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Topological Methods in Cooperative Games

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ABSTRACT. In the following paper, we present a brief survey of the manner in which topological methods (such as fixed point theory) are utilized to prove existence of various outcomes for certain classes of NTU cooperative games.

1 Introduction

Game theory has been defined [10] as "the study of mathematical models of conflict and cooperation between intelligent rational decision makers". In the theory of cooperative games it is assumed that the participants (players) may form groups (coalitions) who can achieve various outcomes, and that the players may reach binding agreements about how to share the spoils. Thus, in a free market, a seller and a buyer reach an agreement about the price and quantity of goods supplied; in a multi-party parliamentary system, one may regard each political party as a player. In the former case we are faced with a two players (or two-person) game, whereas in the second case we have n players (n equals to the number of parties) and the number of possible coalitions may be large.

A natural requirement from a theory is that the predicted outcome(s) are sufficiently stable so that once such an outcome has been reached, no individual and no coalition could gain more by going their own ways. A moment's reflection show that if according to the

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rules of a three-person game any coalition of two or more players may get the entire cake, then *no* agreement can be stable: if x_1, x_2, x_3 denote the fraction of cake promised to the individual players, then the inequalities $x_1+x_2 \geq 1,\ x_1+x_3 \geq 1, x_2+x_3 \geq 1$ (no two-person coalition may improve) are incompatible with the requirement $x_1 + x_2 + x_3 \leq 1$ (the cake is too small). The notion that no non-empty coalition can improve is formalized by the concept of "core" — see Section 2.

In most cases, the participants evaluate differently the same outcome — one person's satisfaction with a piece of cake of size a is in general a function of a, peculiar to that person. The power of a government is hardly a scalar, and while one political party may have security and foreign affairs high on its agenda, another may be mostly interested in socio-economic issues. Situations like these are best modeled by associating with each coalition S a set $V(S)$ of vectors whose components represent the payoffs available to members of S. In the 3-person case mentioned earlier we may associate with each coalition $S \subset \{1,2,3\}$, with S containing at least two players, the set $V(S) = \{x \in R^3 : \sum_{i \in S} x_i \leq 1\}$. (We have inequality rather than equality because players may throw away freely some of the cake.) The general case, where the sets $V(S)$ are not half-spaces, is called NTU (non transferable utility). (Sometimes such a game is called an n-person game without side payments.) Topological methods are required to study the NTU case — linear programming and related methods do not suffice.

Another natural property of a solution concept is that it admits no "asymmetric dependencies". A solution displays an asymmetric dependency if one player needs the presence of a second player to realize his payoff (in the solution), but the second player does not need the presence of the first. In such a case the first player's position is vulnerable. Consider a two-persons divide the cake game. Any division giving the entire cake to one player displays an asymmetric dependency; the player receiving the cake is dependent on the player receiving nothing, whereas the player receiving nothing does not have to join the two-person coalition in order to receive her part of the payoff. On the other hand it is impossible to achieve a division giving a positive amount to each player unless *both* of them join the coalition — there is no asymmetric dependence and the players are *partnered*. If a solution payoff is not partnered, there is an opportunity for one player to demand a larger share of the cake from the other player; a payoff that is not partnered exhibits a potential for instability.

A combinatorial concept of *partnership* will be made precise in Section 2, where the concept of partnered payoffs will be defined as well.

The main purpose of this survey is to present a result about existence of outcomes that cannot be improved upon and admit no asymmetric dependency (i.e., that the partnered core is non-empty). This result, due to Reny and Wooders [11], will be stated precisely and proved in Section 5, besides some more refined results. We will derive game-theoretic results from theorems about closed coverings of simplexes. Those theorems have a long history. The famous Knaster-Kuratowski-Mazurkiewicz (KKM) Theorem, which states that under certain conditions a family of subsets of the simplex has a non-empty intersection, has a number of applications in Game Theory (see [2]). The first proof of the non-emptiness of the core (under suitable assumptions; in general the core may be empty, as in the 3-person game exhibited above) of NTU games was given by Scarf in [13]. The topological nature of the proof is evident and was further elucidated in [14], see also [4] and [15]. In fact, a generalization (KKMS theorem) by Scarf [13] and Shapley [14] of the KKM Theorem is instrumental in proving non-emptiness of the core of NTU games. A further extension by Reny and Wooders [12] looms in the background of their study of partnered cores. Our method involves further extensions of the KKMS theorem (Kannai and Wooders [5]). The relations between the Brouwer fixed point theorem, Sperner's lemma and the KKM and KKMS theorems are well established [2], [4], and non-emptiness of the partnered core seems to be related to topological degree theory [5].

Game theoretic concepts are introduced formally in Section 2. Closed coverings and their relations to multi-valued functions (correspondences) are discussed in Section 3. We describe in Section 4 the simplest degree theory, namely the mod 2 degree theory. (I barely resisted the temptation to call this paper "A crash course in degree theory".) It turns out that this suffices to re-derive all the main results on generalized KKMS theorems and (refined) partnered cores. The topological theorems are proved and applied to game theory in Section 5. Here we follow mainly [5]. Further applications are mentioned briefly in Section 6.

2 Cooperative Games

Let $N = \{1, 2, \cdots, n\}$ be the set of all players. A subset of N is called a *coalition*. An outcome of the game (a *payoff vector*) is simply an n-dimensional vector $x = (x_1, \cdots, x_n)$; the intuitive meaning is that the i-th player "receives" x_i.

In the simplest case (*transferable* utility) the players may transfer "payments" among each other, and the game is given by means of a *characteristic function* (or a worth function), which is simply a real-valued function v defined on the coalitions, such that

$$v(\emptyset) = 0. \tag{1}$$

Usually one requires that a payoff vector satisfies (at least) the following conditions:

$$\sum_{i=1}^{n} x_i = v(N) \tag{2}$$

(*feasibility* and *Pareto-optimality*), and

$$x_i \geq v(\{i\}) \qquad i = 1, \cdots, n \tag{3}$$

(*individual rationality*). The condition (2) incorporates both the requirement that the members of the grand coalition N can actually achieve the outcome x ($\Sigma_{i=1}^n x_i \leq v(N)$ - feasibility) and cannot achieve more ($\Sigma_{i=1}^n x_i \geq v(N)$ - Pareto-optimality). The condition (3) means that no individual can achieve more than the amount allocated to him as a pay-off. Note that individual rationality and feasibility are not necessarily compatible; clearly

$$\sum_{i=1}^{n} v(\{i\}) \leq v(N) \tag{4}$$

is needed.

We are interested in the more general case, in which payoffs cannot be transfered freely between different players (even if they are members of the same coalition). Hence we have always to specify all components of a vector $x = (x_1, \cdots, x_n) \in R^N (= R^n)$. We denote by R^S the subspace of R^N defined by $x_i = 0$ for $i \not\in S$. A coalition S controls the projection x^S of x on R^S given by the restriction of x to the coordinates indexed by the elements of S.

