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On BMO$^{\varphi,p}$ Singularities of Solutions of Complex Vector Fields [1 2]

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ABSTRACT

We study the singularities of solutions in BMO$^{\varphi,p}$ of an complex vector field $L = X + iY$. Necessary and sufficient conditions are established in the plane when $\mathcal{H}_\varphi(\Sigma) = 0$, where Σ is the set where X, Y are linearly dependent and $\mathcal{H}_\varphi$ is the Hausdorff measure defined by φ.

1 Introduction

Besicovitch [Be2] showed that if $\Omega \subset \mathbb{C}$ is an open *bounded* set and $E \subset \Omega$ is a Borel set of null $\mathcal{H}^1$−measure (here $\mathcal{H}^s$ stands for the s−dimensional Hausdorff measure) then any function $u : \Omega \longrightarrow \mathbb{C}$ that is bounded and has complex derivative in $\Omega \setminus E$ agrees with an analytic function in Ω. This result still holds if u is continuous and the set E is σ-finite with respect to $\mathcal{H}^1$. Kaufman [Ka] in a precise way extended these results from the bounded and continuous cases, to functions in BMO and VMO respectively. He proved that if u is in BMO (or VMO) and has complex derivative in $\Omega \setminus E$ then u agree with a holomorphic function in Ω if and only if $\mathcal{H}^1(E) = 0$ (or E is σ-finite with respect to $\mathcal{H}^1$ respectively). Mizuta [Mi] extended the results in [Ka]

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obtaining new sufficient conditions to u agree with an analytic function in Ω. The size of the singularity E in [Mi] is measured by the Hausdorff measure $\mathcal{H}^{\varphi}$ determined by a measure function φ (cf. [Ro]) and the function u belongs to a certain p–space of functions. When $p = \infty$ the p–space defined in [Mi] agree with $\mathrm{BMO}^{\varphi}(\Omega)$, the bounded mean oscillation space defined by the seminorm

$$\|u\|_{\mathrm{BMO}^{\varphi}} = \sup_{B\subset\Omega} r^{n-1}\varphi(r)^{-1}\left[\frac{1}{\mathrm{m}(B)}\inf_{c\in\mathbb{R}}\int_B |u(z)-c|dm(z)\right] \qquad 1.1$$

(here B is a disk contained in Ω). The space $\mathrm{BMO}^{\varphi}(\Omega)$ was first defined by Spanne [Sp].

Let $P(x,D)$ be a linear partial differential operator defined in an open set $\Omega \subset \mathbb{R}^n$ and $K \subset \Omega$ a compact set. We say that K is a *removable* singularity relatively to $P(x,D)$ and Ω, if for any distribution u which satisfy $\mathrm{supp}P(x,D)u \subset K$, it follows $\mathrm{supp}P(x,D)u = \emptyset$. Dolzenko ([Do]) have shown in an early work that if u is Hölder continuous with exponent $0 < s-1 < 1$ then a *necessary* and *sufficient* condition to K to be *removable* relatively to a open subset of the plane Ω and $\overline{\partial}$ is $\mathcal{H}^s(K) = 0$. Later Uy ([Uy]) proved that the result still holds when $s = 1$. The natural correspondence between the $(s-1)$–Hölder space and BMO^{φ}, $\varphi(t) = \phi_s(r) = r^s$, shows that the result in [Ka] for BMO spaces extends the latter to the case when $s = 1$ (see Theorem 5.1, pp 213 in [To]). In [Mi] the condition $\mathcal{H}^{\varphi}(K) = 0$ is proved to be sufficient to a function u complex differentiable in $\Omega \setminus K$ agree with a analytic function in Ω when it belongs to $\mathrm{BMO}^{\varphi}(\Omega)$. Thus it is also necessary at least for the cases when $\varphi(r) = \phi_s(r) = r^s$. This last observation suggest that the bounded mean oscillation spaces defined by 1.1 should be the right place to study a generalization of the original problem settled by Besicovitch. Here we extend the results of [Mi] to a arbitrary nonvanishing complex vector field $\mathrm{L} = \mathrm{X} + i\mathrm{Y}$ and the $\mathrm{BMO}^{\varphi,\mathrm{p}}(\Omega)$ spaces defined below.

A measure function φ is a function defined for all $r \geq 0$, monotonic increasing, upper semicontinuous and positive for $r > 0$. Let us denote by $\mathbf{M}$ the set of all measure functions. When φ and ϕ belongs to $\mathbf{M}$, we write

$$\varphi \sim \phi \quad \text{if} \quad 0 < C_0 = \liminf_{r\to 0}\frac{\varphi(r)}{\phi(r)} \leq \limsup_{r\to 0}\frac{\varphi(r)}{\phi(r)} = C_1 < \infty$$

and

$$\varphi \prec \phi \quad \text{if} \quad \lim_{r\to 0}\phi(r)/\varphi(r) = 0.$$

We say that φ and ϕ are comparable if

$$\varphi \prec \phi, \quad \text{or} \quad \phi \sim \varphi, \quad \text{or} \quad \phi \prec \varphi$$

and they are monotonically comparable if the ratio φ/ϕ is monotonic. If $\varphi \sim \phi$ then the Hausdorff measures $\mathcal{H}^{\varphi}$ and $\mathcal{H}^{\phi}$ are also equivalent in the sense that

$$C_0\mathcal{H}^{\varphi}(E) \leq \mathcal{H}^{\phi}(E) \leq C_1\mathcal{H}^{\varphi}(E) \text{ for all Borel set } E \subset \Omega$$

(see Theorem 41, pp 80, [Ro] for a full converse). Denote by $\mathcal{M}$ the set of equivalence classes $M/\sim$ and denote by $\mathcal{S}$ the subset of $\mathcal{M}$ where each equivalence class has a representative φ which is monotonically comparable to φ_s for all $0 \leq s$ or $\varphi \sim \phi_s$ for some s (ϕ_s is defined by $\phi_s(r) = r^s$). These concepts relative to measure functions were borrowed from §.2 Scales of function, [RT]. Through the paper we will assume that φ is representative of some class in $\mathcal{S}$ and most of the time we will freely assume that it is monotonically comparable to the functions ϕ_s for $s \geq 0$.

The main result in this paper is stated for a locally integrable complex vector field L in the *plane*. A complex vector fields is locally integrable if one can find for any $z \in \Omega$ a relatively open neighborhood $\mathcal{O}$ and a smooth function $Z : \mathcal{O} \to \mathbb{C}$ such that dZ does not vanishes and $dZ(\mathrm{L}) \equiv 0$ in $\mathcal{O}$. Suppose that Σ the closed set where X and Y are linearly dependent is a set of zero Hausdorff $\mathcal{H}_{\varphi}$−measure. Then a function $u \in BMO^{\varphi,p}(\Omega)$ weakly agree with a homogeneous solution of $\mathrm{L}u = 0$ if and only if u is differentiable outside a Borel set $E \subset \Omega$ with $\mathcal{H}^{\varphi}(E) = 0$ and $du(\mathrm{L}) \equiv 0$ in $\Omega \setminus E$. This paper is organized as follows; in Section 2 we introduce the spaces $BMO^{\varphi,p}_{loc}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is an open set and prove the main results. The Appendix is devoted to show that the Cauchy transform of a finite φ−uniform measure is in the space $BMO^{\varphi,p}(\mathbb{C})$. This will be needed in order to establish the necessity of the condition described in the paragraph above as $\Omega \subset \mathbb{R}^2$.

