap U_2$, so that $U_1 \cap U_2 \neq \emptyset$.

Let C_2 be a component of $U_1 \cap U_2$ and let $p_2 \in \overline{C}_2 \cap \partial(U_1 \cap U_2)$. Then $p_2 \notin A_1 \cup A_2$. We can assume that $p_2 \in A_3$.

Let U_3 be an open set such that $A_3 \subseteq U_3 \subseteq \overline{U}_3 \subseteq X \backslash (A_1 \cup A_2)$, and $\overline{U}_3$ is compact. Note that $p_2 \in \partial(U_1 \cap U_2) \cap U_3$, so that $U_1 \cap U_2 \cap U_3 \neq \emptyset$.

Let C_3 be a component of $U_1 \cap U_2 \cap U_3$ and let $p_3 \in \overline{C}_3 \cap \partial(U_1 \cap U_2 \cap U_3)$. Then $p_3 \notin A_1 \cup A_2 \cup A_3$.

Let $H_n = \bigcap_{j=1}^{n} \overline{U}_j$ for each $n \in \mathbb{N}$. Then $\{H_n : n \in \mathbb{N}\}$ is a tower of compact sets, and hence $\bigcap_{n\in\mathbb{N}} H_n \neq \emptyset$. Note that $\bigcap_{n\in\mathbb{N}} H_n = \bigcap_{n\in\mathbb{N}} \overline{U}_n \neq \emptyset$.

Let $p \in \bigcap_{n\in\mathbb{N}} \overline{U}_n$. Then $p \in A_n$ for some $n \in \mathbb{N}$. But $p \in \overline{U}_{n+1} \subseteq X\backslash(A_1 \cup A_2 \cup \cdots \cup A_n) \subseteq X\backslash A_n$. This contradiction proves the theorem. ■

14 ARCS

If X is a connected space and $p \in X$, then p is called a **cutpoint** of X provided $X\backslash\{p\}$ is not connected.

14.1 **Theorem**. *Let $f: X \to Y$ be a homeomorphism of a space X onto a space Y and let $p \in X$. Then p is a cutpoint of X if and only $f(p)$ is a cutpoint of Y.*

14.2 **Exercise**. *The space $\mathbb{R}$ is not homeomorphic to S^1.*

If $\leq$ is a total order on a set X, then for $a < b$ in X:

$(a,b) = \{x \in X : a < x < b\}$
$[a,b) = \{x \in X : a \leq x < b\}$
$(a,b] = \{x \in X : a < x \leq b\}$
$[a,b] = \{x \in X : a \leq x \leq b\}$

14.3 **Theorem**. *Let $\leq$ be a total order on a nondegenerate set X and let $\beta = \{(a,b) : a < b \text{ in } X\} \cup \{(a, \sup X] : a \neq \sup X\} \cup \{[\inf X, b) : b \neq \inf X\}$. Then β is a basis for a unique topology on X.*

The topology generated by β in 14.3 is called the **order topology** on X induced by $\leq$.

If $\leq$ is a total order on a set X, then X is said to be **order dense** if for each $x < y$ in X, there exists $z \in X$ such that $x < z < y$. We say that X is **order complete** provided each nonempty subset of X has a sup and inf.

An **arc** is a space X with a total order such that X has the order topology, X is order dense, and X is order complete.

A **continuum** is a compact connected Hausdorff space.

14.4 **Theorem**. *An arc is a continuum.*

If X is an arc, then $\sup X$ and $\inf X$ are called the **endpoints** of X.

If X is a space and a and b are distinct points of X, then X is said to be **irreducibly connected between a and b** if X is connected and no proper subset of X containing both a and b is connected.

14.5 **Lemma**. *Let X be a continuum which is irreducibly connected between a and b, and let $p \in X\backslash\{a,b\}$. Then $X\backslash\{p\}$ has exactly two components C_a and C_b. Moreover, $\overline{C}_a = C_a \cup \{p\}$ and $\overline{C}_b = C_b \cup \{p\}$.*

Proof. It is immediate that p is a cutpoint of X and that a and b lie in different components of $X\backslash\{p\}$. In view of the Boundary Bumping Theorem, we see that $\overline{C}_a \cap \partial(X\backslash\{p\}) \neq \emptyset$. Since $\partial(X\backslash\{p\}) = \{p\}$, we have that $p \in \overline{C}_a$, and similarly $p \in \overline{C}_b$. Thus $\overline{C}_a \cup \overline{C}_b$ is a connected subset of X containing a and b, and hence $X = \overline{C}_a \cup \overline{C}_b$. Now $C_a \cap \overline{C}_b = \emptyset = \overline{C}_b \cap C_a$.

We claim that $\overline{C}_a \cap \overline{C}_b = \{p\}$. Suppose that $q \neq p$ and $q \in \overline{C}_a \cap \overline{C}_b$. Then $q \notin C_a \cup C_b$, so that in particular $q \neq a$ and $q \neq b$, and hence q is a cutpoint of X, and $X\backslash\{q\} = A|B$, with $C_a \subseteq A$ and $C_b \subseteq B$. Now $\overline{C}_a \subseteq \overline{A}$ and $\overline{C}_b \subseteq \overline{B}$. Thus $p \in \overline{A}$ and $p \in \overline{B}$. From $p \in \overline{A}$, we have that $p \notin B$, since $\overline{A} \cap B = \emptyset$, and similarly $p \notin A$. But since $p \neq q$ and $X\backslash\{q\} = A \cup B$, we must have that $p \in A$ or $p \in B$. This contradiction proves that $\overline{C}_a \cap \overline{C}_b = \{p\}$. Thus $\overline{C}_a = C_a \cup \{p\}$ and $\overline{C}_b = C_b \cup \{p\}$, and C_a and C_b are the two components of $X\backslash\{p\}$. ∎

14.6 **Lemma**. *Let X be a continuum which is irreducibly connected between distinct points a and b. Then each $p \in X\backslash\{a,b\}$ is a cutpoint of X. Define $x \leq y$ in X if either $x = y$ or x lies in the component of $X\backslash\{y\}$ containing a. Then $\leq$ is a total order on X.*

The order $\leq$ in 14.6 is called the **cutpoint order**.

14.7 **Lemma**. *Let X be an arc, $a = \inf X$, and $b = \sup X$. Then X is irreducibly connected between a and b and the order on X is the cutpoint order.*

14.8 **Lemma**. *Let X be a separable arc with E a countable dense subset of X such that* $\inf X$, $\sup X$ $\notin E$. *Let Q denote the set of all rational numbers in* $(0,1)$. *Then there exists a strictly order preserving function $f\colon E \to Q$ from E onto Q.*

Proof. Let $E = \{e_1, e_2, \ldots\}$ and let $Q = \{r_1, r_2, \ldots\}$. Let $\mathcal{A}$ denote the set of all pairs (A, g_A) such that:

(a) $A \subseteq E$;

(b) $e_1 \in A$;

(c) if $e_n \in A$, then $\{e_1, e_2, \ldots, e_n\} \subseteq A$;

(d) $g_A\colon A \to Q$ is a strictly order preserving function;

(e) $g_A(e_1) = r_1$; and

(f) for $1 < m$ in $\mathbb{N}$ and $e_m \in A$, $g(e_m) = r_k$, where $k = \min\{i\colon g_A|\{e_1, e_2, \ldots e_m\}$ and $g_A(e_m) = r_i\}$ is strictly order preserving $\}$.

To see that $\mathcal{A} \neq \emptyset$, let $A = \{e_1\}$ and define $g_A(e_1) = r_1$.

Define $(A, g_A) \leq (B, g_B)$ on $\mathcal{A}$ provided $A \subseteq B$ and $g_B|A = g_A$. Then $\leq$ is a partial order on $\mathcal{A}$.

Let $\mathcal{A}'$ be a maximal chain in $\mathcal{A}$, and let $H = \bigcup\{A\colon (A, g_A) \in \mathcal{A}'\}$.

We claim that $H = E$. Suppose that $E\backslash H \neq \emptyset$ and let $e_s \in E\backslash H$. Then $e_t \notin H$ for $s \leq t \in \mathbb{N}$ and hence H is finite. Let $H = \{e_1, e_2, \ldots, e_m\}$, with

$e_m \in A$ for some $(A, g_A) \in \mathcal{A}'$. From (c), we have that $H = A$. Let $r_j \in Q$ such that $g(e_{m+1}) = r_j$ and $g|H = g_H$, $g\colon H \cup \{e_{m+1}\} \to Q$ is strictly order preserving. Then $(H \cup \{e_{m+1}\}, g) \in \mathcal{A}$; contradicting the maximality of $\mathcal{A}'$. It follows that $E \backslash H = \emptyset$, and $H = E$.

Define $f\colon E \to Q$ so that $f|A = g_A$ for each $(A, g_A) \in \mathcal{A}'$. Then f is strictly order preserving and hence injective. That f is surjective follows from (f). ■

14.9 **Theorem**. *A separable arc is an interval, i.e., homeomorphic to* $I = [0,1]$.

