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From Fibonacci Numbers to Symmetric Functions

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(To the memory of my teacher, Max Dehn)

Introduction

In this essay we are interested in the not so well-known theory that lies behind a very famous and interesting collection of results that have to do with the Fibonacci numbers. The interest in Fibonacci numbers has several facets, their interpretation as a growth sequence is of interest both in mathematics and in physics and biology. Their use in primality testing and their connection with the so-called Golden Ratio have been exploited by number theorists. Recreational mathematics has been much interested in their striking numerical properties such as the fact that, if the numbers are suitably indexed, those indexed by primes are themselves prime, and pairs of numbers indexed by relatively prime numbers are themselves relatively prime; and, so on for results which are constantly being added to.

The Fibonacci sequence has also been generalized to sequences of Fibonacci polynomials in two indeterminants which satisfy the same recursion formula [Ri, p.34]. There are many interesting relations among these polynomials which in turn produce the large number of well-advertised numerical relations [Ri],[HW],[Vo]. And, of course, there is a respected journal devoted to mathematics connected to them, *The Fibonacci Quarterly*. What is not so well-known is that this body of results is a part of a much more general and far-reaching theory that impinges on the theory of multiplicative arithmetic functions, on the theory of equations and symmetric functions and even reaches into representation theory of groups and algebras, and into other extensions of these subjects. That is, the lore of Fibonacci numbers is just the tip of an iceberg. It is this iceberg that we wish to have a look at here.

This is an expository essay and will not be organized into the format of theorem and proof, though many results will be stated and often the key ideas of proofs will be indicated. But the main object of the paper is to make available in a unified way the collection of concepts which are behind some well-known results and to point out the richness of their applications. There are some results here which have only been recently introduced into the literature, and there is implicit in the presentation directions for further research.

The reference list includes elementary sources from which basic theory can be learned, books and articles which give applications of these ideas inside an outside of mathematics, as well as some reference to current research work. The list is hardly exhaustive, but is suggestive and each source contains useful further bibliography.

1 Fibonacci and Lucas Sequences

We begin with the polynomial $p(x) = x^2 - x - 1$. This is one of the more famous polynomials in the history of mathematics. It's roots give a solution to the problem first posed and solved—by a geometrical construction—by the Pythagoreans: construct a rectangle whose width is to its length as its length is to the sum of its length and width, $w : l = l : l + w$. Labelling the two roots of the polynomial α and β and letting α be the positive root, taking $w = 1$, we find that $l = \alpha$ will do the job; or more generally, we choose the ratio of the length to the width to be α. The negative root, β, is the negative reciprocal of α, thus $\mid \beta \mid = \frac{w}{l}$. In the 16th century α was given the name ***golden ratio*** or ***golden section***, possibly due to a remark made by Johannes Kepler (see [Bo],p.53, for a discussion of the Pythagorean discovery and the Kepler remark. For an interesting discussion of the golden section in connection with the mathematics used in the building of medieval cathedrals see [McC]).

A remarkable property of a "golden" rectangle is that it is self-replicating, in the sense that if a square of side l is erected on the rectangle R , then a new rectangle of length $w + l$ and width l is formed which solves the same problem.

Clearly, this process may be continued indefinitely (fig.1), and, in fact, at each stage there are choices. The picture that results is that of a collection of symmetrical "spirals" emmanating from a central point that might remind one of the spiral distribution of seeds in a sunflower—and for good reason. For this is a pattern that is typical of many biological growth processes [Th].

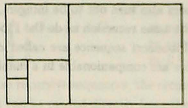

Figure 1

Note that this recursion process gives rise to the equalities

$$\frac{w}{l} = \frac{l}{w+l} = \frac{w+l}{w+2l} = \frac{w+2l}{2w+3l} = \frac{2w+3l}{3w+5l} = \dots = \frac{f_{m-1}w + f_m l}{f_m + f_{m+1}}.$$

The sequence of numbers f_m being $(1, 1, 2, 3, 5, 8, 13, \dots)$ which satisfy the recursion relation $f_{m+1} = f_m + f_{m-1}$, that is, the Fibonacci numbers which arise from the rabbit problem posed by Leonard of Pisa (Fibonacci) in his text, Liber Abaci, first published in 1202.

It will be useful to have a look at some of the relations between the Fibonacci sequence and the golden ratio. For starters we shall look at the powers of the two roots α and β of $p(x)$.

It is clear that we have $\alpha^2 = \alpha + 1$. And it is also clear that by iteration we have

$$\alpha^{m+1} = \phi_m \alpha + \phi_{m-1}, \tag{1.1}$$

and, similarly,

$$\beta^{m+1} = \phi_m \beta + \phi_{m-1}, \tag{1.2}$$

where ϕ_m and ϕ_{m-1} are non-negative integers. It is an easy exercise to show that $\phi_m = f_m$, that is, the coefficients are just the Fibonacci numbers. From this one can deduce a well-known relation between the Fibonacci numbers and the roots α and β of the polynomial $x^2 - x - 1$; namely, using (1.1) and (1.2), we deduce the famous Binet formulas

$$f_m = \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta}. \tag{1.3}$$

Writing $\alpha - \beta = \Delta$, it will then be useful to write (1.3) as

$$f_m = \frac{det\begin{bmatrix} 1 & 1 \\ \alpha^{m+1} & \beta^{m+1} \end{bmatrix}}{\Delta}. \tag{1.4}$$

Δ^2 is just the discriminant of the polynomial $p(x)$.