2 Removable singularities in $BMO^{\varphi,p}$ spaces

Let us fix φ a representative of some class in $\mathcal{S}$. Let $B = B(w,r) \subset \Omega$ be an arbitrary open ball and

$$M_p(u,B) = \left[\frac{1}{m(B)} \inf_{c\in\mathbb{R}} \int_B |u(z) - c|^p dm(z)\right]^{\frac{1}{p}} \quad 2.4$$

Let the space $BMO^{\varphi,p}(\Omega)$, $p \geq 1$, defined as the space of all functions u in $L^p_{loc}(\mathbb{R}^n)$, such that

$$\|u\|_{BMO^{\varphi,p}} = \sup_{B\subset\Omega} r^{n-1}\varphi(r)^{-1}M_p(u,B) < \infty \quad 2.5$$

It is well known that this seminorm turns $BMO^{\varphi,p}(\Omega)$ modulo constants into a

Banach space. In fact, if we choose for $u \in \mathrm{BMO}^{\varphi,p}(\Omega)$ the constant c to be the mean

$$u_B = \frac{1}{\mathrm{m}(B)} \int_B u(z)\mathrm{dm}(z) \qquad 2.6$$

we obtain an equivalent seminorm. Spaces like $\mathrm{BMO}^{\varphi,p}(\Omega)$ were first introduced by Spanne in [Sp]. The spaces $\mathrm{BMO}^{\varphi,p}(\Omega)$ seems to be adequate to study the singularities of solutions of first order differential operators related to the Hausdorff measure $\mathcal{H}_\varphi$. Let $\mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}(\Omega)$ be the space of functions in $L^p_{loc}(\Omega)$ defined as in (2.5), but restricting the supremum to the family $\mathcal{B}$ of balls $B = B(w,r) \subset \Omega$ such that $distance(B, \partial\Omega) \geq c\sqrt{n}r$ for some large constant c. Such space is independent of the choice of c (see [Jo], Lemma 2.3 and [RR] Hilfssatz 2, pp 4). Here we are assuming that c is large enough to implies that $25\sqrt{n}B \subset \Omega$ if $B \subset \Omega$. Observe that $||u||_{\mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}} \leq ||u||_{\mathrm{BMO}^{\varphi,p}}$.

Define the $\varphi, p-$oscillation function of $u \in \mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}(\Omega)$ at (w,r) as

$$\mathrm{O(w,r)} = \sup_{B} \varphi(r)^{-1} r^{n-1} M_p(u,B) \;\; \text{where}\, B = B(w,r) \in \mathcal{B} \qquad 2.7$$

The subspace of $\mathrm{BMO}^{\varphi,p}(\Omega)$ with $\limsup_{r\to 0} \mathrm{O}(w,r) \equiv 0$ uniformly for $w \in \Omega$ will be denoted by $\mathrm{VMO}^{\varphi,p}(\Omega)$.

We can argue as in [Cm] and [Me] to prove the following proposition;

Proposition 2.1 *Let $\Omega \subset R^n$ be open and $u \in \mathrm{L}^1{}_{loc}(\Omega)$. If $r \in [0, \mathrm{diameter}(\Omega)]$ and $\int_0^{\mathrm{diameter}(\Omega)} \varphi(\rho)\rho^{-n}\mathrm{d}\rho < \infty$ then each inequality* (2.8), (2.9), (2.10), *and* (2.11) *below implies the precedent one.*

$$|u(z) - u(w)| \leq C \int_0^r \varphi(\rho)\rho^{-n}d\rho, \quad all\ z, w \in B(x,r) \subset \Omega \qquad 2.8$$

$$|u(z) - u_B| \leq C \int_0^r \varphi(\rho)\rho^{-n}d\rho, \quad all\ z \in B(x,r), B(x,r) \subset \Omega \qquad 2.9$$

$$\frac{1}{\mathrm{m}(B)} \int_B |u(z) - u_B|\mathrm{dm}(z) \leq C\,\varphi(r)r^{1-n}, \quad all\ B(x,r) \subset \Omega \qquad 2.10$$

$$\left[\frac{1}{\mathrm{m(B)}} \int_B |u(z) - u_B|^p \mathrm{dm}(z)\right]^{\frac{1}{p}} \leq C\,\varphi(r)r^{1-n}, \quad all\ B(x,r) \subset \Omega, \qquad 2.11$$

for all $1 \leq p < \infty$.

Proof. The first implication is trivial. Let us prove then (2.10) $\Rightarrow$ (2.9). Consider the sequence of points $w_k = w + 2^{-k}(z - w)$ converging to w and let

$$u_{B_k} = \frac{1}{V(n)(2^{-k}r)^n} \int_{B_k} |u(z)|\mathrm{dm}(z) \qquad 2.12$$

where $V(n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ is the volume of the unitary ball in $\mathbb{R}^n$, m is the Lebesgue measure and $B_k = B(w + 2^{-k}(z-w), r_k)$ with $r_k = 2^{-k}|z-w|$, $k = 1,2,3,...$

Then

$$|u(z) - u(w)| \leq |u(z) - u_{B_1}| + |u_{B_1} - u(w)| \tag{2.13}$$

We apply triangle inequality to the right side of the inequality in (2.13) to obtain

$$|u_{B_1} - u(w)| \leq |u_{B_l} - u(w)| + \sum_{k=1}^{l-1} |u_{B_{k+1}} - u_{B_k}| \tag{2.14}$$

and the last summand in (2.14) can be majored as follows;

$$\sum_{k=1}^{l-1} |u_{B_{k+1}} - u_{B_k}| \leq \frac{1}{V(n)} \sum_{k=1}^{l-1} \frac{1}{(2^{-(k+1)}r)^n} \int_{B_{k+1}} |u(t) - u_{B_k}| dm(t) \tag{2.15}$$

$$\leq \sum_{k=1}^{l-1} \frac{\varphi(2^{-k}r)}{(2^{-k}r)^{n-1}} \frac{2^n}{2^{-k}r\varphi(2^{-k}r)} \int_{B_k} |u(t) - u_{B_k}| dm(t) \tag{2.16}$$

$$\leq C \sum_{k=1}^{l-1} \frac{\varphi(2^{-k}r)}{(2^{-k}r)^{n-1}} \leq C \sum_{k=1}^{l-1} \frac{\varphi(2^{-k}r)}{(2^{-k}r)^n} 2^{-k}r \tag{2.17}$$

$$\leq C \sum_{k=1}^{l} \int_{2^{-k}r}^{2^{-k+1}r} \frac{\varphi(\rho)d\rho}{\rho^n} \leq C \int_0^r \frac{\varphi(\rho)d\rho}{\rho^n} \tag{2.18}$$

Then (2.9) holds and it implies (2.8) trivially since the same estimate holds for the other summand in the right side of 2.13. This proves that

$$|u(z) - u(w)| \leq C \int_0^r \frac{\varphi(\rho)d\rho}{\rho^n} \tag{2.19}$$

at the Lebesgue points of u. Uniform continuity allows to redefine u on all Ω, preserving the modulus of continuity. Note that if $\varphi(r) = r^{n-1+\delta}$, then $BMO^{\varphi,1}$ is the homogeneous δ-Lipschitz space ■

Corollary 2.2 *If* $\int_0^{\text{diameter}(\Omega)} \varphi(\rho)\rho^{-n} d\rho < \infty$ *and* $\varphi(r)r^{-s}$ *is non decreasing for* $s > n-1$ *and* $r \in (0, \text{diameter}(\Omega))$ *then*

$$BMO^{\varphi,1}_{loc}(\Omega) = BMO^{\varphi,p}_{loc}(\Omega) \quad \textit{for all } 1 \leq p < \infty.$$

Proof. Since $\varphi(r)r^{-s}$ is non decreasing it follows

$$\int_0^r \frac{\varphi(\rho)d\rho}{\rho^n} \leq \varphi(r)r^{-s} \int_0^r \frac{d\rho}{\rho^{n-s}} = (s+1-n)^{-1}\varphi(r)r^{1-n}.$$