Proof. Let X be a separable arc, $a = \inf X$, $b = \sup X$, E a countable dense subset of X with $a, b \notin E$, and let Q be the set of rational numbers in $(0,1)$. Let $f\colon E \to Q$ be a strictly order preserving function from E onto Q (14.8). Define $g\colon X \to I$ by $g(x) = \inf f([x,b] \cap E)$ and $g(b) = 1$.

We claim that g is strictly order preserving. Let $u < v$ in X, and let $u < t < v$, for $t \in E$. Let $v' \in [v,b] \cap E$. Then $t < v'$ and $f(t) < f(v')$, so that $f(t) \leq g(v)$. Now let $t' \in E$ such that $u < t' < t$. Then $f(t') < f(t)$, so that $g(u) < f(t)$ and hence $g(u) < g(v)$.

It follows that $g\colon X \to I$ is injective.

We claim that g is surjective. Let $q \in I$ and let $p = \inf f^{-1}([q,1] \cap Q)$. Then $q = g(p)$ and hence g is surjective.

To see that g is continuous, observe that $g^{-1}(c,d) = (g^{-1}(c), g^{-1}(d))$, and hence g is a homeomorphism. ■

14.10 **Lemma**. *If p is a cutpoint of a continuum X and $X \backslash \{p\} = A|B$, then $A \cup \{p\}$ is a continuum.*

Proof. Let $x \in A$ and let C_x be the component of $X \backslash \{p\}$ containing x. Then $\overline{C}_x \cap \partial(X \backslash \{p\}) \neq \emptyset$, so that $p \in \overline{C}_x$. Since $\overline{A} \cap B = \emptyset$, $\overline{C}_x \subseteq \overline{A} = A \cup \{p\}$ (Note that B is open), so that $A \cup \{p\} = \bigcup_{x \in A} \overline{C}_x$ is connected, since $p \in \overline{C}_x$ and $x \in A$. Since B is open in $X \backslash \{p\}$ (and hence in X). $A \cup \{p\} = \overline{A}$ is closed and hence compact. ■

14.11 **Theorem**. *Each non degenerate continuum has at least two non cutpoints.*

Proof. Let X be a non degenerate continuum. If X has no cutpoints, then X has at least two non cutpoints, since X is non degenerate.

Suppose then that X has at least one cutpoint and let p be a cutpoint of X. Then $X \backslash \{p\} = A|B$.

Suppose that each cutpoint of A is also a cutpoint of X. For $x \in A$, let $X \backslash \{x\} = P_x | Q_x$ with $p \in P_x$. Then $Q_x \cup \{x\} = \overline{Q}_x$ is a continuum containing x and not p, so that $\overline{Q}_x \subseteq A$.

Now $\{\overline{Q}_x : x \in A\}$ is partially ordered by inclusion. Let $\{\overline{Q}_x : x \in A_0 \subseteq A\}$ be a maximal chain, and let $Q = \bigcap_{x \in A_0} \overline{Q}_x$. Then Q is a nonempty continuum.

We claim that if $a \in Q_b$, then $\overline{Q}_a \subseteq Q_b$. Let $a \in Q_b$. Then $\overline{P}_b$ is a connected subset of $X \backslash \{a\}$ containing p, and hence $\overline{P}_b \subseteq P_a$. Thus $\overline{Q}_a$ is a connected subset of $X \backslash \{b\}$, and since $a \in \overline{Q}_a \cap Q_b$, we have $\overline{Q}_a \subseteq Q_b$.

We now have that $Q = \bigcap_{x \in A_0} \overline{Q}_x = \bigcap_{x \in A_0} Q_x$. Let $r \in Q$. Then, for $x \in A_0$, $r \in Q_x$, $\overline{Q}_r \subseteq Q_x$, and hence $\overline{Q}_r \subset \overline{Q}_x$ for all $x \in A_0$; contradicting the maximality of the chain. It follows that A contains a non cutpoint of X, and similarly so does B. ■

14.12 **Theorem.** *Let X be a continuum. These are equivalent:*

(a) X has exactly two non cutpoints a and b;

(b) X is irreducibly connected between a and b; and

(c) X is an arc with endpoints a and b.

Proof. (c) **implies** (b) follows from 14.7.

(b) **implies** (a). Suppose that X is irreducibly connected between a and b. Let $p \in X \backslash \{a, b\}$. Then $a, b \in X \backslash \{p\} \subset X$ and hence $X \backslash \{p\}$ is not connected. It follows that p is a cutpoint of X. Then $X \backslash \{a\} = A|B$, $b \in B$, B is open in $X \backslash \{a\}$ (which is open) and hence B is open. Also $\overline{A} \cap B = \emptyset = A \cap \overline{B}$. Let C be the component of B containing b. Then $\overline{C} \subseteq \overline{B}$, and hence $\overline{C} \cap A = \emptyset$. Also by the Boundary Bumping Theorem, $\overline{C} \cap \partial B \neq \emptyset$. Let $p \in \overline{C} \cap \partial B$. Then $p \notin A$ since $p \in \overline{C}$ and $p \notin B$, since $p \in \partial B$ and B is open. Thus $p = a$, so that $\overline{C}$ is a connected subset of X containing a and b, and $\overline{C} = X$. Since $\overline{C} \cap A = \emptyset$, we have $A = \emptyset$, and $X \backslash \{a\}$ is connected. Similarly, $X \backslash \{b\}$ is connected.

(a) **implies** (b) is immediate.

(b) **implies** (c). Suppose that X is irreducibly connected between a and b. Let $\leq$ be the total order on X defined in 14.6. We need to show that:

(1) X is $\leq$ dense;

(2) X is $\leq$ complete; and

(3) X has the $\leq$ topology.

We first show (3): X has the $\leq$ topology.

Let τ be the given topology on X and let $\leq_\tau$ be the topology on X inducted by $\leq$. Let $j: (X, \tau) \to (X, \leq_\tau)$ be the identity map. Observe that $\leq_\tau$ is a Hausdorff topology on X, and that j is bijective. We need only show that j is continuous. In view of 14.5, for $p \in X \backslash \{a, b\}$, we have that $\overline{C}_a = [a, p]$, where C_a is the component of $X \backslash \{b\}$ containing a. Thus $[a, p]$ is τ-closed, so that $(p, b]$ is τ-open. Likewise, $[a, p)$ is τ-open, so that for $q < p$, we have that $(q, p) = [a, p) \cap (q, b]$ is τ-open, and j is continuous.

To prove (1): X is $\leq$ dense, let $x < y$ in X. If $\{z \in X\colon x < z < y\} = \emptyset$, then $X = [a, y) \cup (x, b]$ is a separation of X; contradicting that X is connected. Thus X is $\leq$ dense.

To complete the proof of 14.10, we prove (2): X is $\leq$ complete.

First we show that if K is a connected subset of X, $u \in K$, $v \in K$, then $[u, v] \subseteq K$. Suppose to the contrary that there exists $t \in [u, v]$ which is not in K. Then $u < t < v$, so that u is in the component T of $X \backslash \{t\}$ containing a. Since K is a connected subset of $X \backslash \{t\}$ containing u, we have that $K \subseteq T$. Thus $v \in T$ and $v < t$; contradicting $t < v$. We conclude that $[u, v] \subseteq K$.

Let A be an open subset of X. We will show that $\inf A$ exists in X. If $a \in \overline{A}$, then $a = \inf A$, so we assume that $a \notin \overline{A}$. Let C be the component of a in $X \backslash \overline{A}$, and let $p \in \overline{C} \cap \partial(X \backslash \overline{A}) = \overline{C} \cap \partial A$. Now $C \subseteq X \backslash A$, so that $\overline{C} \subseteq X \backslash A$, since $X \backslash A$ is closed. Since $a, p \in \overline{C}$, by the preceding argument, we have $[a, p] \subseteq \overline{C} \subseteq X \backslash A$, and hence $p \leq x$ for all $x \in A$. Now let $t \in X$ with $p < t$. Then $p \in [a, t)$ and since $p \in \partial A \subseteq \overline{A}$ ($p \notin A$), $[a, t)$ is an open set containing p and hence $[a, t) \cap A \neq \emptyset$. It follows that $x \in [a, t)$ for some $x \in A$, i.e., $x < t$ and t is not a lower bound for A. We conclude that $p = \inf A$, i.e., if A is open, then $\inf A$ exists.

Let $T \subseteq X$. We will show that $\sup T$ exists. Now if $b \in T$, then $b = \sup T$, so we assume that $b \notin T$. Let $B = \{x \in X\colon t < x \text{ for all } t \in T\}$. Now if B is not open, then there exists $x \in B$ such that $(e, b] \cap T \neq \emptyset$ for all e such that $e < x$, so that $x = \sup T$. Assume then that B is open. Then $\inf B = c$ exists from the paragraph above, and $c = \sup T$.

If $T \subseteq X$, to see that $\inf T$ exists, again we can assume that $a \notin T$, and let $D = \{x \in X\colon x < t \text{ for all } t \in T\}$. Then $\inf T = \sup D$. ∎

Let $(A, \leq_A)$ and $(B, \leq_B)$ be totally ordered sets. Define $(a, b) \leq (c, d)$ on $A \times B$ if either $(a, b) = (c, d)$; or $a <_A c$; or $a = c$ and $b <_B d$. Then $\leq$ is a total order on $A \times B$ called the **lexicographic order**.

