Cubo A Mathematical Journal
Vol.05/N$^{\underline{o}}$03 - OCTOBER 2003

Non-Commuting Numbers and Functions

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God made the integers, the rest are works of man.

L. KRONECKER

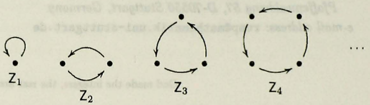
The concept of number, in the first instance that of a natural number, belongs to the fundamentals of mathematics since its earliest manifestation as a component of human culture. When regarded as *ordinal* numbers, natural numbers are abstracted from the process of counting. By way of a second abstraction, disregarding the order of counting, natural numbers represent the *cardinalities* of finite non-empty sets. Cardinal and ordinal numbers do not yet fall apart in the world of finite numbers, and this is not the only respect in which natural numbers are fundamental.

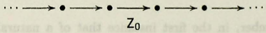
The observation that counting can be inverted soon leads to the discovery of negative numbers, and thus to the ring of integers. It seems to be a popular opinion that rings have to be commutative as long as numbers are in the focus of study. Our article deals with this matter, and our first aim will be to show that the integers have a structural predisposition to non-commutativity, though in an embryonic state, waiting to become unfolded. The following sections will be concerned with unfolding, paving a way that eventually borders on the shores of present research.

1 The Ring $\mathbb{Z}$

Before dealing with non-commutative entities, let us have a closer look upon $\mathbb{Z}$, the ring of integers. Admittedly, this ring is commutative. But why at all is $\mathbb{Z}$ a ring?

The answer will direct our attention to module theory. Here is a quick formalization of counting. Forward and backward counting should be unique, so we consider a set Z with a bijection $\sigma\colon Z \to Z$. For any $a \in Z$, counting one step forward leeds to the element $\sigma(a)$. We assume Z to be *connected*, i. e. $Z \neq \varnothing$, and there is no partition $Z = A \cup B$ into non-empty disjoint sets A, B invariant under σ. Let us call such a system (Z, σ) a *cycle*. The structure of a cycle is easily determined. For any natural number n there is a finite cycle $\mathsf{Z}_n = \{a_1, \ldots, a_n\}$ with $\sigma(a_i) = a_{i+1}$ for $i < n$ and $\sigma(a_n) = a_1$. Furthermore, there is an infinite cycle $\mathsf{Z}_0 = \{\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots\}$ with $\sigma(a_i) = a_{i+1}$ for all $i \in \mathbb{Z}$.





For cycles Z and Z' we define a *homomorphism* $f\colon Z \to Z'$ as a map which respects counting, i. e.

$$f(\sigma a) = \sigma f(a) \tag{1}$$

for all $a \in Z$. The concept of homomorphism applies to all kinds of mathematical structures (groups, rings, partially ordered sets, etc.). It simply means that the operations and relations that constitute the structure are preserved. A bijective homomorphism is also called an *isomorphism*. Obviously, every cycle Z admits an isomorphism $Z \xrightarrow{\sim} \mathsf{Z}_n$ for some $n \in \mathbb{N} := \{0, 1, 2, \ldots\}$. To have a convenient notation, let us fix an element 0 for any cycle Z_n, and denote $\sigma^m(0)$ by m for all $m \in \mathbb{Z}$. (Note that σ is invertible; so negative powers of σ are to be taken as powers of the inverse σ^{-1}.) For the finite cycles Z_n, this leads to an overlap of notation such that 0 coincides with all multiples of n. While synonyms do not oppose logic, this has the advantage that σ can be defined for all cycles Z_n by the same rule $\sigma(m) = m+1$.

For a cycle Z, let $\mathrm{End}(Z)$ denote the set of homomorphisms $Z \to Z$. (Homomorphisms between the same object are also called *endomorphisms*.) Amazingly, $\mathrm{End}(Z)$ is again a cycle. Namely, there is a natural bijection

$$\mathrm{End}(Z) \xrightarrow{\sim} Z \tag{2}$$

which maps $f \in \mathrm{End}(Z)$ to $f(0)$. So the cycle structure of Z can be transported to $\mathrm{End}(Z)$. What is more, $\mathrm{End}(Z)$ is even a group with composition of maps as group operation:

$$(f+g)(m) := f(g(m)). \tag{3}$$

I don't believe in accidents of nature. But here is a case where map composition is commutative. So it is justified to denote the group operation of $\mathrm{End}(Z)$ by $+$. The attentive reader will notice that we have got the collection of cyclic groups $C_n := \mathrm{End}(Z_n)$ in this way.