Formally, it is convenient to define a non-transferable utility n-person game (NTU game) as a set-valued function V defined on the coalitions, such that:

$$V(\emptyset) = \emptyset \tag{5}$$

$$\text{for all } \ S \neq \emptyset, V(S) \ \text{ is a non-empty closed subset of } \ R^N \tag{6}$$

$$\text{if } \ x \in V(S) \ \text{ and } \ y_i \leq x_i \ \text{ for all } \ i \in S, \ \text{ then } \ y \in V(S). \tag{7}$$

The meaning of (7) is that $V(S)$ is a "cylinder", that is, the Cartesian product of a subset of R^S with $R^{N \setminus S}$ (this is done only for technical convenience), and that a coalition S can achieve, along with every vector, all vectors paying less to every member of S (this is a more substantive assumption). A transferable utility game v can be transformed into a non-transferable utility game V by setting

$$V(S) = \{x \in R^N : \sum_{i \in S} x_i \leq v(S)\} \tag{8}$$

for all non-empty coalitions S. This example suggests the following condition, which we will always assume:

There exists a closed set $F \subset R^N$ such that

$$V(N) = \{x \in R^N : \exists y \in F \ \text{ with } \ x_i \leq y_i \ \text{ for all } \ i \in N\} \tag{9}$$

In the NTU case, an outcome of the game (a payoff vector) is once again an n-dimensional vector $x = (x_1, \cdots, x_n)$. Thus, a payoff vector is feasible if

$$x \in V(N) \tag{10}$$

i.e., the members of the grand coalition N can actually achieve the outcome x. By (9), a payoff x is feasible if there exists $y \in F$ with $x_i \leq y_i$ for all $i \in N$. The payoff vector x is individually rational if

$$\text{for no } i \in N \text{ there exists } y \in V(\{i\}) \text{ such that } x_i < y_i. \tag{11}$$

i.e., no individual can achieve, acting alone, more than the amount allocated to him as a payoff. To simplify matters, we will assume that

$$V(\{i\}) = \{x \in R^N : x_i \leq 0\} \tag{12}$$

Hence feasible, individually rational payoff vectors exist only if F contains at least some vectors with non-negative components. We will assume that

$$F \cap \{x \in R^N : x_i \geq 0 \text{ for all } i \in N\} \text{ is a non-empty compact set}. \tag{13}$$

The members of the coalition S can improve their payoffs by their own efforts if there exists a vector $y \in V(S)$ with $x_i < y_i$ for all $i \in S$. By (7) the members of S can improve upon x iff $x \in int\ V(S)$. Hence the core of the game V, defined as the set of all feasible payoff vectors that cannot be improved upon by any coalition, (i.e., no individual and no group can improve) coincides with $V(N) \setminus \cup_{S \subset N} int\ V(S)$. (In the transferable utility case, the members of a coalition S can improve upon x if $\Sigma_{i \in S} x_i < v(S)$, and the core coincides with the set of all feasible payoff such that for all $S \subset N$,

$$\sum_{i \in S} x_i \geq v(S).) \tag{14}$$

It is clear that the set $V(N)$ has to be sufficiently large for the core to be non-empty. (In the transferable utility case $v(N)$ has to be a sufficiently large number.) Analysis of transferable utility games leads, via the theory of linear inequalities, to study certain collections of coalitions, called *balanced* collections (see [4] for a brief survey).

The collection $\mathcal{B}$ of subsets of N is called *balanced* if there exist nonnegative weights $\{\lambda^S\}_{S \in \mathcal{B}}$ such that for every $i \in N$,

$$\sum_{i \in S \in \mathcal{B}} \lambda^S = 1.$$

Every partition of N is a balanced collection, with weights equal to 1. Note that it is possible to write the balancedness condition as

$$\sum_{S \in \mathcal{B}} \lambda^S e^S = e^N \tag{15}$$

where e^S denote the vector in R^N whose i^{th} coordinate is 1 if $i \in S$ and 0 otherwise. (In other words, e^S denotes the indicator function of S.)

It is easy to see that if the core of a transferable utility game is non empty then

$$\sum_{S \in \mathcal{B}} \lambda^S v(S) \le v(N). \tag{16}$$

A game satisfying the condition (16) is called balanced. It follows from duality theory of linear inequalities that a TU game has a non-empty core if and only if the game is balanced (Shapley and Bondareva; see [4]). Motivated by the case of transferable utility, Scarf [13] defined a *balanced game* to be a game V in which the relation

$$\cap_{S \in \mathcal{B}} V(S) \subset V(N) \tag{17}$$

holds for every balanced collection $\mathcal{B}$ of subsets of N, and proved the following

Theorem 2.1 *Every balanced game has a nonempty core.*

We wish to consider a subset of the core consisting of payoff vectors which are also *partnered.* Let $\mathcal{P}$ be a collection of subsets of N. For each i in N let

$$\mathcal{P}_i = \{S \in \mathcal{P} : i \in S\}.$$

We say that $\mathcal{P}$ is *partnered* if for each i in N the set $\mathcal{P}_i$ is nonempty and for every i and j in N the following requirement is satisfied:

$$\text{if } \mathcal{P}_i \subseteq \mathcal{P}_j \text{ then } \mathcal{P}_j \subseteq \mathcal{P}_i;$$

i.e. if all subsets in $\mathcal{P}$ that contain i also contain j then all subsets containing j also contain i. Let $\mathcal{P}[i]$ denote the set of those $j \in N$ such that $\mathcal{P}_i = \mathcal{P}_j$. We say that $\mathcal{P}$ is *minimally partnered* if it is partnered and for each $i \in N$, $\mathcal{P}[i] = \{i\}$.

Let (N, V) be a game and let $x \in R^N$ be a payoff for (N, V). A coalition S is said to *support the payoff* x if $x \in V(S)$. Let $\mathcal{S}(x)$ denote the set of coalitions supporting the payoff x. The payoff x is called a *partnered payoff* if the collection $\mathcal{S}(x)$ has the partnership property. The payoff x is *minimally partnered* if it is partnered and if the set of supporting coalitions is minimally partnered. Note that partnered payoffs need not be feasible.

Let $P(N, V)$ denote the set of all partnered payoffs for the game (N, V). The *partnered core* is denoted by $C^*(N, V)$ and is defined by

$$C^*(N,V) = P(N,V) \bigcap C(N,V)$$

where $C(N, V)$ denotes the core of the game (N, V).

Reny and Wooders [11] proved the following strengthening of Scarf's Theorem, assuming in addition to the assumptions made above that there exists a constant M such that for each $S \subset N$ and $x \in V(S)$ the estimate

$$x_i \leq M \quad \text{for all} \quad i \in S \tag{18}$$

holds.

Theorem 2.2 *Every balanced game has a nonempty partnered core.*

Reny and Wooders proved also that under certain conditions there exist payoffs in $C^*(N, V)$ such that $S(x)$ is *minimally* partnered.

3 Closed Coverings and Correspondences

The relation between closed coverings of the simplex and fixed point theorems goes way back. To fix the notation, let Δ denote the unit simplex in R^N. For every $S \in \mathcal{N}$ define

$$\begin{aligned} \Delta^S &= conv\{e^i : i \in S\}, \text{ and} \\ m_S &= \frac{e^S}{|S|}, \end{aligned}$$

where "*conv*" denotes the convex hull and $|S|$ denotes the number of elements in the set S. The point m_S is called the *barycenter* of Δ^S.

The following result (the Knaster, Kuratowski and Mazurkiewicz Theorem) is a prototype of the theory.