14.13 **Exercise**. Let $X = I \times I$ with the topology induced by the lexicographic order on X. Then X is an arc which is not an interval.

Let $\prec$ be a well ordering of $\mathbb{R} \backslash \mathbb{N}$, let $r \notin \mathbb{R}$ and let $\widetilde{\mathbb{R}} = \mathbb{R} \cup \{r\}$ and extend $\prec$ to $\widetilde{\mathbb{R}}$ by defining:

(a) $1 \prec 2 \prec 3 \prec \cdots$;

(b) $n \prec x$ when $n \in \mathbb{N}$ and $x \in \mathbb{R} \backslash \mathbb{N}$; and

(c) $x \prec r$ for each $x \in \mathbb{R}$.

Note that $\prec$ is a well ordering of $\widetilde{\mathbb{R}}$. For each $a \in \widetilde{\mathbb{R}}$, let $L(a) = \{x \in \widetilde{\mathbb{R}}\colon x \prec a\}$. Note that $L(r)$ is uncountable. Let $\Omega = \inf \{x \in \widetilde{\mathbb{R}}\colon L(x) \text{ is uncountable } \}$, $\omega = \inf \{x \in L(\Omega)\colon \mathbb{N} \subseteq L(x)\}$, and let $\Theta = L(\Omega) \cup \{\Omega\}$.

14.14 **Exercise**.

(a) Θ is uncountable;
(b) $L(x)$ is countable for each $x \in L(\Omega)$;
(c) $\omega = \inf\{x \in \Theta\colon L(x)$ is infinite $\}$; and
(d) $L(\omega) = \mathbb{N}$.

Define $\leq$ on Θ by $x \leq y$ if either $x = y$ or $x \prec y$, and observe that $\leq$ is a total order on Θ. Give Θ the topology induced by $\leq$.

14.15 **Theorem**. *The space Θ is a compact Hausdorff space and is not first countable at Ω.*

14.16 **Theorem**. *The space $L(\Omega)$ is a countably compact normal subspace of Θ.*

Proof. That $L(\Omega)$ is countably compact is a consequence of 14.15.

To see that $L(\Omega)$ is normal, let A and B be disjoint closed subsets of $L(\Omega)$. For each $x \in L(\Omega)$, define $x^+ = \inf\{y \in L(\Omega)\colon x < y\}$. Now for each $a \in A$, let x_a, $p_a \in L(\Omega)$ such that $a \in (x_a, p_a) \subseteq X\backslash B$. Then $x_a < a < a^+ \leq p_a$, so that $a \in (x_a, a^+) \subseteq (x_a, p_a) \subseteq X\backslash B$. Let $U = \bigcup_{a\in A} (x_a, a^+)$. Then U is open and $A \subseteq U$. Also: for each $b \in B$, let $y_b \in L(\Omega)$ such that $b \in (y_b, b^+) \subseteq X\backslash A$. Let $V = \bigcup_{b\in B} (y_b, b^+)$. Then V is open and $B \subseteq V$. It ramains to show that U and V are disjoint.

Suppose that $U \cap V \neq \emptyset$ and let $z \in U \cap V$. Then $z \in (x_a, a^+) \cap (y_b, y^+)$ for some $a \in A$ and $b \in B$. We can assume that $a < b$. Now $x_a < z < a^+$, so that $x_a < z \leq a$ and likewise $y_b < z \leq b$. We have that $y_b < z \leq a < b$, so that $a \in (y_b, b) \subseteq (y_b, b^+)$; contradicting that $(y_b, b^+) \subseteq X\backslash A$. Thus $U \cap V = \emptyset$. ■

The space $(L(\Omega) \times [0, 1)) \cup \{(\Omega, 0)\}$ with the topology induced by the lexicographic order is called the **long line**.

14.17 **Exercise**. The long line is an arc.

The space $[\Theta \times (\mathbb{N} \cup \{\omega\})]\backslash\{(\Omega, \omega)\}$ with the relative product topology on $\Theta \times (\mathbb{N} \cup \{\omega\})$ is called the **Tychonoff plank**.

14.18 **Theorem**. *The space $\Theta \times (\mathbb{N} \cup \{\omega\})$ is a compact Hausdorff space.*

14.19 **Theorem**. *The Tychonoff plank is a locally compact Hausdorff space which is not normal.*

Proof. Let $X = [\Theta \times (\mathbb{N} \cup \omega)]\backslash(\Omega, \omega)$ be the Tychonoff plank. Then, since X is open in $\Theta \times (\mathbb{N} \cup \omega)$, X is locally compact and Hausdorff. Let $A = \{\Omega\} \times \mathbb{N}$ and let $B = (\Theta\backslash\{\Omega\}) \times \{\omega\}$. Then A and B are disjoint closed subsets of X.

Suppose that U and V are disjoint open subsets of X with $A \subseteq U$ and $B \subseteq V$. For each $a \in \mathbb{N}$, let $\bar{a} = \inf\{z \in \Theta\colon [z, \Omega] \times \{a\} \subseteq U\}$ and let $p = \sup_{a\in A} \bar{a}$. Now, since A is countable, $\{\bar{a}\colon a \in A\}$ is countable, and hence

$p < \Omega$. Let $p < q < \Omega$ in Θ. Then $\{q\} \times \mathbb{N} \subseteq U$ and hence $\{q\} \times (\mathbb{N} \cup \{\omega\}) \subseteq \overline{U}$. But $(q, \omega) \in B \subseteq V$. ■

15 PEANO SPACES

If a and b are points of a set S, then a **simple chain from a to b** is a finite collection of sets $\{H_j : j = 1, 2, \ldots, n\}$ of subsets of S such that $a \in H_1$, $b \in H_n$, and $H_i \cap H_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

15.1 **Theorem.** *A space X is connected if and only if for each a and b in X and each open cover $\mathcal{U}$ of X, there exists a finite subcollection of $\mathcal{U}$ which is a simple chain from a to b.*

Proof. Suppose that the condition holds and X is not connected. Then $X = A \cup B$, where A and B are disjoint nonempty open subsets of X. Let $a \in A$ and $b \in B$. Then $\{A, B\}$ is an open cover of X which contains no simple chain from a to b.

Suppose that X is connected and let $a \in X$. Let $\mathcal{U}$ be an open cover of X and let $K = \{x \in X :$ there is a finite subcollection of $\mathcal{U}$ which is a simple chain from a to $x\}$. We claim that K is both open and closed.

Let $p \in K$, and let $H_1, H_2, \ldots, H_n$ be a simple chain in $\mathcal{U}$ from a to p. For $x \in H_n \backslash H_{n-1}$, we have that $H_1, H_2, \ldots, H_n$ is a simple chain in $\mathcal{U}$ from a to x; for $x \in H_n \cap H_{n-1}$, we have that $H_1, H_2, \ldots, H_{n-1}$ is a simple chain in $\mathcal{U}$ from a to x and hence $x \in K$. Thus $H_n \subseteq K$ and K is open.

Let $p \in X \backslash K$ and let $U \in \mathcal{U}$ such that $p \in U$. Suppose that $U \cap K \neq \emptyset$, and let $q \in U \cap K$. Let $H_1, \ldots, H_m$ be a simple chain in $\mathcal{U}$ from a to q, and let $k = \min\{i : H_i \cap U \neq \emptyset\}$. Then $H_1, H_2, \ldots, H_k, U$ is a simple chain in $\mathcal{U}$ from a to p; contradicting that $p \notin K$. Thus $U \cap K = \emptyset$ and $p \in U \subseteq X \backslash K$, $X \backslash K$ is open, and K is closed. Since X is connected, we have $X = K$, since $K \neq \emptyset$. ■

Let E be a subset of a metric space (X, d), $a, b \in E$, and let $\epsilon > 0$. An **ϵ-chain** from a to b in E is a finite set $\{x_1, x_2, \ldots, x_n\}$ of points of E such that $a = x_1$, $b = x_n$, and $d(x_j, x_{j+1}) < \epsilon$ for $j = 1, 2, \ldots, n - 1$.

A subset E of a metric space (X, d) is said to be **well-chained** if for each $a, b \in E$ and each $\epsilon > 0$, there exists an ϵ-chain from a to b in E.

15.2 **Theorem.** *Each connected subset of a metric space is well-chained.*

If X is a metrizable space, then a metric d on X is called an **M-metric** provided the topology on X is determined by d and $N_r(x)$ is connected for each $x \in X$ and each $r > 0$.

15.3 **Theorem.** *Let X be a metrizable space. Then X admits an M-metric if and only if X is connected and locally connected.*

Proof. Suppose that X admits an M-metric ρ. Then for each $p \in X$ and each $\epsilon > 0$, $N_\epsilon(p)$ is connected and hence X is locally connected. To see that X is connected, let x and y be distinct points of X, and let $r = 2\rho(x, y)$. Then $N_r(x)$ is a connected set containing both x and y.