Now a homomorphism between groups is defined as a map f which respects the group operation (say, $+$):

$$f(a+b) = f(a) + f(b). \tag{4}$$

This automatically implies $f(0) = 0$ for the neutral element 0. So let us define

$$\mathbb{Z}_n := \mathrm{End}(C_n) \tag{5}$$

for all $n \in \mathbb{N}$. The group operation on C_n leads to an operation on $\mathbb{Z}_n$, namely $(\forall f, g \in \mathbb{Z}_n)$,

$$(f+g)(m) := f(m) + g(m). \tag{6}$$

Since C_n is abelian, this makes $\mathbb{Z}_n$ into an abelian group. Moreover, the composition of maps leads to another operation on $\mathbb{Z}_n$:

$$(f \cdot g)(m) := f(g(m)). \tag{7}$$

So $\mathbb{Z}_n$ becomes a ring with the identical map as unity. Again, we encounter the "accident of nature" that the operation (7) is commutative! Among these rings, we find $\mathbb{Z}_0 = \mathbb{Z}$, our ring of integers, and its residue class rings $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ which are of basic importance in number theory.

So far, we have shown that the ring $\mathbb{Z}$ emerges from the counting operation by a twofold application of "End". We shall see in a minute that endomorphism rings play a particular part in representation theory. Most of them are non-commutative. We conclude this section with the remark that the ring $\mathbb{Z}$ is distinguished among all (not necessarily commutative) rings.

Proposition 1. *For every ring R there is a unique homomorphism $f\colon \mathbb{Z} \to R$.*

Note that a ring homomorphism is defined as a map f that preserves addition (4) and multiplication, and the unit element, i. e. $f(1) = 1$.

2 Non-Commutative Arithmetics

We may have got a suspicion that commutativity of $\mathbb{Z}$ results from a couple of accidents. To see more, let us try to classify finite abelian groups up to isomorphism. For any such group A, Proposition 1 provides a unique homomorphism

$$f : \mathbb{Z} \to \mathrm{End}(A). \tag{8}$$

This means that for $n \in \mathbb{Z}$ and $a \in A$, there is an element $na := f(n)(a)$ subject to the following rules $(m, n \in \mathbb{Z}, a, b \in A)$:

$$\left.\begin{aligned} n(a+b) &= na + nb \\ (m+n)a &= ma + na \\ (mn)a &= m(na) \\ 1 \cdot a &= a. \end{aligned}\right\} \tag{9}$$

We say that $\mathbb{Z}$ operates on A such that A becomes a $\mathbb{Z}$*-module*. More generally, when a homomorphism (8) is given with $\mathbb{Z}$ replaced by a ring R, then (9) holds for $m, n \in R$. Then A is said to be an R*-module*. We call an R-module M *finitely generated* if there are finitely many elements $x_1, \ldots, x_n \in M$ such that every $x \in M$ can be written in the form

$$x = r_1 x_1 + \cdots + r_n x_n \tag{10}$$

with suitable $r_1, \ldots, r_n \in R$. In particular, every finite abelian group A is finitely generated as a $\mathbb{Z}$-module. For a given prime p, let A_p denote the subgroup of all $x \in A$ such that $p^m x = 0$ for some $m \in \mathbb{N}$. Then there are finitely many primes $p_1, \ldots, p_r$ such that every $x \in A$ has a unique representation

$$x = x_1 + \cdots + x_r \tag{11}$$

with $x_i \in A_{p_i}$. We express this by saying that A is a *direct sum*

$$A = A_{p_1} \oplus \cdots \oplus A_{p_r} \tag{12}$$

of the subgroups A_{p_i}. This reduces our problem to the case of a finite abelian group A with $p^m A = 0$ for some prime p.