Then

$$|u(z) - u(w)| \leq C(s+1-n)^{-1}\varphi(r)r^{1-n} \qquad 2.8'$$

and (2.8') implies (2.11) in a trivial way. ■

From now on we assume that the monotone increasing function φ is *doubling*, it means there exists positive constant b such that $\varphi(2r) \leq b\varphi(r)$. Let $\mathcal{L}^p_w(\mathcal{O})$ be the space of functions in $\mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}(\mathcal{O})$ which satisfies $\mathrm{L}u = 0$ weakly in an open set $\mathcal{O} \subset \Omega$. Let us denote

$$N_p(u,B) = \left[\frac{1}{\mathrm{m}(B)} \inf_{v \in \mathcal{L}^{\checkmark}_{\underline{w}}(\nabla B)} \int_B |u(z) - v(z)|^p \mathrm{dm}(z)\right]^{\frac{1}{p}} \qquad 2.20$$

and define the $\varphi, p-$mean oscillation of u relative to $\mathcal{L}^p(\Omega)$ at (w,r) as

$$\mathrm{A_p}(w,r) = \sup_B \varphi(r)^{-1} r^{n-1} N_p(u,B) \ \text{ where } B = B(w,r) \in \mathcal{B} \qquad 2.21$$

Let $u \in \mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}(\Omega)$ and G be a Borel subset of Ω. Pick a denumerable covering $\{B_j = B(w_j, r_j)\}_{j\in J}$ of G of balls in $\mathcal{B}$ and define

$$\mathcal{A}_{\varphi,p}(u,\mathrm{G}) = \inf \sum_{i=1}^{\infty} \mathrm{A_p}(z_i, 5\sqrt{n}r_i)\varphi(r_i) \qquad 2.22$$

where the infimum is taken over all such denumerable coverings $\{B_i = B(w_i, r_i)\}$ of G. Since φ is doubling we must have;

$$\mathcal{A}_p(u,\mathrm{G}) \leq \inf \sum_{i=1}^{\infty} r_i^{n-1} N_p(u, B_i) \leq C(n,b)\mathcal{A}_p(u,\mathrm{G}) \qquad 2.23$$

where the infimum is again taken over the same coverings in (2.22).

Let $L = \mathrm{X} + i\mathrm{Y}$ be a nonvanishing complex vector field and E a $\sigma-$finite set with respect to $\mathcal{H}_\varphi$. We will prove that

$$\mathcal{A}_p(u, \Omega \setminus \mathrm{E}) = 0 \ \text{ and } \ \limsup_{r\to 0,\, z\in \mathrm{E}} \mathrm{A_p}(\mathrm{z},\mathrm{r}) \in L^1(\mathrm{E}, \mathcal{H}_\varphi\lfloor \mathrm{E}) \qquad 2.24$$

is a sufficient condition on $u \in \mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}(\Omega)$ to ensures that Lu is a measure absolutely continuous with respect to $\mathcal{H}_\varphi\lfloor$E. In particular this implies the Theorem 1 in [KW], there $L = \bar{\partial}$ and $\mathrm{E} = \emptyset$. Also it explain the dichotomy appearing in [Be2] and [Ka].

In the next theorem $L = X + iY$ is a nonvanishing complex vector field defined in Ω and L^t is its formal adjoint. The measure function φ is supposed to be doubling.

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset and $E \subset \Omega$ be a Borel $\sigma-$finite measure set relative to $\mathcal{H}_\varphi$. If $u \in \mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}(\Omega)$ and $\mathcal{A}_p(u, \Omega \setminus E) = 0$ then there exists a absolute constant C such that*

$$\left| \int_\Omega u(z) L^t \psi(z) dm(z) \right| \leq C \|\psi\|_{L^\infty} \int_E \limsup_{r \to 0} A_p(z, r) d\mathcal{H}_\varphi(z) \qquad 2.25$$

where $\psi \in C^1_o(\Omega)$.

Proof. We will denote by C any constant appearing in the proof. The $\limsup_{r\to 0} A_p(z, r)$ is upper semicontinuos and bounded by $\|u\|_{\mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}}$ in Ω. Let us assume without loss of generality that $\|u\|_{\mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}} \leq 1$ and that

$$\int_E \limsup_{r \to 0} A_p(z, r) d\mathcal{H}_\varphi(z) < \infty \qquad 2.26$$

Let $\epsilon > 0$ be given. Since $\mathcal{A}_p(u, \Omega \setminus E) = 0$ it follows from 2.23 that we can find a denumerable family $\{B(w_j, r_j)\}_{j \in J}$ of ball in $\mathcal{B}$ such that it is a covering of of $\Omega \setminus E$ and

$$\sum_{j \in J} r_j^{n-1} N_p(u, 5\sqrt{n} B_j) \leq \epsilon \qquad 2.27$$

It follows from 2.23 that the set

$$E_k = \{z \in E : 2^{-(k+1)} < \limsup_{r \to 0} A_p(z, r) \leq 2^{-k}\} \qquad 2.28$$

has finite $\mathcal{H}_\varphi-$measure for all $k = 0, 1, 2, 3....$ For the same ϵ one can find a denumerable covering of E by balls $B(z_i, r_i) \subset \mathbb{R}^n$, $i \in I$ such that for some subset $I_k \subset I$ and $r_i \leq \epsilon$ we have

$$\sum_{i \in I_k} \varphi(r_i) \leq \mathcal{H}_\varphi(E_k) + \epsilon 2^{-k} \qquad 2.29$$

Let us denote by $\{B(w_k, r_k)\}_{k \in \mathbb{N}}$ the denumerable covering covering of Ω obtained by the union of the coverings of E and $\Omega \setminus E$ described above. We may assume that $\mathbb{N} = J \cup I$ and $J \cap I = \emptyset$. Let $B_k = B(z_k, r_k)$, $k = 1, ..., k(K)$ be a subcovering of $K = \mathrm{supp}\psi$ extracted from $\{B(w_k, r_k)\}_{k \in \mathbb{N}}$ and assume $r_k \geq r_{k+1}$ for $k = 1, ...k(K) - 1$. We inductively select dyadic squares $S_{mk} \in \mathcal{S}$ with disjoint interiors such that, $\cup_{k=1}^{l} B_k \subset \cup_{k=1}^{l} \cup_{m=1}^{m(k)} S_{mk}$, for all $1 \leq k \leq l \leq k(K)$, $m(k) \leq 3^n$, and

$$\frac{1}{2\sqrt{n}} < \frac{r_k}{\mathrm{diam}(S_{mk})} \leq \frac{1}{\sqrt{n}} \qquad 2.30$$

Now $6/5\, S_{mk} \subset 5\sqrt{n} B_k$ if $6/5\, S_{mk} \cap B_k \neq \emptyset$. We can also find smooth functions ψ_{mk}, $k \leq k(K)$, $m \leq m(k)$, such that

$$\mathrm{supp}\psi_{mk} \subset 6/5\, S_{mk} \text{ and } \Psi(z) = \sum \psi_{mk}(z) = 1 \qquad 2.31$$

in a neighborhood of K and $\Psi(z)=0$ outside a ϵ−neighborhood of K. Moreover

$$\|\psi_{mk}\|_{L^q} \le Cr_{m,k}^{n/q} \quad \text{and} \quad \|\mathrm{L}^t\psi_{mk}\|_{L^q} \le Cr_{mk}^{(n-q)/q} \tag{2.32}$$

