Cubo A Mathematical Journal
Vol.05/Nº03 - OCTOBER 2003

A Direct Method of Optimization and its Application to a Class of Differential Games

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Abstract. Many problems in economics and engineering can be posed as dynamic optimization problems involving the extremization of an integral over a given class of functions subject to prescribed end conditions. These problems are usually addressed via the Calculus of Variations or the Maximum Principle of optimal control theory, applying necessary conditions to obtain candidate optimal solutions, and then assuring optimality via sufficient conditions if available. These methods are variational in that they employ the comparison of solutions in a neighborhood of the optimal one.

A different approach was first proposed in the 1960's and more recently expanded in Refs. 1-3. This approach permits the direct derivation of global extrema for some classes of dynamic optimization problems without the use of comparison techniques. Instead, it employs coordinate transformations and the imposition of a functional identity. The direct method is readily applicable to a class of open-loop differential games, as shown in Ref. 4.

PART I. The Fundamental Problem of the Calculus of Variations

The basic problem

Given

$$I(y(\cdot)) := \int_a^b F[x, y(x), y'(x)]dx \tag{1}$$

with $F(\cdot) : [a,b] \times R^{2n} \to R$ continuous, and $()' := \frac{d()}{dx}$, extremize $I(y(\cdot))$ over all piecewise smooth (pws) $y(\cdot) : [a,b] \to R^n$ satisfying prescribed end conditions

$$y(a) = y_a, \qquad y(b) = y_b. \tag{2}$$

A basic lemma

The following lemma was deduced in Refs. 1-3, together with various corollaries:

Lemma *Let $y = z(x, \tilde{y})$ be a transformation having a unique inverse $\tilde{y} = \tilde{z}(x, y)$, with $z(\cdot)$ of class C^1 on $[a,b] \times R^n$, such that there is a one-to-one correspondence*

$$y(x) \Longleftrightarrow \tilde{y}(x) \tag{3}$$

for all pws $y(\cdot)$ satisfying (2) and all pws $\tilde{y}(\cdot)$ satisfying

$$\tilde{y}(a) = \tilde{z}(a, y_a), \qquad \tilde{y}(b) = \tilde{z}(b, y_b) \tag{4}$$

Let $H(\cdot) : [a,b] \times R^n \to R$ be a C^1 function such that the functional identity

$$F[x, y(x), y'(x)] - F[x, \tilde{y}(x), \tilde{y}'(x)] = \frac{dH[x, \tilde{y}(x)]}{dx} \tag{5}$$

is met for all pws $\tilde{y}(\cdot)$ satisfying (4), where $y(x) = z[x, \tilde{y}(x)]$.

If $\tilde{y}^*(\cdot)$ extremizes

$$I(\tilde{y}(\cdot)) = \int_a^b F[x, \tilde{y}(x), \tilde{y}'(x)]dx \tag{6}$$

over all pws $\tilde{y}(\cdot)$ satisfying (4), then $y^*(\cdot)$ with $y^*(x) = z[x, \tilde{y}^*(x)]$ extremizes $I(y(\cdot))$ over all pws $y(\cdot)$ satisfying (2).

The relation to Carathéodory's Basic Theorem

The following remark is quoted from Ref. 5:

The theory for direct sufficient conditions for solving problems in the calculus of variations and optimal control is not nearly as developed as the theory for necessary conditions or that of the direct methods for the existence of minimizers (or maximizers). Indeed, most results of this type may be traced back to the earliest roots with very little change. In this paper we report on some recent developments on two such methods appearing in Carlson (2002a).[1] These two methods are compared and contrasted, and finally combined to arrive at what is apparently a new method. The first of these two methods appears in 1935 in

[1] Ref. 6

the book by Carathéodory (1982)[2] while the second is due to Leitmann (1967).[3] These two methods have striking similarities, yet are clearly different. The first perturbs the objective functional to obtain a new functional which has the same set of minimizers (or maximizers) as the original problem. The feasible set is the same for both problems. In the second method, a transformation is defined which establishes a one-to-one correspondence between the original set of feasible trajectories and a new set of trajectories. With this transformation, the original problem is replaced by a new one in which the objective functional remains the same but the feasible set is replaced by the new trajectories.