Since A is finitely generated, there exists a surjective homomorphism

$$p : F_0 \to A \tag{13}$$

with F_0 *free*, i. e. $F_0 = A_1 \oplus \cdots \oplus A_r$, $A_i \cong C_0$ (the infinite cyclic group). The *kernel* $F_1 := \{x \in F_0 \mid p(x) = 0\}$ of p is again free. So we obtain a pair $\binom{F_0}{F_1}$ of finitely generated free abelian groups with F_1 a subgroup of F_0, and $p^m F_0 \subset F_1$. Conversely, every such pair gives rise to a finite abelian group A with an epimorphism (13) having F_1 as kernel.

The pairs $\binom{F_0}{F_1}$ can be interpreted as modules over the ring

$$\Lambda = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ p^m\mathbb{Z} & \mathbb{Z} \end{pmatrix}. \tag{14}$$

The elements of Λ are matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with entries in $\mathbb{Z}$ such that c is divisible by p^m. A pair $\binom{F_0}{F_1}$ with $p^m F_0 \subset F_1 \subset F_0$ becomes a Λ-module via ordinary matrix multiplication. The condition that F_0 and F_1 are free means that $\binom{F_0}{F_1}$ is a Λ-*lattice*, i. e. a Λ-module that is free as a $\mathbb{Z}$-module. With respect to multiplication, Λ itself is a Λ-lattice. For this reason, Λ is said to be an *order*. There is one indecomposable Λ-lattice, namely $\binom{\mathbb{Z}}{\mathbb{Z}}$, which corresponds to the abelian group $A = 0$. To get rid of that Λ-lattice, we replace Λ by the overorder

$$\Lambda' = \begin{pmatrix} \mathbb{Z} & p^{-1}\mathbb{Z} \\ p^m\mathbb{Z} & \mathbb{Z} \end{pmatrix}. \tag{15}$$

Proposition 2. *Up to isomorphism, there is a one-to-one correspondence between finite abelian groups A with $p^m A = 0$ and Λ'-lattices. The indecomposable Λ'-lattices are*

$$\binom{\mathbb{Z}}{p\mathbb{Z}}, \binom{\mathbb{Z}}{p^2\mathbb{Z}}, \dots, \binom{\mathbb{Z}}{p^m\mathbb{Z}}. \tag{16}$$

So the problem to determine finite $\mathbb{Z}$-modules has turned into a problem on lattices over a ring Λ' which is certainly non-commutative.

3 Curves and Orders

A plane algebraic curve C is a set of points $(x, y) \in \mathbb{R}^2$ satisfying a polynomial equation $f(x, y) = 0$. For example, let $C_{(m)}$ be given by the polynomial $f_m(x, y) = y^2 - x^{2m+1}$, $m \in \mathbb{N}$. So $C_{(0)}$ is a parabola, and $C_{(1)}$ the Neil parabola[1]. For $m \geqslant 1$, the curve $C_{(m)}$ has a singularity at the origin O. A singularity can be regarded as an infinitesimal distortion. To reveal its structure, we may introduce a t-axis perpendicular to the plane and take $t = y/x$ as an additional equation. Then we obtain a space curve $\widetilde{C}_{(m)}$ such that the projection along the t-axis yields a bijection between $\widetilde{C}_{(m)}$ and $C_{(m)}$ except $x = 0$ (where t is undefined). Excluding $x = 0$, the curve $\widetilde{C}_{(m)}$ is given by the equation $t^2 - x^{2m-1} = 0$, whence $\widetilde{C}_{(m)}$ is of the form $C_{(m-1)}$. So after m steps, the singularity disappears.

This so-called σ-*process* of desingularization can be interpreted in terms of ring theory. Let $\mathbb{R}[x, y]$ denote the ring of polynomials f in two variables x, y with real coefficients. We

[1] W. Neil determined the arc length of $C_{(1)}$ in 1657.