This is essentially contained in the basic lemmas Lemma 3.1 and Lemma 3.2 in [HP]. Let $v_k \in \mathcal{L}^p_w(5\sqrt{n}B_k)$, then

$$\sum_{k=1}^{k(K)} \sum_{m\le m(k)} \int_{5\sqrt{n}B_k} |u(z)-v_k(z)||(\psi \mathrm{L}^t\psi_{mk} - \psi_{mk}\mathrm{L}\psi)(z)|dm(z) \tag{2.33}$$

$$\le \|\psi\|_{\mathrm{L}^\infty} \sum_{k=1}^{k(K)} \int_{5\sqrt{n}B_k} |u(z)-v_k(z)| \sum_{m\le m(k)} |\mathrm{L}^t\psi_{mk}(z)|dm(z) +$$

$$\|\mathrm{L}\psi(z)\|_{\mathrm{L}^\infty} \sum_{k=1}^{k(K)} \int_{5\sqrt{n}B_k} |u(z)-v_k(z)| \sum_{m\le m(k)} |\psi_{mk}(z)|dm(z) \le$$

$$\|\psi\|_{\mathrm{L}^\infty} \sum_{k=1}^{k(K)} \sum_{m\le m(k)} \|[u(z)-v_k(z)]\chi_{5\sqrt{n}B_k}\|_{\mathrm{L}^p} \|\mathrm{L}^t\psi_{mk}(z)\|_{\mathrm{L}^q} +$$

$$\|\mathrm{L}\psi(z)\|_{\mathrm{L}^\infty} \sum_{k=1}^{k(K)} \sum_{m\le m(k)} \|[u(z)-v_k(z)]\chi_{5\sqrt{n}B_k}\|_{\mathrm{L}^p} \|\psi_{mk}(z)\|_{\mathrm{L}^q} \le$$

$$C\Bigg[\|\psi\|_{\mathrm{L}^\infty} \sum_{k=1}^{k(K)} \sum_{m\le m(k)} r_k^{(n-q)/q} \|[u(z)-v_k(z)]\chi_{5\sqrt{n}B_k}\|_{\mathrm{L}^p} +$$

$$\|\mathrm{L}\psi(z)\|_{\mathrm{L}^\infty} \sum_{k=1}^{k(K)} \sum_{m\le m(k)} r_k^{n/q} \|[u(z)-v_k(z)]\chi_{5\sqrt{n}B_k}\|_{\mathrm{L}^p}\Bigg] \tag{2.34}$$

The last inequality ia a consequence of the inequalities (2.32). It follows from (2.21)-(2.22) that (2.34) is bounded by

$$C\Bigg[\|\psi\|_{\mathrm{L}^\infty} \sum_{j\in J} \varphi(r_j)\mathrm{A}_p(w_j, 5\sqrt{n}\, r_j) + \|\mathrm{L}\psi(z)\|_{\mathrm{L}^\infty} \sum_{j\in J} r_j\varphi(r_j)\mathrm{A}_p(w_j, 5\sqrt{n}\, r_j)$$

$$\|\psi\|_{\mathrm{L}^\infty} \sum_{i\in I} \varphi(r_i)\mathrm{A}_p(w_i, 5\sqrt{n}\, r_i) + \|\mathrm{L}\psi(z)\|_{\mathrm{L}^\infty} \sum_{i\in I} r_i\varphi(r_i)\mathrm{A}_p(w_i, 5\sqrt{n}\, r_i)\Bigg] \tag{2.35}$$

We know from (2.27) that

$$\|\psi\|_{\mathrm{L}^\infty} \sum_{j\in J} r_k^{(n-q)/q} \|[u(z)-v_k(z)]\chi_{5\sqrt{n}B_k}\|_{\mathrm{L}^p} +$$

$$\|L\psi(z)\|_{L^\infty} \sum_{j\in J} r_k^{n/q} \|[u(z)-v_k(z)]\chi_{5\sqrt{n}B_k}\|_{L^p} \leq$$

$$(\|\psi\|_{L^\infty} + \mathrm{diam}(\Omega)\|L\psi(z)\|_{L^\infty}) \sum_{j\in J} r_j^{n-1} N_p(u, 5\sqrt{n}B_j)$$

$$\leq (\|\psi\|_{L^\infty} + \mathrm{diam}(\Omega)\|L\psi(z)\|_{L^\infty})\epsilon \qquad 2.36$$

The upper semicontinuity of $\limsup_{r\to 0} A_p(z,r)$ implies that $A_p(w_i, 5\sqrt{n}\,r_i) < 2^{-k+1}$ for all $i \in I_k$, if ϵ is small enough. It follows from 2.21 that

$$\sum_{i\in I_k} \varphi(r_i) A_p(w_i, 5\sqrt{n}\; r_i) \leq 4 \sum_{i\in I_k} \varphi(r_i) 2^{-(k+1)} \leq 4(\mathcal{H}_\varphi(E_k) + 2^{-k}\epsilon) 2^{-(k+1)}$$

$$\leq 4 \int_{E_k} \limsup_{r\to 0} A_p(z,r) d\mathcal{H}_\varphi(z) + 2^{-k+1}\epsilon \qquad 2.37$$

Consequently

$$\sum_{i\in I} \varphi(r_i) A_p(w_i, 5\sqrt{n}\; r_i) \leq C\left(\int_E \limsup_{r\to 0} A_p(z,r) d\mathcal{H}_\varphi(z) + \epsilon \right) \qquad 2.38$$

A similar inequality as in (2.37) is true if we change φ by ϕ where $\phi(r) = r\varphi(r)$, for all $r > 0$. Since $E_\infty = \{z \in E : \limsup_{r\to 0} A_p(z,r) = 0\}$ has σ−finite $\mathcal{H}_\varphi$−measure and and $\limsup_{r\to 0} A_p(z,r) = 0$ the same type of argument can be repeated to shows that $\sum_{i\in I\setminus\cup I_k} \varphi(5\sqrt{n}\; r_i) A_p(w_i, 5\sqrt{n}\; r_i)$ becomes arbitrarily small when $\epsilon \to 0$ and $r_i \leq \epsilon$.

Putting together (2.36) and (2.38) we have

$$\left| \int_\Omega u(z) L^t\psi(z) dm(z) \right| \leq C \Big[\epsilon (\|\psi\|_{L^\infty} + \mathrm{diam}(\Omega)\|L\psi(z)\|_{L^\infty}) +$$

$$\|\psi\|_{L^\infty} \int_E \limsup_{r\to 0} A_p(z,r) d\mathcal{H}_\varphi(z) + \|L^t\psi\|_{L^\infty} \int_E \limsup_{r\to 0} A_p(z,r) d\mathcal{H}_\phi(z) + \epsilon \Big] \qquad 2.39$$

Since ϵ is arbitrary and $\mathcal{H}_\phi(E) = 0$ (with $\phi(r) = r\varphi(r)$), it follows that

$$\left| \int_\Omega u(z) L^t\psi(z) dm(z) \right| \leq C\|\psi\|_{L^\infty} \int_E \limsup_{r\to 0} A_p(z,r) d\mathcal{H}_\varphi(z) \qquad 2.40$$

The Riesz representation theorem together 2.40 implies that Lu is a Radon measure. If $\mathcal{O}$ is an open subset of Ω then the inequality 2.40 applies for $\mathcal{O}$ and $\mathcal{O} \cap E$ in the place of Ω and E respectively. It follows that $Lu(\mathcal{O} \cap E) = 0$ if $\mathcal{H}_\varphi(\mathcal{O} \cap E) = 0$. This implies that Lu is absolutely continuous with respect to the measure $\mathcal{H}_\varphi \lfloor E$. ∎

Remark. The same argument can be applied to a partial differential operator $P(x,D)$ of arbitrary order. If m is the order of $P(x,D)$ then we change the power r^{n-1} to r^{n-m} in 2.5 to obtain an analogous result.