From (1.3) it is natural to look at the numbers

$$\frac{\alpha^{m+1} + \beta^{m+1}}{\alpha + \beta} = \alpha^{m+1} + \beta^{m+1}. \tag{1.5}$$

(since $\alpha + \beta = 1$). These numbers also turn out to be integers, the so-called Lucas numbers $(2, 1, 3, 4, 7, \ldots)$, which satisfy the same recursion as do the Fibonacci numbers [HW, p.148]. The Lucas sequence and the Fibonacci sequence are called *companion sequences*. As we shall see later, the two sequences are companionable in a number of ways.

2 Linear Recursion and Arithmetic Sequences

The construction made here can be considerably generalized and put into a setting in which it is seen to be a special case of very general ideas which reach out to many other parts of mathematics. But before we go all the way, let us first look at a natural intermediate generalization of these definitions which are well-known in number theory.

Consider the generic second degree (monic) polynomial $x^2 - t_1 x - t_2 = p(x; t_1, t_2)$, where the coefficients, t_1, t_2 are parameters. Let $A(t_1, t_2) = A$ and $B(t_1, t_2) = B$ be the roots of $P(x)$, so that $A + B = t_1$ and $AB = -t_2$. As in (1.1) and (1.2), we can easily write the powers of the roots to the first and zero-th powers, thus

$$A^{m+1} = \hat{S}_{(m)} A - \hat{S}_{(m,1)} \tag{2.1}$$

and

$$B^{m+1} = \hat{S}_{(m)} B - \hat{S}_{(m,1)}, \tag{2.2}$$

where, for now, $\hat{S}_{(m)}$ and $\hat{S}_{(m,1)}$ are rather elaborate names for whatever functions of t_1 and t_2 that occur when we compute A^{m+1} and B^{m+1} iteratively, starting with $A^2 = t_1 A + t_2$, $B^2 = t_1 B + t_2$. As with α and β, we can make the computation

$$\frac{A^{m+1} - B^{m+1}}{A - B} = \hat{S}_{(m)}. \tag{2.3}$$

As (2.1) or (2.2) suggest, $\hat{S}_{(m)}$ is one of a family of polynomials. For reasons that will be apparent later, we single it out and give it the special notation $F_m(t_1, t_2) = F_m$, or if we need to make the number of variables clear we shall also write $\hat{S}_{(m)} = F_{k,m}$, where in this case $k = 2$.

We could also have considered the sequence of polynomials $A^m + B^m = F_m t_1 - 2\hat{S}_{(m,1)}$, which we shall rename $G_m(t_1, t_2) = G_m$ $(= G_{2,m})$. Both of the sequences $\{F_m\}$ and $\{G_m\}$ will look tamer if we point out that

$$F_m = \begin{cases} t_1 F_{m-1} + t_2 F_{m-2}, & m > 0 \\ 1, & m = 0 \\ 0, & m < 0 \end{cases}$$

and

$$G_m = \begin{cases} t_1G_{m-1} + t_2G_{m-2}, & m > 0 \\ 1, & m = 0 \\ 0, & m < 0. \end{cases}$$

That is, both $\{F_m\}$ and $\{G_m\}$ are recursive sequences, the recursion relation being the same for the two sequences, facts that can be proved directly from the definition. Moreover, when $t_1 = 1 = t_2$, then $\{F_m(1,1)\}$ is just the sequence of Fibonacci numbers and $\{G_m(1,1)\}$ is just the sequence of Lucas numbers; moreover, $\{F_m(2,1)\}$ is the sequence of Pell numbers, while $\{G_m(2,1)\}$ is the companion sequence of the Pell numbers.

$\{F_m(3,-2)\}$ and $\{G_m(3,-2)\}$ are the sequences $\{2^m - 1\}$ and the sequence of companion numbers $\{2^m + 1\}$. The Mersenne primes belong to the sequence $\{F_m(3,-2)\}$, while the Fermat primes belong to the sequence $\{G_m(3,-2)\}$.(For an interesting discussion of primes in sequences see [Ri, p.41 ff]).

Since the $F-$ and $G-$ sequences are recursive, they give rise to (linear) recursive sequences of integers for all pairs of integer values of t_1 and t_2 . Of course, some sequences are more interesting than others. Their theory was developed by E. Lucas [1878] and expanded by D.H. Lehmer in [1930]. Such sequences are variously called Lucas and or Lehmer sequences [Ri]. They have been applied in a number of ways in mathematics, for example, for primality testing in number theory [Ri, ch. 2] (also see [MacH 3] and [MacH, 1].

It is the polynomial $p(x)$ that sets the whole machinery in motion. In the Lucas-Lehmer case, this is a second degree polynomial. But we needn't stop there. In fact, what we have seen so far is just the tip of an iceberg, one manifestation of a much more general state-of-affairs.

3 Generalized Fibonacci and Generalized Lucas Polynomials

Let us definite the *core-polynomial* to be the generic polynomial of degree k, $p(x;t_1,\ldots,t_k) = x^k - t_1x^{k-1} - \ldots - t_k$. We'll worry about the field of coefficients later. We want to find a relation between the roots of $p(x)$ and the coefficients that generalizes the spirit of the Binet formulas, (2.3), which we can write in matrix matrix form as in (1.4):

$$F_m = \frac{det\begin{bmatrix} 1 & 1 \\ A^{m+1} & B^{m+1} \end{bmatrix}}{\Delta}, \tag{3.1}$$

where Δ again means the difference of the roots, or rather, we should say, one of the square-roots of the discriminant of the core-polynomial.