have already seen in §1 that for some non-zero $n \in \mathbb{N}$, a formal identification $n \equiv 0$ turns $\mathbb{Z}$ into a finite ring $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. Similarly, we may form the ring $\mathcal{O}_f := \mathbb{R}[x,y]/f\mathbb{R}[x,y]$ by stipulating $f = 0$. (Thus $g, h \in \mathbb{R}[x,y]$ have to be identified whenever $g - h$ is a multiple of f.) Every point (x_0, y_0) on the curve C with equation $f(x,y) = 0$ gives rise to a ring homomorphism P: $\mathcal{O}_f \to \mathbb{R}$ with $P(g) := g(x_0, y_0)$, and vice versa. The curve C can thus be recovered from the ring $\mathcal{O}_f$ as the collection $\operatorname{Spec} \mathcal{O}_f$ of ring homomorphisms P, also called the *spectrum* of $\mathcal{O}_f$. In other words, the ring $\mathcal{O}_f$ is just an algebraic substitute for the curve C. Its elements $g \in \mathcal{O}_f$ can be regarded as polynomial functions on C. There is still some ambiguity caused by the fact that C can also be given by the equation $f(x,y)^2 = 0$. However, the ring $\mathcal{O}_{f^2}$ contains the non-zero element f satisfying $f^2 = 0$. As a function on C, such an element vanishes identically. A commutative ring without elements $a \neq 0$ satisfying $a^2 = 0$ is said to be *reduced*.

To analyse C in the near of a singularity at O, we replace $\mathbb{R}[x,y]$ by the ring $\mathbb{C}[[x,y]]$ of power series over $\mathbb{C}$ to get the local ring $\widehat{\mathcal{O}}_f$ which describes the vicinity of O. Note that $\operatorname{Spec} \widehat{\mathcal{O}}_f$ consists of just one point. We already defined an order as a ring Λ which is finitely generated and free as a $\mathbb{Z}$-module. The ring $\widehat{\mathcal{O}}_{f_m}$ is free as a $\mathbb{C}[[x]]$-module with generators 1 and y. Therefore, it can be regarded as an *R-order* with $R := \mathbb{C}[[x]]$. It is not hard to show that there are exactly $m+1$ indecomposable $\widehat{\mathcal{O}}_{f_m}$*-lattices* (i. e. finitely generated R-free $\widehat{\mathcal{O}}_{f_m}$-modules), up to isomorphism. One step of desingularization leads from $\widehat{\mathcal{O}}_{f_m}$ to the overorder $\widehat{\mathcal{O}}_{f_{m-1}}$. By this process, we loose just one indecomposable $\widehat{\mathcal{O}}_{f_m}$-lattice, namely $\widehat{\mathcal{O}}_{f_m}$ itself, while the other indecomposables can be regarded as $\widehat{\mathcal{O}}_{f_{m-1}}$-lattices. Proceeding in this way, we end up with the regular ring $\widehat{\mathcal{O}}_{f_0} \cong \mathbb{C}[[y]]$.

Virtually the same consideration applies to the non-commutative $\mathbb{Z}$-order (14). By Proposition 2, Λ has exactly $m+1$ indecomposables, and there is a series $\Lambda = \Lambda_m \subset \Lambda_{m-1} \subset \cdots \subset \Lambda_1 \subset \Lambda_0$ of overorders ending with Λ_0, a non-commutative regular order. Here, a $\mathbb{Z}$-order Λ is said to be *regular* if every Λ-lattice is *projective*, i. e. a direct summand of a free Λ-lattice $\Lambda \oplus \cdots \oplus \Lambda$. (For commutative $\mathbb{C}[x]$-orders this property says that the corresponding curve has no singular points.) Generalizing reduced commutative rings, we call Λ *semiprime* if there is no non-zero $g \in \Lambda$ with $ghg = 0$ for all $h \in \Lambda$. Now the existence of a desingularization generalizes to

Proposition 3 (see [16]). *Every semiprime $\mathbb{Z}$-order has a regular overorder.*

In the following section, we will consider orders Λ corresponding to surfaces instead of curves. For arbitrary dimensions, there is an analogue of a Λ-lattice called a *Cohen-Macaulay module*.