We now introduce the concept of $\mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}(\Omega)$ subspace tangent to $\mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}(\Omega)$ in a point.

Definition 2.3 *We will say that $\mathcal{L}^p_w(\Omega)$ is tangent to $u \in \mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}(\Omega)$ at a point $w \in \Omega$ if*

$$N_p(u, B(w,r)) \leq r\,o(r) \tag{2.41}$$

The Theorem 1 in [K] is contained by the Corollary 2.4 below. Compare also with the results of Theorem 4.1 in [HP] for $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$.

Corollary 2.4 *Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset and $\mathrm{E} \subset \Omega$ be the Borel set of points where $\mathcal{L}^p_w(\Omega)$ is not tangent to $u \in \mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}(\Omega)$. Then u is a weak solution of $Lu = 0$ in Ω if $\mathcal{H}_\varphi(\mathrm{E}) = 0$ or $u \in \mathrm{VMO}^{\varphi,p}(\Omega)$ and E is a set of $\sigma-$finite $\mathcal{H}_\varphi-$measure.*

Proof. Let $\Omega \setminus \mathrm{E}$ be the set where $\mathcal{L}^p_w(\Omega)$ is tangent to u. It follows from (2.41) that for each $\epsilon > 0$ one can find a denumerable covering of E by balls B_j in Ω such that $N_p(u, B(w_j, r_j)) \leq r_j\,o(r_j)$ with $o(r_j) \leq \epsilon$ and the balls $\{B_j(w_j, 5^{-1}r_j)\}$ are disjoint. Then

$$\inf \sum_{i=1}^{\infty} r_i^{n-1} N_p(u, B_i) \leq \inf \sum_{i=1}^{\infty} r_i^n o(r_i) \leq 5\mathrm{volume}(\Omega)\epsilon \tag{2.42}$$

Then (2.26) and (2.42) implies that

$$\mathcal{A}_p(u, \Omega \setminus \mathrm{E}) = 0 \tag{2.43}$$

Now Corollary 2.4 follows from (2.43) and (2.25)in Theorem 2.3. ∎

We say that $\mathrm{L} = \mathrm{X} + i\mathrm{Y}$ is a locally integrable vector field in Ω if for any point $w \in \Omega$ there exists a ball $B_0 = B(w, r_0) \subset \Omega$ and a smooth function Z defined in B_0 such that $\mathrm{dZ} \neq 0$ and $\mathrm{dZ}(L) \equiv 0$. Without loss of generality we may assume that $\mathrm{Z}(w) = 0$. If $u \in \mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}(\Omega)$ is differentiable at w and $du(L)(w) = 0$ then for some $c \in \mathbb{C}$;

$$|u(z) - u(w) - c\mathrm{Z}(z)| \leq r\,o(r) \tag{2.44}$$

with $\lim_{r\to 0} o(r) = 0$ and for all $z \in B(w,r)$ with $r \leq r_0$. It follows easily that (2.44) implies (2.41) at w.

Definition 2.5 *Let* E *be a Borel set in* Ω. *Define by* $\mathcal{L}^p(\Omega \setminus \mathrm{E})$ *the linear subspace of functions* u *in* $\mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}(\Omega)$ *whose are differentiable in* $\Omega \setminus \mathrm{E}$ *and such that* $du(\mathrm{L}) \equiv 0$ *there.*

Observe that when E is closed $\mathcal{L}^p(\Omega \setminus \mathrm{E}) \subset \mathcal{L}^p_w(\Omega \setminus \mathrm{E})$

Corollary 2.6 *Let* $\Omega \subset \mathbb{R}^2$ *be an open set and* Σ *be the set of points in* Ω *where* X *is linearly dependent with* Y. *Let* $u \in \mathcal{L}^p(\Omega \setminus \mathrm{E})$, $1 \le p \le \min\{\frac{2}{\dim \mathrm{E}}, 2\}$, *be given (where* $\dim \mathrm{E}$ *is the Hausdorff dimension of* E*). Then*

i) *If* $\mathcal{H}_\varphi(\Sigma) = 0$ *then* $\mathcal{L}^p(\mathrm{V} \setminus \mathrm{E}) = \mathcal{L}^p_w(\mathrm{V})$ *for all open subset* $\mathrm{V} \subset \Omega$ *if and only if* $\mathcal{H}_\varphi(\mathrm{E}) = 0$.

ii) *If* Σ *is* σ*-finite relatively to* $\mathcal{H}_\varphi$ *then* $\mathrm{VMO}^{\varphi,p}_{\mathrm{loc}}(\Omega) \cap \mathcal{L}^p(\mathrm{V} \setminus \mathrm{E}) = \mathrm{VMO}^{\varphi,p}_{\mathrm{loc}}(\Omega) \cap \mathcal{L}^p_w(\mathrm{V})$ *for all open subset* $\mathrm{V} \subset \Omega$ *if and only if* E *is* σ*-finite relatively to* $\mathcal{H}_\varphi$.

Proof. The hypothesis implies that $\mathcal{L}^p_w(\Omega)$ is tangent to u at the points in the complement of E. Then we may apply Corollary 2.3 to prove one of the implications in i) and ii). To prove the converse implication in i) we must observe that $\mathcal{H}_\varphi(\Sigma) = 0$ and $\mathcal{H}_\varphi(\mathrm{E}) > 0$ implies that $\mathcal{H}_\varphi(\mathrm{E} \setminus \Sigma) > 0$. It follows from Theorem 3, Ch II in [Ca] and the results in [Be3] that we can find a compact set $K \subset \mathrm{E} \setminus \Sigma$ such that $\mathcal{H}_\varphi(K) > 0$. Shrinking K if it is necessary we may assume that there exists a function Z, defined in a neighborhood V of K such that $d\mathrm{Z} \neq 0$ and $d\mathrm{Z}(L) \equiv 0$. Such function is a local diffeomorphism since K is away from Σ, thus we may assume that Z is a diffeomorphism in V with Jacobian bounded below by a positive number. The bilipschitz nature of Z in V would imply that, if $B(\mathrm{Z}(z), r) \subset \mathrm{Z}(V)$ then $B(z, \sqrt{n}c^{-1}r) \subset \mathrm{Z}^{-1}(B(\mathrm{Z}(z), r)) \subset B(z, \sqrt{n}cr) \subset \mathrm{V}$ for some positive constant c. Since φ is doubling the pull back of $\mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}(\mathrm{Z}(\mathrm{V}))$ by Z is $\mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}(\mathrm{V})$. Also if $0 < \mathcal{H}_\varphi(K) < \infty$ then $0 < \mathcal{H}_\varphi(\mathrm{Z}(K)) < \infty$. Theorem 1, Ch II in [Ca] assures the existence of a φ-uniform measure μ supported by $\mathrm{Z}(K)$. It follows from Theorem 3.1 that the Cauchy transform $\mathcal{C}(\mu)$ is in $\mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}(\mathbb{C})$. In particular it is in $\mathrm{BMO}^{\varphi,p}_{\mathrm{loc}}(\mathrm{Z}(\mathrm{V}))$. This completes the proof of i). If E is of non σ-finite $\mathcal{H}_\varphi$-measure the results in [Be4] shows that one can find a function ϕ such that $\lim_{r \to 0} \phi(r)\varphi^{-1}(r) = 0$ and E is of non σ-finite $\mathcal{H}_\phi$-measure, thus a set of positive $\mathcal{H}_\phi$- measure. Now we proceed exactly as before to find a finite ϕ-uniform measure μ supported in a compact subset $\mathrm{Z}(K) \subset \mathbb{C}$. Then $\mathcal{C}(\mu) \circ \mathrm{Z} \in \mathrm{BMO}^{\phi,p}_{\mathrm{loc}}(\mathrm{V}) \subset \mathrm{VMO}^{\varphi,p}_{\mathrm{loc}}(\mathrm{V})$. This completes the proof of ii). ∎

Remark. What we can say if $\mathcal{H}_\varphi(\Sigma) > 0$? This is indeed a difficult question. If L is analytic then $\mathrm{int}\Sigma = \emptyset$. In this case we have a complete satisfactory answer in [HT] when $\varphi(r) = r$. When $\Omega \subset \mathbb{R}^n$ and $n > 2$ it is an open question to find the extent of

Corollary 2.5 valid when the orbits (in the sense of Sussman [Su]) defined by the real and imaginary parts of L are two dimensional.