Let $R_1, ..., R_k$ be the roots of $p(x; t_1, ..., t_k)$. We proceed as in the quadratic case: we find an expression for the powers of a root in terms of the first $k-1$ powers of that root, as in (2.1) and (2.2). Again by iteration, this turns out to be

$$R_i^m = \sum_{j=0}^{k-1} (-1)^j \hat{S}_{(n-k+1,1^j)} R_i^{k-j-1}, j = 1, ..., k, \tag{3.2}$$

where the $\hat{S}_{(n-k+1,1^j)}$ are functions of $t_1, ..., t_k$. That there is such an expression is obvious from the fact that the $R's$ are roots of a polynomial of degree k (with coefficients $t_1, ..., t_k$). What, in particular, the $\hat{S}_{(n-k+1,1^j)}$ are is yet to be divulged; but, in any case, using (3.2), we can form the determinant

$$det \begin{bmatrix} 1 & \cdots & 1 \\ R_1 & \cdots & R_k \\ \vdots & \vdots & \vdots \\ R_1^{k-1} & \cdots & R_k^{k-1} \\ R_1^m & \cdots & R_k^m \end{bmatrix}. \tag{3.3}$$

Using the multilinearity of the determinant function and (3.3), and letting

$$\hat{S}_{(m-k+1)} = F_{m-k+1}$$

we can compute that

$$F_{m-k+1} = \frac{det \begin{bmatrix} 1 & \cdots & 1 \\ R_1 & \cdots & R_k \\ \vdots & \vdots & \vdots \\ R_1^{k-1} & \cdots & R_k^{k-1} \\ R_1^m & \cdots & R_k^m \end{bmatrix}}{\Delta}, \tag{3.4}$$

where$\Delta^2 = discriminant\ p(x; t_1, ..., t_k)$

From (3.4), we get that

$$F_m = \frac{det \begin{bmatrix} 1 & \cdots & 1 \\ R_1 & \cdots & R_k \\ \vdots & \vdots & \vdots \\ R_1^{k-1} & \cdots & R_k^{k-1} \\ R_1^{m+k-1} & \cdots & R_k^{m+k-1)} \end{bmatrix}}{\Delta}. \tag{3.5}$$

Now, of course, the $F's$ are functions of the $t's$, while the determinants on the right hand side of (3.4) and (3.5) are functions of the $R's$ which are implicitly functions of the $t's$.

So these equations mean that if we choose particular values for the $t's$, then we determine particular roots, $R's$, and the two numbers on either side of (3.5), say, are the same.

We now have a reasonable generalization of the Binet formulas which relate the roots of the core polynomial to its coefficients, and for each set of values of the coeficient vector $\mathbf{t} = (\mathbf{t}_1, ..., \mathbf{t}_k)$, we have a recursively given arithmetical sequence (of degree k-1), $F_m(\mathbf{t})$, determined by a recursive set of polynomials. The recursion is given by

$$F_{k,n} = F_m(\mathbf{t}) = \sum_{\mathbf{j}=1}^{\mathbf{k}} \mathbf{t_j F_{n-j}(t)}. \tag{3.6}$$

The companion polynomials giving the *co-sequence* (i.e., companion sequence) are determined by the recursion

$$G_{k,n} = G_n = \sum_{j=1}^{n-1} t_j G_{n-j} + nt_n. \tag{3.7}$$

Thus we have a sequence and a co-sequence for each $k, k = 1, 2, ...$. Since increasing k only adds new terms to the old and leaves the first k polynomials in each sequence unchanged, we shall often omit the subscript k from our notation. That is, the length of the polynomials is allowed to increase without bound. This causes no harm because the $k-$ sequences can be recovered by truncation, that is, by letting $t_j = 0$ for $j > k$. This has the effect of picking out a particular generic polynomial $p(x)$ of degree k, that is, a particular core polynomial.

The polynomials F_n play an important role in the theory of multiplicative arithmetic functions [MacH 3]. In [MacH 2] the polynomials in the $f-$sequence are called Generalized Fibonacci Polynomials, while the polynomials in the $G-$sequence are called Generalized Lucas Polynomials. (These terms are not always used consistently in the literature.) Clearly, the sequences described in Section 2.0 are just the polynomials $F_{2,n}$ and $G_{2,n}$ described in this section, that is, we have generalized from the case where $p(x)$ is of degree 2.We should also point out that (3.6) can be deduced from (3.5). We take (3.7) to be a definition. One which we would like to interpret in terms of the roots of $P(x; t_1, ..., t_k)$.

Before doing this, however, let us have a look at the first several of these polynomials. Using (3.6) we can begin to compute the F-sequence as follows:

$$F_0 = 1$$
$$F_1 = t_1$$
$$F_2 = t_1^2 + t_2$$
$$F_3 = t_1^3 + 2t_1t_2 + t_3$$
$$F_4 = t_1^4 + 3t_1^2t_2 + t_2^2 + 2t_1t_3 + t_4$$

$$F_5 = t_1^5 + 4t_1^3t_2 + 3t_1t_2^2 + 2t_2t_3 + 3t_1^2t_3 + 2t_1t_4 + t_5$$
$$F_6 = t_1^6 + 5t_1^4t_2 + 6t_1^2t_2^2 + t_2^3 + 4t_1^3t_3 + t_3^2 + 6t_1t_2t_3 + 3t_1^2t_4 + 2t_2t_4 + 2t_1t_5 + t_6.$$

Similarly, we can use (3.7) to produce the first six terms of the G-sequence:

$$G_0 = 1$$
$$G_1 = t_1$$
$$G_2 = t_1^2 + 2t_2$$
$$G_3 = t_1^3 + 3t_1t_2 + 3t_3$$
$$G_4 = t_1^4 + 4t_1^2t_2 + 2t_2^2 + 4t_1t_3 + 4t_4$$
$$G_5 = t_1^5 + 5t_1^3t_2 + 5t_1t_2^2 + 5t_2t_3 + 5t_1^2t_3 + 5t_1t_4 + 5t_5$$
$$G_6 = t_1^6 + 6t_1^4t_2 + 9t_1^2t_2^2 + 6t_2^3 + 6t_1^3t_3 + 6t_3^2 + 12t_1t_2t_3 + 6t_1^2t_4 + 6t_2t_4 + 6t_1t_5 + 6t_6.$$

There is much to be learned by examining the form and the structure of these polynomials, but let us first give the postponed answer to the question put just ahead of the lists. What do the truncated polynomials represent in terms of the roots of the core polynomial $p(x)$? With a little work, we can compute the answer using (3.5). It turns out to be the following expression:

$$F_{k,n} = \sum R_1^{s_1}...R_k^{s_k}, \quad \text{where} \quad \sum s_i = n, s_i \in \{0,1,2,...\}.$$

These are famously known as the *Complete Symmetric Polynomials* (CSP); they are simply the polynomials which are the sums of all monomials of total degree n; they are homogeneous symmetric functions.