Theorem 3.1 (KKM Theorem): *Let $C^i, 1 \leq i \leq n$ be a family of closed subsets of Δ such that for every $T \subset N$,*

$$\cup_{j \in T} C^j \supset \Delta^T. \tag{19}$$

Then $\cap_{i=1}^n C^i \neq \emptyset$.

(Note that the condition (19) implies in particular that the family $C^i, 1 \leq i \leq n$ forms a covering of Δ — choose $T = N$. A related result states that if $\Delta \subset \cup_{i=1}^n C^i$ and for every i, $\Delta^{N\setminus\{i\}} \subset C^i$, then $\cap_{i=1}^n C^i \neq \emptyset$.) The KKM Theorem is closely related to the Brouwer fixed point Thereom and to many existence results in Game Theory. The following generalization [4], [14] is essential for dealing with cores of NTU games.

Theorem 3.2 (KKMS Theorem): *Let $\{C^S\}$ be a family of closed subsets of Δ indexed by the non-empty coalitions such that for every $T \subset N$,*

$$\cup_{S \subset T} C^S \supset \Delta^T. \tag{20}$$

Then there exists a balanced collection B such that $\cap_{S \in B}^k C^S \neq \emptyset$.

In analogy to the terminology used for payoffs, we say that a coalition S *supports* the point $x \in \Delta$ if $x \in C^S$, and we denote by $S(x)$ the set of coalitions supporting x. Reny and Wooders [12] obtained the following extensions of the KKMS Theorem:

Theorem 3.3 a) *Let $\{C^S\}$ be a family of closed subsets of Δ indexed by the non-empty coalitions such that the condition (20) is satisfied. Then there exists $x^* \in \Delta$ such that the supporting collection for x^*, $S(x^*) \equiv \{S \in \mathcal{N} : x^* \in C^S\}$, is balanced and partnered.*

b) *If the set*

$$\{x^* \in \Delta : S(x^*) \text{ is balanced and partnered}\}$$

is at most countable, then at least one $x^* \in \Delta$ renders the supporting collection $S(x^*)$ balanced and minimally partnered.

The study of intersection properties of closed coverings motivates the study of certain correspondences (set-valued functions). The geometric idea is very simple: Let $\{C^S\}$ be a covering of Δ. We may label every $x \in \Delta$ by the collection of coalitions $L(x) = \{S : x \in C^S\}$ and associate with x the convex set $F(x) = conv\{m_S : S \in L(x)\}$. Due to certain technicalities concerning boundary behavior we will have to modify this construction a bit (see Proposition 1).

Let F be a correspondence from X to Y, i.e., for every $x \in X$, the "value" $F(x)$ is a subset of Y. If X and Y are topological spaces, we say that F is *upper-hemicontinuous* is the *graph* $G(F)$ of F (i.e., the set of pairs (x, y) such that $y \in F(x)$) is closed in the product topology of $X \times Y$. If X and Y are metric spaces then F is upper-hemicontinuous if and only if for every sequence $\{x_n, y_n\}$ such that $y_n \in F(x_n)$ and $x_n \to \bar{x}$, $y_n \to \bar{y}$, we have $\bar{y} \in F(\bar{x})$. In this paper we will consider only the case where X and Y are subsets of a Euclidean space R^M, and in most applications $X = Y = \Delta$.

The following is essentially well known (stronger results will be stated and proved in Section 5).

Theorem 3.4 *Let $F(x)$ be a correspondence from Δ into the closed convex subsets of Δ such that*

$$F \text{ is upper} - \text{hemicontinuous}; \tag{21}$$

and

$$\text{For all } T \subset N, \text{if } x \in \Delta^T \text{ then } F(x) \subset \Delta^T. \tag{22}$$

Then there exists $x \in \Delta$ such that $m_N \in F(x)$.

Example: The one-dimensional simplex $\Delta \subset R^2$ may be parametrized as $\{(1-x, x) : x \in [0,1]\}$. The barycenter $m_N = (0.5, 0.5)$ corresponds to $x = 0.5$. Set $F(x) = e^1$ if $x \in [0, 0.5]$, $F(0.5) = \Delta$, $F(x) = e^2$ if $x \in [0.5, 1]$. Then the assumptions of Theorem 3.4 are satisfied, and $m_N \in F(0.5)$. If, however, we modify F by setting $F(0.5) = \{0, 1\}$, then F is no longer convex valued (it still satisfies conditions (21) and (22)), but m_N is not contained in any set $F(x)$.

The condition (22) restricts the behavior of F on the boundary $\partial\Delta$ of Δ. (We denote here and in the sequel the boundary of a set A by ∂A.) The annoying technical details are taken care of by the following Proposition.

Proposition 1. Let $\{C^S : S \subseteq N\}$ be a family of closed subsets of Δ satisfying (20). Then there exist a simplex Δ' contained in the interior of Δ, a homeomorphism φ of Δ onto Δ' and a correspondence F from Δ into the closed convex subsets of Δ satisfying (21), and (22), and such that

$$\text{for all } y \in \Delta',\ F(y) = conv\{m_S : \varphi^{-1}(y) \in C^S\}, \tag{23}$$

and

$$\text{if } m_N \in F(x) \text{ then } x \in \Delta'. \tag{24}$$

Moreover,

$$F \text{ assumes a finite number of distinct values.} \tag{25}$$

Proof. Set $f^i = \frac{e^N + e^i}{n+1}$, $i = 1, \ldots, n$, $\Delta' = conv\{f^i : i \in N\}$. Then the simplex Δ' is contained in the interior of Δ. Every $x \in \Delta$ may be written is the form $x = \sum_{i=1}^n x_i e^i$ where $\sum_{i=1}^n x_i = 1$, $x_i \geq 0$ for all $i \in N$. Set $\varphi(x) = \sum_{i=1}^n x_i f^i$. Then φ is a homeomorphism of Δ onto Δ'. Note that $\varphi_i(x) = \frac{1+x_i}{n+1}$ for all $i \in N$. Let $\eta : \Delta \to \Delta$ be defined by

$$\eta_i(y) = \frac{\max\left((n+1)y_i - 1, 0\right)}{\sum_{j \in N} \max\left((n+1)y_j - 1, 0\right)} \text{ for all } i \in N.$$

Then the continuous function η coincides with φ^{-1} on Δ' and $\eta(\Delta\backslash\Delta') \subset \partial\Delta$. We define a labeling function

$$L(y) = \{S : \eta(y) \in C^S \text{ and } (n+1)y_i \geq 1 \text{ for all } i \in S\}, \tag{26}$$

and a correspondence

$$F(y) = conv\{m_S : S \in L(y)\}.$$

For every y set $T = \{i : (n+1)y_i \geq 1\}$. Then $\eta(y) \in \Delta^T$. By (20) there exists a coalition $S \subset T$ such that $\eta(y) \in C^S$, and $S \subset T$ implies that $L(y)$ is not empty. The upper-hemicontinuity of F follows from the continuity of η and the closedness of each C^S. F may assume at most 2^{2^n} distinct values. If $y \in \Delta^T$ then $\{i : (n+1)y_i \geq 1\} \subset T$ so that $S \subset T$ whenever $S \in L(y)$. Hence $m_S \in \Delta^T$ and thus $F(y) \subset \Delta^T$. If $y \notin \Delta'$ then $(n+1)y_i < 1$ for at least one index i. Hence $i \notin S$ if $S \subset L(y)$, and $m_N \notin F(y)$. ■

Observe that the collection $\mathcal{B}$ is balanced if and only if $m_N \in conv\{m_S : S \in \mathcal{B}\}$. The KKMS Theorem follows immediately from this observation, Theorem 3.4, and Proposition 1. In fact, let $m_N \in F(y)$ and consider the collection $\mathcal{B} = L(y)$. Then $\eta(y) \in \cap_{S \subset \mathcal{B}} C^S$ and $m_N \in conv\{m_S : S \in \mathcal{B}\}$.