Suppose that (X, d) is a connected and locally connected metric space and let $\mathcal{U} = \{U : U \subseteq X$ and U is open and connected $\}$. Then $\mathcal{U}$ is a basis for a topology on X (5.19). For $a, b \in X$, define $\rho(a, b) = \inf\{\operatorname{diam} U : U \in \mathcal{U},\ a, b \in U\}$. Then ρ is the desired M-metric on X. ∎

Recall that $I = [0, 1]$ with the usual topology.

A Hausdorff space X is called a **Peano space** if there exists a continuous surjection $f : I \to X$.

15.4 **Theorem.** *Each Peano space is compact connected locally connected and metrizable.*

A space X is said to be **arcwise connected** if for each pair a, b of distinct points of X, there exists an embedding $g : I \to X$ such that $g(0) = a$ and $g(1) = b$.

If $\mathcal{A} = \{A_1, \ldots, A_n\}$ and $\mathcal{B} = \{B_1, \ldots, B_n\}$ are simple chains in a space X, then $\mathcal{B}$ **simply refines** $\mathcal{A}$ provided:

(1) Each B_i is contained in some A_r; and

(2) If $B_i \cup B_k \subseteq A_r$ for $i < k$, then $B_j \subseteq A_{r'}$ for all $i < j < k$ where $|r' - r| \leq 1$.

15.5 **Lemma.** *Let X be a space, $p, q \in X$, and $\mathcal{A} = \{A_1, \ldots, A_n\}$ a simple chain of connected open sets from p to q in X. Let C be a family of open sets such that each member of $\mathcal{A}$ is a union of members of C. Then there is a simple chain of members of C from p to q which simply refines $\mathcal{A}$.*

15.6 **Theorem.** *Each locally compact connected locally connected metric space is arcwise connected.*

Proof. Let X be a locally compact connected locally connected metric space and let p and q be distinct points of X. Let X be given an M-metric (15.3).

Let C_1 be a simple chain from p to q such that each link of C_1 is open, connected, has compact closure, and has diameter less than 1.

Let C_{n+1} be a simple chain from p to q such that each link is open, connected, has compact closure, has diameter less than $\frac{1}{n+1}$, and such that C_{n+1} simply refines C_n.

For each $n \in \mathbb{N}$, let $A_n = \bigcup\{\overline{L} : L \in C_n\}$. Then A_n is a compact connected subset of X containing p and q for each $n \in \mathbb{N}$. Also note that $\{A_n : n \in \mathbb{N}\}$ is a tower. Let $A = \bigcap_{n \in \mathbb{N}} A_n$. Then A is a subcontinuum of X containing p and q.

Let $x \in A\backslash\{p,q\}$. We claim that x is a cutpoint of A. For each $n \in \mathbb{N}$, let $P_n = \bigcup\{L\colon L \in C_n$ and L precedes the one or two links of C_n containing $x\}$ and let $F_n = \bigcup\{L\colon L \in C_n$ and L follows the one or two links of C_n containing $x\}$. Let $P = \bigcup_{n\in\mathbb{N}} P_n$ and $F = \bigcup_{n\in\mathbb{N}} F_n$. Then P and F are open, $P \cap A \neq \emptyset$, $F \cap A \neq \emptyset$, and $A\backslash\{x\} = (P \cap A)|(F \cap A)$, so that x is a cutpoint of A. Thus p and q are the only non cutpoints of A (14.12), and hence A is an arc with endpoints p and q (14.10). Since A is compact and metric, A is separable and hence an interval (14.9). ■

15.7 **Theorem**. *Each Peano space is arcwise connected.*

If (X,d) is a metric space and $\mathcal{U}$ is an open cover of X, then $\delta > 0$ is called a **Lebesque number** for $\mathcal{U}$ if for each $E \subseteq X$ with $\operatorname{diam} E < \delta$, there exists $U \in \mathcal{U}$ such that $E \subseteq U$.

15.8 **Theorem**. *Each open cover of a compact metric space has a Lebesgue number.*

Proof. Let X be a compact metric space and let $\mathcal{U}$ be an open cover of X.

Suppose that $\mathcal{U}$ does not have a Lebesgue number. Then for each $n \in \mathbb{N}$, there exists $A_n \subseteq X$ such that $\operatorname{diam} A_n < \frac{1}{n}$ and A_n is not contained in any member of $\mathcal{U}$. Let $p_n \in A_n$ for each $n \in \mathbb{N}$. Then $p_n \xrightarrow{f} p$ for some $p \in X$. Let $U \in \mathcal{U}$ such that $p \in U$, and let $r > 0$ such that $N_r(p) \subseteq U$. Let $k \in \mathbb{N}$ such that $\frac{1}{k} < \frac{r}{2}$ and $p_k \in N_{\frac{r}{2}}(p)$. Let $y \in A_k$. Then $d(p,y) \leq d(p,p_k) + d(p_k,y) < \frac{r}{2} + \frac{1}{k} = r$, so that $A_k \subseteq N_r(p) \subseteq U$; which is a contradiction. ■

15.9 **Lemma**. *Let (X,d) be a compact locally connected metric space and let $\epsilon > 0$. Then there exists $\delta > 0$ such that if $x,y \in X$ and $d(x,y) < \delta$, then $\{x,y\}$ is contained in an open connected set of diameter less than ϵ.*

Proof. For each $p \in X$ let U_p be an open connected set containing p with $\operatorname{diam} U_p < \epsilon$ (since X is locally connected). Then $\mathcal{U} = \{U_p\colon p \in X\}$ is an open cover of X. Let $\delta > 0$ be a Lebesgue number for $\mathcal{U}$. Then for $x,y \in X$ with $d(x,y) < \delta$, we have $\operatorname{diam}\{x,y\} < \delta$ so that $\{x,y\} \subseteq U_p$ for some p. ■

15.10 **Lemma**. *Let (X,d) be a compact connected locally connected metric space and let $\epsilon > 0$. Then there exists $\delta > 0$ such that if $p,q \in X$ and $0 < d(p,q) < \delta$, then there exists and embedding $g\colon I \to X$ such that $g(0) = p$, $g(1) = q$, and $\operatorname{diam} g(I) < \epsilon$.*

Proof. Let $\delta > 0$ be as in 15.9, and let U be an open connected set of diameter less than ϵ containing p and q, where $0 < d(p,q) < \delta$. Then U is locally compact, connected, locally connected and metric, and hence U is arcwise connected (15.6). ■

15.11 **The Hahn-Mazurkiewicz Theorem**. *A space X is a Peano space if and only if X is compact connected locally connected and metrizable.*

Proof. If X is a Peano space, then X has these properties (15.4).

Suppose that (X,d) is a compact connected locally connected metric space. Let $\{\epsilon_n\}$ be a decreasing sequence of positive real numbers such that $\epsilon_n \xrightarrow{e} 0$ and if $p,q \in X$ and $0 < d(p,q) < \epsilon_n$, then there exists an embedding $h\colon I \to X$ such that $h(0) = p$, $h(1) = q$, and $\operatorname{diam} h(I) < \frac{1}{n}$ (15.10).

Let $K \subset I$ denote the Cantor set and let $f\colon K \to X$ be a continuous surjection (12.14). We claim that f can be extended to a continuous function $g\colon I \to X$.

Since $f\colon K \to X$ is uniformly continuous (6.30), there is a decreasing sequence $\{\delta_n\}$ of positive real numbers converging to 0 such that if $s,t \in K$ and $|s-t| < \delta_n$, then $d(f(s),f(t)) < \epsilon_n$.

Let $\{P_j = (a_j,b_j)\colon j \in \mathbb{N}$ denote the collection of open interval components of $I\backslash K$, and let $\alpha_j = b_j - a_j$ for each $j \in \mathbb{N}$.

Let $\mathcal{A}_0 = \{P_j\colon \delta_1 \leq \alpha_j\}$ and for $n \in \mathbb{N}$, let $\mathcal{A}_n = \{P_j\colon \delta_{n+1} \leq \alpha_j < \delta_n\}$. Note that $\mathcal{A}_0$ and each $\mathcal{A}_n$ is at most finite.

For each $P_j \in \mathcal{A}_0$, define $g_j\colon \overline{P}_j = [a_j,b_j] \to X$ so that $g_j(\overline{P}_j)$ is an arc from $f(a_j)$ to $f(b_j)$ if $f(a_j) \neq f(b_j)$ (15.6), and $g(\overline{P}_j) = f(a_j)$ if $f(a_j) = f(b_j)$.

For each $P_j \in \mathcal{A}_n$ ($n \in \mathbb{N}$), note that $d(f(a_j),f(b_j)) < \epsilon_n$. Define $g_j\colon \overline{P}_j \to X$ so that $g_j(\overline{P}_j)$ is an arc from $f(a_j)$ to $f(b_j)$ with $\operatorname{diam} g_j(\overline{P}_j) < \frac{1}{n}$ if $f(a_j) \neq f(b_j)$, and $g_j(\overline{P}_j) = f(a_j)$ if $f(a_j) = f(b_j)$.

Now define $g\colon I \to X$ so that $g|\overline{P}_j = g_j$ and $g|K = f$. Note that if $x \in \overline{P}_j \cap K$, then $g_j(x) = f(x)$, so that g is well-defined.