For the $G-$polynomials the answer is:

$$G_{k,n} = R_1^n + ... + R_k^n. \tag{3.8}$$

These polynomials are also famous and are known as the *Power Symmetric Polynomials* (PSP). How we deduced this is not so clear from what we have done, though it could be proved using (3.7), i.e. by induction; however, we shall be able to see this result clearly when we discuss symmetric polynomials.

4 Symmetric Polynomials

A *symmetric polynomial*, say in the letters $R_1, ..., R_k$, is one for which any exchange of the names for the letters leaves the polynomial unchanged. Another way of saying this, using our notation, is that any permutations of the indices leaves the polynomial unchanged.

Thus $R_1^2 + R_2^2$ remains unchanged (as a polynomial) if the indices 1 and 2 are interchanged. Similarly $R_1^3+R_2^3+R_3^3+14R_1R_2R_3$ remains unchanged under any permutation of the indices, e.g., for the permutation which takes (123) to (231), or any of the other five permutations of the three indices. That is a symmetric polynomial written in terms of the R_j is invariant under the action of the symmetric group of degree k, $Sym(k)$, on its indexing set.

It is well-known that the symmetric polynomials with, say rational coefficients, of degree n form a ring, usually written as Λ^n and if all of the symmetric polynomials are considered, they too form a ring Λ. If we allow our coefficients to come from a field, then these rings become algebras. In this essay we will generally be interested in either the integers or the rationals as coefficients. The complete symmetric polynomials form a basis of the ring Λ over the integers and the power symmetric polynomials form a basis for the algebra Λ over the rationals.

The elementary symmetric polynomials (ESP) also form a basis of Λ over the integers. The *elementary symmetric polynomials* are just those polynomials in $R_1, \ldots, R_k$ obtained by taking the sums of the products of the R_i one at a time, two at a time, up until k at a time:

$$e_{k,j} = \sum_{1 \leq i_1 < \ldots < i_j} R_{i_1} \ldots R_{i_j}.$$

This fact, that the elementary symmetric polynomials are an integer basis for the ring of symmetric polynomials, is one of two fundamental theorems concerning symmetric polynomials. The other fundamental theorem concerns the relationship of the roots of a polynomial and the symmetric functions. We shall restate them for future reference:

FT1. Every symmetric polynomial can be written as an expression in sums and products of elementary symmetric polynomials with integer coefficients (see [Wa, p.78] for discussion and proof).

FT2. Let $p(x) = x^k + a_1x^{k-1} + \ldots + a_k$ be a monic polynomial of degree k with roots $R_1, \ldots, R_k$, then the coefficients of $p(x)$ are elementary symmetric functions of the roots; precisely, $e_{k,j} = (-1)^j a_j$.

To see that FT2 is true, just write the polynomial, using a consequence of the fundamental theorem of algebra, as a product of linear factors involving its roots, then multiply out. a_1 is just equal to the sum of the roots, i.e.,is the *trace* of $p(x)$, and a_k is the product of the roots, i.e., the *norm* of $p(x)$. FT2. applied to the generic polynomial $p(x; t_1, \ldots, t_k)$ tells us that

$$t_j = (-1)^{j+1} e_{k,j}. \tag{4.1}$$

(We shall ignore the subscript k when it is clear from the context.) Thus (4.1) determines a transformation (in fact, an involution) from what we might call the *root-basis* for the ring of symmetric functions to the ESP-basis; thus our $F-$ and $G-$polynomials can can be thought of as the CSP's and the PSP's rewritten from the root basis in the ESP-basis. Of course, they are no longer symmetric polynomials. Following G. Polya [Po] (also see [Wa, p.78]), we shall call them *isobaric* polynomials.

We should remark at this point that we now have a tool to check (3.8).

5 Isobaric Polynomials

If we look at the tables in section 3.0 of Generalized Fibonacci and Generalized Lucas polynomials we shall see that the monomials in each of the polynomials F_n and G_n is of the form $A_\alpha t_1^{\alpha_1}...t_k^{\alpha_k}$ (which we shall write $\mathbf{A}_\alpha \mathbf{t}^\alpha$) where

$$\sum_{j=1}^{k} j\alpha_j = n. \tag{5.1}$$

But this is the same as saying

$$(1^{\alpha_1}, 2^{\alpha_2}, ..., k^{\alpha_k}) \tag{5.2}$$

is a decomposition of the number n, i.e., that $1+...+1+2+...+2+...+k+...+k = n$ where 1 occurs α_1 times, 2 occurs α_2 times, and so on. For example, $(2^3, 3^4)$ is a decomposition of 18; $18 = 2+2+2+3+3+3+3$.

It turns out that the involution determined by equation (4.1) takes homogeneous symmetric polynomials of total degree n to isobaric polynomials of *isobaric degree* n and in such a way that the monomials are an encoding of all possible decompositions of the number n; thus the maximal number of monomials in such an isobaric polynomial is equal to the number of decompositions of n. The truncations at k are determined by the decompositions of n into parts whose maximal entry is k [HW, Ch.19].