4 The Desingularization Quiver[2]

The curve singularities considered in §3 belong to the classes $A_2, A_4, A_6 \ldots$ among the so-called *simple* singularities. The other classes are $A_1, A_3, A_5, \ldots, D_4, D_5, D_6, \ldots, E_6, E_7, E_8$, according to a pattern that arises in various branches of mathematics. For example, the class E_8 consists of the singularities (at O) of the form

$$f_n(z_1, \ldots, z_n) = z_1^5 + z_2^3 + z_3^2 + \cdots + z_n^2 = 0. \tag{17}$$

This class of singularities is interesting in several respects. For $n = 3$, equation (17) describes a complex surface $S \subset \mathbb{C}^3$. It was known to invariant theorists of the 19th century [13] that the symmetry group G of the icosahedron is a homomorphic image of the *binary icosahedral group* $\widehat{G}$, a group of 2×2 matrices, such that the elements of $\widehat{G}$ are mapped in pairs to the elements of G. They found three $\widehat{G}$-invariant polynomials $\Phi_{12}, \Phi_{20}, \Phi_{30} \in \mathbb{C}[x, y]$ which generate the ring $\mathbb{C}[x, y]^{\widehat{G}}$ of $\widehat{G}$-invariant polynomials in $\mathbb{C}[x, y]$. Furthermore, these polynomials are related by the equation

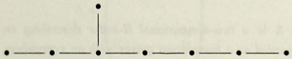
$$\Phi_{12}^5 + \Phi_{20}^3 + \Phi_{30}^2 = 0, \tag{18}$$

which shows that $\mathbb{C}[x, y]^{\widehat{G}}$ is isomorphic to the ring $\mathcal{O}_{f_3}$ (over $\mathbb{C}$) of a type E_8 surface.

Among all the rings $\widehat{\mathcal{O}}_f$ of an isolated surface singularity $f = 0$, only $\widehat{\mathcal{O}}_{f_3}$ has the property that its elements admit a unique factorization into prime elements [3].. The intersection of the zero set $f_5 = 0$ with the sphere $S^9 = \{z \in \mathbb{C}^5 \mid |z| = 1\}$ is topologically equivalent with the 7-dimensional sphere S^7, but its differential structure differs from that of S^7.

In §3 we applied the σ-process to singular curves lying on a regular surface. The same process leads to a desingularization of simple surface singularities. After finitely many steps we obtain a surjection $\pi\colon \widetilde{X} \to X$ of surfaces with $\widetilde{X}$ regular. Then $\pi^{-1}(O) = \bigcup_{i=1}^m L_i$ with lines L_i which may be assumed to intersect transversally. The *desingularization graph*, given by the vertices $1, \ldots, m$ with an edge between different i and j whenever $L_i \cap L_j \neq \varnothing$, determines the type of singularity. For example, the desingularization graph of the surface S with equation $f_3 = 0$ is of the form

E_8 :



(19)

There is a module-theoretic interpretation of this graph. By the above, the ring $\widehat{\mathcal{O}}_{f_3}$ can be regarded as a subring of $\mathbb{C}[[x, y]]$, generated by the invariants $\Phi_{12}, \Phi_{20}, \Phi_{30}$. As an $\widehat{\mathcal{O}}_{f_3}$-module, $\mathbb{C}[[x, y]]$ decomposes into $\widehat{\mathcal{O}}_{f_3}$ plus eight indecomposable modules M_i which can be

[2]Oriented graphs (= collections of arrows) were called *quivers* by P. Gabriel (1970) who used them to attack problems in representation theory.

associated with the lines L_i and thus with the nodes of E_8. Amazingly, $\widehat{\mathcal{O}}_{f_3}$ and $M_1, \ldots, M_8$ form a complete set of indecomposable Cohen-Macaulay modules over $\widehat{\mathcal{O}}_{f_3}$. Furthermore, the homomorphisms between the M_i can be read off from the desingularization graph. Namely, let us call a non-isomorphism $M_i \to M_j$ *irreducible* if it cannot be written as a non-trivial composition $M_i \to \bigoplus M_k \to M_j$. Then for different nodes i, j, there exists an irreducible homomorphism $M_i \to M_j$ if and only if i and j are connected by an edge.

For a wide class of rings Λ (e. g., orders), a concept of Cohen-Macaulay module is available. Then the irreducible homomorphisms between indecomposable Cohen-Macaulay modules constitute an oriented graph $\mathbb{A}(\Lambda)$. For the ring $\widehat{\mathcal{O}}_f$ of a simple surface singularity $f = 0$, the arrows in $\mathbb{A}(\widehat{\mathcal{O}}_f)$ occur in pairs $i \rightleftarrows j$, and these pairs, except the one connected with $\widehat{\mathcal{O}}_f$, correspond to the edges of the desingularization graph. So in dimension two, the graph $\mathbb{A}(\Lambda)$ should provide a picture of a (possibly non-commutative) "singularity".