Since $g|K = f$, and f is surjective, it follows that g is surjective. It remains to show that g is continuous.

To see that g is continuous let $p \in I$. We show that g is continuous at p.

If $p \in P_j$ for some j, then $g|P_j = g_j$ and P_j is open, it is clear that g is continuous at p.

Suppose that $p \in K$ and consider two cases:

Case 1. $p \notin \overline{P}_j$ for all $j \in \mathbb{N}$.

Let $\epsilon > 0$ and consider the open set $N_\epsilon(g(p))$ in X. Let $n \in \mathbb{N}$ such that $\epsilon_n < \frac{\epsilon}{2}$ and $\frac{1}{n} < \frac{\epsilon}{2}$. Let U be an open interval containing p such that $\operatorname{diam} U < \frac{\delta_n}{2}$ and such that if $\operatorname{diam} \overline{P}_j \geq \delta_n$, then $P_j \cap U = \emptyset$. We claim that $g(U) \subseteq N_\epsilon(g(p))$. Let $t \in U$. If $t \in K$, then $|t-p| < \delta_n$, so that $d(f(t),f(p)) < \epsilon_n < \epsilon$ and thus, since in this case $f(t) = g(t)$ and $f(p) = g(p)$, $d(g(t),g(p)) < \epsilon$ and $g(t) \in N_\epsilon(g(p))$. On the other hand, if $t \in I\backslash K$, then $t \in P_i$ for some $i \in \mathbb{N}$, and $\operatorname{diam} g_i(\overline{P}_i) < \frac{1}{n}$. Now either a_i or b_i is in U. We can assume that $b_i \in U$. Now $|p - b_i| < \delta_n$, so that $d(f(p),f(b_i)) < \epsilon_n$, i.e., $d(g(p),g(b_i)) < \epsilon_n < \frac{\epsilon}{2}$ and $d(g(b_i),g(t)) \leq \frac{1}{n} < \frac{\epsilon}{2}$, and hence $d(g(p),g(t)) < \epsilon$, so that $g(t) \in N_\epsilon(g(p))$. Thus $g(U) \subseteq N_\epsilon(g(p))$.

Case 2. $p \in \overline{P}_j$ for some $j \in \mathbb{N}$. Then either $p = a_j$ or $p = b_j$. We can assume that $p = b_j$. Let $V = (s, b_j]$ be open in $\overline{P}_j$ so that $g(V_j) \subseteq N_\epsilon(g(p))$ and let $n \in \mathbb{N}$ such that $\epsilon_n < \frac{\epsilon}{2}$ and $\frac{1}{n} < \frac{\epsilon}{2}$, and let U be an open interval containing p such that diam $U < \frac{\delta_n}{2}$ and such that diam $\overline{P}_j \geq \delta_n$ implies $P_i \cap U = \emptyset$ if $i \neq j$. Let $W = (U \cap V) \cup ([b_j, 1] \cap U)$ and proceed as in Case 1. ∎

16 LIMIT SPACES

A **projective system** $(X_\alpha, f_\alpha^\beta, D)$ is a directed set D with a collection $\{X_\alpha : \alpha \in D\}$ of Hausdorff spaces, and continuous functions $f_\alpha^\beta : X_\beta \to X_\alpha$ such that if $\alpha \leq \beta \leq \gamma$ in D, then $f_\alpha^\gamma = f_\alpha^\beta \circ f_\beta^\gamma$, and $f_\alpha^\alpha = 1_{X_\alpha}$ for each $\alpha \in D$. Each X_α is called a **factor space** and each f_α^β is called a **bonding map**.

Let $(X_\alpha, f_\alpha^\beta, D)$ be a projective sysem of spaces and let $P = \prod_{\alpha \in D} X_\alpha$. For $\alpha \leq \beta$ in D, let $S_\alpha^\beta = \{x \in P : f_\alpha^\beta \circ \pi_\beta(x) = \pi_\alpha(x)\}$, where $\pi_\beta : P \to X_\beta$ is projection. Let $S_\beta = \bigcap\{S_\alpha^\beta : \alpha \leq \beta\}$ for each $\beta \in D$, and let $X = \bigcap\{S_\beta : \beta \in D\}$. The space X with the relative topology of P is called the **projective limit** of the system $(X_\alpha, f_\alpha^\beta, D)$ and is denoted $\varprojlim X_\alpha$.

16.1 **Theorem.** *If $(X_\alpha, f_\alpha^\beta, D)$ is a projective system of spaces, then $\varprojlim X_\alpha$ is a closed subspace of $\prod_{\alpha \in D} X_\alpha$.*

16.2 **Theorem.** *The projective limit of nonempty compact spaces is nonempty and compact.*

16.3 **Theorem.** *The projective limit of compact connected spaces is compact and connected.*

16.4 **Theorem.** *Let $\{(X_\alpha, \tau_\alpha) : \alpha \in A\}$ be a family of disjoint spaces, let $X = \bigcup_{\alpha \in A} X_\alpha$, and let $\tau = \{U : U \subseteq X$ and $U \cap X_\alpha \in \tau_\alpha$ for each $\alpha \in A\}$. Then (X, τ) is a space.*

The space (X, τ) in 16.4 is called the **topological sum** of the spaces $\{(X_\alpha, \tau_\alpha) : \alpha \in A\}$ and is denoted $X = \sum_{\alpha \in A} X_\alpha$.

16.5 **Theorem.** *Let $\{X_\alpha : \alpha \in A\}$ be a family of disjoint spaces and let $X = \sum_{\alpha \in A} X_\alpha$. If $f : X \to Y$ is a function from X into a space Y, the f is continuous if and only if $f|X_\alpha$ is continuous for each $\alpha \in A$. Moreover, each inclusion $j_\alpha : X_\alpha \to X$ is an open and closed embedding of X_α into X.*

16.6 **Theorem**. *The sum of a family of paracompact Hausdorff spaces is a paracompact Hausdorff space.*

16.7 **Theorem**. *Let X be a locally compact Hausdorff space. Then X is paracompact if and only if X is the sum of a family of locally compact σ-compact Hausdorff spaces.*

Proof. Suppose that X is the sum of a family $\mathcal{A}$ of locally compact σ-compact Hausdorff spaces. Then each member of $\mathcal{A}$ is paracompact (4.30), and hence X is paracompact (16.6).

Suppose that X is paracompact. Since X is locally compact, there exists and open cover $\{U_\alpha : \alpha \in A\}$ of X such that $\overline{U}_\alpha$ is compact for each $\alpha \in A$. Let $\{V_\alpha : \alpha \in A\}$ be a locally finite open refinement such that $V_\alpha \subseteq U_\alpha$ for each $\alpha \in A$. Now for each $\alpha \in A$, $\overline{V}_\alpha$ is compact, so that $F_\alpha = \{\beta \in A : V_\alpha \cap V_\beta \neq \emptyset\}$ is finite (or $\emptyset$ in the case that $V_\alpha = \emptyset$). Let $R = \{(x, y) \in X \times X :$ thereexists $V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_n}$ with $x \in V_{\alpha_1}$, $y \in V_{\alpha_n}$, and $V_{\alpha_j} \cap V_{\alpha_{j+1}} \neq \emptyset$ for $j = 1, 2, \ldots, n-1\}$. Then R is an equivalence relation on X. Let $\{X_\gamma : \gamma \in \Gamma\}$ be the family of R-classes in X. Then each X_γ is open (and hence locally compact) and $X = \sum_{\gamma \in \Gamma} X_\gamma$. It remains to show that each X_γ is σ-compact. Note that X_γ is also closed, since $X \backslash X_\gamma$ is open. Now choose $V_{\gamma_0} \subseteq X_\gamma$, let $Q_1 = \bigcup\{V_\alpha : \alpha \in F_{\alpha_0}\}$ and for $n \geq 2$, let $Q_n = \bigcup\{V_\alpha : V_\alpha \cap Q_{n-1} \neq \emptyset\}$. Note that Q_n is a finite union, so that $\overline{Q}_n$ is compact, $X_\gamma = \bigcup_{n \in \mathbb{N}} Q_n$ and hence X_γ is σ-compact. ∎

16.8 **Theorem**. *Let X and Y be disjoint spaces, A a closed subspace of X, $f : A \to Y$ a continuous function, $S = X + Y$, and let $\rho = \{(a, b) \in S \times S$: either $a = b$, $f(a) = b$, $f(b) = a$, or $f(a) = f(b)\}$. Then ρ is an equivalence relation on S.*

The quotient space S/ρ in 16.8 is called the **adjunction space** obtained by adjoining X to Y with f. It is denoted $X \bigcup_{(f,A)} Y$.

16.9 **Theorem**. *Let X and Y be disjoint spaces, A a closed subspace of X, $f : A \to Y$ a continuous function, $S = X \bigcup_{(f,A)} Y$, and let $\pi : X + Y \to S$ be the natural quotient map. Then:*

(1) $\pi|Y : Y \to S$ is a closed embedding;

(2) If X and Y are compact, then S is compact; and

(3) If X and Y are connected, then S is connected.