Corollaries

The Lemma has a corollary from which all subsequent corollaries follow, namely,

Corollary 1 *The functional identity (5) implies the identity*

$$F[x, z(x,\tilde{y}), \frac{\partial z(x,\tilde{y})}{\partial x} + \sum_{i=1}^{n} \frac{\partial z(x,\tilde{y})}{\partial \tilde{y}_i}\tilde{p}_i] - F(x,\tilde{y},\tilde{p}) \equiv \frac{\partial H(x,\tilde{y})}{\partial x} + \sum_{i=1}^{n} \frac{\partial H(x,\tilde{y})}{\partial \tilde{y}_i}\tilde{p}_i \tag{7}$$

on $(a,b) \times R^{2n}$.

An important consequence of Corollary 1 is

Corollary 2[4] *The left-hand-side of identity (7) is linear in the* $\tilde{p}_i$, *that is, it is of the form*

$$\theta(x,\tilde{y}) + \sum_{i=1}^{n} \psi_i(x,\tilde{y})\tilde{p}_i \tag{8}$$

and

$$\frac{\partial H(x,\tilde{y})}{\partial x} = \theta(x,\tilde{y}),\ \frac{\partial H(x,\tilde{y})}{\partial \tilde{y}_i} = \psi_i(x,\tilde{y}) \tag{9}$$

on $[a,b] \times R^n$.

Other corollaries and remarks, as well as numerous examples, can be found in Refs. 1-3. In addition, the extension of the direct method to a class of differential games is treated in Ref. 4.

Differential side conditions

Given

$$I(y(\cdot)) = \int_a^b F[x, y(x), y'(x)]dx$$

[2] Ref. 7

[3] Ref. 8

[4] Note that Corollary 2 implies that the identity (7) is met for x on the closed interval, and hence that the functional identity (5) is satisfied for *all pws* $\tilde{y}(\cdot)$ regardless of end conditions.

extremize $I(y(\cdot))$ over all *pws* $y(\cdot) : [a, b] \rightarrow R^n$ satisfying prescribed conditions

$$s_i[x, y(x), y'(x)] = 0, \qquad i = 1, 2, \cdots, m < n$$

with $s_i(\cdot)$ continuous on $[a, b] \times R^{2n}$, and

$$y(a) = y_a, \qquad y(b) = y_b.$$

The infinite horizon case

Given

$$I(y(\cdot)) := \int_a^\infty F[x, y(x), y'(x)]dx$$

with $F(\cdot) : [a, \infty) \times R^{2n} \rightarrow R$ continuous, extremize $I(y(\cdot))$ over all *pws* $y(\cdot) : [a, \infty) \rightarrow R^n$ satisfying given initial condition

$$y(a) = y_a.$$

These extensions to the basic problem are treated in Ref. 3. Inequality constraints are treated in Ref. 9.

Part II. Open-loop Differential Games with Separated Dynamics

Statement of the problem

In order to simplify the notation, we restrict the discussion to two-player games; however, the discussion is equally applicable to many-player games. Furthermore, we make an obvious change of notation in order to adhere to the notation usually employed in the theory of differential games.

Consider two players who compete over time. Each player $i = 1, 2$ chooses a sequence of actions so as to extremize a given integral

$$\underset{u(\cdot)}{\text{extr}} \left\{ J_1 = \int_{t_1}^{t_2} I_1[t, x(t), y(t), u(t), v(t)]dt \right\} \tag{1}$$

given $v(\cdot)$, and

$$\underset{v(\cdot)}{\text{extr}} \left\{ J_2 = \int_{t_1}^{t_2} I_2[t, x(t), y(t), u(t), v(t)]dt \right\} \tag{2}$$

given $u(\cdot)$, and subject to the dynamic constraints of the form

$$\frac{dx(t)}{dt} =: x'(t) = u(t)l_1[t, x(t)] + m_1[t, x(t)], \tag{3}$$

$$\frac{dy(t)}{dt} =: y'(t) = v(t)l_2[t, y(t)] + m_2[t, y(t)] \tag{4}$$

with continuous and non-zero $l_i(\cdot)$ and continuous $m_i(\cdot)$, and the boundary conditions