Since R-orders Λ are finitely generated over a commutative ring R, the concept of *isolated* singularity can be generalized to R-orders. If an R-order Λ has this property, then $\mathbb{A}(\Lambda)$ looks like a piece of knitting. For each vertex M which is non-projective as a Λ-module, there exists a unique vertex τM such that for each arrow $X \to M$ there is an arrow $\tau M \to X$ and vice versa. The vertices which are not of the form τM are dual to the projective modules and are called *injective*. So $\mathbb{A}(\Lambda)$ is made up of *meshes*

$$\begin{array}{ccccc} & & M_1 & & \\ & \nearrow & & \searrow & \\ \tau M & & \vdots & & M \\ & \searrow & & \nearrow & \\ & & M_s & & \end{array} \qquad (20)$$

with possible multiplicities among the M_i. This important structure, discovered by Auslander and Reiten in the early seventies, furnished with Gabriel's term for a graph with arrows, has become well-known as the *Auslander-Reiten quiver*. More generally, an oriented graph Q with vertex set Q_0 and a bijection $\tau\colon\ Q_0 \setminus \mathcal{P} \xrightarrow{\sim} Q_0 \setminus \mathcal{I}$ leading to a mesh structure as described above is called a *translation quiver*. Similar to Auslander-Reiten quivers, the vertices in $\mathcal{P}$ (resp. $\mathcal{I}$) are called *projective* (resp. *injective*). For *two-dimensional* R-orders Λ (i. e. those with $\dim R = 2$), even the projective and injective vertices in $\mathbb{A}(\Lambda)$ are connected by meshes.

Proposition 4. *Let Λ be a two-dimensional R-order describing an isolated singularity. Then $\mathbb{A}(\Lambda)$ can be regarded as a translation quiver with an everywhere defined bijection τ.*

This holds since Auslander-Reiten quivers are in a sense "two-dimensional". By contrast, if $\dim \Lambda \neq 2$, the projective vertices, i. e. the direct summands of Λ, give rise to exceptions.

5 Non-Commutative Curves

By a striking result of Buchweitz, Greuel, Schreyer [4], and Knörrer [14], the simple singularities $f = 0$ can be characterized by the property that $\mathbb{A}(\widehat{\mathbb{O}}_f)$ is finite. By analogy, a non-commutative singularity, given by a ring Λ, should be regarded as "simple" if the number of vertices in $\mathbb{A}(\Lambda)$ is finite. One of the basic still unsolved problems in representation theory is to determine the orders Λ with this property (see [17], p. 2, for one-dimensional orders). Before non-commutative singularities can be tackled, an understanding of regular rings Λ is needed. For a curve, surface, or higher-dimensional complex variety X given by an equation $f = 0$, regularity at O simply means that the ring $\widehat{\mathbb{O}}_f$ is isomorphic to a power series ring $\mathbb{C}[[z_1, \ldots, z_n]]$. Modern algebraic geometry has brought about a redefinition of regularity with no explicit reference to coordinates $z_1, \ldots, z_n$, nor to a base field like $\mathbb{C}$. In this way, the geometric behaviour of arbitrary commutative rings can be studied.

Indeed, our considerations in §1 cicle around the spectrum of $\mathbb{Z}$. Namely, Spec $\mathbb{Z}$ consists of the ring homomorphisms $P_p: \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ with p prime (i. e. with $\mathbb{Z}/p\mathbb{Z}$ a field), and $P_0: \mathbb{Z} \to \mathbb{Q}$. (Note that every homomorphism $\mathbb{Z} \to F$ into a field F factors through some P_p or P_0.) In this view, the integers can be regarded as "functions" on Spec $\mathbb{Z}$. Namely, every non-zero $n \in \mathbb{Z}$ with factorization

$$n = \pm p_1^{n_1} \cdots p_r^{n_r} \tag{21}$$

into primes p_i is uniquely determined by its values n_i at P_{p_i} and ± 1 at P_0.