If X is a space and $\{A_\alpha : \alpha \in D\}$ is a net of subsets of X, then:

$$\overline{\lim} A_\alpha = \{x \in X : \text{if } U \text{ is an open set containing } x, \text{then } U \cap A_\alpha \neq^f \emptyset\}$$

and

$$\underline{\lim}\, A_\alpha = \{x \in X : \text{if } U \text{ is an open set containing } x, \text{then } U \cap A_\alpha \neq^e \emptyset\}$$

If $\underline{\lim}\, A_\alpha = \overline{\lim}\, A_\alpha$, then we say that the limit of the net A_α exists and write $\lim A_\alpha = \underline{\lim}\, A_\alpha = \overline{\lim}\, A_\alpha$.

If X is a space, then $\underline{\lim}\, A_\alpha \subseteq \overline{\lim}\, A_\alpha$. If X is a compact Hausdorff space and A_α is a net of closed nonempty subsets of X, then $\overline{\lim}\, A_\alpha \neq \emptyset$.

Let X be a compact Hausdorff space and let X^* denote the family of all closed subsets of X. For U and V open in X, let $K(U,V) = \{T \in X^* : T \subseteq U \text{ and } T \cap V \neq \emptyset\}$. Then $\{K(U,V) : U \text{ and } V \text{ open in } X\}$ is a subbasis for a topology on X^* called the **Vietoris topology**.

16.10 **Lemma**. *Let X be a compact Hausdorff space and let A_α be a net in X^*. Then $A = \lim A_\alpha$ if and only if $A_\alpha \overset{e}{\to} A$ in the Vietoris topology on X^*.*

16.11 **Theorem**. *Let X be a compact Hausdorff space. Then X^* with the Vietoris topology is a compact Hausdorff space.*

17 FUNCTION SPACES

If X and Y are spaces, then Y^X denotes the set of all functions from X into Y. Let f_α be a net in Y^X and let $f \in Y^X$.

Starting with some concept of convergence of nets in Y^X one can induce a topology on Y^X by declaring that a subset C in Y^X is closed if whenever f_α is a net in C and $f_\alpha \overset{e}{\to} f$, then $f \in C$.

The net f_α is said to **converge pointwise** to f provided $f_\alpha(x) \overset{e}{\to} f(x)$ in Y for each $x \in X$. This is denoted $f_\alpha \overset{pc}{\to} f$. The topology on Y^X induced by pointwise convergence is denoted τ_{pc} and is called the **topology of pointwise convergence**.

The net f_α is said to **converge continuously** to f provided that for each net $x_\beta \overset{e}{\to} x$ in X, $f_\alpha(x_\beta) \overset{e}{\to} f(x)$ in Y. This is denoted $f_\alpha \overset{cc}{\to} f$. The topology on Y^X induced by continuous convergence is denoted τ_{cc} and is call the **topology of continuous convergence**.

It is clear that if $f_\alpha \overset{cc}{\to} f$, then $f_\alpha \overset{pc}{\to} f$. A simple argument can be used to establish that $\tau_{pc} \subseteq \tau_{cc}$.

Let X and Y be spaces. For $A \subseteq X$ and $B \subseteq Y$, define $N(A,B) = \{f \in Y^X : f(A) \subseteq B\}$.

The topology on Y^X for which $\{N(x,U) : x \in X \text{ and } U \text{ is open in } Y\}$ is a subbasis is denoted τ_{po} and is called the **point-open** topology.

The topology on Y^X for which $\{N(K,U)\colon K$ is a compact subset of X and U is open in $Y\}$ is a subbasis is denoted τ_{co} and is called the **compact-open** topology.

It is a simple exercise to show that $\tau_{po} \subseteq \tau_{co}$.

17.1 **Theorem.** *Let X and Y be spaces. Then τ_{po} is the product topology on Y^X.*

17.2 **Lemma.** *Let X and Y be spaces and let f_α be a net in Y^X. Then $f_\alpha \overset{pc}{\to} f$ if and only if $f_\alpha \overset{e}{\to} f$ in the topology of pointwise convergence on Y^X.*

17.3 **Theorem.** *Let X and Y be spaces. Then $\tau_{po} = \tau_{pc}$, i.e., the topology of pointwise convergenece, the point-open topology and the product topology on Y^X are all the same.*

If X and Y are spaces and $x \in X$, then the function $e_x\colon Y^X \to Y$ defined by $e_x(f) = f(x)$ for $f \in Y^X$ is called the **evaluation map** determined by x.

17.4 **Theorem.** *Let X and Y be spaces and let τ be a topology on Y^X. Then e_x is τ-continuous for every $x \in X$ if and only if $\tau_{po} \subseteq \tau$.*

Observe that $e_x\colon Y^X \to Y$ is τ_{co}-continuous for each $x \in X$.

If X and Y are spaces, then $C(X,Y)$ denoted the set of all continuous functions from X into Y. We will consider this in the relative topology on Y^X for the compact-open and point-open topologies.

17.5 **Theorem.** *Let X and Y be locally compact Hausdorff spaces. Then $\tau_{co}|C(X,Y) = \tau_{cc}|C(X,Y)$, i.e., the topology of continuous convergence and the compact-open topoolgy on Y^X are the same on $C(X,Y)$.*

If X is a space and (Y,d) is a metric space, then a net f_α in Y^X is said to **converge uniformly** to $f \in Y^X$ provided that for each $\epsilon > 0$, there exists β such that $\beta \leq \alpha$ implies $d(f_\alpha(x), f(x)) < \epsilon$ for each $x \in X$.

17.6 **Theorem.** *Let X be a space, Y a metric space and f_α a net of continuous functions in Y^X which converge uniformly to $f \in Y^X$. Then f is continuous.*

Proof. Let $p \in X$ and let W be an open set containing $f(p)$. Then there exists $\epsilon > 0$ such that $N_\epsilon(f(p)) \subseteq W$. Now there exists β such that $f_\beta(x) \in N_{\frac{\epsilon}{3}}(f(x))$ for all $x \in X$, i.e., $d(f_\beta(x), f(x)) < \frac{\epsilon}{3}$ for all $x \in X$. Thus for each $x \in X$ we have $d(f(x), f(p)) \leq d(f(x), f_\beta(x)) + d(f_\beta(x), f_\beta(p)) + d(f_\beta(p), f(p)) < d(f_\beta(x), f_\beta(p)) + \frac{2}{3}\epsilon$. Since f_β is continuous, there exists an open set U in X such that $f_\beta(U) \subseteq N_{\frac{\epsilon}{3}}(f_\beta(p))$ with $p \in U$. Thus $d(f_\beta(x), f_\beta(p)) < \frac{\epsilon}{3}$ for each $x \in U$. It follows that $d(f(x), f(p)) < \epsilon$ for each $x \in U$, so that $f(U) \subseteq N_\epsilon(f(p)) \subseteq W$, and f is continuous. ■

17.7 **Lemma.** *If (X,d) is a metric space, A is a compact subset of X, and B is a closed subset of X such that $A \cap B = \emptyset$, then $d(A,B) > 0$.*

17.8 **Theorem.** *Let X be a space, Y a metric space, f_α a net in $C(X,Y)$,* *and $f \in C(X,Y)$. Then $f_\alpha \xrightarrow{e} f$ in the τ_{co} topology on $C(X,Y)$ if and only if $f_\alpha|E$ converges uniformly to $f|E$ for each compact subset E of X.*

Proof. Suppose that $f_\alpha|E$ converges uniformly to $f|E$ for each compact subset E of X. We want to show that $f_\alpha \xrightarrow{e} f$ in the τ_{co} topology. Let $N(E,U)$ be a subbasic τ_{co}-open set containing f, with E compact and U open. Then $f(E) \subseteq U$, so that $f(E) \cap (Y \backslash U) = \emptyset$. Since E is compact and f is continuous, $f(E)$ is compact, and hence $d(f(E), Y\backslash U) = \epsilon > 0$. Now there exists β such that $\beta \leq \alpha$ implies that $d(f_\alpha(x), f(x)) < \epsilon$ for every $x \in E$. Thus $f_\alpha(x) \notin Y\backslash U$ when $\beta \leq \alpha$ for every $x \in E$. It follows that $f_\alpha(x) \in U$ for every $x \in E$, when $\beta \leq \alpha$, so that $f_\alpha(E) \subseteq U$ when $\beta \leq \alpha$, i.e., $f_\alpha \in^e N(E,U)$. We conclude that $f_\alpha \xrightarrow{e} f$ in τ_{co}.