3 Appendix: The Cauchy Transform of an uniform φ–uniform measure

Recall that φ is a representative of some class in $\mathcal{S}$. We will assume that the measure function φ verifies the following conditions:

(i) there exists a positive constant $b = b(\varphi)$ such that $\varphi(2r) \leq b\varphi(r)$ for all $r > 0$(doubling property)

(ii) $\varphi(r)r^{-n}$ is non increasing in $(0, \infty)$ (condition A in [Mr])

Consider

$$\mathcal{H}^{\epsilon}_{\varphi}(\mathrm{E}) = \inf_{\{B_k\}} \sum_k \varphi(diam B_k) \qquad 3.1$$

where $\{B_k\}_{k \in \mathbb{N}}$ run over all ϵ-covers of E and $0 < \epsilon \leq \infty$ with elements belonging to a family of sets $\mathcal{F}$, that is $\mathrm{E} \subset \bigcup_k B_k$, $diam B_k \leq \epsilon$. The family of $\mathcal{F}$ may be a family of open, closed or convex sets. Since φ is doubling, we obtain a measure comparable with the original Hausdorff measure.

The φ-Hausdorff measure $\mathcal{H}_\varphi$, defined for Borel sets $\mathrm{E} \subset \mathbb{R}^n$ by

$$\mathcal{H}_\varphi(\mathrm{E}) = \lim_{\epsilon \to 0} \mathcal{H}^{\epsilon}_{\varphi}(\mathrm{E}) \qquad 3.2$$

When $\varphi(0) > 0$ the measure $\mathcal{H}_\varphi$ will be a multiple of $\mathcal{H}_{\varphi_0}$ with $\varphi_0 \equiv 1$ the zero dimensional Hausdorff measure which corresponds to the *counting measure*. Note that the Hausdorff measure depends only on the behavior of the function φ near zero. We say that a Borel measure μ in the plane is an ***uniform φ-Hausdorff measure*** if and only if there exists $c = c(\mu)$ such that

$$|\mu(B(y,r))| \leq c\varphi(r) \text{ for all } r < diam(\mathrm{supp}\mu) \text{ and all } y \in \mathrm{supp}\mu \qquad 3.3$$

The Theorem 3 in §II of [Ca] assures that any Borel set $\mathrm{E} \subset \mathbb{C}$ with positive $\mathcal{H}_\varphi$-measure contains a closed set F such that $0 < \mathcal{H}_\varphi(F) < 0$. Since $\mathbb{R}^n$ is σ–compact we may change closed by compact in the last statement. The Theorem 1 in §II of [Ca] asserts that (3.3) implies $\mu(\mathrm{E}) \leq c\mathcal{H}^{\infty}_{\varphi}(\mathrm{E})$ and consequently an uniform φ-Hausdorff measure is absolutely continuous with respect to $\mathcal{H}_\varphi$. Also the same Theorem assures the existence of a constant C depending only on the dimension n such that for every compact set $K \subset \mathbb{C}$ there is a φ–uniform measure μ such that $\mu(K) \geq C\mathcal{H}^{\infty}_{\varphi}(K)$. When $\mathrm{E} \subset \mathbb{C}$ is a Borel set which has no σ–finite $\mathcal{H}_\varphi$–measure it is proved in [Be4] that there exists a monotone increasing function ϕ, such that $\lim_{r\to 0}\phi(r)\varphi(r)^{-1} = 0$ and $E \subset \mathbb{C}$ has no σ−finite $\mathcal{H}_\phi$−measure. Combined with the previous result this implies that there exists a ϕ−uniform measure supported in some compact subset of E. For a general account on Hausdorff measures see Rogers [Ro]. Throughout this section we will denote by $M^\varphi(E)$ the space of the φ-uniform finite measures μ in Ω concentrated in E, that is $\mu = \mu\lfloor E$ (recall that for an arbitrary measure μ, $\mu\lfloor E\,(A) = \mu(E \cap A)$ for any measurable set A). It will be useful to consider the norm

$$\|\mu\|_{M^\varphi(E)} = |\mu|(E) + \sup_{w\in E,\, r>0} \frac{|\mu|(B(w,r))}{\varphi(r)} \qquad 3.4$$

for such measures. With respect to this norm $M^\varphi(E)$ is a Banach space. Suppose that E is a σ-finite Borel set with respect to the $\mathcal{H}_\varphi$ measure. The Cauchy transform $C(\mu)$ of a finite Borel measure $\mu \in M^\varphi(\mathbb{C})$ is defined *a.e.* in $\mathbb{C}$ by

$$C(\mu)(z) = \int_{\mathbb{C}} (\zeta - z)^{-1} d\mu(\zeta), \quad z \in \mathbb{C} \qquad 3.5$$

It can be continuously extended to the Riemman sphere $\hat{\mathbb{C}}$ since $\lim_{z\to\infty} C(\mu)(z) = 0$, and it is differentiable at ∞, where $C(\mu)'(\infty) = \lim_{z\to\infty}\int_{\mathbb{C}} z(\zeta - z)^{-1}d\mu(\zeta) = -\mu(\mathbb{C})$. Recall that we are considering functions φ satisfying the condition that $r^{-2}\varphi(r)$ is non increasing (Condition (iii) above). We are assuming that φ belongs to a irreducible, maximal, and strongly dense scale of functions (cf § 2. Scale of functions, [RT]). Then the Hausdorff dimension of sets with finite $\mathcal{H}_\varphi$ measure will be equal to $2 - \delta$, where

$$\delta = \inf\{\,\delta' \in [0,2] : \varphi(r)r^{-2+\delta'} \text{ is non increasing in } [0,\infty)\} \qquad 3.6$$

The Cauchy transform $C(\mu)$ of a finite measure μ is always in $L^p{}_{loc}(\mathbb{C})$ for $1 \le p < 2$. One can guess that φ−uniformity of a finite measure μ will somehow be reflected in the growing behavior in discs of means of its Cauchy transform. Indeed we will show that the Cauchy transform of a finite measure μ which is φ−uniform is in $BMO^{\psi,p}(\mathbb{C})$ where ψ is defined by

$$\psi(r) = \begin{cases} \varphi(r) \text{ if } 2(p-1)/p < \delta \le 2 \\ \log(r^{-1}R)^{1/p}\varphi(r) \text{ if } \delta = 2(p-1)/p \text{ where } R = \operatorname{diam}(\operatorname{supp}\mu) < \infty \end{cases} \qquad 3.7$$

When $\delta = 0$ it follows from (3.1) that φ is non increasing and consequently $\varphi(r)$ is constant (because φ is always increasing monotone). The case when $\varphi(r) \equiv 0$ leads us to the null measure. Let us assume without loss of generality that $\lim_{r\to 0}\varphi(r) = 1$ and let $\mu \in M^\varphi(E)$ be a given measure. Since μ is finite there exists a denumerable set of points $E = \{w_n\}_{n\in\mathbb{N}}$, such that

$$d\mu(\zeta) = f(\zeta)d\mathcal{H}^0\lfloor\mathrm{E} \text{ and } \sum_{j=1}^{\infty}|f(w_j)| < \infty.$$