The decomposition given in (5.2) is sometimes indicated by a *shape* of the sort illustrated in fig.2, this being the shape for $(4, 3, 1^2) = \lambda$.

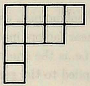

Figure 2

λ is called either a *Ferrer's diagram* or a *Young diagram* [Yo]. Such a shape is completely determined once we know the "exponents" α_j of the decomposition (5.2). We can regard an isobaric polynomial as a polynomial whose indeterminants are the shapes determined by the total degree n.

This relation among symmetric functions, decompositions of natural numbers, and shapes is one which penetrates deeply into the theory and application of symmetric functions (see, e.g., [Ha], [Mi], [Ma]). We shall talk more about these shapes in just a bit, but first let us go back to the coefficients in the representations of the powers of the roots R_j of the core polynomial $p(x)$ in (3.2).

These coefficients, which were themselves polynomials in $t_1, ..., t_k$, were denoted by symbols of the form $\hat{S}_{(m,1^j)}$. The notation suggests that there is a polynomial $S_{(m,1^j)}$, and, indeed there is. We shall use the hat-notation to indicate the image of a symmetric polynomial in the root-basis under the involution which takes it to the ESP-basis, that is to the resulting isobaric polynomial. We shall call this image an *isobaric reflect* (of whatever the symmetric preimage is).

So what is the symmetric preimage $S_{(m,1^j)}$ of $\hat{S}_{(m,1^j)}$? It is just the *Schur* symmetric polynomial *determined by* a shape of the form shown in fig. 3. in this case the shape determine by $(4, 1^5)$.

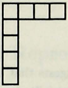

Figure 3

Such a shape is called, for obvious reasons, a *hook* diagram. But what is a Schur polynomial?

6 Schur Symmetric Polynomials

It will be convenient to write $\alpha = (\alpha_1, ...\alpha_k)$ and to use the notation $\alpha \vdash n$ to mean that the shape $\lambda = (1^{\alpha_1}, 2^{\alpha_2}, ..., k^{\alpha_k})$ is a decomposition of n . Then given any shape λ the isobaric reflect of the Schur polynomial determined by that shape is

$$\hat{S}_\lambda = det[F_{\alpha_i - i + j}], 1 \leq i, j \leq n. \tag{6.1}$$

The involution then can be used to write this expression as a symmetric polynomial in the root-basis. However, the reflect will be of more interest to us.

Two remarks are in order here. The first is that there is also a determinant directly in terms of the $t_j's$. And the second is that both of these determinants are versions of what is known in the literature as the Jacobi-Trudi formulas [Ma].

Now we can answer the question concerning the coefficients of the formula (3.2). These coefficients are just Schur-hook reflects.

I suppose that a healthy "so what?" is in order here; we should try to give some justification of what is so special about Schur-hooks, and about Schur polynomials at all. First we give a table, computed using (6.1) showing the Schur-hook reflects for $n = 6$ in terms of $t's$ and $F's$. From (6.1) we see that our convention introduced in Section 2.0 of denoting Schur-hook reflects of the form $\hat{S}_{(m)}$ by F_m is justified. So using (6.1) and (4.1) and doing a little work we get the following table:

Hook Schur Reflects for $n = 6$

$$\begin{array}{lcr}
\hat{S}_{(6)} & = & t_1\hat{S}_{(5)} + t_2\hat{S}_{(4)} + t_3\hat{S}_{(3)} + t_4\hat{S}_{(2)} + t_5\hat{S}_{(1)} + t_6\hat{S}_{(0)} \\
\hat{S}_{(5,1)} & = & -(t_2\hat{S}_{(4)} + t_3\hat{S}_{(3)} + t_4\hat{S}_{(2)} + t_5\hat{S}_{(1)} + t_6\hat{S}_{(0)}) \\
\hat{S}_{(4,1^2)} & = & t_3\hat{S}_{(3)} + t_4\hat{S}_{(2)} + t_5\hat{S}_{(1)} + t_6\hat{S}_{(0)} \\
\hat{S}_{(3,1^3)} & = & -(t_4\hat{S}_{(2)} + t_5\hat{S}_{(1)} + t_6\hat{S}_{(0)}) \\
\hat{S}_{(2,1^4)} & = & t_5\hat{S}_{(1)} + t_6\hat{S}_{(0)} \\
\hat{S}_{(1^6)} & = & -(t_6\hat{S}_{(0)}).
\end{array}$$

In this table $\hat{S}_{(0)} = 1$. The table suggests that there is a recursion relation among the Schur hooks. This is the case. Namely, the following relation holds:

$$\hat{S}_{(m-j,1^j)} = (-1)^j \sum_{i=j+1}^{m} t_i\hat{S}_{(m-j)}, \tag{6.2}$$

Moreover, from (6.1), we see that $\hat{S}_{(n)}$ is just F_n, so that the F-polynomials are, in fact, Schur hooks induced by Young diagrams consisting of a row with n squares. ($\hat{S}_{(1^m)}$ is a Schur hook induced by a *column* of m squares.) So, in particular, every Schur hook can be written in terms of $F's$ and $F's$ are Schur hooks, which already gives Schur-hooks a certain status in the world. (There are recursion formulas for Schur polynomials in general, again in terms of F-polynomials, but they are a bit more complicated and not necessary for us to look at here [Ma].)

Before explaining why the Schur polynomials are of interest in mathematics (and in physics), let us gather up a few more examples of interesting sequences that we get by evaluating some of these sequences at the integers, in fact at $(1,1,\ldots)$.