$$x(t_1) = x_1, \quad x(t_2) = x_2, \tag{5}$$

$$y(t_1) = y_1, \quad y(t_2) = y_2. \tag{6}$$

In the differential game (1)–(6) $[x(t), y(t)]$ is the vector of state variables of players 1 and 2, respectively and $[u(t), v(t)]$ is the vector of control variables. Player 1 chooses his actions in terms of the control $u(t)$ and player 2 chooses $v(t)$. $I_i[t, x(t), y(t), u(t), v(t)]$ is the integrand of the objective function of player i. The important characteristic of the differential game (1)–(4) is that neither the opponent's state nor his control variable enter the state equation of either player. Differential games with this property are referred to as games with separated state equations.

In a game with separated state equations a given control function of the rival results in a given state variable of the rival. For instance let us look at player 1 who treats $v(\cdot)$ as given. For a given control $v(\cdot)$ there exists a unique solution of the state equation (4) so that the statements "for a given $v(\cdot)$" and "a given $y(\cdot)$" are equivalent. If that is the case, then the dynamic game can be restated as a problem of the Calculus of Variations by substituting (3) and (4) into (1) and (2) to obtain

$$\operatorname*{extr}_{x(\cdot)} \left\{ J_1 = \int_{t_1}^{t_2} I_1 \left\{ t, x(t), y(t), \frac{x'(t) - m_1[t, x(t)]}{l_1[t, x(t)]}, \frac{y'(t) - m_2[t, y(t)]}{l_2[t, x(t)]} \right\} dt \right\}$$

given $y(\cdot)$, and

$$\operatorname*{extr}_{y(\cdot)} \left\{ J_2 = \int_{t_1}^{t_2} I_2 \left\{ t, x(t), y(t), \frac{x'(t) - m_1[t, x(t)]}{l_1[t, x(t)]}, \frac{y'(t) - m_2[t, y(t)]}{l_2[t, x(t)]} \right\} dt \right\}$$

given $x(\cdot)$, and subject to the boundary conditions (5) and (6).

In what follows we will be dealing with problems that can be reduced to the following two player games with scalar state variables and finite time horizon. For the sake of convenience we restrict out attention to minimization problems

$$\min_{x(\cdot)} \left\{ J_1 = \int_{t_1}^{t_2} F_1[t, x(t), x'(t), y(t), y'(t)]dt \right\} \tag{7}$$

for a given $y(\cdot)$, and

$$\min_{y(\cdot)} \left\{ J_2 = \int_{t_1}^{t_2} F_2[t, x(t), x'(t), y(t), y'(t)]dt \right\} \tag{8}$$

for a given $x(\cdot)$, and subject to the prescribed boundary conditions

$$x(t_1) = x_1, \quad x(t_2) = x_2, \tag{9}$$

$$y(t_1) = y_1, \quad y(t_2) = y_2. \tag{10}$$

Definition *A time path of state variables $[x^*(\cdot), y^*(\cdot)]$ is called an open-loop Nash equilibrium of the game (7)–(10) if and only if $x^*(\cdot)$ minimizes*

$$J_1 = \int_{t_1}^{t_2} F_1[t, x(t), x'(t), y^*(t), y^{*\prime}(t)]dt$$

subject to the boundary condition (9), and $y^*(\cdot)$ minimizes

$$J_2 = \int_{t_1}^{t_2} F_2[t, x^*(t), x^{*\prime}(t), y(t), y'(t)]dt$$

subject to the boundary condition (10).

Lemma *Let $x = z_1(t, \tilde{x})$ be a transformation having a unique inverse $\tilde{x} = \tilde{z}_1(t, x)$ for all $t \in [t_1, t_2]$ such that there is a one-to-one correspondence*

$$x(t) \Longleftrightarrow \tilde{x}(t)$$

for all $x(\cdot)$ satisfying boundary condition (9) and $\tilde{x}(\cdot)$ satisfying

$$\tilde{x}(t_1) = \tilde{z}_1(t_1, x_1), \quad \tilde{x}(t_2) = \tilde{z}_2(t_2, x_2).$$