Suppose that $f_\alpha \xrightarrow{e} f$ in τ_{co} in $C(X,Y)$. Let E be a compact subset of X and let $\epsilon > 0$. Then $f(E)$ is compact. Let $\{p_1, p_2, \ldots, p_k\}$ be a finite $\frac{\epsilon}{3}$-net for $f(E)$. Define $S_j = N_{\frac{\epsilon}{3}}(p_j)$ and $G_j = N_{\frac{2\epsilon}{3}}(p_j)$ for $1 \leq j \leq k$. Then $\overline{S}_j \subseteq G_j$ for $1 \leq j \leq k$. Now $f(E) \subseteq \bigcup_{j=1}^{k} \overline{S}_j$, so that $E \subseteq \bigcup_{j=1}^{k} f^{-1}(\overline{S}_j)$. Let $E_j = E \cap f^{-1}(\overline{S}_j)$ for $1 \leq j \leq k$. Then E_j is compact, $E = \bigcup_{j=1}^{k} E_j$, and $f(E_j) \subseteq \overline{S}_j \subseteq G_j$. Now $f(E_j) \subseteq G_j$, so that $f \in N(E_j, G_j)$ for $1 \leq j \leq k$. Let $K = \bigcap_{j=1}^{k} N(E_j, G_j)$. Then K is open in τ_{co}-topology and $f \in K$. Thus there exists β such that $f_\alpha \in K$ when $\beta \leq \alpha$, and hence $f_\alpha(E_j) \subseteq G_j$ for $1 \leq j \leq k$.

Now let $x \in E$ and let $\beta \leq \alpha$. Then $x \in E_m$ for some $1 \leq m \leq k$, so that $f_\alpha(x) \in f_\alpha(E_m) \subseteq G_m$ and $d(f_\alpha(x), p_m) < \frac{2\epsilon}{3}$; Also $f(x) \in f(E_m) \subseteq \overline{S}_m$, so that $d(f(x), p_m) \leq \frac{\epsilon}{3}$. We obtain that $d(f_\alpha(x), f(x)) < \epsilon$. ∎

In view of 17.8, the compact-open topology on $C(X,Y)$ is sometimes referred to as the topology of compact convergence.

If X is a space and Y is a metric space, then a function $f\colon X \to Y$ is said to be **bounded** provided $f(X)$ is a bounded subset of Y.

17.9 **Theorem**. *If $f\colon X \to Y$ is a continuous function from a compact space X into a metric space Y, then f is bounded.*

17.10 **Exercise**. Let X be a compact space and $f\colon X \to \mathbb{R}$ a continuous function. Then there exists $p, q \in X$ such that $f(p) = \sup f(X)$ and $f(q) = \inf f(X)$.

If (X,d) and (Y,e) are metric spaces, and $f\colon X \to Y$ is a function, then f is said to be **uniformly continuous** if for each $\epsilon > 0$, there exists $\delta > 0$ such that $e(f(a), f(b)) < \epsilon$ whenever $a, b \in X$ and $d(a,b) < \delta$.

An equivalent formulation for uniform continuity of $f\colon X \to Y$ is that for each $\epsilon > 0$, there exists $\delta > 0$ such that $f(N_\delta(x)) \subseteq N_\epsilon(f(x))$ for each $x \in X$.

17.11 **Theorem.** *Let X be a compact metric space and Y a metric space. Then each continuous function $f: X \to Y$ is uniformly continuous.*

Proof. Let $\epsilon > 0$. Since X is compact and f is continuous, $f(X)$ is compact. Let $\{y_1, y_2, \ldots, y_n\}$ be a finite $\frac{\epsilon}{2}$-net for $f(X)$. Then $f(X) \subseteq \bigcup_{j=1}^{n} N_{\frac{\epsilon}{2}}(y_j)$ and $\{f^{-1}(N_{\frac{\epsilon}{2}}(y_j)) \colon 1 \le j \le n\}$ is an open cover of X. For $p \in X$, let $\delta_p > 0$ such that $N_{2\delta_p}(p) \subseteq f^{-1}(N_{\frac{\epsilon}{2}}(y_j))$ for some j. Now $\{N_{\delta_p}(p) \colon p \in X\}$ is an open cover of X. Let $N_{\delta_{p_1}}(p_1), \ldots, N_{\delta_{p_k}}(p_k)$ be a finite subcover, and let $\delta = \min\{\delta_{p_1}, \ldots, \delta_{p_k}\}$.

Suppose that $a, b \in X$ and $d(a,b) < \delta$. Now $a \in N_{\delta_{p_i}}(p_i)$ for some i, so that $d(a, p_i) < \delta_{p_i}$ and $d(a,b) < \delta_{p_i}$. We obtain that $d(a,b) < 2\delta_{p_i}$ and $a, b \in N_{2\delta_{p_i}}(p_i)$. Thus $f(a), f(b) \in N_{\frac{\epsilon}{2}}(y_j)$ for some j, and hence $d(f(a), f(b)) < \epsilon$. ■

17.12 **Theorem.** *Let X be a compact space, (Y,d) a metric space. For $f, g \in C(X,Y)$, define $\rho(f,g) = \sup\{d(f(x), g(x)) \colon x \in X\}$. Then ρ is a metric on $C(X,Y)$.*

The metric ρ on $C(X,Y)$ in 17.12 is called the **sup metric** and the topology on $C(X,Y)$ induced by this metric is denoted τ_{sm} and is called **the sup metric topology**.

17.13 **Theorem.** *Let X be a compact space, Y a metric space, f_α a net in $C(X,Y)$, and $f \in C(X,Y)$. Then f_α converges uniformly to f if and only if $f_\alpha \xrightarrow{c} f$ in the sup metric topology τ_{sm} on $C(X,Y)$.*

17.14 **Theorem.** *Let X be a compact space and Y a metric space. Then $\tau_{co} = \tau_{sm}$ on $C(X,Y)$.*

Proof. We first show that $\tau_{co} \subseteq \tau_{sm}$. Let $N(K,U)$ be a subbasic τ_{co}-open set, with K a compact subset of X and U an open subset of Y. Let $f \in N(K,U)$. Then $f(K) \subseteq U$. For each $p \in f(K)$, there exists $\epsilon_p > 0$ such that $N_{2\epsilon_p}(p) \subseteq U$. Let $N_{\epsilon_{p_1}}(p_1), \ldots, N_{\epsilon_{p_n}}(p_n)$ cover $f(K)$, and let $\epsilon = \min\{\epsilon_{p_1}, \ldots, \epsilon_{p_n}\}$. We claim that $N_\epsilon(f) \subseteq N(K,U)$. Let $g \in N_\epsilon(f)$. Then $d(f,g) < \epsilon$. We want to show that $g \in N(K,U)$, i.e., $g(K) \subseteq U$. Let $x \in K$. Then $f(x) \in N_{\epsilon_{p_j}}(p_j)$ for some j, so that $d(f(x), p_j) < \epsilon_{p_j}$. Now $d(f,g) < \epsilon$ implies that $d(f(x), g(x)) < \epsilon \le \epsilon_{p_j}$ and hence $d(g(x), p_j) < 2\epsilon_{p_j}$, so that $g(x) \in N_{2\epsilon_{p_j}}(p_j) \subseteq U$. Thus $\tau_{co} \subseteq \tau_{sm}$.

We need to show that $\tau_{sm} \subseteq \tau_{co}$. Let $N_\epsilon(f)$ for $\epsilon > 0$ be a basic open set in τ_{sm}. Let $\{p_1, p_2, \ldots, p_n\}$ be an $\frac{\epsilon}{2}$-net for $f(X)$. We claim that $f \in \bigcap_{j=1}^{n} N(f^{-1}(\overline{N_{\frac{\epsilon}{8}}(p_j)}), N_{\frac{\epsilon}{4}}(p_j)) \subseteq N_\epsilon(f)$. Now, $f \in \bigcap_{j=1}^{n} N(f^{-1}(\overline{N_{\frac{\epsilon}{8}}(p_j)}), N_{\frac{\epsilon}{4}}(p_j))$, since $f[f^{-1}(\overline{N_{\frac{\epsilon}{8}}(p_j)})] = \overline{N_{\frac{\epsilon}{8}}(p_j)} \subseteq N_{\frac{\epsilon}{4}}(p_j)$ for $1 \le j \le n$.

Let $g \in \bigcap_{j=1}^{n} N(f^{-1}(\overline{N_{\frac{\epsilon}{8}}(p_j)}), N_{\frac{\epsilon}{4}}(p_j))$, and let $x \in X$. Since $f(X) \subseteq \bigcup_{j=1}^{n} N_{\frac{\epsilon}{8}}(p_j)$, we have that $x \in f^{-1}(\overline{N_{\frac{\epsilon}{8}}(p_m)})$ for some $m = 1, 2, \ldots, n$, so that $g(x) \in N_{\frac{\epsilon}{4}}(p_m)$ and $f(x) \in \overline{N_{\frac{\epsilon}{8}}(p_m)} \subseteq N_{\frac{\epsilon}{4}}(p_m)$ and we obtain $d(g(x), p_m) < \frac{\epsilon}{4}$, $d(f(x), p_m) < \frac{\epsilon}{4}$, and hence $d(g(x), f(x)) < \frac{\epsilon}{2}$. Since $d(g(x), f(x)) < \frac{\epsilon}{2}$ for each $x \in X$, $d(g, f) \leq \frac{\epsilon}{2} < \epsilon$.

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18 DECOMPOSITION SPACES

A **decomposition** of a topological space X is a collection $\mathcal{D}$ of pairwise disjoint subsets of X whose union is X.