For untruncated polynomials we have

$$F_m(1,1,...) = 2^{m-1}, m > 0 \tag{6.3}$$

and

$$G_m(1,1,...) = 2^m - 1, m \geq 0. \tag{6.4}$$

From (6.3) and (6.4) and from Euclid's results on even perfect numbers, we have that, whenever m is a prime and $G_m(1,1,...)$ is a prime, then $F_m(1,1,...)G_m(1,1,...)$ is an even perfect number [LeV].

The pattern for F_m continues for hooks:

$$\hat{S}_{(m-j,1^j)}(1,1,...) = 2^{m-j-1} \tag{6.5}$$

but,

$$\hat{S}_{(\lambda)}(1,1,...) = 0 \tag{6.6}$$

if λ is not a hook.

Thus (6.5) and (6.6) tell us that we can distinguish hook Schur polynomials from non-hook Schur polynomials by evaluation!

But why are Schur symmetric polynomials important?

7 Schur Polynomials and Representation Theory

Schur polynomials, named after Issai Schur (1875-1941) were not brought into the world by Schur, but rather by Georg Frobenius (1849-1917) (see[Ma, p.61]) who was also responsible both for representation theory (using matrices) and character theory of finite groups, though , possibly he did not appreciate the connection betwen the two. Schur did see the relation of the polynomials named after him to the character theory of the finite symmetric groups and alternating groups (among other things). And, as it happens, this relation to character theory is the source of their importance, and it turns out, not just to the character theory of symmetric groups. But we are getting ahead of ourselves.

Though this is not the place for an exposition of representation theory and character theory—there are many good sources for this subject ([Se], [Hi], [Ro], [Fu], [Sr]), —we shall sketch enough of the ideas in order to see how the concepts in this paper apply.

A (faithful) representation of a group is merely an isomorphic copy of the group, preferably one that tells us something about the group. We are really interested in *linear* representations, which is what Frobenius was interested in. A linear representation of a group (for simplicity we restrict ourselves to finite groups) is a group of linear transformations of a vector space (for us, over the complex numbers, though this is not the most general case), isomorphic to the given group. In turn, since a linear transformation can be represented by a matrix, we can think of the representation as a group of matrices. The representation is *irreducible* if the vector space, regarded as an algebra over the group ring of its linear transformations, is not decomposable into a direct sum of subalgebras. It turns out that there are exactly as many irreducible representations of a group as there are conjugacy classes of elements in the group itself. (Two elements a and b in a group are *conjugate* if there is a element g in the group such that $a = g^{-1}bg$. Conjugacy is an equivalence relation.)

A *character* of a representation is then the map that assigns to each matrix in the group of representations its trace, i.e. the sum of its diagonal elements. It is important for the theory that similar matrices have the same trace—traces are not affected by basis changes of the underlying vector space. The irreducible (complex) characters of a group are then just the characters of the irreducible representations.

Suppose we consider the symmetric group of degree n, $Sym(n)$, that is, we consider the isomorphism class of a permutation group on n elements. Suppose that $Sym(n)$ has m conjugacy classes. Two elements in a permutation group are conjugate iff and only if they have the same cycle structure. But the number of different cycle structures in a permutation group is just the number of ways that n can be written as sums of natural numbers less than or equal to n, that is, it is equal to the number of *decompositions* of n. And with each decomposition we can associate a Young diagram, that is a shape consisting of n squares . This gives us m shapes. With each shape we can associate a Schur function (6.1), and with each Schur function we can associate a character in an effective way. Now there are always exactly m Schur functions of degree n (and isobaric Schur functions of isobaric degree n). These m Schur functions of degree n contain all of the information necessary to compute the *character table* of $Sym(n)$.

The key to this computation is *the Frobenius Character Theorem* [Fu, p.93] which gives us the irreducible character associated with a particular shape arising from the decomposition of n.

We give an example computing the character table of $Sym(3)$, the symmetric group of degree 3. For this we need the Schur reflects of isobaric polynomials of degree 3. There are three decompositions of 3, namely, $(1^3), (2,1), (3)$.

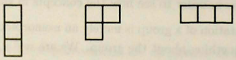

Figure 4

Let us first do the shape (1,2) $(1+2=3)$ with two squareson the top row and one on the bottom; that is, we want to compute the irreducible character associated with the Schur function $S_{(2,1)}$. The scheme is this: we first use (6.1) to find $S_{(2,1)}$ in terms of $F-$polynomials. We then write each $F-$polynomial in the determinant in terms of the $G-$polynomials (recall that the $G-$polynomials are also a basis). The formula for this is

$$F_\lambda = \sum_{\mu\vdash\lambda} \frac{1}{z(\mu)} G_\mu,$$

where $G_\mu = G_1^{\mu_1}...G_k^{\mu_k}$, and $z(\mu) = \prod j^{\mu_j}\mu_j!$.

Next, we expand the determinant, multiplying out completely so that we have a sum of products of $G_j^{\mu_j}$, and for each $G_j^{\mu_j}$, substitute $(1^{\mu_j}, 2^{\mu_2}, ..., k^{\mu_k})$. Now the right hand side of our formula is in terms of $(1^{\mu_j}, 2^{\mu_2}, ..., k^{\mu_k}) = \mu$. This is just the cycle structure of an element in a conjugacy class of $Sym(n)$, one such symbol for each class. In the case of $Sym(3)$, there are 3 classes (the class of cycles of length 3, the class with a product of two cycles, one of length one and one of length 2, and a single one-cycle, the identity element, with class sizes 2 and 3 and 1, respectively).

Finally we multiply each monomial in this expression by $\frac{n}{c[\mu]}$, a number determined by the size of the conjugacy class $c[\mu]$. This is just the value of this character at each class of $Sym(n)$. This will be just the coefficient of the appropriate decomposition $(1^{\mu_j}, 2^{\mu_2}, ..., k^{\mu_k}) = \mu$.