The Cauchy transform is

$$C(\mu)(z) = \int_{\mathbb{C}} f(\zeta)(\zeta - z)^{-1}\, d\mathcal{H}^0\lfloor\mathrm{E}(\zeta) = \sum_{j=1}^{\infty} f(w_j)(w_j - z)^{-1} \qquad 3.8$$

Integrating on a disc $B = B(w, r)$, we obtain

$$\int_B |C(\mu)(z)|^p \mathrm{dm}(z) = \int_B \left|\sum_{j=1}^{\infty} f(w_j)(w_j - z)^{-1}\right|^p \mathrm{dm}(z) \le$$

$$\left[\sum_{j=1}^{\infty}|f(w_j)|\right]^{-1} \int_B \sum_{j=1}^{\infty} |f(w_j|^p|w_j - z|^{-p}\mathrm{dm}(z) \le$$

$$\le \frac{2\pi|\mu|^{p-1}(\mathrm{E})}{2-p} r^{2-p} \le C\|\mu\|_{\mathrm{M}^{\varphi}(\mathrm{E})} r^{2-p}\varphi(r)^p \qquad 3.9$$

Hence in $C(\mu) \in \mathrm{BMO}^{\varphi,p}(\mathbb{C})$ if $1 \le p < 2$. The function $\varphi(r)$ plays the role of a constant at the right hand side of (3.4).

We will assume in the next Theorem 3.1 that $\mu \in \mathrm{M}^{\varphi}(\mathrm{E})$ has compact support when $\delta = 2(p-1)/p$. The constant δ is a constant related to φ and defined in (3.6). Also we will denote $B(w, 2r)$ by $2B$ and set $D = \mathbb{C} \setminus 2B$.

Theorem 3.1 *Let* $\mathrm{E} \subset \Omega$ *be a Borel set and* $\mathcal{H}_{\varphi}\lfloor\mathrm{E}$ *be* σ*−finite. Let* $\mu \in \mathrm{M}^{\varphi}(\mathrm{E})$ *with* $1 \le p \le \min\left\{\frac{2}{2-\delta}, 2\right\}$. *Then there exists a constant* $C > 0$ *for all* $B = B(w, r) \subset \mathbb{C}$ *such that*

$$\int_B |C(\mu)(z) - C(\mu\lfloor D)(w)|^p \mathrm{dm}(z) \le C(p)\, \|\mu\|^p_{\mathrm{M}^{\varphi}(\mathrm{E})}\, r^{2-p}\psi^p(r) \qquad 3.10$$

In particular $C(\mu) \in \mathrm{BMO}^{\psi,p}(\mathbb{C})$,.

Proof. Let $\mu \in \mathrm{M}^{\varphi}(\mathrm{E})$ be a given measure. In view of (3.4) we may assume without loss of generality that $c = \|\mu\|_{\mathrm{M}^{\varphi}(\mathrm{E})}$ for the constant c in (3.3). Consider the Cauchy transform of the measure $\mu\lfloor\mathbb{C}\setminus 2B$.

$$C(\mu\lfloor\mathbb{C}\setminus 2B)(w) = \int_{\mathbb{C}\setminus 2B} (\zeta - w)^{-1} d\mu(\zeta), \quad w \in \mathbb{C} \qquad 3.11$$

Let us suppose first that, $2(p-1)/p < \delta \leq 2$ and write $\mathcal{C}(\mu)(z)$ as the sum

$$\mathcal{C}(\mu)(z) = \int_{2B} (\zeta - z)^{-1} d\mu(\zeta) + \int_{\mathbb{C}\setminus 2B} (\zeta - z)^{-1} d\mu(\zeta) \tag{3.12}$$

Hence

$$\int_B |\mathcal{C}(\mu)(z)|^p \mathrm{dm}(z) \leq$$

$$\int_B \left| \int_{\mathbb{C}\setminus 2B} (\zeta - z)^{-1} d\mu(\zeta) \right|^p \mathrm{dm}(z) + \int_B \left| \int_{2B} (\zeta - z)^{-1} d\mu(\zeta) \right|^p \mathrm{dm}(z) \tag{3.13}$$

We will show that both summands in (3.13) are finite.

Let $z = w + \rho e^{i\theta}$, then

$$\int_B \left| \int_{\mathbb{C}\setminus 2B} (\zeta - z)^{-1} d\mu(\zeta) \right|^p \mathrm{dm}(z) \leq |\mu|^{p-1}(\mathbb{C}) \int_B \int_{\mathbb{C}\setminus 2B} |\zeta - z|^{-p} d|\mu|(\zeta) \mathrm{dm}(z) \leq$$

$$|\mu|^{p-1}(\mathbb{C}) \int_{\mathbb{C}\setminus 2B} \int_0^{2\pi} \int_0^{r} r^{-p} \rho d\rho d\theta d|\mu|(\zeta) = |\mu|^{p-1}(\mathbb{C}) \int_{\mathbb{C}\setminus 2B} \pi r^{2-p} d|\mu|(\zeta) \leq$$

$$\pi r^{2-p} |\mu|^{p-1}(\mathbb{C}) \leq \pi r^{2-p} |\mu|^{p-1}(\mathbb{C}) \tag{3.14}$$

and

$$\int_B \left| \int_{2B} (\zeta - z)^{-1} d\mu(\zeta) \right|^p \mathrm{dm}(z) \leq |\mu|^{p-1}(2B) \int_B \int_{2B} |\zeta - z|^{-p} d|\mu|(\zeta) \mathrm{dm}(z) \leq$$

$$|\mu|^{p-1}(2B) \int_{2B} \int_B |\zeta - z|^{-p} \mathrm{dm}(z) d|\mu|(\zeta) \leq |\mu|^{p-1}(2B) \int_{2B} \left(\int_B \rho^{1-p} d\rho d\theta \right) d|\mu|(\zeta)$$

$$\leq \frac{2\pi}{2-p} r^{2-p} |\mu|^p(2B) \leq \frac{2\pi}{2-p} r^{2-p} c^p b^p \varphi^p(r) \text{ for all } z \in B \tag{3.15}$$

Now we estimate the mean,

$$\int_B \left| \int_{\mathbb{C}} (\zeta - z)^{-1} d\mu(\zeta) - \int_{\mathbb{C}\setminus 2B} (\zeta - w)^{-1} d\mu(\zeta) \right|^p \mathrm{dm}(z) =$$

$$\int_B \left| \int_{2B \cup \mathbb{C}\setminus 2B} (\zeta - z)^{-1} d\mu(\zeta) - \int_{\mathbb{C}\setminus 2B} (\zeta - w)^{-1} d\mu(\zeta) \right|^p \mathrm{dm}(z) \leq \frac{2\pi}{2-p} r^{2-p} c^p b^p \varphi^p(r)$$

$$+ \int_B \left| \int_{\mathbb{C}\setminus 2B} (\zeta - z)^{-1} d\mu(\zeta) - \int_{\mathbb{C}\setminus 2B} (\zeta - w)^{-1} d\mu(\zeta) \right|^p \mathrm{dm}(z). \tag{3.16}$$

Let us apply an analogue of the Minkowiski inequality (see [HLP]) to estimate the second summand in (3.16).