If for a given $y^*(\cdot)$ the transformation $x = z_1(t, \tilde{x})$ is such that there exists a functional identity of the form

$$F_1[t, x(t), x'(t), y^*(t), y^{*\prime}(t)] - F_1[t, \tilde{x}(t), \tilde{x}'(t), y^*(t), y^{*\prime}(t)] = \frac{d}{dt}H^1[t, \tilde{x}(t)] \tag{11}$$

then, if $\tilde{x}^*(\cdot)$ yields an extremum of J_1 with $\tilde{x}^*(\cdot)$ satisfying the transformed boundary conditions, $x^*(\cdot)$ with $x^*(t) = z_1[t, \tilde{x}^*(t)]$ yields an extremum for J_1 with the boundary conditions (9).

Let $y = z_2(t, \tilde{y})$ be a transformation having a unique inverse $\tilde{y} = \tilde{z}_2(t, y)$ for all $t \in [t_1, t_2]$ such that there is a one-to-one correspondence

$$y(t) \Longleftrightarrow \tilde{y}(t)$$

for all $y(\cdot)$ satisfying boundary condition (10) and $\tilde{y}(\cdot)$ satisfying

$$\tilde{y}(t_1) = \tilde{z}_2(t_1, y_1), \quad \tilde{y}(t_2) = \tilde{z}_2(t_2, y_2).$$

If for a given $x^*(\cdot)$ the transformation $y = z_2(t, \tilde{y})$ is such that there exists a functional identity of the form

$$F_2[t, x^*(t), x^{*\prime}(t), y(t), y'(t)] - F_2[t, x^*(t), x^{*\prime}(t), \tilde{y}(t), \tilde{y}'(t)] = \frac{d}{dt} H^2[t, \tilde{y}(t)] \qquad (12)$$

then, if $\tilde{y}^*(\cdot)$ yields an extremum of J_2 with $\tilde{y}^*(\cdot)$ satisfying the transformed boundary conditions, $y^*(\cdot)$ with $y^*(t) = z_2[t, \tilde{y}^*(t)]$ yields an extremum for J_2 with the boundary conditions (10). Moreover then $[x^*(\cdot), y^*(\cdot)]$ with $x^*(t) = z_1[t, \tilde{x}^*(t)]$, and $y^*(t) = z_2[t, \tilde{y}^*(t)]$ is an open-loop Nash equilibrium.

Corollary 1 *The integrands $F_i(\cdot)$ and the transformation in the Lemma must be such that the left-hand sides of (11) and (12) are linear in $\tilde{x}'(t)$ and $\tilde{y}'(t)$, respectively.*

Corollary 2 *For integrands $F_i(\cdot)$ of the form*

$$\begin{aligned} F_1[t, x(t), x'(t), y^*(t), y^{*\prime}(t)] &= a_1[t, x(t), y^*(t)][x'(t)]^2 + b_1[t, x(t), y^*(t)]x'(t) \\ &\quad + c_1[t, x(t), y^*(t)], \qquad (13) \\ a_1[t, x, y^*(t)] &\neq 0 \ \forall (t, x) \in [t_1, t_2] \times R \\ F_2[t, x^*(t), x^{*\prime}(t), y(t), y'(t)] &= a_2[t, y(t), x^*(t)][y'(t)]^2 + b_2[t, y(t), x^*(t)]y'(t) \\ &\quad + c_2[t, y(t), x^*(t)], \qquad (14) \\ a_2[t, y, x^*(t)] &\neq 0 \ \forall (t, y) \in [t_1, t_2] \times R \end{aligned}$$

the class of admissible transformations must satisfy the following partial differential equations

$$\left[\frac{\partial z_1(t, \tilde{x})}{\partial \tilde{x}}\right]^2 a_1[t, z_1(t, \tilde{x}), y^*(t)] = a_1[t, \tilde{x}, y^*(t)], \qquad (15)$$

$$\left[\frac{\partial z_2(t, \tilde{y})}{\partial \tilde{y}}\right]^2 a_2[t, z_2(t, \tilde{y}), x^*(t)] = a_2[t, \tilde{y}, x^*(t)]. \qquad (16)$$