So let's do the computation for the character deternined by $S_{(2,1)}$. By (6.1) $\hat{S}_{(2,1)} = F_1F_2 - F_3$, and writing each F_j that appears in this determinant in the $G-$basis gives us $F_1 = G_1$, $F_2 = \frac{1}{2}G_1^2 + \frac{1}{2}G_2$, and $F_3 = \frac{1}{3!}G_1^3 + \frac{1}{2}G_1G_2 + \frac{1}{3}G_3$. So that after substituting in the determinant and expanding we have the expression

$$\frac{1}{3}G_1^3 + 0G_1G_2 - \frac{1}{3}G_3.$$

Substituting the cycle structure symbols for the G's gives

$$\frac{1}{3}(1^3) + 0(1,2) - \frac{1}{3}(3).$$

And, finally, multiplying each term by the class size adjuster $\frac{n}{c[\mu]}$, we end up with $2(1^3) + 0(2,1) - (3)$, that is just the second row of the character table of $Sym(3)$ below:

Class	1	3	2
Schur	(1^3)	$(1,2)$	(3)
(3)	1	1	1
$(2,1)$	2	0	-1
(1^3)	1	-1	1

This algorithm can be simplified somewhat; however, the particular prescription given here traces through the isomorphisms that map the Schur polynomial to the appropriate character function by way of the Frobenius character theorem. There are three such steps, the Jacobi-Trudi formula which represents the Schur polynomial as isobaric polynomials in the F−basis, the map which takes this result and represents it in terms of the G−basis, and then the involution which takes this version to a symmetric polynomial in the CSP-basis, which is just the equation involved in the Frobenius Character Theorem [Macd or Fulton]. Below we have also listed the isobaric versions of the Schur polynomials for $n = 3$ in the ESP- or t-basis. The character determined by the shape (3) (first row of table), and by (1^3), (third row of table), are constructed using the same procedure.

$$\begin{aligned}\hat{S}_{(3)} &= t_1^3 + 2t_1t_2 + t_3\\ \hat{S}_{(2.1)} &= \quad -t_1t_2 - t_3\\ \hat{S}(1^3) &= \qquad\qquad t_3.\end{aligned}$$

Applications of character theory to other parts of mathematics can be found in [Sr]. Character theory was used extensively in the recent classification of the finite simple groups. It is also important in the physics of elementary particles [Hi], [Ro], [Ha], [Mi].

It should be mentioned here that the application of symmetric functions to representation theory is by no means confined to the symmetric groups. One of the most active areas of algebra nowadays is the search for representations of algebraic systems, especially those that are important in physical theory, e.g., in Lie Algebras. The symmetric functions play an important role in this enterprise. (For a glimpse see [Fu]).

8 Weighted Isobaric Polynomials

The Generalized Fibonacci Polynomials, i.e., the F−sequence and the Generalized Lucas Polynomials, i.e., the G−sequence, belong to a much larger class of isobaric polynomials, that is those that can be usefully organized into sequences. The sequences that we have in mind are those that are determined by assigning a *weight* ω_j to the indeterminants, t_j $j = 1, 2, ..., k, ...$ To motivate this idea, we point out that the coefficients of F_m are uniquely determined by the exponent vector $\alpha = (\alpha_1, ..., \alpha_k)$ of the monomial term $t_1^{\alpha_1}...t_k^{\alpha_k}$. This is a result of the recursion in (3.6). Because of (3.7), a similar result holds for the polynomials G_m. In the case of F_m we have that, for a given α, the coefficient is just the multinomial coefficient

$$\binom{\sum \alpha_i}{\alpha_i}_{i=1,...,k}. \tag{8.1}$$

In the case of G_m the analogous result is

$$\binom{\sum \alpha_i - 1}{\alpha_i} n. \tag{8.2}$$

For an arbitrary *weighted isobaric polynomials*, the coefficient of the α term is

$$A_\alpha = \binom{\sum \alpha_i}{\alpha_i} \frac{\sum_{i=1}^k \alpha_i \omega_i}{\sum_{i=1}^k \alpha_i}.$$

The weight vector being given by $\omega = (\omega_1, \omega_2, ...)$. Thus a member of a sequence of weighted isobaric polynomials is a polynomial of the form $P_{\alpha,\omega} = \sum_{\alpha \vdash n} A_\alpha t^\alpha$.

Weighted Isobaric Polynomials for $n = 0$ to $n = 5$

$$P_{0,\omega} = 1$$

$$P_{1,\omega} = \omega_1 t_1$$

$$P_{2,\omega} = \omega_1 t_1^2 + \omega_2 t_2$$

$$P_3 = \omega_1 t_1^3 + (\omega_1 + \omega_2) t_1 t_2 + \omega_3 t_3$$

$$P_4 = \omega_1 t_1^4 + (2\omega_1 + \omega_2) t_1^2 t_2 + \omega_2 t_2^2 + (\omega_1 + \omega_3) t_1 t_3 + \omega_4 t_4$$

$$P_5 = \omega_1 t_1^5 + (3\omega_1 + \omega_2) t_1^3 t_2 + (\omega_1 + 2\omega_2) t_1 t_2^2 + (\omega_2 + \omega_3) t_2 t_3 + (2\omega_1 + \omega_3) t_1^2 t_3$$
$$+ (\omega_1 + \omega_4) t_1 t_4 + \omega_5 t_5$$

$$P_6 = \omega_1 t_1^6 + (4\omega_1 + \omega_2) t_1^4 t_2 + \omega_2 t_2^3 + 3(\omega_1 + \omega_2) t_1^2 t_2^2 + (3\omega_1 + \omega_3) t_1^3 t_3 + \omega_3 t_3^2$$
$$+ 2(\omega_1 + \omega_2 + \omega_3) t_1 t_2 t_3 + (2\omega 1 + \omega_4) t_1^2 t_4 + (\omega_2 + \omega_4) t_2 t_4 + (\omega_1 + \omega_5) t_1 t_5 + \omega_6 t_6.$$