Stronger results about coverings (such as Theorem 3.3) will follow from topological theorems on closed coverings that are refinements of Theorem 3.4. The conclusions (24) and (25) will be used as well.

4 Topological Methods

Let Ω be a bounded open subset of R^m (while the theory may be extended to unbounded sets, we need only the bounded case) and let f be a continuous function mapping $\bar{\Omega}$ (the closure of Ω) into R^m. We say that f is a *piecewise linear* mapping if there exists a finite family $\{P_k\}_{k \in K}$ of hyperplanes such that for every $x \in \Omega \setminus \cup_{k \in K} P_k$ there exists a neighborhood $\mathcal{U}$ of x, a matrix A and a point $b \in R^m$ with $f(y) = Ay + b$ for all $y \in \mathcal{U}$. The hyperplanes P_k decompose Ω to open connected subdomains $\Omega_1, \ldots, \Omega_j$. (Observe that A and b are constant on every Ω_i.) Let p be a point which is contained neither in the image of the boundary of Ω, nor of any m-1-dimensional set $\Omega \cap P_k$: $p \notin f(\partial\Omega) \cup_{k \in K} f(\Omega \cap P_k)$. (Note that $p \notin \partial\Omega_i$ for all $1 \leq i \leq j$, and if the matrix A representing f is singular in Ω_i, then $p \notin f(\Omega_i.)$ Denote by $n(p, f, \Omega)$ the number of sets $f(\Omega_i)$ containing p.

Proposition 2. Let $p, q \in R^m$, $p, q \notin f(\partial\Omega) \cup_{k \in K} f(\Omega \cap P_k)$, such that p and q may be joined by a polygonal line not intersecting $f(\partial\Omega)$. Then

$$n(p,f,\Omega) = n(q,f,\Omega) \bmod 2. \tag{27}$$

Proof. We may assume, without loss of generality, that the sets $f(\Omega_i)$ are open for all $1 \le i \le j$ (i.e., the matrices representing f on Ω_i are all regular). In fact, given p and q, there exists a piecewise linear function f' mapping $\bar{\Omega}$ to R^m such that (1) the matrices representing f on Ω_i are all regular ; (2) for every i, $p \in f'(\Omega_i)$ if and only if $p \in f(\Omega_i)$, and $q \in f'(\Omega_i)$ if and only $q \in f(\Omega_i)$. Then $n(p,f,\Omega) = n(p,f',\Omega)$ and $n(q,f,\Omega) = n(q,f',\Omega)$, so that it suffices to prove (27) for f'.

It follows that we may join p and q by a polygonal line l such that l intersects every m-1-dimensional set $f(P_k \cap \Omega)$ in a finite number of points and if $k \ne k'$ then $l \cap f(P_k \cap P_{k'} \cap \Omega) = \emptyset$. In particular, no point of l is contained in the image of more than one hyperplane P_k. The value of the integer $n(x,f,\Omega)$ may change, as x moves along l form p to q, only if x crosses an m-1-dimensional set $f(S)$ where $S \subset \partial\Omega_i$ for some i. Observe that by assumption $S \not\subset \partial\Omega$. Hence S is the common boundary of precisely two m dimensional domains Ω_i and $\Omega_{i'}$. If $f(\Omega_i) \cap f(\Omega_{i'}) = \emptyset$ then $n(x,f,\Omega)$ is unchanged when x crosses $f(S)$. If $f(\Omega_i) \cap f(\Omega_{i'}) \ne \emptyset$ then $n(x,f,\Omega)$ changes by 2. In either case, the *parity* of $n(x,f,\Omega)$ remains fixed. ∎

If $p \notin f(\partial\Omega)$ then p is in the closure of the set of points q for which $n(q,f,\Omega)$ is well defined, and the residue class mod 2 (the parity) of $n(q,f,\Omega)$ is constant near p. We denote this residue class by $d(p,f,\Omega)$, and call it the *degree (mod 2) of f at p relative to* Ω. We will denote the residue class of the even integers by 0, and the class of odd integers by 1.

We now analyze the behavior of the degree when f is deformed.

Proposition 3. Let $f_t(x)$ be a continuous map of $\bar{\Omega} \times [0,1]$ into R^m. Let $\{P_k\}_{k\in K}$ be a finite family of hyperplanes such that $f_t(x)$ is a piecewise linear mapping with respect to this family for each $t \in [0,1]$. If $p \notin \cup_{t\in[0,1]} f_t(\partial\Omega)$, then $d(p,f_0) = d(p,f_1)$.

Proof. Consider first an arbitrary piecewise linear map g of $\bar{\Omega}$ and a point $q \in R^m$ such that $q \notin g(\partial\Omega) \cup g(\cup_{k\in K} P_k \cap \Omega)$. If $q \in g(\bar{\Omega})$, then $q \in g(\Omega_i)$ where Ω_i one of the open connected sub-domains in the decomposition of Ω by the planes P_k. Let $g(x) = Ax + b$ for $x \in \Omega_i$ where A is a regular matrix and $b \in R^m$. If g' is another piecewise linear mapping of $\bar{\Omega}$ into R^m such that $g'(x) = A'x + b'$ for x near $\bar{x} = g^{-1}(q) \cap \Omega_i$, where A' is regular and $b' \in R^m$, then

$$\begin{aligned}(A')^{-1}(q-b') &= A^{-1}(q-b) + [(A')^{-1} - A^{-1}](q-b') + A^{-1}(b-b') = \\ &\bar{x} + [(A')^{-1} - A^{-1}](q-b') + A^{-1}(b-b').\end{aligned} \tag{28}$$

If $\max_{x\in\bar\Omega}|g(x) - g'(x)|$ is sufficiently small, then both $\|(A')^{-1} - A^{-1}\|$ and $|b - b'|$ are small, so that $(A')^{-1}(q - b')$ is sufficiently close to $\bar{x}$. Hence $n(q, g, \Omega) = n(q, g', \Omega)$.