If the coefficients of $[x'(t)]^2$ and $[y'(t)]^2$ are nonzero functions of t only, i.e., $a_1[t, y^*(t)]$ and $a_2[t, x^*(t)]$, then

$$\frac{\partial z_1(t, \tilde{x})}{\partial \tilde{x}} = \pm 1,$$

$$\frac{\partial z_2(t, \tilde{y})}{\partial \tilde{y}} = \pm 1,$$

so that admissible transformations must be of the form

$$x = z_1(t, \tilde{x}) = \pm \tilde{x} + f(t),$$

$$y = z_1(t, \tilde{y}) = \pm \tilde{y} + g(t).$$

Example Consider the following simple linear quadratic differential game

$$\min_{x(\cdot)} \left\{ J_1 = \int_{t_1}^{t_2} [x^2(t) + y^2(t) + [x'(t)]^2 dt \right\} \tag{17}$$

given $y(\cdot)$, and

$$\min_{y(\cdot)} \left\{ J_2 = \int_{t_1}^{t_2} [x^2(t) + y^2(t) + [y'(t)]^2 dt \right\} \tag{18}$$

given $x(\cdot)$, and subject to the boundary conditions (9) and (10). Now we apply the Lemma of Part II in order to deduce directly the open-loop Nash equilibrium $[x^*(\cdot), y^*(\cdot)]$ of the game (17)–(18).

Applying Corollary 2, and choosing the + sign, we have transformations of the form

$$x(t) = \bar{x}(t) + f(t), \tag{19}$$

$$y(t) = \bar{y}(t) + g(t). \tag{20}$$

Using these transformations in the functional identities (11) and (12), the identities resulting from (11) and (12) (recall Corollary 2 of Part I) imply

$$H_t^1(t, \bar{x}) = 2\bar{x}f(t) + f^2(t) + f'^2(t),$$

$$H_{\bar{x}}^1(t, \bar{x}) = 2f'(t),$$

and

$$H_t^2(t, \bar{y}) = 2\bar{y}g(t) + g^2(t) + g'^2(t),$$

$$H_{\bar{y}}^2(t, \bar{x}) = 2g'(t).$$

Now, if we apply the identities

$$H_{t\bar{x}}^1(t, \bar{x}) \equiv H_{\bar{x}t}^1(t, \bar{y}), \; H_{t\bar{y}}^2(t, \bar{y}) \equiv H_{\bar{y}t}^2(t, \bar{y}),$$

we obtain a differential equation for $f(\cdot)$ and $g(\cdot)$, respectively,

$$f''(t) - f(t) = 0,$$

$$g''(t) - g(t) = 0.$$

The general solutions of these equations are given by

$$f(t) = c_1 e^t + c_2 e^{-t},$$

$$g(t) = d_1 e^t + d_2 e^{-t},$$

where c_i and d_i are constants of integration.

With (17) and (18) for $(\tilde{x}, \tilde{y})$ it is easily seen that $\tilde{x}^*(t) \equiv 0$ is the best response of player 1 to $y^*(\cdot)$, and $\tilde{y}^*(t) \equiv 0$ is the best response of player 2 to $x^*(\cdot)$. Hence, we get

$$x^*(t) = f(t) = [c_1 e^t + c_2 e^{-t}],$$

$$y^*(t) = g(t) = [d_1 e^t + d_2 e^{-t}],$$

as the open-loop Nash equilibrium for the original problem, where the constants of integration are determined through the boundary conditions (9) and (10). This solution corresponds exactly to the one derived via the Maximum Principle.

Conclusions

We have used coordinate transformations as introduced in Ref. 1 and extended in Ref. 3 to derive open-loop Nash equilibria for finite time horizon differential games. We presented the general theory for the case of differential games with separated state equations (see Ref. 4), and discussed a simple example. In Ref. 4, one of the strengths of coordinate transformations was demonstrated by means of a transboundary pollution game. Even without global curvature assumptions on the functions of the model we were able to derive a Nash equilibrium. While this property of coordinate transformations is very attractive there are many open issues for future research. In particular we plan to generalize the theory of coordinate transformations beyond the class of differential games with separated state equations and to explore its use in deriving closed-loop Nash equilibria.

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