The Weighted Isobaric Polynomials (WIPs) provide a source for recursive sequences (in fact, all recursive sequences are accounted for in this way). The recursion is given by the formulas

$$P_{(k,n,\omega)} = \sum_{j=1}^{n-1} t_j P_{n-j} + \omega_n t_n. \tag{8.3}$$

The weight vectors for the Generalized Fibonacci Sequence and for the Generalized Lucas Sequence, as can be inferred from (8.1), (8.2) and (8.3), are, respectively, given by $\omega_j = 1$ for all j, and $\omega_j = j$ for all j. Other famous WIP-sequences are the hook-Schur reflects, of which, of course, F_m is one. The weight vector for the hook reflects associated with the decomposition $(n-r, 1^r)$ is $\omega_{(r)} = (-1)^{(r+1)}(0, \ldots 0, 1, 1, \ldots)$ with $r\,0's$, and the rest $1's$. For generating functions for the weighted isobaric polynomials see [MT1].

There are some very interesting algebraic facts about these WIP-polynomials. For example it turns out that addition of isobaric polynomials of the same isobaric degree gives the same result as taking the isobaric polynomial that is associated with the sum of their weight vectors. That is we can add isobaric polynomials by adding their weight vectors at the same time maintaining isobaric degree.

$$P_{n,\omega} + P_{n,\mu} = P_{n,\omega+\mu}. \tag{8.4}$$

Thus, we can see from (8.4) that under addition the isobaric polynomials (or rather sequences of isobaric polynomials) have the same algebraic structure as the weight vectors under vector addition. This has as consequences that:

The set of WIP-polynomials of isobaric degree n is a free Z − module.

Hence,

The set of all weighted sequences of WIP-polynomials is a free graded Z-module.

Morever, it can be proved that every WIP can be written uniquely as a linearly combination of of hook-reflects, that is,

The weighted sequences of hook reflects constitute a basis for this module.

These statements are proved in [MT1]. There is much interesting and useful algebraic structure among the isobaric polynomials that we haven't discussed here; e.g., a useful operation that ties this structure to the algebra associated with multiplicative arithmetic functions is the convolution product (see, e.g., [MacH1] [MT1] [MT2]).

In Section 2.0 and in (6.3) and (6.4) we looked at the evaluations of $F_{(2,m)}$, of $G_{(2,m)}$ at (1,1) and of F_m and G_m at (1,1,..) and found that we got some interesting sequences of numbers. We can also easily see that $F_{k,m}$ gives us sequences that are recursively dependent on the previous k terms, generalizing the Fibonacci numbers recursive dependency on the previous two; that is, the numbers are defined recursively by $f_{n+1} = \sum_{j=1}^{n} f_j$. We might wonder what the analogous results are for more general cases. From (6.3) and (6.4) together with (8.1) and (8.2) we can deduce these two interesting results:

$$\sum_{\alpha \vdash n} \binom{\sum \alpha_i}{\alpha_i} = 2^{n-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} \quad n > 0$$

$$\sum_{\alpha \vdash n} \binom{\sum \alpha_i}{\alpha_i} n = 2^n - 1 = \sum_{i=0}^{n-1} \binom{n-1}{i} - 1, n \geq 0,$$

which relate the numbers on the left hand sides involving decompositions of n to the binomial coefficients and the Pascal triangle.

If we ask similar questions of the weighted isobaric sequences in general, we have that

$$P_{n,\omega}(1,1) = \sum_{j=1}^{n-1} 2^{n-j-1}\omega_j + \omega_n$$

and, hence that

$$\sum_{\alpha \vdash n} A_\alpha = \sum_{\alpha \vdash n} \binom{\sum \alpha_i}{\alpha_i} \frac{\sum_{i=1}^{k} \alpha_i \omega_i}{\sum_{i=1}^{k} \alpha_i} = \sum_{j=1}^{n-1} 2^{n-j-1}\omega_j + \omega_n.$$

At the end of section 1.0, we mentioned that the generalized Fibonacci polynomials and the generalized Lucas polynomials, i.e., the $F-$ and $G-$polynomials, are companionable in a number of ways. Here is another interesting relation between them.

$$\frac{\delta G_n}{\delta t_j} = F_{n-j}$$

thus F_j is a derivative of G_n for every integer j such that $0 \leq j \leq n$. It is quite likely that there is no other pair of WIP-sequences with this derivative relationship. It would be useful to know this for sure. It would also be useful to have a clear definition of what ought to be meant by a *companion* sequence.

Coda

At the centre of the spider web lies the *core*-polynomial, that is, the generic monic polynomial in one indeterminant. The web radiates to the ring of symmetric functions in k variables and their reflects in the ESP-basis giving the isobaric polynomials. These include all of the k-variable Schur functions or, rather, their isobaric reflects, giving a direct connection to the character table of the symmetric group of degree k, which is, in fact, the Galois group of the core polynomial.

And they include the (recursive) sequences of weighted isobaric polynomials, all of which can be expressed in terms of hook-Schur reflects. When the $t's$ are evaluated at some vector with k integer components, that is when the core-polynomial is specialized to a polynomial with integer coefficients, then the WIP sequences give us all recursive sequences of integers, including the Fibonacci sequence and the Lucas sequence.(Various fields of coefficients are of interest also.) There are also field extensions related to the core polynomial which are important [MacH1].

The web extends even farther than we have indicated in this essay as can be surmised from [Fu] or [Ma].

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