It follows from the continuity of f_t that if $q \in R^m$ is not contained in $f_{t_0}(\partial\Omega) \cup f_{t_0}(\cup_{k\in K} P_k \cap \Omega)$ then there exists a positive ϵ such that $n(q, f_t, \Omega)$ is well defined if $|t - t_0| < \epsilon$, and $n(q, f_t, \Omega) = n(q, f_{t_0}, \Omega)$ for those values of t. Let $t_0 \in [0, 1]$ be arbitrary. Then $d(p, f_{t_0}, \Omega) = d(q, f_{t_0}, \Omega) = n(q, f_{t_0}, \Omega) \bmod 2$ for all q near p for which q is not contained in $f_{t_0}(\partial\Omega) \cup f_{t_0}(\cup_{k\in K} P_k \cap \Omega)$. It follows from Proposition 2 that for each t such that $|t - t_0| < \epsilon$, $d(q, f_t, \Omega)$ is independent of q. Hence $d(p, f_t, \Omega) = d(p, f_{t_0}, \Omega)$ if $|t - t_0| < \epsilon$. The compactness of $[0, 1]$ implies that $d(p, f_0, \Omega) = d(p, f_1, \Omega)$. ∎

We can now define the mod 2 degree for a general continuous function $f : \bar\Omega \to R^m$ and a point $p \notin f(\partial\Omega)$: let g be a piecewise linear mapping of $\bar\Omega$ such that $|g(x) - f(x)| < dist(p, f(\partial\Omega))$ for all $x \in \bar\Omega$. If g_1 and g_2 are two such mappings, then Proposition 3 implies that $d(p, g_1, \Omega) = d(p, g_2, \Omega)$. Hence $d(p, g, \Omega)$ is the same for all such mappings g, and we set $d(p, f, \Omega)$ to be this common value.

Example: Consider the function $f(x) = x^2$ with $\Omega = (-1, 2)$. Then $f(\partial\Omega) = \{1, 4\}$. Clearly, $d(p, f, \Omega)$ vanishes if $p < 1$ as $p \notin f(\Omega)$ if $p < 0$, and there are two solutions to the equation $f(x) = p$ for $p \in (0, 1)$. The degree is not defined for $p = 1$, and $d(p, f, \Omega) = 1$ if $p \in (1, 4)$, reflecting the unique solvability there. If $p > 4$ there is no solution in Ω and $d(p, f, \Omega) = 0$. The reader may wish to illustrate the proof of Proposition 2 by considering the (geometrically similar) piecewise linear map f defined on $\bar\Omega$ by $f(x) = -x$ for $x \in [-1, 0]$ and $f(x) = 2x$ for $x \in (0, 2]$.

The basic properties of the degree are summarized in the following theorem:

Theorem 4.1 *Let Ω be a bounded open subset of R^m, f a continuous function mapping $\bar\Omega$ into R^m. Then*

(I) $d(p, I, \Omega) = 1$ if I is the identity map of Ω and $p \in \Omega$.

(II) If $f_t(x)$ is a continuous map of $\bar\Omega \times [0, 1]$ into R^m and $p \notin \cup_{t\in[0,1]} f_t(\partial\Omega)$ $(0 \le t \le 1)$, then $d(p, f_t, \Omega)$ is independent of $t \in [0, 1]$.

(III) If $\{\Omega_i\}_{i=1}^k$ are pairwise disjoint open subset of Ω such that $f(x) \neq p$ whenever $x \in \bar\Omega \setminus \cap_{i=1}^k \Omega_i$, then

$$d(p, f, \Omega) = \sum_{i=1}^{k} d(p, f, \Omega_i). \tag{29}$$

(IV) If p and q may be joined by a polygonal line not intersecting $f(\partial\Omega)$, then $d(p, f, \Omega) = d(q, f, \Omega)$.

(V) If $d(p, f, \Omega) = 1$ then $p \in f(\Omega)$.

Proof. The statements (I), (II), (III), and (IV) follow at once from the definitions and the preceding Propositions if f (or f_t) are piecewise linear, and in the general case by a simple approximation argument. To prove (V), note that $d(p, f, \Omega)$ is defined only if $p \notin f(\partial\Omega)$. If in addition $p \notin f(\Omega)$, then $dist(p, f(\bar{\Omega}) > 0$. If f' is a sufficiently close piecewise linear approximation of f on $\bar{\Omega}$, then $dist(p, f'(\bar{\Omega}) > 0$ as well. Thus $n(p, f', \Omega) = 0$ so that $d(p, f', \Omega) = 0$. Hence $d(p, f, \Omega) = 0$, contradicting the assumption. ■

Remark: It is possible to construct, with a little more effort, an integer valued degree. The mod 2 degree suffices for our purposes.

The theory developed so far is sufficiently strong for many applications, including e.g. Theorem 3.1. Extensions for convex valued correspondences are well known. In fact, they are treated extensively in the book [7]. Degree theory for convex valued correspondences suffice for proving results such as Theorems 2.1, 3.2, and 3.4. It appears that a natural context for studying issues involving partnered collections is degree theory for certain classes of non-convex valued correspondences. Such theories were constructed long ago and are well known to the initiated, but they never became very popular with the masses.

We are now going to describe how to extend degree theory to a simple class of non-convex valued correspondences, namely to correspondences whose values are compact and contractible. Recall that a subset A of R^m is *contractible* if for every point $a \in A$ there exists a continuous map $f_t(x)$ of $A \times [0,1]$ into A such that $f_0(x) \equiv a$ and $f_1(x) \equiv x$. (Clearly, all convex set are contractible; more generally, star-shaped sets are contractible.) The crucial technical step in the construction of degree is a lemma (Proposition 4 below) which enables approximating the graph of a correspondence by the graph of a single valued continuous function.

Let Ω be a bounded open subset of R^m. A correspondence F defined on $\bar{\Omega}$ whose values are compact contractible subsets of R^m is *admissible* if 1) F is upper-hemicontinuous and 2) for every $\epsilon > 0$ there exists a continuous single-valued function $f_\epsilon : \bar{\Omega} \to R^m$ such that for every $x \in \bar{\Omega}$ there exist $x_\epsilon \in \bar{\Omega}$ and $y \in F(x_\epsilon)$ with $|x - x_\epsilon|^2 + |f(x) - y|^2 < \epsilon^2$. In other words, $G(f) \subset B_\epsilon(G(F))$ where for every $Y \subset R^k$ we set $B_\epsilon(Y) = \cup_{y \in Y}\{x : |x - y| < \epsilon\}$.

Let F be admissible, and set $F(\partial\Omega) = \cup_{x \in \partial\Omega} F(x)$. Note that $\partial\Omega$ is compact. The upper-hemicontinuity of F implies that $F(\partial\Omega)$ is closed (actually compact). If $p \notin F(\partial\Omega)$ then $dist(p, F(\partial\Omega)) = \rho > 0$. Let f be a single valued continuous function from Ω into R^m such that $G(f) \subset B_{\rho/2}(G(F))$. We set $d(p, F, \Omega) = d(p, f, \Omega)$. It follows from Theorem 4.1(II) that $d(p, F, \Omega)$ is well defined for admissible correspondences. Moreover, the basic properties of the degree carry over.

Theorem 4.2 *Let Ω be a bounded open subset of R^m, F an admissible correspondence defined on $\bar{\Omega}$ whose values are compact contractible subsets of R^m. Then*

(I) $d(p, I, \Omega) = 1$ if I is the identity map of Ω and $p \in \Omega$.

(II) If $F_t(x)$ is a upper-hemicontinuous map of $\bar{\Omega} \times [0,1]$ whose values are compact contractible subsets of R^m such that for each $t \in [0,1]$ F_t is admissible, and $p \notin \cup_{t\in[0,1]} F_t(\partial\Omega)$ $(0 \leq t \leq 1)$, then $d(p, F_t, \Omega)$ is independent of $t \in [0,1]$.