$$\int_B \left| \int_{\mathbb{C}\setminus 2B} (\zeta - z)^{-1} d\mu(\zeta) - \int_{\mathbb{C}\setminus 2B} (\zeta - w)^{-1} d\mu(\zeta) \right|^p \mathrm{dm}(z) \leq$$

$$\int_B \left| \int_{\mathbb{C}\setminus 2B} |(\zeta - z)^{-1} - (\zeta - w)^{-1}| d|\mu|(\zeta) \right|^p dm(z) \leq$$

$$\left(\int_{\mathbb{C}\setminus 2B} \left[\int_B |(\zeta - z)^{-1} - (\zeta - w)^{-1}|^p dm(z) \right]^{1/p} d|\mu|(\zeta) \right)^p =$$

$$\left(\int_{\mathbb{C}\setminus 2B} \left[\int_B |(z - w)\left((\zeta - z)(\zeta - w)\right)^{-1}|^p dm(z) \right]^{1/p} d|\mu|(\zeta) \right)^p. \qquad 3.17$$

Let $z = \zeta + \varrho e^{i\vartheta}$ with $\mathrm{Arg}(\zeta - w) = 0$. Then $2r \leq w$ and $r + \varrho \leq 2r + w$. Now we may dominate the integral inside the bracket in (3.17) by

$$\int_B |r(\varrho e^{i\vartheta}|\zeta - w|)^{-1}|^p \varrho d\varrho d\vartheta \leq r^p \int_B |\zeta - w|^{2-p} |\zeta - w|^{-2} \varrho^{1-p} d\varrho d\vartheta$$

$$\leq \alpha^{-p} r^p \int_B (r + \varrho)^{2-p} |\zeta - w|^{-2} \varrho^{1-p} d\varrho d\vartheta \leq$$

$$r^p |\zeta - w|^{-2} \int_{w-r}^{w+r} \int_{-\arcsin(r/w)}^{\arcsin(r/w)} (r + \varrho)^{2-p} \varrho^{1-p} d\varrho d\vartheta \qquad 3.18$$

Observe that $0 < \arcsin(r/w) \leq (r/w) \leq \pi/2$ and $1 \leq p < 2$. Then

$$(r + \varrho)^{2-p} \varrho^{1-p} (r/w) \leq (2r + w)^{2-p} (r + w)^{1-p} (r/w) \leq$$

$$2[r^{2-p}(r + w)^{1-p} + (r + w)^{3-2p}](r/w) \leq$$

$$2[r^{2-p} w^{1-p} (1 + r/w)^{1-p} + w^{3-2p}(1 + r/w)^{3-2p}](r/w) \leq C r^{3-2p}$$

since $1 \leq 1 + r/w \leq 1 + 2^{-1}$. Then the quantity inside the parenthesis in (3.17) is dominated by

$$C c r^{\frac{4-p}{p}} \int_{\mathbb{C}\setminus 2B} |\zeta - w|^{-2/p} d|\mu|(\zeta) \qquad 3.19$$

The function $|\mu|(B_\rho(w))$ is monotone increasing in ρ and bounded by $|\mu|(\mathrm{E})$. It follows that

$$d|\mu|(B_\rho(w)) = f^* d\rho + \nu$$

for some $f^* \in L^\infty([0, \infty])$ and a measure $\nu \perp d\rho$ supported on $\{\rho_j\}$ for $j = 1, 2, 3, \ldots$ with $\rho_j \neq 0$.

Then for values of $r \neq \rho_j$ and for all $\delta' < \delta$ we have

$$\int_{\mathbb{C}\setminus 2B} |\zeta - w|^{-2/p} d|\mu|(\zeta) = \int_{2r}^{\infty} \rho^{-2/p} d|\mu|(B_\rho(w))$$

$$= \rho^{-2/p}|\mu|(B_\rho(w))\Big|_{2r}^{\infty} + (2/p)\int_{2r}^{\infty}\rho^{-(2+p)/p}|\mu|(B_\rho(w))d\rho \le$$

$$-(2r)^{-2/p}|\mu|(B_{2r}(w)) + \int_{2r}^{\infty} 2c\varphi(\rho)\rho^{-(2+p)/p}d\rho \le$$

$$\int_{2r}^{\infty} 2c\varphi(\rho)\rho^{\delta'-2/p}\frac{d\rho}{\rho^{1+\delta'}} \le 2c\varphi(2r)(2r)^{\delta'-2/p}\int_{2r}^{\infty}\frac{d\rho}{\rho^{1+\delta'}} =$$

$$2c\varphi(2r)(2r)^{[\delta'+2(p-1)/p]-2}\int_{2r}^{\infty}\frac{d\rho}{\rho^{1+\delta'}} = 2c\delta'^{-1}\varphi(2r)(2r)^{-2/p}, \tag{3.20}$$

is verified if $1 \le p < 2$ and $2(p-1)/p < \delta \le 2$.

Combining (3.19) and (3.20) we estimate the second summand in (3.16) as

$$\int_B\left|\int_{\mathbb{C}\setminus 2B}(\zeta - z)^{-1}d\mu(\zeta) - \int_{\mathbb{C}\setminus 2B}(\zeta - w)^{-1}d\mu(\zeta)\right|^p \mathrm{dm}(z) \le Cr^{2-p}c^p b^p\varphi^p(r).$$

Consequently

$$\int_B|C(\mu)(z) - C(\mu\lfloor\mathbb{C}\setminus 2B)(w)|^p\mathrm{dm}(z) \le \frac{2\pi}{2-p}r^{2-p}|\mu|^p(2B) + Cr^{2-p}c^p b^p\varphi^p(r)$$

$$\le C(p)\,\|\mu\|^p_{\mathrm{M}^\varphi(\mathrm{E})}\,r^{2-p}\,\varphi^p(r)$$

for all $r \neq \rho_j$, $j = 1, 2, 3, \ldots$. By density it is also true for all $r > 0$.

If $\delta = 2(p-1)/p$, we proceed as before until we reach (3.19). Assume that $\mathrm{diam}(\mathrm{supp}\mu) \le R$ for some $R > 0$ and denote by d the distance from w to $\mathrm{supp}\mu$. If $2r > d + R$ then

$$\int_{\mathbb{C}\setminus 2B}|\zeta - w|^{-2/p}d|\mu|(\zeta) = 0,$$

otherwise

$$\int_{\mathbb{C}\setminus 2B}|\zeta - w|^{-2/p}d|\mu|(\zeta) = \int_{\max\{2r,d\}}^{d+R}\rho^{-2}d|\mu|(B_\rho(w))$$

$$= \rho^{-2/p}|\mu|(B_\rho(w))\Big|_{\max\{2r,d\}}^{d+R} + \int_{\max\{2r,d\}}^{d+R} 2\rho^{-(2+p)/p}|\mu|(B_\rho(w))d\rho \le$$

$$\int_{\max\{2r,d\}}^{d+R} 2c\varphi(\rho)\rho^{-(2+p)/p}d\rho \le 2c\varphi(2r)(2r)^{[2(p-1)/p]-2}\int_{\max\{2r,d\}}^{d+R}\frac{d\rho}{\rho} \le$$

$$2c\varphi(2r)(2r)^{-2/p}\log\left([2r]^{-1}R\right) \le 2cb\varphi(r)r^{-2/p}\log\left(r^{-1}R\right)$$

Then

$$\int_B|C(\mu)(z) - C(\mu\lfloor\mathbb{C}\setminus 2B)(w)|^p\mathrm{dm}(z) \le Cr^{2-p}\,\|\mu\|^p_{\mathrm{M}^\varphi}\,\log(r^{-1}R)\varphi^p(r)$$

This implies that $C(\mu) \in \mathrm{BMO}^{\psi,p}(\mathbb{C})$ and $\|C(\mu)\|_{\mathrm{BMO}^{\psi,p}} \leq C(p)\|\mu\|_{M^{\varphi}}$, where $\psi(r) = \varphi(r)$ if $2(p-1)/p < \delta \leq 2$ and $\psi(r) = \log(r^{-1}R)^{1/p}\varphi(r)$ if $\delta = 2(p-1)/p$ ∎

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