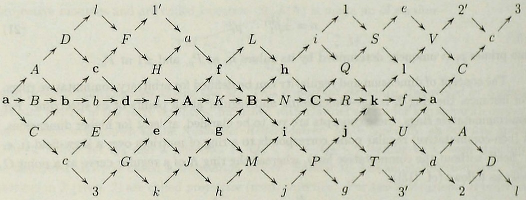
The concept of dimension and regularity can be defined for arbitrary commutative rings. For instance, the ring $\mathbb{Z}$ is one-dimensional and regular at all points of its spectrum. For non-commutative rings, these concepts are yet to be clarified, at least for higher dimensions. A (non-commutative) regular point corresponds to a ring of matrices over a skew-field (i. e. a "field" without the commutative law), whereas the ring Λ of a regular curve at a point O looks as follows (cf. [15]).

$$\Lambda = \begin{pmatrix} V & M & \cdots & M \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & M \\ V & \cdots & \cdots & V \end{pmatrix} \tag{22}$$

Here V is a discrete valuation ring in a skew-field with maximal ideal M. Discrete valuations can be generalized to higher dimensions and thus lead to a class of regular rings of type A_∞^∞ [19]. Presently, non-commutative regular rings are a highly active field of research, with important contributions made by Artin, Lusztig, Stafford, Tate, Van den Bergh, and many others. A discussion of those results would go beyond the scope of this article.

Up to dimension two, one can say that regularity is now well-understood. So let us embark for a trip towards the ocean of singularities. Here even dimension zero offers an abundance of singularities. A simple singularity of dimension 0 is specified by a *representation-finite* ring, that is, a ring Λ with only finitely many isomorphism classes of indecomposable Λ-modules such that every Λ-module is a direct sum of finitely generated modules. (It is known that this concept is left-right symmetric.) For K-orders Λ with K a field (these are better known as K*-algebras*), the possible finite Auslander-Reiten quivers $\mathbb{A}(\Lambda)$ have been determined by Igusa, Todorov [7, 8] and Brenner [2]. For two-dimensional orders, an exceptional class according to Proposition 4, the corresponding problem was solved by Reiten and Van den Bergh [18].

Just around the millennium, the finite Auslander-Reiten quivers belonging to one-dimensional orders have been characterized in a trilogy of papers [9, 10, 11] by a young Japanese mathematician, Osamu Iyama. Beyond that, with the methods developed there he was able to prove Auslander's conjecture on finite representation dimension of 0-dimensional orders [1], and Solomon's second conjecture [5] on ζ-functions of 1-dimensional orders. A typical Auslander-Reiten quiver of a one-dimensional $\mathbb{C}[[x]]$-order, $\Lambda = \begin{pmatrix} \mathbb{C}[[x]] & x\mathbb{C}[[x]] & x\mathbb{C}[[x]] \\ x^2\mathbb{C}[[x]] & \mathbb{C}[[x]] & x^2\mathbb{C}[[x]] \\ x^3\mathbb{C}[[x]] & x^3\mathbb{C}[[x]] & \mathbb{C}[[x]] \end{pmatrix}$, is given as follows.



The ends are to be put together with a twist; so $\mathbb{A}(\Lambda)$ looks like a Möbius strip. Taken in itself, i. e. as a 0-dimensional object, the singular point belongs to a 3×3 matrix ring A over the field $K = \mathbb{C}((x))$ of fractional power series. Thus Λ is a subring of A, and every Λ-lattice E generates an A-module KE. Since there is only one indecomposable A-module S, up to isomorphism, we have $KE \cong S^{\rho(E)}$ with a number $\rho(E)$ which is called the *rational rank* of E. The vertices of $\mathbb{A}(\Lambda)$ are depicted by four types of symbols: e. g. $a, 1, 1'$; A; a; **A**, according to the rational ranks 1; 2; 3; 4. The numbers 1, 2, 3 (resp. 1', 2', 3') refer to the projective (resp. injective) indecomposables.