(III) If $\{\Omega_i\}_{i=1}^k$ are pairwise disjoint open subset of Ω such that $p \notin F(x)$ whenever $x \in \bar{\Omega} \setminus \cap_{i=1}^k \Omega_i$, then

$$d(p, F, \Omega) = \sum_{i=1}^{k} d(p, F, \Omega_i). \tag{30}$$

(IV) If p and q may be joined by a polygonal line not intersecting $F(\partial\Omega)$, then $d(p, F, \Omega) = d(q, F, \Omega)$.

(V) If $d(p, F, \Omega) = 1$ then there exists $x \in \Omega$ such that $p \in F(x)$.

Proof. The statements (I), (III), and (IV), follow at once from the definition of degree for admissible correspondences and from the relevant parts of Theorem 4.1. For proving statement (II) an additional simple compactness argument is required as well. To prove (V), note that if $p \notin F(\partial\Omega)$ and $d(p, F, \Omega) = 1$ then there exists a sequence $\{f_n\}$ of continuous single-valued functions from $\bar{\Omega}$ to R^m such that $p \notin f_n(\partial\Omega)$, $d(p, f_n, \Omega) = 1$, and $G(f_n) \subset B_{1/n}(G(F))$. By Theorem 4.1(V) there exists a sequence $\{x_n\}$ such that $x_n \in \Omega$ and $f_n(x_n) = p$ for every n. We may assume, by compactness of $\bar{\Omega}$, that $x_n \to \bar{x} \in \bar{\Omega}$. Moreover, for every n there exists $(x_n^*, y_n) \in G(F)$ such that $|x_n - x_n^*|^2 + |y_n - p|^2 < 1/n$. Then $x_n^* \to \bar{x}$, $y_n \to p$, and $p \in F(\bar{x})$ by upper-hemicontinuity of F. By assumption $p \notin F(\partial\Omega)$. Hence $\bar{x} \in \Omega$. ■

In the rest of this papers all functions and correspondences will be defined on subsets of the n-1-dimensional hyperplane $H \subset R^n$ affinely generated by Δ. This hyperplane is affinely equivalent to R^{n-1} so that the preceding theory applies (with $m = n - 1$). We quote now without proof the following Lemma (a special case of a more general result established in [8]; the simpler convex case may be found in the book [7]).

Proposition 4. (Approximation Lemma): Let F be an upper-hemicontinuous correspondence defined in Δ such that for every $y \in \Delta$, $F(y)$ is a contractible and compact subset of H. Then for every $\epsilon > 0$ there exists a continuous single-valued function $f \to H$ such that $G(f) \subset B_\epsilon(G(F))$.

Thus, if F satisfies the assumptions of Proposition 4, then F is an admissible correspondence for every $\Omega \subset int(\Delta)$.

To illustrate how the preceding theory may be applied, we prove the following fixed point theorem, a special case of the Eilenberg-Montgomery fixed point Theorem:

Theorem 4.3 *Let F be an upper-hemicontinuous correspondence defined in Δ such that for every $y \in \Delta$ $F(y)$ is a contractible and compact subset of Δ. Then there exists a point $x \in \Delta$ such that $x \in F(x)$.*

Proof. We may assume that $x \notin F(x)$ for every $x \in \partial\Delta$. Set $F_t(x) = x - tF(x) + tm_N$ for $t \in [0,1]$ (i.e., $F_t(x) = \{y : \exists z \in F_t(x) \text{ such that } y = x - tz + tm_N\}$). By assumption, $m_N \notin F_1(\partial\Delta)$. Moreover, if $t < 1$ and $x \in \partial\Delta$ then $x_1 = 0$ for some $i \in N$ and for every $y \in F(x)$ we have $y_i \geq 0$. Hence $x_i - ty_i + \frac{t}{n} < \frac{1}{n}$, so that $x - tF(x) + tm_N$ cannot be equal to m_N. Thus $F_t(x)$ satisfies all assumptions of Theorem 4.2(II) so that $d(m_N, F_1, int(\Delta)) = d(m_N, F_0, int(\Delta))$. But $F_0(x) = x$, and by Theorem 4.2(I) $d(m_N, F_0, int(\Delta)) = 1$, so that $d(m_N, F_1, int(\Delta)) = 1$. Hence $d(m_N, x - F(x) + m_N, int(\Delta)) = 1$, and by Theorem 4.2(V) there exists a point $x \in int(\Delta)$ with $m_N \in x - F(x) + m_N$ or $x \in F(x)$. ■

We obtain as special cases the fixed point theorems of Kakutani (if F is convex valued) and of Brouwer (if F is single-valued).

The following formulation of the Brouwer fixed point theorem is useful for many applications:

Theorem 4.4 *Let K be a topological space homeomorphic to a simplex $\Delta \subset R^n$, and let f be a continuous (single-valued) mapping of K into itself. Then there exists a point $x \in K$ with $x = f(x)$.*

Proof. Let $h : \Delta \to K$ be the homeomorphism map, i.e., h is one-one onto and h^{-1} is continuous. Define $\tilde{f} : \Delta \to \Delta$ by $\tilde{f}(x) = h^{-1}(f(h(x)))$. Then $\tilde{f}$ is a continuous (single-valued) mapping of Δ into itself. As such, there exists a point $y \in \Delta$ such that $y = \tilde{f}(y)$ or $y = h^{-1}(f(h(y)))$. Hence $h(y) = f(h(y))$. ■

In particular, every continuous self-map of a compact convex subset of R^n has a fixed point.

5 Main Results

We begin by refining the concept of balancedness. We say that the collection $\mathcal{B}$ of subsets of N is *strictly balanced* if all weights λ^S in (15) can be chosen to be positive. It is easy to show that a strictly balanced collection of sets is partnered.

Proposition 5. Let $\mathcal{B}$ be a strictly balanced family of subsets of N. Then $\mathcal{B}$ is partnered.

Proof. Suppose that $\mathcal{B}$ is strictly balanced but not partnered. Then there exists $i, j \in N$ such that for all $S \in \mathcal{B}$ with $i \in S$ it holds that $j \in S$, but there exists $T \in \mathcal{B}$ with $j \in T$, $i \notin T$. Let $\{\omega_S : S \in \mathcal{B}\}$ denote a set of strictly positive balancing weights for $\mathcal{B}$. Since the weights ω_S on all the sets in $\mathcal{B}$ are strictly positive, $\sum_{S:\, i \in S} \omega_S < \sum_{S:\, j \in S} \omega_S = 1$. This is a contradiction. ■

Note that there exist partnered collections (even minimally partnered) which are balanced, but not strictly balanced ([5]).

We can now formulate a strengthening of Theorem 2.2.