A decomposition $\mathcal{D}$ of a space X is said to be **upper semi-continuous** at $D \in \mathcal{D}$ if for each open set U in X containing D, there exists an open set V in X such that $D \subseteq V \subseteq V$, and if $D' \in \mathcal{D}$ such that $D' \cap V \neq \emptyset$, then $D' \subseteq U$. If $\mathcal{D}$ is upper semi-continuous at each of its members, then $\mathcal{D}$ is called an **upper semi-continuous decomposition** of X.

Observe that a decomposition $\mathcal{D}$ of a space X determines an equivalence relation on X by declaring that each element of the decomposition is an equivalence class. We let $X/\mathcal{D}$ denote the quotient space and $\phi\colon X \to X/\mathcal{D}$ the natural map.

If $\mathcal{D}$ is a decomposition of a space X and V is a subset of X, denote $\mathcal{D}_0(V) = \bigcup\{D \in \mathcal{D}\colon D \subseteq V\}$. If $A \subseteq X$, then $Sat\,(A) = \phi^{-1}\phi(A) = \bigcup\{D \in \mathcal{D}\colon D \cap A \neq \emptyset\} = X \backslash \mathcal{D}_0(X \backslash A)$.

If $\mathcal{D}$ is a decompostion of a space X, then the **graph** of $\mathcal{D}$ is defined $\text{Graph}(\mathcal{D}) = \{(x, y) \in X \times X\colon \phi(x) = \phi(y)\}$.

18.1 **Lemma.** *Let $\mathcal{D}$ be a decompositon of a space X. These are equivalent:*

(1) $\mathcal{D}$ is upper semi-continuous;

(2) For each open set W in X, $\mathcal{D}_0(W)$ is open; and

(3) $\phi\colon X \to X/\mathcal{D}$ is a closed map.

If further, X is a compact Hausdorff space and each $D \in \mathcal{D}$ is closed, then these are equivalent to

(4) $\text{Graph}(\mathcal{D})$ is closed in $X \times X$; and

(5) $X/\mathcal{D}$ is Hausdorff.

A decomposition $\mathcal{D}$ of a space X is said to be **lower semi-continuous** at $D \in \mathcal{D}$ if for each $p, q \in D$ and each open set V such that $p \in V$, there exists an open set W with $q \in W$ such that if $D' \in \mathcal{D}$ and $D' \cap W \neq \emptyset$, then $D' \cap V \neq \emptyset$. We say that $\mathcal{D}$ is **lower semi-continuous** if it is lower semi-continuous at each of its members.

18.2 **Theorem.** *Let $\mathcal{D}$ be a decomposition of a space X. Then $\mathcal{D}$ is lower semi-continuous if and only if the natural map $\phi: X \to X/\mathcal{D}$ is open.*

A decomposition $\mathcal{D}$ of a space X is **continuous** at $D \in \mathcal{D}$ provided $\mathcal{D}$ is both upper and lower semi-continuous at D.

Let X be a space, Γ a directed set, and let $\{A_\alpha: \alpha \in \Gamma\}$ a collection of subsets of X. Define

$$\limsup A_\alpha = \{x \in X: x \in V \text{ open } \Rightarrow V \cap A_\alpha \neq^f \emptyset\}$$

and

$$\liminf A_\alpha = \{x \in X: x \in V \text{ open } \Rightarrow V \cap A_\alpha \neq^e \emptyset\}$$

Note that $\liminf A_\alpha \subseteq \limsup A_\alpha$ and that both are closed in X.

If $\mathcal{D}$ is a decomposition of a space X, then for each $x \in X$, let D_x denote the member of $\mathcal{D}$ containing x.

18.3 **Theorem.** *Let $\mathcal{D}$ be a decompositon of a space X into closed subsets:*

(1) If X is a compact Hausdorff space such that for each net $x_\alpha \overset{e}{\to} x$ in X, $\limsup D_{x_\alpha} \subseteq D_x$, then $\mathcal{D}$ is a upper semi-continuous decomposition.

(2) If X is a T_3-space and $\mathcal{D}$ is upper semi-continuous, then for each net $x_\alpha \overset{e}{\to} x$ in X, $\limsup D_{x_\alpha} \subseteq D_x$.

18.4 **Theorem.** *Let $\mathcal{D}$ be a decomposition of a space X. Then $\mathcal{D}$ is lower semi-continuous if and only if for each net $x_\alpha \overset{e}{\to} x$, $D_x \subseteq \liminf D_{x_\alpha}$.*

18.5 **Corollary.** *Let $\mathcal{D}$ be an upper semi-continuous decomposition of closed subsets of a T_3-space X. Then $\mathcal{D}$ is continuous if and only if for each net $x_\alpha \overset{e}{\to} x$ in X, $\limsup D_{x_\alpha} = D_x = \liminf D_{x_\alpha}$.*

Let X be a space and let $C(X)$ denote the set of all closed nonempty subsets of X. For open subsets U and V of X let

$$N(U, V) = \{A \in C(X): A \subseteq U, A \cap V \neq \emptyset\}.$$

Then $\{N(U, V): U, V \text{(open)} \subseteq X\}$ is a subbase for a topology on $C(X)$ called the **Vietoris topology**.

We will hereafter (for the remainder of section 18) assume that $C(X)$ is endowed with the Vietoris topology.

If A_α is a net of subsets of a space X, then we write $A = \lim A_\alpha$ to denote that $A = \liminf A_\alpha = \limsup A_\alpha$.

18.6 **Theorem.** *Let A_α be a net of closed nonempty subsets of a space X. Then $A_\alpha \overset{e}{\to} A$ in the Vietoris topolgy on $C(X)$ if and only if $A = \lim A_\alpha$.*

Recall that a continuum is a compact connected Hausdorff space.

18.7 **Theorem.** *Let X be a space*

(1) If X is T_3, then $C(X)$ is Hausdorff;

(2) If X is a compact Hausdorff space, then $C(X)$ is a compact Hausdorff space;

(3) If X is a locally compact Hausdorff space, then $C(X)$ is a locally compact Hausdorff space; and

(4) If X is a continuum, then $C(X)$ is a continuum.

If (X, d) is a metric space, $\epsilon > 0$, and $A \subseteq X$, then $N_\epsilon(A) = \{x \in X: d(x, A) < \epsilon\}$.

For subsets A and B of X, we define $d^*(A, B) = \inf\{\epsilon: A \subseteq N_\epsilon(B)$ and $B \subseteq N_\epsilon(A)\}$.

18.8 **Theorem.** *Let (X, d) be a compact metric space, then d^* is a metric on $C(X)$ such that the metric topology is the same as the Vietoris topology.*

The metric d^* in 18.8 is called the Hausdorff metric on $C(X)$.

19 FILTERS

If X is a set, then a family of nonempty subsets $\mathcal{F}$ of X is called a **filter** in X provided:

(1) If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$ and

(2) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.

A filter is closed under finite intersections and supersets.

If X is a space and $\mathcal{F}$ is a filter in X, then $\mathcal{F}$ is said to **converge** to $x \in X$ provided that each neighborhood of x is a member of $\mathcal{F}$.

19.1 **Theorem.** *If X is a space and U is a subset of X, then U is open if and only if U is a member of each filter which converges to a point of U.*

19.2 **Theorem.** *Let X be a space, $A \subseteq X$, and let $x \in X$. Then x is a limit point of A if and only if $A \setminus \{x\}$ is a member of some filter which converges to x.*

19.3 **Theorem.** *Let X be a space, $x \in X$, and let Φ_x be the collection of all filters which converge to x. Then $\bigcap\{\mathcal{F}: \mathcal{F} \in \Phi_x\}$ is a local basis at x.*

19.4 **Theorem.** *Let X be a space, $x \in X$, and $\mathcal{F}$ a filter in X which converges to x. If $\mathcal{G}$ is a filter in X which contains $\mathcal{F}$, then $\mathcal{G}$ converges to x.*

19.5 **Theorem.** *If X is a set and x_α is a net in X, then $\mathcal{F} = \{A: A \subseteq X$, and $x_\alpha \in^e A\}$ is a filter in X.*

19.6 **Theorem.** *Let X be a set, $\mathcal{F}$ a filter in X, and let $D = \{(x, F): F \in \mathcal{F}$ and $x \in F\}$. Define $(x, F) \leq (y, G)$ in D provided $G \subseteq F$. Let $f(x, F) = x$. Then $\mathcal{F}$ is precisely the family of all sets A such that the net $\{f(x, F): (x, F) \in D\}$ is eventually in A.*

19.7 **Theorem.** *Let X and Y be spaces, and $f: X \to Y$ a function. Then f is continuous at $a \in X$ if and only if for each filter $\mathcal{F}$ in X which converges* *to A, the filter $f(\mathcal{F})$ converges to $f(a)$.*

An **ultrafilter** is a maximal filter.

19.8 **Theorem.** *Let $\mathcal{F}$ be an ultrafilter in X.*

(1) If $A \cup B \in \mathcal{F}$, then either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

(2) If $A \subseteq X$, then either $A \in \mathcal{F}$ or $(X \backslash A) \in \mathcal{F}$.

19.9 **Theorem.** *A space X is compact if and only if each ultrafilter in X converges.*