By Iyama's criterion, finite Auslander-Reiten quivers of orders are characterized essentially by two conditions, (I) and (II), formulated in terms of *ladders*. For a vertex M let $\vartheta(M)$ denote the middle term $M_1 \oplus \ldots \oplus M_s$ in the mesh (20). For projective M we take those M_i with an arrow $M_i \to M$ and put $\tau M := 0$. Dually, $\vartheta^-(M) := M_1' \oplus \cdots \oplus M_t'$ according to the arrows $M \to M_i'$. We extend ϑ to direct sums via $\vartheta(M \oplus N) = \vartheta(M) \oplus \vartheta(N)$ and similarly for τ. Now we consider homomorphisms $f\colon M \to N$ such that $\vartheta N = M \oplus N'$ and define $M' := \tau N$. For suitable f this process can be repeated and leads to a *ladder*

$$\begin{array}{ccccccc} M''' & \longrightarrow & M'' & \longrightarrow & M' & \longrightarrow & M \\ \cdots \ \downarrow{\scriptstyle f'''} & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ N''' & \longrightarrow & N'' & \longrightarrow & N' & \longrightarrow & N \end{array} \tag{23}$$

Iyama's condition (I) says that for each injective vertex I, the ladder of $I \to \vartheta^- I$ ends with an arrow $\vartheta P \to P$ with P projective, and that the $\vartheta^- I, (\vartheta^- I)', (\vartheta^- I)'', \ldots$ (for all I) exhaust the whole collection of vertices. Condition (II) states, roughly speaking, that there are enough projective vertices. Of course, such a condition is necessary since (I) trivially holds if there are no projective vertices at all.

The reader may check condition (I) for the above $\mathbb{C}[[x]]$-order Λ. For example, the arrow $1' \to \vartheta^- 1'$ leads to a ladder

$$\begin{array}{ccccccccccccccccc} gQR\mathrm{j} & \longrightarrow & TS\mathrm{k} & \longrightarrow & Uc\mathrm{l} & \longrightarrow & \mathrm{a}V & \longrightarrow & BC & \longrightarrow & \mathrm{b} & \longrightarrow & \mathrm{c} & \longrightarrow & F & \longrightarrow & 1' \\ \cdots \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{kk}T & \longrightarrow & 3'1fU & \longrightarrow & dV\mathrm{a} & \longrightarrow & A2'C & \longrightarrow & \mathrm{b}c & \longrightarrow & bE & \longrightarrow & \mathrm{d} & \longrightarrow & H & \longrightarrow & a \end{array}$$

which, after 23 steps, ends with the arrow $i \to 1$. Similarly, the ladders starting with the second resp. third injective vertex end with the corresponding projective vertex after 29 resp. 39 steps.

For each mesh (20) in $\mathbb{A}(\Lambda)$, the relation $\rho(M) + \rho(\tau M) = \sum \rho(M_i)$ holds. If M is projective (injective), then $\rho(M) = \vartheta M$ (resp. $= \vartheta^- M$). Therefore, ρ is said to be an *additive function* on $\mathbb{A}(\Lambda)$. For orders of dimension 0 or 2, a characterization of finite Auslander-Reiten quivers can be given in terms of additive functions. Therefore, Iyama's work led to the problem ([12], 7.4) whether such a characterization is possible in dimension one. Last year, this question could be answered in the affirmative [20].

We already mentioned that 0-dimensional simple singularities admit a definition by means of representation-finite rings. The now existing combinatorial description of finite Auslander-Reiten quivers of one-dimensional orders Λ conveys a deep structural insight into the categories of Λ-lattices. This suggests a one-dimensional analogue of representation-finite rings which precisely specifies (non-commutative) simple curve singularities. We call these rings *lattice-finite*. Their Auslander-Reiten quivers are characterized as follows [20].

Theorem. *A finite translation quiver satisfying Iyama's condition (II) (enough projectives) occurs as an Auslander-Reiten quiver of a lattice-finite ring if and only if it admits an additive function with values in the set of natural numbers.*

Much more could be said about non-commutativity. But let us look back on our starting point. Is commutativity of $\mathbb{Z}$ a happy accident? I think we proved just the opposite. From a philosophical point of view, accidents adhere to essentials. By Proposition 1, the ring $\mathbb{Z}$ is initial. Number theorists agree that $\mathbb{Z}$ is the most fundamental, also the "hardest" ring. It is the essential ring in which all accidental non-commutative rings are rooted. In other words, commutativity is the essence, whereas non-commutative rings make up the embellishment. The reader who wants to know more about non-commutative rings is thus well-advised to acquire a solid knowledge in commutative algebra first (there are excellent textbooks, e. g., [6]), which makes a subsequent study of non-commutative rings all the more enjoyable.

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