Theorem 5.1 *Let (N, V) be a balanced game satisfying (18). Then there is a point y in the core whose collection of supporting coalitions $S(y) = \{S \subset N : y \in V(S)\}$ is strictly balanced.*

Theorem 5.1 follows in turn form the following strengthening of Theorem 3.3a:

Theorem 5.2 *Let $\{C^S : S \subset N\}$ be a family of closed subsets of Δ such that (20) is satisfied. Then there exists $x \in \Delta$ such that the supporting collection $S(x) = \{S : x \in C^S\}$ is strictly balanced.*

Proof of Theorem 5.1. Set $\tilde{e}_i = -nMe^i$ for $i \in N$, and define the simplex $\tilde{\Delta}^S$ to be the convex hull of $\{\tilde{e}_i\}_{i \in S}$. The linear function $h(x) = \sum x_i \tilde{e}_i$ maps Δ^S homeomorphically onto $\tilde{\Delta}^S$. Set

$$t(x) = \sup\{t : x + t(1, \cdots, 1) \in \cup_{S \subset N} V(S)\} \ . \tag{31}$$

By (7) and (18) the supremum in (31) is finite and is actually a maximum, and defines a continuous function of $x \in R^N$. Set now

$$\tilde{C}^S = \{x \in \tilde{\Delta}^N : x + t(x)(1, \cdots, 1) \in V(S)\} \ . \tag{32}$$

The sets $\tilde{C}_S$ are closed by continuity of t and (6). We want to show that (20) is satisfied (for $C^T = h^{-1}(\tilde{C}^T)$). Let $x \in \tilde{C}^S \cap \tilde{\Delta}^T$. We will show that $S \subset T$. (For all $x \in \tilde{\Delta}^T$ there exists at least one $S \subset N$ such that $x \in \tilde{C}^S$. Hence (20) follows.) If $T = N$ there is nothing to prove; we may assume therefore that $T \neq N$. Since $x \in \tilde{\Delta}^T$ we have $x_j \leq -nM/|T| < -M$ for at least one $j \in S$. Taking $S = \{j\}$ in (31) we obtain

$$t(x) > M \quad . \tag{33}$$

Combining (33) with (18), we find that $x_i < 0$ for all $i \in S$. On the other hand, $x \in \tilde{\Delta}^T$ implies $x_i = 0$ for $i \notin T$. Hence $S \subset T$.

It follows from Theorem 5.2 that there exists a point $x \in \Delta^N$ such that the supporting collection $\mathcal{S}(x) = \{S : x \in C^S\}$ is strictly balanced. However $\mathcal{S}(x) = \{S : h(x) \in \tilde{C}^S\}$. It follows that the point $y = h(x) + t(h(x))(1, \cdots, 1)$ belongs to $\cap_{S \in \mathcal{S}} V(S)$, but not to $\cup_{S \subset N} \text{int}\, V(S)$. By (17), $y \in V(N)$. Hence y is in the core, and the collection supporting y is strictly balanced. ■

Note that Scarf's theorem (Theorem 2.1) about the non-emptiness of the core without the boundedness assumption (18) follows from the existence statements of Theorems 5.1 or 2.2 by a simple limiting argument utilizing (13), see for example [4].

To complete the proof of Theorem 5.1, we have to prove Theorem 5.2. The latter Theorem follows from the following variant of Theorem 3.4:

Theorem 5.3 *Let $F(x)$ be a correspondence from Δ into the closed convex subsets of Δ such that F satisfies (21) and (22). Assume in addition that F satisfies (25). Then there exists $x \in \Delta$ such that $m_N \in rel\ int(F(x))$.*
(As usual, $rel\ int(K)$ means the interior of K in the affine submanifold spanned by K.)

Proof of Theorem 5.2. Let F be the map whose existence is stated in Proposition 1. In particular F satisfies the conditions (21), (22), and (25). By Theorem 5.3 there exists $y \in \Delta$ such that $m_N \in rel\ int(F(y))$, and by (23) and (24) there exists $x \in \Delta$ ($x = \varphi^{-1}(y)$) such that

$$m_N \in rel\ int[conv\{m_S : x \in C^S\}]. \tag{34}$$

Clearly, $\Sigma := \{S : x \in C^S\}$ is balanced. Moreover, it is *strictly* balanced. In fact, let $S \in \Sigma$, $S \neq N$ (without loss of generality, $\Sigma \neq \{\text{N}\}$) and let ℓ_S denote the line joining m_N and m_S. Then m_N is contained in the interior of the interval $\ell_S \cap conv\{m_S\}_{S \in \Sigma}$. Hence there exists an $a_S \in conv\{m_S\}_{S \in \Sigma}$ and positive numbers α_S, β_S such that $\alpha_S + \beta_S = 1$ and $m_N = \alpha_S m_S + \beta_S a_S$. We may average these equations with positive weights over $S \in \Sigma$, $S \neq N$ and obtain m_N as a convex combination of the points m_S, $S \in \Sigma$, with positive weights for each $S \neq N$. ■

Proof of Theorem 5.3. Note first that if y is not in the relative interior of a convex set K, then removing an open ball $B(y,\delta)$ of radius δ centered at y from K results in a nonempty closed contractible set as long as $\delta > 0$ is sufficiently small. It follows from (21) and (25) that if $m_N \in rel\ int(F(x))$ for no $x \in \Delta$, then there exists a $\delta > 0$ such that $F(x)\backslash B(m_N,\delta)$ is nonempty and contractible for all $x \in \Delta$. Moreover the openness of $B(m_N,\delta)$ implies that the correspondence $x \to G(x) = F(x)\backslash B(m_N,\delta)$ is upper-hemicontinuous. For any $t \in [0,1]$ and $x \in \Delta$, set $G_t(x) = tx + (1-t)G(x)$. If $x \in \partial\Delta$ then $x \in \Delta^T$ where T is a proper subset of N and $G(x) \subset F(x) \subset \Delta^T$. Hence $m_N \notin G_t(x)$ if $x \in \partial\Delta$, and $G_t(x)$ satisfies the conditions of Theorem 4.2(II), so that $d(m_N, G_t, int(\Delta))$ is independent of t. But Theorem 4.2(I) implies that $d(m_N, G_1, int(\Delta)) = d(m_N, I, int(\Delta)) = 1$. By Theorem 4.2(V) there exists a point $x \in int(\Delta)$ such that $m_N \in G(x) = F(x)\backslash B(m_N,\delta)$, a contradiction. ∎

Remark: Theorem 3.4 may be proved by a similar (actually a direct) degree argument, establishing that $d(m_N, F, int(\Delta)) = 1$. In fact, the same method proves that Theorem 3.4 continues to hold if we assume only that the values of $F(x)$ are contractible, rather than convex.

For minimal partnership we have the following family of results. The topological concept of zero dimension is useful here. Recall that a topological space X has *dimension zero* if for every $p \in X$ there is an arbitrarily small open set with empty boundaries containing p. It is well known (cf. [3], p. 147) that among compact spaces the zero-dimensional spaces and the totally disconnected spaces are identical.

Theorem 5.4 *Let $F(x)$ be a correspondence from Δ into the closed convex subsets of Δ satisfying (21), (22), and (25). Assume also that:*

$$\text{The closure of the set } \{x : m_N \in rel\ int(F(x))\} \text{ is zero-dimensional.} \tag{35}$$

Then there exists $x \in \Delta$ such that $F(x)$ has non-empty (n-1)-dimensional interior and $m_N \in int(F(x))$.

(By "int" we mean the "interior in the topology on the hyperplane H".)

Proof. Denote by $\bar{X}$ the closure of the set $\{x : m_N \in rel\ int(F(x))\}$. By assumption, $\bar{X}$ is zero-dimensional. This means that for every $\varepsilon > 0$, the set $\bar{X}$ may be covered by a finite number of disjoint open sets whose diameter is less than ε. Let $\{D_{i,m}\}_{i=1}^{P_m}$ denote such a collection of sets with $diam\ (D_{i,m}) < \frac{1}{m}$, $\bar{X} \subseteq \cup_{i=1}^{P_m} D_{i,m}$ and $D_{i,m} \cap D_{j,m} = \emptyset$ for $i \neq j$. Then $D_{i,m} \cap \bar{X}$ is both open and closed in $\bar{X}$, so that $\partial D_{i,m} \cap \bar{X} = \emptyset$.

With δ as in the proof of Theorem 5.3, set $\delta(x) = \min[dist(x,\bar{X}), \delta]$, and define an upper-hemicontinuous correspondence G by

$$G(x) = F(x)\backslash B(m_N, \delta(x)). \tag{36}$$

Then $m_N \notin G(x)$ if $x \notin \bar{X}$. A similar argument to the one used in the proof of Theorem 5.3 enables us to infer from (22) that

$$d(G, int(\Delta), m_N) = 1. \tag{37}$$

By construction, $m_N \notin G(y)$ for all $y \in \partial D_{i,m}$, $1 \leq i \leq P_m$. Note also that by Proposition 4, $G(x)$, defines over any open subset of $int(\Delta)$, is admissible. Hence $d(G, D_{i,m}, m_N)$ is well defined and

$$\sum_{i=1}^{P_m} d(G, D_{i,m}, m_N) = d(G, int(\Delta), m_N). \tag{38}$$

It follows from (38) and (37) that there exists $i_0 = i_0(m)$ such that $d(G, D_{i_0(m),m}, m_N) \neq 0$. By compactness there exists $\bar{x} \in \bar{X}$ and a sequence $D_{i_0(m),m}$ of neighborhoods (with $D_{i_0(m),m} \cap \bar{X}$ compact) such that $\bar{x} = \cap_{m=1}^{\infty} D_{i_0(m),m}$. For each m, $d(G, D_{i_0(m),m}, m_N) \neq 0$ implies the existence of an (n-1)-dimensional ball B_m, centered at m_N, such that

$$B_m \subseteq \bigcup\nolimits_{x \in D_{i_0(m),m}} G(x) \subseteq \bigcup\nolimits_{x \in D_{i_0(m),m}} F(x). \tag{39}$$

Set $\underline{a}_i = e^i - m_N$, $1 \leq i \leq n$. Fix for a moment $\underline{a}_j$ for an index $1 \leq j \leq n$. By (25) there exists a positive number δ_j such that if $m_N + \varepsilon \underline{a}_j \in F(y)$ for a certain $y \in \Delta$ and a positive ε (no matter how small), then $m_N + \delta_j \underline{a}_j \in F(y)$. By (39) there exists a sequence x^m converging to $\bar{x}$ and a sequence of positive real numbers ε_m such that $m_N + \delta_j \underline{a}_j \in F(x^m)$. By the upper-hemicontinuity $m_N + \delta_j \underline{a}_j \in F(\bar{x})$. The convexity of $F(\bar{x})$ and the spanning property of $\underline{a}_1, ..., \underline{a}_n$ imply that m_N is an interior point of $F(\bar{x})$. ■

Theorem 5.4 implies the following result about closed covering.

Theorem 5.5 *Let $\{C^S : S \subset N\}$ be a family of closed subsets of Δ such that (20) is satisfied. Assume that the closure of the set*

$$\{x : \{S : x \in C^S\} \text{ is strictly balanced}\}$$

is zero-dimensional. Then there exists $x \in \Delta$ such that the collection $\{S : x \in C^S\}$ is minimally partnered and strictly balanced.

Proof. Let F be the map whose existence is stated in Proposition 1. Note in particular that F satisfies (22), and the assumptions imply that (35) is satisfied as well. Hence there exists (by Theorem 5.4) $x \in \Delta$ such that $m_N \in int(D(x))$ [where $D(x) = conv(m_S : x \in C^S)$]. If

$\Sigma = \{S : x \in C^S\}$ is not minimally partnered, then there exists a pair i, j such that for every $S \in \Sigma$ either i and j both belong to S, or neither belongs. Hence for all $y \in D(x)$, $y_i = y_j$. Thus $int(D(x))$ is empty, a contradiction. ■

Comparing the assumptions of Theorem 5.5 with those of Reny and Wooders (1998) (Theorem 3.3b above), it appears that neither is stronger than the other. On the one hand a countable set (as assumed in Theorem 3.3b) may be dense and hence have closure of positive dimension; on the other hand, a set of dimension zero (as assumed in Theorem 5.5) may be uncountable (for example, a Cantor set on a line). Examples are provided in [5]. The following is a generalization of Theorem 3.3b.

Theorem 5.6 *Let $\{C^S : S \subset N\}$ be a family of closed subsets of Δ such that (20) is satisfied. If the set $\{x^* \in \Delta : \mathcal{S}(x^*)$ is balanced and partnered$\}$ is zero dimensional, then at least one $x^* \in \Delta$ renders $\mathcal{S}(x^*)$ strictly balanced and minimally partnered. Moreover, $m_N \in int\ [conv\{m_S\}_{S \in \mathcal{S}(x^*)}]$.*

The proof of Theorem 5.6 involves ideas found in the previous proofs as well as essential refinements (see [5]), and will not be reproduced here.

Theorems 5.5 and 5.6 induce theorems on the set of minimally partnered core outcomes of games, in the same manner as Theorem 5.1 was derived from Theorem 5.2. In particular, the results of [11] may be re-derived and refined. That some care is needed may be seen by an example (constructed in [5]) of a 12 players game (a marriage and adoption game) whose partnered core is homeomorphic to the Cantor set.

6 Other Applications

We mention briefly several applications of the theory and the methods outlined in previous sections.

1) Cooperative game theory influenced, and was influenced by, the theory of *market equilibrium*. A distribution of resources is in the *core* of an economy if no individual (and no group) may achieve, acting on her (their) own, preferable results for herself (its members). Sometimes a core distribution is associated with a *price* system. Existence can be established using fixed point methods similar to those expounded here [9].

2) As explained in the introduction, a natural requirement from a theory is that the predicted outcome(s) are sufficiently stable so that once such an outcome has been reached,

no individual could gain more by making a different choice. This applies to *non-cooperative* games as well, and leads to the celebrated concept of **Nash equilibrium point**– a concept that has become very popular recently [1]. Here each participant's strategy is the "best response" to the others' choices of strategies. Existence of Nash equilibria is obtained from the Brouwer fixed point theorem, or, more transparently, from the Kakutani fixed point theorem [2], [9].

3) If $d(p, f, \Omega) = 1$ (as is the case in all applications discussed here and in the rest of the present paper), than it seems plausible that for "typical" f and p, the number of solutions of the equation $f(x) = p$ in Ω is odd. This intuition leads to *genericity* results.

4) One of the few instances in which integer valued degree (rather than mod 2) is required, is in the proof of the fundamental theorem of algebra [3].

5) Fixed point theory and degree theory may be extended to certain families of functions defined on infinite dimensional spaces. The theory provides existence theorems for ordinary and partial differential equations [6], [7]. Degree theory enables "counting" the number of solutions in many cases.

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